AN-SPS: Adaptive Sample Size Nonmonotone Line Search Spectral Projected Subgradient Method for Convex Constrained Optimization Problems

Nataša Krklec Jerinkić[*], Tijana Ostojić [†]

August 22, 2022

Abstract

We consider convex optimization problems with possibly nonsmooth objective function in the form of mathematical expectation. The proposed framework (AN-SPS) employs Sample Average Approximations (SAA) to approximate the objective function which is either unavailable or too costly to compute. The sample size is chosen in an adaptive manner which eventually pushes SAA error to zero almost surely (a.s.). The search direction is based on a scaled subgradient and a spectral coefficient, both related to the SAA function. The step size is obtained via nonmonotone line search over a predefined interval which yields theoretically sound, but practically faster algorithm. The method retains feasibility by projecting the resulting points onto the feasible set. The a.s. convergence of AN-SPS method is proved without the assumptions of bounded feasible set or bounded iterates. Preliminary numerical results on Hinge loss problems reveal the advantages of the proposed adaptive scheme. Moreover, a study of different nonmonotone line search strategies combined with different spectral coefficients within AN-SPS framework is also conducted, yielding some hints for the future work.

Key words: Nonsmooth Optimization, Spectral Projected Gradient, Sample Average Approximation, Adaptive Variable Sample Size Strategies, Non-monotone Line Search.

---

[*]Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia. e-mail: `natasa.krklec@dmi.uns.ac.rs`

[†]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia. e-mail: `tijana.ostojic@uns.ac.rs`

1

1 Introduction

The problem. We consider convex constrained optimization problem with the objective function in the form of mathematical expectation, i.e.,

$$\min_{x \in \Omega} f(x) = E(F(x, \xi)), \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a convex set, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and convex function with respect to x, and $\xi : \mathcal{A} \to \mathbb{R}^m$ is random vector on a probability space $(\mathcal{A}, \mathcal{F}, P)$. We assume that it is possible to find the exact projection onto feasible set, so a typical representative of Ω would be n-dimensional ball, nonegativity constraints, or generic box constraints. We do not impose smoothness of F, so we are dealing with nondifferentiable functions F in general. This framework covers many important optimization problems, [8], [31], [32], [39], such as Hinge loss within machine learning framework. Moreover, it is known that general constrained optimization problems may be solved through penalty methods where the relevant subproblems are often transformed into nonnegativity constrained problems by introducing the slack variables, or semi-smooth unconstrained problems. Both cases fall into the framework that we consider provided that the objective function is convex.

Variable sample size schemes. The objective function in (1) is usually unavailable or too costly to be evaluated directly [36]. For instance, there are many applications where the analytical form of the mathematical expectation can not be attained. Moreover, there are also online training problems (e.g. optimization problems that come from time series analysis) where the sample size grows as the time goes by. However, even if sample size is finite and we are dealing with finite sum problem, working with the full sample throughout the whole optimization process is usually too costly or even unnecessary. This is the reason why Variable Sample Size (VSS) schemes have been developed over the past few decades overlapping with the Big Data era [2], [3], [13], [15], [21], [25], [28], [30], to name just a few. The idea is to work with Sample Average Approximation (SAA) functions

$$f_{\mathcal{N}}(x) = \frac{1}{N} \sum_{i \in \mathcal{N}} f_i(x), \tag{2}$$

where $f_i(x) = F(x, \xi_i)$ and $\xi_i, i = 1, 2, ...$ are usually assumed to be independent and identically distributed (i.i.d.) [36]. The sample size $N = |\mathcal{N}|$ is varied across the iterations, allowing cheaper approximations whenever possible.

Nonmonotone line search. Line search methods are known as a powerful tool in classical optimization, especially in smooth deterministic case. They provide global convergence with good practical performance. However, in stochastic nonsmooth framework, it is very hard to analyze them.

In the stochastic case, line search yields biased estimators which complicates the classical analysis and seeks for alternative approaches [9], [16], [20], [29], [33]. In the nonsmooth framework, even if strong convexity holds, lower bounding the step size is very hard. In [20], the steps are bounded from below, but not uniformly since they depend on the tolerance parameter. On the other hand, predefined step size sequence such as the harmonic one is enough to guaranty the convergence under the standard assumptions [6], [19], even in the mini-batch or SA (Stochastic Approximation) framework [35], [38]. Unfortunately, this choice usually yields very slow convergence in practice [6]. SPS framework [24] proposes combination of the line search and predefined sequence by performing the line search on predefined intervals, keeping the method both fast and theoretically sound.

Classical Armijo line search needs descent direction in order to be well defined. While in smooth optimization it is easy to determine it, in the nonsmooth case it is a much more challenging task [20], [42]. Moreover, allowing more freedom to the step size selection may be beneficial, epsecialy when the search directions are of spectral type [5], [24], [27]. Finally, having in mind that the VSS schemes work with approximate functions, nonmonotone line search seems like a reasonable choice in this setup.

Spectral coefficients. Although the problem is not smooth, including some second order information seems to be beneficial according to the existing works [23], [26], [42]. Moreover, spectral-like methods proved to be efficient in the stochastic framework with increasing accuracy [4], [22]. We present a framework that allows different spectral coefficients to be combined with subgradient directions. Following [10], we test different choices of Barzilai-Borwein (BB) rules in stochastic environment.

AN-SPS algorithm. Within this paper, we propose AN-SPS framework - Adaptive sample size Nonmonotone line search Spectral Projected Subgradient method. It assumes subgradient directions, not necessarily descent, which may be combined with spectral coefficients. Both subgradients and spectral coefficients are calculated by employing SAA functions that vary across the iterations in general. In this setup, it is good to have a safeguard for the spectral coefficients to make sure that the resulting coefficients are positive and bounded. We also allow different nonmonotone line search rules, although the method's construction allows monotone rule as well. The step size follows the idea of the SPS framework - line search over predefined intervals.

One of the key points lies in the adaptive sample size strategy. Roughly speaking, the main idea is to balance two types of errors - the one that measures how far is the iterate from the current SAA function's constrained optimum and the one that estimates the SAA error. More precisely, we present an adaptive strategy which determines when to switch to the next level of accuracy and prove that this pushes the sample size to infinity (or to the full sample size in finite sum case). In the SPS framework, the con-

vergence result was proved under the assumption of the sample size increase at each iteration, while in AN-SPS this is a consequence of the algorithm's construction rather than the assumption.

We believe than one more important advantage with respect to SPS is the proposed scaling of the subgradient direction. The scaling strategy is not new in general [7], but it is a novelty within the SPS framework. One of the most important consequences of this modification is that the convergence result is proved without boundedness assumptions - we do not impose any assumption on uniformly bounded subgradients, feasible set, nor the iterates. Instead, we prove that AN-SPS provides bounded sequence of iterates under a mild sample size growth condition.

The main result - almost sure convergence of the whole sequence of iterates - is proved under rather standard conditions for stochastic analysis. Moreover, in the finite sum problems case, the convergence is deterministic and it comes solely under the convexity assumption and bounded local cost functions from below.

Preliminary numerical tests on Hinge loss problems and common data sets for machine learning show the advantages of the proposed adaptive VSS strategy. We also present the results of a study that investigates how different spectral coefficients combine with different nonmonotone rules.

Contributions. This paper may be seen as a continuation of the work presented in [24] and further development of algorithm LS-SPS (Line Search Spectral Projected Subgradient Method for Nonsmooth Optimization) proposed therein. In this light, the main contributions of this work are the following:

i) Adaptive sample size strategy is proposed and we prove that this strategy pushes the sample size to infinity (or to the maximal sample size for finite sum case);

ii) We show that the scaling can relax the boundednes assumptions on subgradients, iterates and feasible set;

iii) The LS-SPS is generalized in a sense that we allow different nonmonotone line search rules. Although important for practical behavior of the algorithm, this change does not effect the convergence analysis and it is investigated mainly through numerical experiments;

iv) Considering the spectral coefficients, we investigate different strategies for its formulation [10] in stochastic framework. Different combinations of spectral coefficient and nonmonotone rules are evaluated within numerical experiments conducted on machine learning Hinge loss problems.

Paper organization. The algorithm is presented in Section 2. Convergence analysis is conducted in Section 3, while preliminary numerical results

4

are reported in Section 4. Section 5 is devoted to the conclusions and some proofs are delegated to the Appendix.

2   The Method

In this section, we state the proposed AN-SPS framework algorithm. The sample used to approximate the objective function via (2) at iteration k is denoted by $\mathcal{N}_k$, while N_k denotes its cardinality. We also set the SAA error measure to $h(N) = 1/N$ for unbounded sample size case and $h(N) = (N_{max} - N)/N_{max}$ for the case where the full sample size is N_{max}. Other choices are eligible as well, but we keep these ones for simplicity. Let us denote by $P_\Omega(z)$ the exact, orthogonal projection of $z \in \mathbb{R}^n$ onto Ω. Furthermore, we define the upper bound of predefined interval for the line search by $\bar{\alpha}_k = \min\{1, C_2/k\}$, where $C_2 > 0$ is can be arbitrary large.

Algorithm 1: AN-SPS (**A**daptive Samle Size **N**onmonotone Line Search **S**pectral **P**rojected **S**ubgradient Method)

- S0 *Initialization.* Given $N_0, m \in \mathbb{N}$, $x_0 \in \Omega$, $C_2 > 0$, $0 < \underline{\zeta} \leq \overline{\zeta} < \infty$, $\zeta_0 \in \left[\underline{\zeta}, \overline{\zeta}\right]$. Set $k = 0$ and $F_0 = f_{\mathcal{N}_0}(x_0)$

- S1 *Search direction.* Choose $\bar{g}_k \in \partial f_{\mathcal{N}_k}(x_k)$. Set $q_k = \max\{1, \|\bar{g}_k\|\}$, $v_k = \bar{g}_k/q_k$ and $p_k = -\zeta_k v_k$.

- S2 *Step size.* Choose finitely many points $\{\tilde{\alpha}_k^1, ..., \tilde{\alpha}_k^m\}$ from the interval $(\frac{1}{k}, \bar{\alpha}_k]$ with $\tilde{\alpha}_k^m = \bar{\alpha}_k$. If the condition

$$f_{\mathcal{N}_k}(x_k + \tilde{\alpha}_k^j p_k) \leq F_k - \eta \tilde{\alpha}_k^j \|p_k\|^2 \tag{3}$$

  is satisfied for some $j \in \{1, ..., m\}$, set $\alpha_k = \tilde{\alpha}_k^{j_{max}}$ where $\tilde{\alpha}_k^{j_{max}}$ is the largest point that satisfies (3). Else, set $\alpha_k = \frac{1}{k}$.

- S3 *Main update.* Set $z_{k+1} = x_k + \alpha_k p_k$, $x_{k+1} = P_\Omega(z_{k+1})$, $s_k = x_{k+1} - x_k$ and $\theta_k = \|s_k\|$.

- S4 *Spectral coefficient update.* Determine $\zeta_{k+1} \in [\underline{\zeta}, \bar{\zeta}]$.

- S5 *Sample size update.* If $\theta_k < h(N_k)$, choose $N_{k+1} > N_k$ and a new sample $\mathcal{N}_{k+1}$. Else, $\mathcal{N}_{k+1} = \mathcal{N}_k$.

- S6 *Nonmonotone line search update.* Determine $F_{k+1} \geq f_{\mathcal{N}_{k+1}}(x_{k+1})$.

- S7 *Iteration update.* Set $k := k + 1$ and go to S1.

First, notice that the initialization and step S3 ensure feasibility of the iterates. In step S1, we choose an arbitrary subgradient of the current approximation function $f_{\mathcal{N}_k}$ at point x_k. Further, scaling with q_k implies that $\|v_k\| \leq 1$. Moreover, boundedness of the spectral coefficient ζ_k yields

5

uniformly bounded search directions p_k. This is very important from the theoretical point of view since it helps us overcome boundeness assumptions mentioned in Introduction.

For the step size selection, we practically use a backtracking-type procedure over the predefined interval $(\frac{1}{k}, \bar{\alpha}_k]$. Notice that C_2 can be arbitrary large, so that in practice $\bar{\alpha}_k$ is equal to 1 in most of the iterations. However, the upper bound C_2/k is needed to ensure theoretical convergence result. The lower bound, $1/k$, is known as a good choice from the theoretical point of view, and often bad choice in practice. Thus, roughly speaking, the line search checks if larger, but still theoretically sound steps are eligible. Since the backtracking checks the Armijo-like condition (3) in at most m points over the interval, the procedure is well defined since if the backtracking fails, the step size is set to $1/k$. This allows us to use nondescent directions and practically arbitrary nonmonotone (or monotone) rule determined by the choice of F_k. For instance, F_k can be set to $f_{\mathcal{N}_k}(x_k) + 0.5^k$, but various other choices are possible as well. The choice of nonmonotone rule does not effect the theoretical convergence of the algorithm, but it can be very important in practice as we will show in Numerical results section.

We will test the performance of some choices for the spectral coefficients, where from theoretical point of view the only requirement is the safeguard stated in the Step S4 of the algorithm - ζ_k must remain within positive, bounded interval $[\underline{\zeta}, \bar{\zeta}]$.

Finally, adaptive sample size strategy is determined within step S5. The overall step length θ_k may be considered as a measure of stationarity related to the current objective function approximation $f_{\mathcal{N}_k}$. In particular, we will show that, if the sample size is fixed, θ_k tends to zero and the sequence of iterates is approaching a minimizer of the current SAA function (see the proof of Theorem 3.1 in the sequel). When θ_k is relatively small (smaller then the measure of SAA error $h(N_k)$), we decide that the two errors are in balance and that we should improve the level of accuracy by enlarging the sample. Notice that step S5 allows completely different sample $\mathcal{N}_{k+1}$ in general with respect to $\mathcal{N}_k$ in the case where the sample size is increased. However, if the sample size is unchanged, the sample is unchanged, i.e., $\mathcal{N}_{k+1} = \mathcal{N}_k$, which allows non-cumulative samples to fit within the proposed framework as well.

AN-SPS algorithm detects the iteration within which the sample size needs to be increased, but it allows a full freedom in the choice of the subsequent sample size as long as it is larger than the current one. After some preliminary tests, we end up with the following selection: when the sample size is increased, it is done as

$$N_{k+1} = \lceil \max\{(1 + \theta_k)N_k, 1.1N_k\} \rceil. \tag{4}$$

Although some other choices such as direct balancing of θ_k and $h(N_{k+1})$ seemed more intuitive, they were all outperformed by the choice (4). Dis-

regarding the safeguard part where, in case of $\theta_k = 0$, the sample size is increased by 10%, the relation becomes

$$1 + \frac{N_{k+1} - N_k}{N_k} \approx 1 + \theta_k.$$

Thus, the relative increase of the sample size is balanced with the stationarity measure. Furthermore, since we know that in these iterations $\theta_k < h(N_k)$, we obtain that the relative increase is bounded above by $h(N_k)$. Apparently, this helps the algorithm to overcome the problems caused by non-beneficiary fast growth of the sample size.

3 Convergence analysis

This section is devoted to convergence analysis of the proposed method. One of the main results lies in Theorem 3.1 where we prove that $h(N_k)$ tends to zero. This means that the sample size tends to infinity in unbounded sample case, while in finite sum case it means that the full sample is eventually reached. After that, we show that we can relax the common assumption of uniformly bounded subgradients stated in the convergence analysis in [24]. Normalized subgradients have been used in the literature, but they represent a novelty with respect to SPS framework. Hence, we need to show that this kind of scaling does not deteriorate the relevant convergence results. We state the boundedness of iterates within Proposition 3.2. Although the convergence result stated in Theorem 3.3 mainly follows from the analysis of SPS [24] (see Theorem 3.1 therein), we provide the proof in the Appendix since it is based on different foundations. Therefore, we show that AN-SPS retains almost sure convergence under relaxed assumptions with respect to LS-SPS proposed in [24], while, on the other hand, it brings more freedom to the choice of nonmonotone line search and the spectral coefficient. Finally, we formalize the conditions needed for the convergence in the finite sum case within Theorem 3.4. We start the analysis by stating the conditions on the function under the expectation in problem (1).

Assumption A 1. *Assume that $F(\cdot, \xi)$ is continuous, convex and bounded from below on Ω with a constant C for any given ξ.*

Notice that this assumption implies that all sample functions f_{N_k} are also convex, continuous and bounded from below with C. We state the first main result below.

Theorem 3.1. *Suppose that Assumption A1 holds and that Ω is closed and convex. Then the sequence $\{N_k\}_{k \in \mathbb{N}}$ generated by AN-SPS satisfies*

$$\lim_{k \to \infty} h(N_k) = 0. \tag{5}$$

Proof. First we show that retaining the same sample pushes θ_k to zero[1]. Assume that $N_k = N$ for all $k \geq \tilde{k}$ and some $N < \infty$, $\tilde{k} \in \mathbb{N}$. According to step S5 of AN-SPS algorithm, this means that $\mathcal{N}_k = \mathcal{N}_{\tilde{k}} := \mathcal{N}$ for all $k \geq \tilde{k}$. Let us show that this implies boundedness of $\{x_k\}_{k \in \mathbb{N}}$. Notice that for all k the step size and the search direction are bounded, more precisely, $\alpha_k \leq \bar{\alpha}_k \leq 1$ and

$$\|p_k\| = \|\zeta_k v_k\| \leq \bar{\zeta}\|v_k\| \leq \bar{\zeta}.$$

Thus, the $\tilde{k}$ initial iterates must be bounded, i.e., there must exist $C_{\tilde{k}}$ such that $\|x_k\| \leq C_{\tilde{k}}$ for all $k = 0, 1, ..., \tilde{k}$. Now, let us observe the remaining sequence of iterates, i.e., $\{x_{\tilde{k}+j}\}_{j \in \mathbb{N}}$. Let $x_{\mathcal{N}}^*$ be an arbitrary solution of the problem $\min_{x \in \Omega} f_{\mathcal{N}}(x)$. Notice that the convexity of $f_{\mathcal{N}}$ and the fact that $\bar{g}_k \in \partial f_{\mathcal{N}}(x_k)$ for all $k \geq \tilde{k}$ and $x \in \mathbb{R}^n$ imply

$$-\bar{g}_k^T (x_k - x) \leq f_{\mathcal{N}}(x) - f_{\mathcal{N}}(x_k).$$

Then, by using nonexpansivity of the projection operator and the fact that $x_{\mathcal{N}}^* \in \Omega$, for all $k \geq \tilde{k}$ we obtain

$$
\begin{aligned}
\|x_{k+1} - x_{\mathcal{N}}^*\|^2 &= \|P_\Omega(z_{k+1}) - P_\Omega(x_{\mathcal{N}}^*)\|^2 \qquad (6)\\
&\leq \|z_{k+1} - x_{\mathcal{N}}^*\|^2 = \|x_k - \alpha_k \zeta_k v_k - x_{\mathcal{N}}^*\|^2\\
&= \|x_k - x_{\mathcal{N}}^*\|^2 - 2\alpha_k \zeta_k \frac{1}{q_k} \bar{g}_k^T (x_k - x_{\mathcal{N}}^*) + \alpha_k^2 \zeta_k^2 \|v_k\|^2\\
&\leq \|x_k - x_{\mathcal{N}}^*\|^2 + 2\alpha_k \frac{\zeta_k}{q_k}(f_{\mathcal{N}_k}(x_{\mathcal{N}}^*) - f_{\mathcal{N}_k}(x_k)) + \alpha_k^2 \bar{\zeta}^2\\
&\leq \|x_k - x_{\mathcal{N}}^*\|^2 + \alpha_k^2 \bar{\zeta}^2.
\end{aligned}
$$

In the last inequality we use the fact that $\mathcal{N}_k = \mathcal{N}$ for all $k \geq \tilde{k}$. Thus,

$$f_{\mathcal{N}_k}(x_{\mathcal{N}}^*) - f_{\mathcal{N}_k}(x_k) = f_{\mathcal{N}}(x_{\mathcal{N}}^*) - f_{\mathcal{N}}(x_k) \leq 0$$

and since $\alpha_k \zeta_k / q_k > 0$ we obtain the result. Furthermore, by using the induction argument, we obtain that for every $p \in \mathbb{N}$ there holds

$$
\begin{aligned}
\|x_{\tilde{k}+p} - x_{\mathcal{N}}^*\|^2 &\leq \|x_{\tilde{k}} - x_{\mathcal{N}}^*\|^2 + \bar{\zeta}^2 \sum_{j=0}^{p-1} \alpha_{\tilde{k}+j}^2 \leq \|x_{\tilde{k}} - x_{\mathcal{N}}^*\|^2 + \bar{\zeta}^2 \sum_{j=0}^{\infty} \alpha_j^2\\
&\leq \|x_{\tilde{k}} - x_{\mathcal{N}}^*\|^2 + \bar{\zeta}^2 C_2^2 \sum_{j=0}^{\infty} \frac{1}{k^2} := \bar{C}_{\tilde{k}} < \infty.
\end{aligned}
$$

Thus, we conclude that the sequence of iterates must be bounded, i.e., there exists a compact set $\bar{\Omega} \subseteq \Omega$ such that $\{x_k\}_{k \in \mathbb{N}} \subseteq \bar{\Omega}$. Since the function $f_{\mathcal{N}}$ is convex due to Assumption A1, there follows that $f_{\mathcal{N}}$ is locally Lipschitz

---

[1]This part of the proof uses the elements of the analysis in [24], but it also brings new steps and thus we provide it in a complete form.

continuous. Moreover, it is (globally) Lipschitz continuous on the compact set $\bar{\Omega}$. Let us denote the corresponding Lipschitz constant by $L_{\bar{\Omega}}$. Then, we know that $\|g\| \leq L_{\bar{\Omega}}$ holds for any $g \in \partial f_{\mathcal{N}}(x)$ and any $x \in \bar{\Omega}$ (see for example [34] or [37]). Having in mind that $\bar{g}_k \in \partial f_{\mathcal{N}}(x_k)$ for all $k \geq \tilde{k}$, we conclude that $\|\bar{g}_k\| \leq L_{\bar{\Omega}}$ for all $k \geq \tilde{k}$. Now, we prove that

$$\liminf_{k \to \infty} f_{\mathcal{N}}(x_k) = f_{\mathcal{N}}^*, \tag{7}$$

where $f_{\mathcal{N}}^*$ is the optimal value of problem $\min_{x \in \Omega} f_{\mathcal{N}}(x)$. Suppose the contrary, i.e., there exists $\varepsilon_{\mathcal{N}} > 0$ such that for all $k \geq \tilde{k}$ there holds $f_{\mathcal{N}}(x_k) - f_{\mathcal{N}}^* \geq 2\varepsilon_{\mathcal{N}}$. Since the iterates are feasible and the function $f_{\mathcal{N}}$ is continuous, there exists $\tilde{y}_{\mathcal{N}} \in \Omega$ such that $f_{\mathcal{N}}(\tilde{y}_{\mathcal{N}}) = f_{\mathcal{N}}^* + \varepsilon_{\mathcal{N}}$. Furthermore, this implies that

$$f_{\mathcal{N}}(x_k) - f_{\mathcal{N}}(\tilde{y}_{\mathcal{N}}) = f_{\mathcal{N}}(x_k) - f_{\mathcal{N}}^* - \varepsilon_{\mathcal{N}} \geq 2\varepsilon_{\mathcal{N}} - \varepsilon_{\mathcal{N}} = \varepsilon_{\mathcal{N}},$$

and thus

$$-\bar{g}_k^T (x_k - \tilde{y}_{\mathcal{N}}) \leq f_{\mathcal{N}}(\tilde{y}_{\mathcal{N}}) - f_{\mathcal{N}}(x_k) \leq -\varepsilon_{\mathcal{N}}.$$

Following the same steps as in (6) and using the previous inequality, we conclude that for all $k \geq \tilde{k}$ there holds

$$
\begin{aligned}
||x_{k+1} - \tilde{y}_{\mathcal{N}}||^2 &\leq ||z_{k+1} - \tilde{y}_{\mathcal{N}}||^2 \tag{8} \\
&\leq ||x_k - \tilde{y}_{\mathcal{N}}||^2 - 2\alpha_k \zeta_k \frac{1}{q_k} \bar{g}_k^T (x_k - \tilde{y}_{\mathcal{N}}) + \alpha_k^2 \zeta_k^2 ||v_k||^2 \\
&\leq ||x_k - \tilde{y}_{\mathcal{N}}||^2 - 2\alpha_k \frac{\zeta_k}{q_k} \varepsilon_{\mathcal{N}} + \alpha_k^2 \bar{\zeta}^2 \\
&\leq ||x_k - \tilde{y}_{\mathcal{N}}||^2 - 2\alpha_k \frac{1}{q_k} \underline{\zeta} \varepsilon_{\mathcal{N}} + \alpha_k^2 \bar{\zeta}^2.
\end{aligned}
$$

Now, using the fact that

$$q_k = \max\{1, \|\bar{g}_k\|\} \leq \max\{1, L_{\bar{\Omega}}\} := q,$$

we conclude that for all $k \geq \tilde{k}$ there holds

$$||x_{k+1} - \tilde{y}_{\mathcal{N}}||^2 \leq ||x_k - \tilde{y}_{\mathcal{N}}||^2 - 2\alpha_k \frac{1}{q} \underline{\zeta} \varepsilon_{\mathcal{N}} + \alpha_k^2 \bar{\zeta}^2 = ||x_k - \tilde{y}_{\mathcal{N}}||^2 - \alpha_k (\frac{2}{q} \underline{\zeta} \varepsilon_{\mathcal{N}} - \alpha_k \bar{\zeta}^2).$$

Since $\alpha_k \leq C_2/k$, there holds $\lim_{k \to \infty} \alpha_k = 0$ and thus there must exist $\bar{k} \geq \tilde{k}$ such that $\alpha_k \bar{\zeta}^2 \leq \frac{1}{q} \underline{\zeta} \varepsilon_{\mathcal{N}} := \underline{\varepsilon}_{\mathcal{N}}$. Therefore, we have

$$||x_{k+1} - \tilde{y}_{\mathcal{N}}||^2 \leq ||x_k - \tilde{y}_{\mathcal{N}}||^2 - \alpha_k \underline{\varepsilon}_{\mathcal{N}}.$$

Moreover, for any $p \in \mathbb{N}$ there holds

$$||x_{\bar{k}+p} - \tilde{y}_{\mathcal{N}}||^2 \leq ||x_{\bar{k}} - \tilde{y}_{\mathcal{N}}||^2 - \underline{\varepsilon}_{\mathcal{N}} \sum_{j=0}^{p-1} \alpha_{\bar{k}+j}$$

9

and letting $p \to \infty$ we obtain the contradiction since $\sum_{k=0}^{\infty} \alpha_k \geq \sum_{k=0}^{\infty} 1/k = \infty$. Thus, we conclude that (7) must hold. Therefore there exists $K_1 \subseteq \mathbb{N}$ such that $\lim_{k \in K_1} f_{\mathcal{N}}(x_k) = f_{\mathcal{N}}^*$ and since the iterates are bounded, there exists $K_2 \subseteq K_1$ and a solution $\tilde{x}_{\mathcal{N}}^*$ of the problem $\min_{x \in \Omega} f_{\mathcal{N}}(x)$ such that

$$\lim_{k \in K_2} x_k = \tilde{x}_{\mathcal{N}}^*.$$

Now, we show that the whole sequence of iterates converges. Let $\{x_k\}_{k \in K_2} := \{x_{k_i}\}_{i \in \mathbb{N}}$. Following the steps of (6) we obtain that the following holds for any $s \in \mathbb{N}$

$$||x_{k_i+s} - \tilde{x}_{\mathcal{N}}^*||^2 \leq ||x_{k_i} - \tilde{x}_{\mathcal{N}}^*||^2 + \bar{\zeta}^2 \sum_{j=0}^{s-1} \alpha_{k_i+j}^2.$$

Since $\sum_{j=k_i}^{\infty} \alpha_j^2$ is the residual of convergent sum, for any $s \in \mathbb{N}$ there holds

$$||x_{k_i+s} - \tilde{x}_{\mathcal{N}}^*||^2 \leq a_i, \text{ where } \lim_{i \to \infty} a_i = 0.$$

Thus, for all $s \in \mathbb{N}$ we have

$$\limsup_{i \to \infty} ||x_{k_i+s} - \tilde{x}_{\mathcal{N}}^*||^2 \leq \limsup_{i \to \infty} a_i = \lim_{i \to \infty} a_i = 0.$$

Since $s \in \mathbb{N}$ is arbitrary, there follows $\limsup_{k \to \infty} ||x_k - \tilde{x}_{\mathcal{N}}^*||^2 = 0$, i.e.

$$\lim_{k \to \infty} x_k = \tilde{x}_{\mathcal{N}}^*,$$

and the step S3 of AN-SPS algorithm implies

$$\lim_{k \to \infty} \theta_k = \lim_{k \to \infty} ||s_k|| = \lim_{k \to \infty} ||x_{k+1} - x_k|| = 0.$$

This completes the first part of the proof, i.e., we have just proved that if the sample is kept fixed, the sequence $\{\theta_k\}_{k \in \mathbb{N}}$ tends to zero.

Finally, we prove the main result (5). Assume the contrary. Since the sequence $\{h(N_k)\}_{k \in \mathbb{N}}$ is nonincreasing, this means that we assume the existence of $\bar{h} > 0$ such that $h(N_k) \geq \bar{h}$ for all $k \in \mathbb{N}$. This further implies that there exist $N < N_\infty$ and $\tilde{k} \in \mathbb{N}$ such that $N_k = N$ for all $k \geq \tilde{k}$, where $N_\infty = \infty$ in unbounded sample case and N_∞ coincides with the full sample size in bounded sample (finite sum) case. Thus, according to S5 of AN-SPS algorithm, there holds that $\theta_k \geq h(N_k) = h(N) \geq \bar{h} > 0$ for all $k \geq \tilde{k}$, since we would have an increase of the sample size N otherwise. On the other hand, we have just proved that if the sample size is fixed, then $\lim_{k \to \infty} \theta_k = 0$, which is in contradiction with $\theta_k \geq \bar{h} > 0$. Thus, we conclude that $\lim_{k \to \infty} h(N_k) = 0$, which completes the proof. $\qquad\square$

Next, we analyze the conditions that provide sequence of bounded iterates generated by AN-SPS algorithm. Let us define the SAA error sequence as follows [24]

$$\bar{e}_k = |f_{\mathcal{N}_k}(x_k) - f(x_k)| + |f_{\mathcal{N}_k}(x^*) - f(x^*)|, \tag{9}$$

where x^* is an arbitrary solution of (1). The proof of the following proposition is similar to the proof of Proposition 3.4 of [24], but the conditions are relaxed since we have $N_k \to \infty$ as a consequence of the Theorem 3.1. Moreover, scaling of the subgradients relaxes the assumption of uniformly bounded $\bar{g}_k$ sequence. Although the modifications are mainly technical, we provide the proof in the Appendix for the sake of completeness. The condition (9) states the sample size growth under which we achieve bounded iterates. For instance, in cumulative sample case, $N_k = k$ is sufficient to ensure this condition. Although we believe that the condition is not too strong, it is still an assumption and not the consequence of the algorithm, so this issue remains as an open question for the future work.

Proposition 3.2. *Suppose that Ω is closed and convex, Assumption A1 holds and $\{x_k\}$ is a sequence generated by Algorithm AN-SPS. Then there exists a compact set $\bar{\Omega} \subseteq \Omega$ such that $\{x_k\} \subseteq \bar{\Omega}$ provided that*

$$\sum_{k=0}^{\infty} \bar{e}_k/k \le C_4 < \infty. \tag{10}$$

As it can be seen from the proof, $\bar{\Omega}$ stated in the previous proposition depends only on x_0 and given constants, so it can be (theoretically) determined independently of the sample path. However, since we consider unbounded sample in general, we need the following assumption.

Assumption A 2. *For every $x \in \Omega$ there exists a constant L_x such that $F(x, \xi)$ is locally Lipschitz-L_x continuous for any ξ.*

This assumption implies that each SAA function is locally Lipschitz continuous with a constant that depends only on a point x and not on a random vector ξ. In bounded sample case this is obviously satisfied under assumption A1, while in general it holds for a certain class of functions - when ξ is separable from x for instance. Next, we prove the almost sure convergence of AN-SPS algorithm under the stated assumptions. Notice that (10) does not necessary imply that $\lim_{k \to \infty} \bar{e}_k = 0$. Thus, we add a common assumption in stochastic analysis in order to ensure a.s. convergence of the sequence $\{\bar{e}_k\}_{k \in \mathbb{N}}$.

Assumption A 3. *The function F is dominated by a P-integrable function on any compact subset of $\mathbb{R}^n$.*

Under the stated assumptions, the Uniform Law of Large Numbers (ULLN) implies (Theorem 7.48 in [36])

$$\lim_{N \to \infty} \sup_{x \in S} |f_N(x) - f(x)| = 0 \quad \text{a.s.} \tag{11}$$

for any compact subset $S \subseteq \mathbb{R}^n$. This will further imply the a.s. convergence of the sequence $\bar{e}_k$. Notice that $\lim_{k \to \infty} \bar{e}_k = 0$ is satisfied in bounded sample case, as well as (10), since AN-SPS achieves the full sample eventually. In that case, the assumptions A2 and A3 are not needed for the convergence result.

Remark: The following theorem states a.s. convergence of the proposed method. Although it follows the same steps, the proof differs from the proof of Theorem 3.1 of [24] in several places. Under the stated assumptions we prove that the sample size tends to infinity and that the iterates remain within a compact set. After that, the proof follows the steps of the proof in [24] completely, except for the scaling of the subgradient in step S1 of AN-SPS algorithm. This alters the inequalities, but the Assumption A2 implies that q_k can be uniformly bounded from above and below, thus the main flow remains the same. We state the proof in the Appendix for completeness.

Theorem 3.3. *Suppose that Assumptions A1-A3 and* (10) *hold and that Ω is closed and convex. Then the sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by AN-SPS converges to a solution of problem* (1) *almost surely.*

Finally, we state the result for finite sum problem as an important class of (1)

$$\min_{x \in \Omega} \frac{1}{N} \sum_{i=1}^{N} f_i(x). \tag{12}$$

As we mentioned before, Assumption A3 is redundant in this case as well as (10) since $\bar{e}_k = 0$ for all k large enough. Moreover, Assumption A2 is also satisfied due to the fact that there are only finitely many functions f_i. At the end, notice that under Assumption A1, the full sample is eventually achieved and the proof of Theorem 3.1 also reveals that the convergence is deterministic. We summarise this in the next theorem.

Theorem 3.4. *Suppose that Assumption A1 holds and that Ω is closed and convex. Then the sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by AN-SPS converges to a solution of problem* (12).

4   Numerical results

Within this section we test the performance of AN-SPS algorithm on well known binary classification data sets listed in Table 1. The problem that we

consider is a constrained finite sum problem with L_2-regularized hinge loss local cost functions, i.e.,

$$\min_{x \in \Omega} f_N(x) := \delta ||x||^2 + \frac{1}{N} \sum_{i=1}^{N} \max\{0, 1 - z_i x^T w_i\}, \ \Omega := \{x \in \mathbb{R}^n \ : ||x||^2 \leq \frac{1}{\delta}\},$$

where $\delta = 10$ is the regularization parameter, $w_i \in \mathbb{R}^n$ are the atributes and $z_i \in \{1, -1\}$ are the corresponding labels.

	Data set	N	n
1	SPLICE [40]	3175	60
2	MUSHROOMS [18]	8124	112
3	ADULT9 [40]	32561	123
4	MNIST [41]	70000	784

Table 1: Properties of the data sets used in the experiments.

AN-SPS algorithm is implemented with the following parameters: $C_2 = 100, \eta = 10^{-4}, m = 2, N_0 = \lceil 0.1N \rceil$. The initial point x_0 is chosen randomly from Ω. We use the method proposed in [42, Algorithm 2, p. 1155] with $B_k = I$ to find a descent direction $-\bar{g}_k$ which is further scaled as in step S1 of AN-SPS algorithm, i.e., $p_k = -\zeta_k \bar{g}_k / q_k$. The sample size is updated according to step S5 of AN-SPS and (4). Recall that the sample size is increased only if $\theta_k < h(N_k)$.

We use cumulative samples, i.e., $\mathcal{N}_k \subseteq \mathcal{N}_{k+1}$ and thus, following the conclusions in [4], we calculate the spectral coefficients based on $s_k = x_{k+1} - x_k$ and the subgradient difference $y_k = \tilde{g}_k - \bar{g}_k$, where $\tilde{g}_k \in \partial f_{\mathcal{N}_k}(x_{k+1})$. This choice requires additional costs with respect to the choice of $\tilde{g}_k = \bar{g}_{k+1}$, but it diminishes the influence of the noise since the difference is calculated on the same approximate function. Furthermore, we test four different choices for the spectral coefficient (see [10] and the references therein for more details):

- Barzilai-Borwein 1 (BB1) [1]:

$$\lambda_k^{BB1} = \frac{s_k^T s_k}{s_k^T y_k}$$

- Barzilai-Borwein 2 (BB2) [1]:

$$\lambda_k^{BB2} = \frac{y_k^T s_k}{y_k^T y_k}$$

- Alternating Barzilai-Borwein (ABB) [44]:

$$\lambda_k := \begin{cases} \lambda_k^{BB2}, & \frac{\lambda_k^{BB2}}{\lambda_k^{BB1}} < 0.8, \\ \lambda_k^{BB1}, & \text{otherwise.} \end{cases}$$

13

- Alternating Barzilai-Borwein - minimum (ABBmin) [12]:

$$\lambda_k := \begin{cases} \min\{\lambda_j^{BB2} : j = \max\{1, k - m_a\}, ..., k\}, & \frac{\lambda_k^{BB2}}{\lambda_k^{BB1}} < 0.8, \\ \lambda_k^{BB1}, & \text{otherwise.} \end{cases}$$

For all the considered choices we take a safeguard

$$\zeta_k = \min\{\overline{\zeta}, \max\{\underline{\zeta}, \lambda_k\}\}, \quad \underline{\zeta} = 10^{-4}, \overline{\zeta} = 10^4.$$

Since the fixed step size such as $\alpha_k = 1/k$ was already addresed in [24] where the results show that it was clearly outperformed by the line search LS-SPS method, we focus our attention on adaptive step size rules. Regarding the nonmonotone rule, we also test four choices (see [21] and the references therein for more details):

- Maximum (MAX) [14]:

$$F_k = \max_{i \in [\max\{1, k-5\}, k]} f_{\mathcal{N}_i}(x_i),$$

- Convex combination (CCA) [43]:

$$F_k = \max\{f_{\mathcal{N}_k}(x_k), D_k\}, \; D_{k+1} = \frac{\eta_k q_k}{q_{k+1}} D_k + \frac{1}{q_{k+1}} f_{\mathcal{N}_{k+1}}(x_{k+1}),$$

$$D_0 = f_{\mathcal{N}_0}(x_0), \; q_{k+1} = \eta_k q_k + 1, \; q_0 = 1, \; \eta_k = 0.85.$$

- Monotone rule (MON):
$$F_k = f_{\mathcal{N}_k}(x_k)$$

- Additional term (ADA) [17]:

$$F_k = f_{\mathcal{N}_k}(x_k) + \frac{1}{2^k}.$$

In order to find the best combination of the strategies proposed above, we track the objective function value and plot it against FEV - number of scalar products, which serves as a measure of computational cost. All the plots are in the log scale. In the first phase of the experiments, we test AN-SPS with different combinations of spectral coefficients and nonmonotone rules, on four different data sets. The results reveal the benefits of the ADA rule in almost all cases, as it can be seen on representative graphs on MNIST data set (Figure 1). In particular, as expected, more "nonmonotonicity" usually yielded better results when combined with the spectral directions.

Furthermore, in order to see the benefits of the adaptive sample size strategy, we compare AN-SPS with: 1) heuristic (HEUR) where the sample
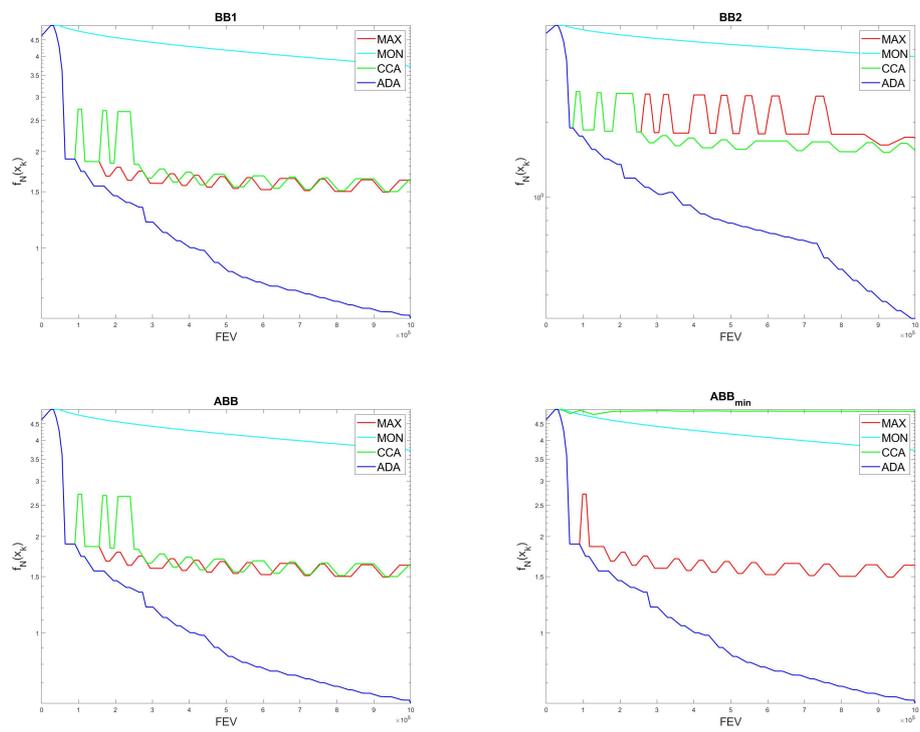
Figure 1: AN-SPS algorithm with different nonmonotone rules and spectral coefficients. Objective function value against computational cost (FEV). MNIST data set.

size is increased at each iteration by $N_{k+1} = \lceil \min\{1.1N_k, N\} \rceil$; 2) fixed sample strategy (FULL) where $N_k = N$ at each iteration. We do the same tests for the HEUR and FULL to find the best performing combinations of BB and line search rules. Finally, we compare the best performing algorithms of each sample size strategy. The results for all the considered data sets are presented in Figure 2 and they show clear advantages of the adaptive sample size strategy in terms of computational costs.
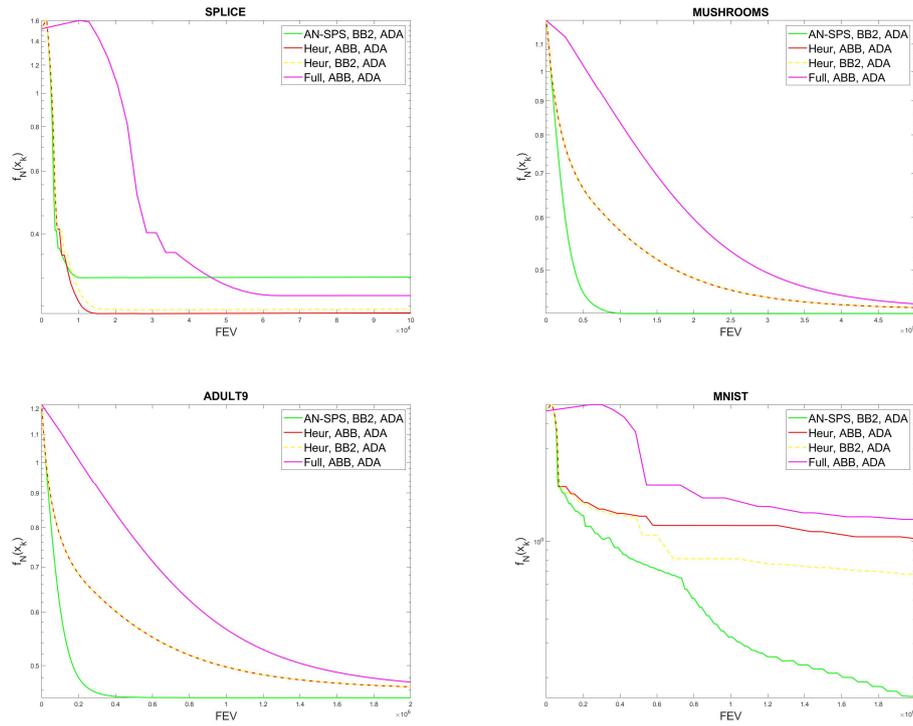

Figure 2: Comparison of the best performing combinations of spectral coefficients and nonmonotone rules of AN-SPS, HEUR and FULL sample size strategies.

5   Conclusions

We provide an adaptive sample size algorithm for constrained nonsmooth convex optimization problems where the objective function is in the form of mathematical expectation and the feasible set allows exact projections. The method allows arbitrary (negative) subgradient direction related to the SAA function which is further scaled and multiplied by the spectral coefficient. The coefficient can be defined in various ways and the only theoretical requirement is to keep it bounded away from zero and from infinity which can be accomplished by using the standard safeguard rule. The scaling is important from theoretical point of view since it helps us to avoid boundedness assumptions in the convergence analysis. We prove that the method pushes sample size to infinity and ensures that the SAA error tends to zero. On the other hand, numerical study on Hinge loss problems shows that the adaptive strategy is efficient in terms of computational costs. Moreover, we prove that the almost sure convergence towards a solution of the original problem is attained under common assumptions in stochastic environment. Furthermore, in the finite sum case, the convergence is deterministic and it is achieved under reduced assumptions (convex and continuous local cost functions bounded from below). Since the spectral coefficients are employed, we propose nonmonotone line search over the predefined intervals, although monotone line search rule is eligible from theoretical point of view. Numerical study also examines the performance of different line search rules and spectral coefficients. Preliminary results provide some hints for the future work which may include adaptive nonmonotone strategies and inexact projections.

Availability statement The datasets analysed during the current study are available in the the MNIST database of handwritten digits [41], LIBSVM Data: Classification (Binary Class) [40] and UCI Machine Learning Repository [18].

Declarations

Conflict of interest The authors declare no competing interests.

References

[1] J. Barzilai, J. M. Borwein, Two-point step size gradient method, *IMA J. Numerical Analysis, 8(1) (1988), pp. 141–148,* https://doi.org/10.1093/imanum/8.1.141

[2] F. Bastin, C. Cirillo, P.L. Toint, An adaptive Monte Carlo algorithm for computing mixed logit estimators, *Comput. Manag. Sci. 3(1) (2006), pp. 55-79, https://doi.org/10.1007/s10287-005-0044-y.*

[3] S. Bellavia, N. Krejić, N. Krklec Jerinkić, Subsampled Inexact Newton methods for minimizing large sums of convex function, *IMA Journal of Numerical Analysis 40(4) (2020), pp. 2309-2341, https://doi.org/10.1093/imanum/drz027.*

[4] S. Bellavia, N. Krklec Jerinkić, G. Malaspina, Subsampled nonmonotone spectral gradient methods, *Communications in Applied and Industrial Mathematics 11(1) (2020), pp. 19-34, DOI: 10.2478/caim-2020-0002.*

[5] E.G. Birgin, J.M. Martínez, M. Raydan, Nonmonotone Spectral Projected Gradients on Convex Sets, *SIAM Journal on Optimization. 10 (2000) pp. 1196-1211, https://doi.org/10.1137/S1052623497330963.*

[6] S. Boyd, A. Mutapcic, Stochastic subgradient methods, *Lecture Notes for EE364b, Stanford University, (2008).*

[7] J. V. Burke, A. S. Lewis, M. L. Overton, A robust gradient sampling algorithm for nonsmooth, nonconvex optimization, *SIAM Journal on Optimization 15(3) (2005), pp. 751-779, https://doi.org/10.1137/030601296.*

[8] V. Cevher, S. Becker, M. Schmidt, Convex Optimization for Big Data: Scalable, randomized, and parallel algorithms for big data analytics, *IEEE Signal Processing Magazine 31(5) (2014), pp. 32-43, DOI: 10.1109/MSP.2014.2329397.*

[9] D. di Serafino, N. Krejić, N. Krklec Jerinkić, M. Viola, LSOS: Line-search Second-Order Stochastic optimization methods for nonconvex finite sums, *arXiv preprint arXiv:2007.15966. (2020).*

[10] D. Di Serafino, V. Ruggiero, G. Toraldo, L. Zanni, On the steplength selection in gradient methods for unconstrained optimization, *Applied Mathematics and Computation 318 (2018), pp. 176-195, https://doi.org/10.1016/j.amc.2017.07.037.*

[11] E. D. Dolan, J. J. Moré, Benchmarking optimization software with performance profiles, *Math. Program., Ser. A 91 (2002), pp. 201-213, https://doi.org/10.1007/s101070100263.*

[12] G. Frassoldati , L. Zanni , G. Zanghirati, New adaptive stepsize selections in gradient methods, J. Ind. Manag. Optim. 4 (2) (2008), pp. 299–312, DOI: 10.3934/jimo.2008.4.299.

[13] M.P. FRIEDLANDER, M. SCHMIDT, Hybrid deterministic-stochastic methods for data fitting, *SIAM Journal on Scientific Computing 34(3) (2012), pp. 1380-1405, https://doi.org/10.1137/110830629.*

[14] L. GRIPPO, F. LAMPARIELLO, S. LUCIDI, A nononotone line search technique for Newton's method, *SIAM Journal on Numerical Analysis 23(4) (1986), pp. 707-716, https://doi.org/10.1137/0723046.*

[15] T. HOMEM-DE-MELLO, Variable-Sample Methods for Stochastic Optimization, *ACM Trans. Model. Comput. Simul. 13(2) (2003), pp. 108–133, https://doi.org/10.1145/858481.858483.*

[16] A. N. IUSEM, A. JOFRÉ, R. I. OLIVEIRA, P. THOMPSON, Variance-based extragradient methods with line search for stochastic variational inequalities, *SIAM Journal on Optimization 29(1) (2019), pp. 175–206, https://doi.org/10.1137/17M1144799.*

[17] D. H. LI, M. FUKUSHIMA, A derivative-free line search and global convergence of Broyden-like method for nonlinear equations, *Opt. Methods Software 13 (2000), pp. 181-201, DOI:10.1080/10556780008805782.*

[18] M. LICHMAN, UCI machine learning repository, *https://archive.ics.uci.edu/ml/index.php, (2013).*

[19] A. JALILZADEH, A. NEDIĆ, U. V. SHANBHAG, F. YOUSEFIAN, A Variable Sample-Size Stochastic Quasi-Newton Method for Smooth and Nonsmooth Stochastic Convex Optimization, *IEEE Conference on Decision and Control (CDC), Miami Beach, FL, (2018), pp. 4097-4102, doi: 10.1109/CDC.2018.8619209.*

[20] K. C. KIWIEL, Convergence of the gradient sampling algorithm for nonsmooth nonconvex optimization, *SIAM Journal on Optimization 18(2) (2007), pp. 379-388, https://doi.org/10.1137/050639673.*

[21] N. KREJIĆ, N. KRKLEC JERINKIĆ, Nonmonotone line search methods with variable sample size, *Numer. Algorithms 68(4) (2015), pp. 711-739, https://doi.org/10.1007/s11075-014-9869-1.*

[22] N. KREJIĆ, N. KRKLEC JERINKIĆ, Spectral projected gradient method for stochastic optimization, *Journal of Global Optimization, 73 (2018), pp. 59–81, https://doi.org/10.1007/s10898-018-0682-6.*

[23] N. KREJIĆ, N. KRKLEC JERINKIĆ, T. OSTOJIĆ, Minimizing Nonsmooth Convex Functions with Variable Accuracy, *arXiv:2103.13651v3, (2021).*

[24] N. KREJIĆ, N. KRKLEC JERINKIĆ, T. OSTOJIĆ, Spectral Projected Subgradient Method for Nonsmooth Convex Optimization Problems, *arXiv:2203.12681, (2022).*

[25] N. KREJIĆ, N. KRKLEC JERINKIĆ, A. ROŽNJIK, Variable sample size method for equality constrained optimization problems, *Optimization Letters, 12(3) (2018), pp. 485–497, https://doi.org/10.1007/s11590-017-1143-8.*

[26] M. LORETO, A. CREMA, Convergence analysis for the modified spectral projected subgradient method,*Optimization Letters, 9(5) (2015), pp. 915-929, https://doi.org/10.1007/s11590-014-0792-0.*

[27] M. LORETO, Y. XU, D. KOTVAL, A numerical study of applying spectral-step subgradient method for solving nonsmooth unconstrained optimization problems, *Computers & Operations Research, 104 (2019), pp. 90-97, https://doi.org/10.1016/j.cor.2018.12.006.*

[28] N. KREJIĆ, J. M. MARTINEZ, Inexact Restoration approach for minimization with inexact evaluation of the objective function, *Mathematics of Computation 85 (2016), pp. 1775-1791, https://doi.org/10.1090/mcom/3025.*

[29] N. KREJIĆ, Z. LUŽANIN, Z. OVCIN, I. STOJKOVSKA, Descent direction method with line search for unconstrained optimization in noisy environment, *Optimization Methods and Software 30(6) 2015 , pp. 1164-1184, https://doi.org/10.1080/10556788.2015.1025403.*

[30] N. KRKLEC JERINKIĆ, A. ROŽNJIK, Penalty variable sample size method for solving optimization problems with equality constraints in a form of mathematical expectation, *Numerical Algorithms, 83 (2020), pp. 701-718https://doi.org/10.1007/s11075-019-00699-6.*

[31] K. MARTI, Stochastic optimization methods, *Springer, Heidelberg, third ed. (2015), Applications in engineering and operations research, https://doi.org/10.1007/978-3-662-46214-0.*

[32] L. MARTINEZ, R. ANDRADE, E. G. BIRGIN, J. M. MARTINEZ, Packmol: A package for building initial configurations for molecular dynamics simulations, *Journal of Computational Chemistry, 30 (2009), pp. 2157-2164, https://doi.org/10.1002/jcc.21224.*

[33] C. PAQUETTE, K. SCHEINBERG, A stochastic line search method with expected complexity analysis, *SIAM Journal on Optimization 30(1) (2020), pp. 349-376, https://doi.org/10.1137/18M1216250.*

[34] B. POLYAK, Introduction to Optimization, *Optimization Software, Inc., (1987).*

[35] H. ROBBINS, D. SIEGMUND, A convergence theorem for non negative almost supermartingales and some applications, *In Optimiz-*

ing methods in statistics (1971), pp. 233-257, Academic Press, https://doi.org/10.1016/B978-0-12-604550-5.50015-8.

[36] A. SHAPIRO, D. DENTCHEVA, A. RUSZCZYNSKI, Lectures on Stochastic Programming: Modeling and Theory. *MPS-SIAM Series on Optimization (2009).*

[37] N. SHOR, Minimization Methods for Non-differentiable Functions, *Springer Series in Computational Mathematics, Springer, (1985).*

[38] J. C. SPALL, Introduction to Stochastic Search and Optimization: Estimation, Simulation, and Control, *John Wiley & Sons (2005).*

[39] D. VICARI, A. OKADA, G. RAGOZINI, C. WEIHS, EDS., Analysis and Modeling of Complex Data in Behavioral and Social Sciences, *Springer, Cham, (2014), https://doi.org/10.1007/978-3-319-06692-9.*

[40] https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/binary.html

[41] http://yann.lecun.com/exdb/mnist/

[42] J. YU, S. VISHWANATHAN, S. GUENTER, N. SCHRAUDOLPH, A Quasi-Newton Approach to Nonsmooth Convex Optimization Problems in Machine Learning, *Journal of Machine Learning Research 11 (2010), pp. 1145-1200.*

[43] H. ZHANG, W. W. HAGER, A nonmonotone line search technique and its application to unconstrained optimization, *SIAM Journal on Optimization 4 (2004), pp. 1043-1056, https://doi.org/10.1137/S1052623403428208.*

[44] B. ZHOU , L. GAO , Y.H. DAI , Gradient methods with adaptive step-sizes, *Comput. Optim. Appl. 35 (1) (2006), pp.69–86, https://doi.org/10.1007/s10589-006-6446-0.*

6   Appendix

Proof of Proposition 3.2.

Proof. Let x^* be an arbitrary solution of the problem (1). Following the

steps of (6) and the definition (9) we obtain for all $k = 0, 1, ..$

$$
\begin{aligned}
||x_{k+1} - x^*||^2 &= ||P_\Omega(z_{k+1}) - P_\Omega(x_\mathcal{N}^*)||^2 &(13)\\
&\leq ||x_k - x^*||^2 + 2\alpha_k \frac{\zeta_k}{q_k}(f_{\mathcal{N}_k}(x_\mathcal{N}^*) - f_{\mathcal{N}_k}(x_k)) + \alpha_k^2 \bar\zeta^2 \\
&\leq ||x_k - x^*||^2 + 2\alpha_k \frac{\zeta_k}{q_k}(f(x^*) - f(x_k) + \bar e_k) + \alpha_k^2 \bar\zeta^2 \\
&\leq ||x_k - x^*||^2 + 2\alpha_k \frac{\zeta_k}{q_k}(f(x^*) - f(x_k)) + 2\alpha_k \frac{\zeta_k}{q_k}\bar e_k + \alpha_k^2 \bar\zeta^2 \\
&\leq ||x_k - x^*||^2 + 2\alpha_k \bar\zeta \bar e_k + \alpha_k^2 \bar\zeta^2,
\end{aligned}
$$

where we use the fact that x_k is feasible and thus $f(x^*) - f(x_k) \leq 0$ and that $q_k \geq 1$. Further, by the induction argument and the fact that $\alpha_k \leq C_2/k$ we obtain

$$
||x_k - x^*||^2 \leq ||x_0 - x^*||^2 + 2C_2\bar\zeta \sum_{k=0}^{\infty} \frac{\bar e_k}{k} + \bar\zeta^2 \sum_{k=0}^{\infty} \frac{C_2^2}{k^2} \leq C_5 < \infty.
$$

This completes the proof. □

Proof of Theorem 3.3

Proof. First, notice that Theorem 3.1 implies that $\lim_{k\to\infty} N_k = \infty$ in ubounded sample case. Moreover, Proposition 3.2 implies that $\{x_k\} \subseteq \bar\Omega$. Furthermore, Assumption A2 implies that for any $\mathcal{N}$ we have locally Lipschitz-L_x continuous function $f_\mathcal{N}(x)$. Thus, there exists a constant L such that $f_\mathcal{N}$ is Lipschitz-L continuous on $\bar\Omega$ for any $\mathcal{N}$. This further implies that $||\bar g_k|| \leq L$ for each k and

$$
1 \leq q_k \leq \max\{1, L\} := \bar q. \tag{14}
$$

Denote by $\mathcal{W}$ the set of all possible sample paths of AN-SPS algorithm. First we prove that

$$
\liminf_{k\to\infty} f(x_k) = f^* \quad \text{a.s.} \tag{15}
$$

Suppose that $\liminf_{k\to\infty} f(x_k) = f^*$ does not happen with probability 1. In that case there exists a subset of sample paths $\tilde{\mathcal{W}} \subseteq \mathcal{W}$ such that $P(\tilde{\mathcal{W}}) > 0$ and for every $w \in \tilde{\mathcal{W}}$ there holds

$$
\liminf_{k\to\infty} f(x_k(w)) > f^*,
$$

i.e., there exists $\varepsilon(w) > 0$ small enough such that $f(x_k(w)) - f^* \geq 2\varepsilon(w)$ for all k. Since f is continuous on feasible set Ω, there exists $\tilde y(w) \in \Omega$ such that $f(\tilde y(w)) = f^* + \varepsilon(w)$. This further implies

$$
f(x_k(w)) - f(\tilde y(w)) = f(x_k(w)) - f^* - \varepsilon(w) \geq 2\varepsilon(w) - \varepsilon(w) = \varepsilon(w).
$$

Let us take an arbitrary $w \in \tilde{\mathcal{W}}$. Denote $z_{k+1}(w) := x_k(w) + \alpha_k(w)p_k(w)$. Notice that nonexpansivity of orthogonal projection and the fact that $\tilde{y} \in \Omega$ together imply

$$||x_{k+1}(w) - \tilde{y}(w)|| = ||P_\Omega(z_{k+1}(w)) - P_\Omega(\tilde{y}(w))|| \leq ||z_{k+1}(w) - \tilde{y}(w)||. \quad (16)$$

Furthermore, using (14) and the fact that $\bar{g}_k$ is subgradient of convex function $f_{\mathcal{N}_k}$, $\bar{g}_k \in \partial f_{\mathcal{N}_k}(x_k)$, we have $f_{\mathcal{N}_k}(x_k) - f_{\mathcal{N}_k}(\tilde{y}) \leq \bar{g}_k^T(x_k - \tilde{y})$ and dropping the w in order to facilitate the reading we obtain

$$
\begin{aligned}
||z_{k+1} - \tilde{y}||^2 &= ||x_k + \alpha_k p_k - \tilde{y}||^2 = ||x_k - \alpha_k \zeta_k v_k - \tilde{y}||^2 \\
&= ||x_k - \tilde{y}||^2 - 2\alpha_k \zeta_k \frac{\bar{g}_k^T}{q_k}(x_k - \tilde{y}) + \alpha_k^2 \zeta_k^2 ||v_k||^2 \\
&\leq ||x_k - \tilde{y}||^2 + 2\alpha_k \frac{\zeta_k}{q_k}(f_{\mathcal{N}_k}(\tilde{y}) - f_{\mathcal{N}_k}(x_k)) + \alpha_k^2 \zeta_k^2 \\
&\leq ||x_k - \tilde{y}||^2 + 2\alpha_k \frac{\zeta_k}{q_k}(f(\tilde{y}) - f(x_k) + 2e_k) + \alpha_k^2 \zeta_k^2 \\
&\leq ||x_k - \tilde{y}||^2 - 2\alpha_k \frac{\zeta_k}{q_k}(f(x_k) - f(\tilde{y})) + 4e_k \alpha_k \bar{\zeta} + \alpha_k^2 \bar{\zeta}^2 \\
&\leq ||x_k - \tilde{y}||^2 - 2\alpha_k \frac{\underline{\zeta}}{\bar{q}}\varepsilon + 4e_k \alpha_k \bar{\zeta} + \alpha_k^2 \bar{\zeta}^2 \\
&= ||x_k - \tilde{y}||^2 - \alpha_k \left(2\frac{\underline{\zeta}}{\bar{q}}\varepsilon - 4e_k \bar{\zeta} - \alpha_k \bar{\zeta}^2 \right). \quad (17)
\end{aligned}
$$

Since, $\{x_k\} \subseteq \bar{\Omega}$, ULLN under the stated assumptions implies $\lim_{k \to \infty} e_k(w) = 0$ for almost every $w \in \mathcal{W}$. Since $P(\tilde{\mathcal{W}}) > 0$, there must exist a sample path $\tilde{w} \in \tilde{\mathcal{W}}$ such that

$$\lim_{k \to \infty} e_k(\tilde{w}) = 0.$$

This further implies the existence of $\tilde{k}(\tilde{w}) \in \mathbb{N}$ such that for all $k \geq \tilde{k}(\tilde{w})$ we have

$$\alpha_k(\tilde{w})\bar{\zeta}^2 + 4e_k(\tilde{w})\bar{\zeta} \leq \varepsilon(\tilde{w})\frac{\underline{\zeta}}{\bar{q}} \quad (18)$$

because step S2 of AN-SPS algorithm implies that $\lim_{k \to \infty} \alpha_k = 0$ for any sample path. Furthermore, since (17) holds for all $w \in \tilde{\mathcal{W}}$ and thus for $\tilde{w}$ as well, from (16)-(18) we obtain

$$||x_{k+1}(\tilde{w}) - \tilde{y}(\tilde{w})||^2 \leq ||z_{k+1}(\tilde{w}) - \tilde{y}(\tilde{w})||^2 \leq ||x_k(\tilde{w}) - \tilde{y}(\tilde{w})||^2 - \alpha_k(\tilde{w})\varepsilon(\tilde{w})\frac{\underline{\zeta}}{\bar{q}}$$

and

$$||x_{k+s}(\tilde{w}) - \tilde{y}(\tilde{w})||^2 \leq ||x_k(\tilde{w}) - \tilde{y}(\tilde{w})||^2 - \varepsilon(\tilde{w})\frac{\underline{\zeta}}{\bar{q}}\sum_{j=0}^{s-1}\alpha_j(\tilde{w}).$$

Letting $s \to \infty$ yields a contradiction since $\sum_{k=0}^{\infty} \alpha_k \geq \sum_{k=0}^{\infty} 1/k = \infty$ for any sample path and we conclude that (15) holds.

Now, let us prove that

$$\lim_{k \to \infty} x_k = x^* \quad \text{a.s.} \tag{19}$$

Notice that (10) implies that $\sum_{k=0}^{\infty} \alpha_k e_k < \infty$ since $\alpha_k \leq C_2/k$. Since (15) holds, we know that

$$\liminf_{k \to \infty} f(x_k(w)) = f^*, \tag{20}$$

for almost every $w \in \mathcal{W}$. In other words, there exists $\overline{\mathcal{W}} \subseteq \mathcal{W}$ such that $P(\overline{\mathcal{W}}) = 1$ and (20) holds for all $w \in \overline{\mathcal{W}}$. Let us consider arbitrary $w \in \overline{\mathcal{W}}$. We will show that $\lim_{k \to \infty} x_k(w) = x^*(w) \in X^*$ which will imply the result (19). Once again let us drop w to facilitate the notation. Let $K_1 \subseteq \mathbb{N}$ be a subsequence of iterations such that

$$\lim_{k \in K_1} f(x_k) = f^*.$$

Since $\{x_k\}_{k \in K_1} \subseteq \{x_k\}_{k \in \mathbb{N}}$ and $\{x_k\}_{k \in \mathbb{N}}$ is bounded, there exist $K_2 \subseteq K_1$ and $\tilde{x}$ such that

$$\lim_{k \in K_2} x_k = \tilde{x}.$$

Then, we have

$$f^* = \lim_{k \in K_1} f(x_k) = \lim_{k \in K_2} f(x_k) = f(\lim_{k \in K_2} x_k) = f(\tilde{x}).$$

Therefore, $f(\tilde{x}) = f^*$ and we have $\tilde{x} \in X^*$. Now, we show that the whole sequence of iterates converges. Let $\{x_k\}_{k \in K_2} := \{x_{k_i}\}_{i \in \mathbb{N}}$. Following the steps of (17) and using the fact that $f(x_k) \geq f(\tilde{x})$ for all k, we obtain that the following holds for any $s \in \mathbb{N}$

$$
\begin{aligned}
||x_{k_i+s} - \tilde{x}||^2 & \leq & ||x_{k_i} - \tilde{x}||^2 + 4\overline{\zeta} \sum_{j=0}^{s-1} e_{k_i+j} \alpha_{k_i+j} + \overline{\zeta}^2 \sum_{j=0}^{s-1} \alpha_{k_i+j}^2 & (21) \\
& \leq & ||x_{k_i} - \tilde{x}||^2 + 4\overline{\zeta} \sum_{j=0}^{\infty} e_{k_i+j} \alpha_{k_i+j} + \overline{\zeta}^2 \sum_{j=0}^{\infty} \alpha_{k_i+j}^2 & (22) \\
& = & ||x_{k_i} - \tilde{x}||^2 + 4\overline{\zeta} \sum_{j=k_i}^{\infty} e_j \alpha_j + \overline{\zeta}^2 \sum_{j=k_i}^{\infty} \alpha_j^2. & (23)
\end{aligned}
$$

Due to the fact that $\sum_{j=k_i}^{\infty} e_j \alpha_j$ and $\sum_{j=k_i}^{\infty} \alpha_j^2$ are the residuals of convergent sums, for any $s \in \mathbb{N}$ there holds

$$||x_{k_i+s} - \tilde{x}||^2 \leq a_i, \quad \text{where} \quad \lim_{i \to \infty} a_i = 0.$$

Thus, for all $s \in \mathbb{N}$ we have

$$\limsup_{i \to \infty} ||x_{k_i+s} - \tilde{x}||^2 \leq \limsup_{i \to \infty} a_i = \lim_{i \to \infty} a_i = 0.$$

Since $s \in \mathbb{N}$ is arbitrary, there follows $\limsup_{k \to \infty} ||x_k - \tilde{x}||^2 = 0$, i.e. $\lim_{k \to \infty} x_k = \tilde{x}$ which completes the proof. $\qquad\square$