A note on quadratic constraints with indicator variables: Convex hull description and perspective relaxation

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Abstract

In this paper, we study the mixed-integer nonlinear set given by a separable quadratic constraint on continuous variables, where each continuous variable is controlled by an additional indicator. This set occurs pervasively in optimization problems with uncertainty and in machine learning. We show that optimization over this set is NP-hard. Despite this negative result, we discover links between the convex hull of the set under study, and a family of polyhedral sets studied in the literature. Moreover, we show that although perspective relaxation in the literature for this set fails to match the structure of its convex hull, it is guaranteed to be a close approximation.

1 Introduction

In this paper, given $Z \subseteq \{0,1\}^n$, we study set

$$X \stackrel{\text{def}}{=} \{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times Z : \|\boldsymbol{x}\|_2^2 \leq 1,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0\},$$

where $\boldsymbol{e}$ is a vector of 1s and "$\circ$" denotes the Hadamard (entry-wise) product of vectors. Set X is non-convex due to the binary constraints encoded by Z, as well as the complementarity constraints $\boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0$ linking the continuous and binary variables. Observe that arbitrary separable quadratic constraints of the form $\sum_{i=1}^n (d_i x_i)^2 \leq b$ can be modeled with X as well through the change of variables $\bar{x}_i \stackrel{\text{def}}{=} (d_i/\sqrt{b}) x_i$. Note that since any $(\boldsymbol{x}, \boldsymbol{z}) \in X$ satisfies $|x_i| \leq 1$, the complementarity constraints can be linearized as the big-M constraints

$$|x_i| \leq z_i, \quad i = 1, \ldots, n. \tag{1}$$

Our overall goal is to understand and characterize the convex hull of X, denoted as $\text{conv}(X)$. Throughout the paper, for simplicity, we use the following convention for division by 0: $a/b = 0$ if $a = b = 0$, and $a/b = \infty$ $(-\infty)$ if $b = 0$ and $a > 0$ (or $a < 0$).

1.1 Applications

Set X arises pervasively in practice. We now discuss three settings where it plays a key role.

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Sparse PCA Set X arises directly in sparse principal component analysis problems [11, 12, 17, 25], a fundamental problem in statistics which can be formulated as

$$\max \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \tag{2a}$$

$$\text{s.t. } \|\boldsymbol{x}\|_2^2 \leq 1,\ \|\boldsymbol{z}\|_1 \leq k,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0, \tag{2b}$$

where $\boldsymbol{\Sigma} \succeq 0$ and $k \in \mathbb{Z}_+$ is a parameter controlling the sparsity of the solution. Observe that the feasible region given by constraints (2b) corresponds exactly to X with set $Z = \{\boldsymbol{z} \in \{0,1\}^n : \|\boldsymbol{z}\|_1 \leq k\}$. Thus, understanding $\text{conv}(X)$ is critical to designing better convex approximations of (2).

General convex quadratic constraints Given $\boldsymbol{\Sigma} \succeq 0$, consider the system of inequalities

$$\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y} \leq b,\ \boldsymbol{y} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0,\ \boldsymbol{y} \in \mathbb{R}^n,\ \boldsymbol{z} \in Z \subseteq \{0,1\}^n. \tag{3}$$

System (3) arises for example in mean-variance optimization problems [5], where the quadratic constraint is used to impose an upper bound on the risk (variance) of the solution. While system (3) involves a non-separable quadratic constraint, a study of set $\text{conv}(X)$ can be still used to construct strong convex relaxations. Indeed, if $\boldsymbol{\Sigma} = \boldsymbol{D} + \boldsymbol{R}$ where $\boldsymbol{R} \succeq 0$, $\boldsymbol{D} \succ 0$ and diagonal, then we can reformulate system (3) by introducing additional variables $(x_0, \boldsymbol{x}) \in \mathbb{R}^{n+1}$ as

$$\sum_{i=0}^{n} x_i^2 \leq 1,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0,\ x_0(1 - z_0) = 0,\ \boldsymbol{z} \in Z \tag{4a}$$

$$z_0 = 1,\ \sqrt{(\boldsymbol{y}'(\boldsymbol{R}/b)\boldsymbol{y})} \leq x_0,\ \sqrt{(D_{ii}/b)}|y_i| \leq x_i \text{ for } i = 1, \ldots, n, \tag{4b}$$

where constraints (4a) correspond precisely to X and constraints (4b) are convex and SOCP-representable. Therefore, convex relaxations for system (3) can be obtained by strengthening constraints (4a) using $\text{conv}(X)$.

Robust optimization Consider a robust optimization problem of the form

$$\min_{\boldsymbol{y} \in Y} \max_{\boldsymbol{a} \in \mathcal{U}} \boldsymbol{a}'\boldsymbol{y}, \tag{5}$$

where vector $\boldsymbol{y}$ are the decision variables, set $Y \subseteq \mathbb{R}^n$ is the (possibly non-convex) feasible region and set $\mathcal{U} \subseteq \mathbb{R}^n$ is an uncertainty set corresponding to the objective coefficients. Robust optimization (5) is a fundamental tool to tackle decision-making under uncertainty problems. Two popular choices for the uncertainty set $\mathcal{U}$, each with its own merits and disadvantages, are: the approach of Ben-Tal and Nemirovski [7], where $\mathcal{U}$ is an ellipsoid; and the approach of Bertsimas and Sim [8], where only a small subset of the coefficients $\boldsymbol{a}$ are allowed to change while satisfying box constraints.

Thus, a natural uncertainty set inspired by the aforementioned two approaches allows few coefficients to change and imposes ellipsoidal constraint on the changing coefficients, that is, set

$$\mathcal{U} \overset{\text{def}}{=} \left\{ \boldsymbol{a} \in \mathbb{R}^n : \exists (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times \{0,1\}^n \text{ s.t. } \boldsymbol{a} = \tilde{\boldsymbol{a}} + \boldsymbol{x},\ \sum_{i=1}^{n} (d_i x_i)^2 \leq b,\ \boldsymbol{z} \in Z,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0 \right\}, \tag{6}$$

where $\tilde{\boldsymbol{a}}$ are the nominal values for the coefficients. The uncertainty set $\mathcal{U}$ is appropriate for example when changes in coefficients $\boldsymbol{a}$ are caused by rare events, and the change in the coefficients (when such changes occur) can be accurately modeled with a Gaussian distribution. A natural candidates for set Z is $Z = \{\boldsymbol{z} \in Z : \sum_{i=1}^n c_i z_i \leq k\}$, where c_i is a coefficient related to how likely the objective coefficients associated with variable y_i are likely to change. More sophisticated options of set Z can also be envisioned to capture more complex relationships on the support of perturbed coefficients.

Since set $\mathcal{U}$ is non-convex, solving (5) can be difficult and require sophisticated approaches [9]. Nonetheless, understanding $\text{conv}(X)$ may lead to the possibility of using standard duality approaches to obtain deterministic counterparts of (5). We further discuss this problem in §5.

1.2 Perspective relaxation and outline

A closely related set to X that is well understood in the literature is the mixed-integer epigraph of a separable quadratic function with indicators, that is, $X_{\text{epi}} \stackrel{\text{def}}{=} \{(\boldsymbol{x}, \boldsymbol{z}, t) \in \mathbb{R}^n \times Z \times \mathbb{R} : \|\boldsymbol{x}\|_2^2 \leq t,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0\}$. Its convex hull can be described via the *perspective relaxation* $\text{cl conv}(X_{\text{epi}}) = \{(\boldsymbol{x}, \boldsymbol{z}, t) \in \mathbb{R}^n \times \text{conv}(Z) \times \mathbb{R} : \sum_{i=1}^n x_i^2/z_i \leq t\}$, see [2, 10, 13, 15] for the case $Z = \{0,1\}^n$ and [6, 20, 21, 22, 23] for cases with more general constraints. Thus, a natural convex relaxation for set X is also given by the perspective relaxation

$$R_{\text{persp}} \stackrel{\text{def}}{=} \left\{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times \text{conv}(Z) : \sum_{i=1}^n x_i^2/z_i \leq 1\right\}. \tag{7}$$

However, it is unclear to what extent relaxation R_{persp} coincides with $\text{conv}(X)$: Are they the same? Does the structure of R_{persp} even "match" $\text{conv}(X)$? Is R_{persp} a strong relaxation? How can it be improved?

All these questions can be precisely answered for polyhedral sets: for example, an inequality is necessary for a polyhedron if it is facet-defining.However, since $\text{conv}(X)$ is in general non-polyhedral, it is unclear (to date) how to formally answer the aforementioned questions. Ideally, one would like to explicitly compute $\text{conv}(X)$ and compare it with R_{persp}. Unfortunately, as we show in §2, optimization over set X is NP-hard even when $Z = \{0,1\}^n$. Thus, an explicit computation of $\text{conv}(X)$ is unlikely. This result immediately implies that $R_{\text{persp}} \neq conv(X)$, but does not provide insights into answering the remaining questions.

In this paper, we close this gap in the literature. In §3 we characterize provide an implicit description of $\text{conv}(X)$ in the original space of variables as the intersection of a family of polyhedral sets; the description is implicit in the sense that the polyhedra are not explicitly given, but rather defined themselves as convex hulls of a particular class of mixed-integer sets. Interestingly, this family of polyhedra is well-studied in the literature. In §4 we review how to obtain facet-defining inequalities for the underlying polyhedral sets, and also show that R_{persp} corresponds to using a strong nonlinear relaxation of these polyhedral sets. In §5 we propose an approximate deterministic counterpart of the robust optimization problem (5) with discrete uncertainty (6), and in §6 we present computations with this proposed formulation.

2 NP-hardness

In this section we show that optimization of a linear function over set X is NP-hard. This result indicates that a compact explicit computation of $\text{conv}(X)$ is unlikely to be possible.

Consider the optimization problem

$$\min_{\boldsymbol{x},\boldsymbol{z}} \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} \tag{8a}$$

$$\text{s.t. } \|\boldsymbol{x}\|_2^2 \le 1 \tag{8b}$$

$$\boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0 \tag{8c}$$

$$\boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{z} \in Z. \tag{8d}$$

Proposition 1. *Problem* (8) *is NP-hard even if* $Z = \{0,1\}^n$.

Proof. Consider problem (8) where vector $\boldsymbol{z}$ is fixed, and let $S = \{i \in \{1,\dots,n\} : z_i = 1\}$ (assume $S \neq \emptyset$). Then, for this choice of $\boldsymbol{z}$, problem (8) reduces to

$$\epsilon_S = \min_{\boldsymbol{x}} \sum_{i\in S} c_i + \sum_{i\in S} a_i x_i \tag{9a}$$

$$\text{s.t. } \sum_{i\in S} x_i^2 \le 1 \tag{9b}$$

$$\boldsymbol{x} \in \mathbb{R}^S. \tag{9c}$$

Since the Lagrangian dual of problem (9) has no duality gap (as Slater condition holds), an optimal objective value ϵ_S can be computed as

$$\begin{aligned}
\epsilon_S &= \sum_{i\in S} c_i + \max_{\lambda\ge 0} \min_{\boldsymbol{x}\in\mathbb{R}^S} \sum_{i\in S} a_i x_i + \lambda \sum_{i\in S} x_i^2 - \lambda \\
&= \sum_{i\in S} c_i + \max_{\lambda\ge 0} -\frac{1}{4\lambda}\sum_{i\in S} a_i^2 - \lambda && (\because 2x_i^* = -a_i/\lambda) \\
&= \sum_{i\in S} c_i - \sqrt{\sum_{i\in S} a_i^2}. && (\because \lambda^* = \tfrac{1}{2}\sqrt{\textstyle\sum_{i\in S} a_i^2})
\end{aligned}$$

In other words, the optimal vector $\boldsymbol{z}$ of (8) can be found by either setting $\boldsymbol{z} = \boldsymbol{0}$ (with objective value $\epsilon_\emptyset = 0$), or by solving the optimization problem

$$\min_{\boldsymbol{z}\in Z} \sum_{i=1}^n c_i z_i - \sqrt{\sum_{i=1}^n a_i^2 z_i}. \tag{10}$$

Finally, as the partition problem can be reduced to problem (10) with $Z = \{0,1\}^n$ (see [1]), problem (8) is NP-hard even in this case. □

Remark 1. If $\boldsymbol{c} = \boldsymbol{0}$ but there is a constraint of the form $\|\boldsymbol{z}\|_1 = k$, then (10) can be solved by sorting. Polynomial-time solvability of this case suggests that it may be possible to construct a convex relaxation that guarantees integrality of the solutions under these conditions. In other words, it may be possible to characterize the convex hull of the set

$$Y = \left\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_0 \le k,\ \|\boldsymbol{x}\|_2^2 \le 1\right\},$$

where $\|\boldsymbol{x}\|_0 = \sum_{i=1}^n \mathbb{1}_{\{x_i\neq 0\}}$ is the cardinality of the support of $\boldsymbol{x}$. Indeed, set Y is permutation-invariant, and its convex hull $\text{conv}(Y)$ is described in [16], or projection of the perspective relaxation (i.e., $\text{conv}(Y) = \text{proj}_{\boldsymbol{x}}(R_{persp})$; see Appendix A for the detailed proof). Note however that these relaxations are not ideal for X, i.e., solutions of linear optimization problems over $\text{conv}(Y)$ do not coincide with the solutions of optimization problems over X if $\boldsymbol{c} \neq \boldsymbol{0}$. □

Remark 2. We point that ideal relaxations for X may be substantially more effective than relaxations such as Y described in Remark 1 even if $\boldsymbol{c} = \mathbf{0}$. For example, consider an optimization of the form

$$\min_{\boldsymbol{x},\boldsymbol{z}} f(\boldsymbol{x}, \boldsymbol{z}) \tag{11a}$$

$$\text{s.t. } \|\boldsymbol{x}\|_2^2 \leq 1,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0,\ \boldsymbol{e}^\top \boldsymbol{z} \leq \boldsymbol{k} \tag{11b}$$

$$\boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{z} \in \{0,1\}^n, \tag{11c}$$

where $f : \mathbb{R}^n \times [0,1]^n \to \mathbb{R} \cup \{\infty\}$ is a convex function on the extended real line. Function f can be used to capture several structural constraints of the optimization problem, e.g., setting $f(\boldsymbol{x}, \boldsymbol{z}) = 0$ if $\boldsymbol{Ax} + \boldsymbol{Gz} \leq b$ and $f(\boldsymbol{x}, \boldsymbol{z}) = \infty$ otherwise, allows for capturing arbitrary polyhedral constraints. Ideally, we would like to compute the convex hull of the whole set (which includes interplay between f and X), but realistically one instead computes relaxations for special substructures, such as the ones induces by X. In particular, given any relaxation R of X, consider the relaxation of (11) given by

$$\begin{aligned}
\min_{(\boldsymbol{x},\boldsymbol{z}) \in R} f(\boldsymbol{x}, \boldsymbol{z}) &\Leftrightarrow \min_{\boldsymbol{x^1},\boldsymbol{z^1},\boldsymbol{x^2},\boldsymbol{z^2}} f(\boldsymbol{x^1}, \boldsymbol{z^1}) \text{ s.t. } \boldsymbol{x^1} = \boldsymbol{x^2},\ \boldsymbol{z^1} = \boldsymbol{z^2},\ (\boldsymbol{x^2}, \boldsymbol{z^2}) \in R \\
&\Leftrightarrow \max_{\boldsymbol{\lambda},\boldsymbol{\mu}} \min_{\boldsymbol{x^1},\boldsymbol{z^1},\boldsymbol{x^2},\boldsymbol{z^2}} f(\boldsymbol{x^1}, \boldsymbol{z^1}) + \boldsymbol{\lambda}^\top (\boldsymbol{x^2} - \boldsymbol{x^1}) + \boldsymbol{\mu}^\top (\boldsymbol{z^2} - \boldsymbol{z^1}) \text{ s.t. } (\boldsymbol{x^2}, \boldsymbol{z^2}) \in R,
\end{aligned} \tag{12}$$

where the last equivalence assumes that strong duality holds. Note that the problem decouples in $(\boldsymbol{x^1}, \boldsymbol{z^1})$ and $(\boldsymbol{x^2}, \boldsymbol{z^2})$, and we see that the best relaxations of X are those that are tight when the objective is given by the optimal (and unknown) $\boldsymbol{\lambda^*}, \boldsymbol{\mu^*}$. Note that $\text{conv}(X)$ is thus always a good relaxation, since it is tight for any linear objective, whereas relaxations such as Y may not be tight unless $\boldsymbol{\mu^*} = \mathbf{0}$. We present in Appendix B a detailed example illustrating this point. □

3 Implicit convexification

From Proposition 1, we know that an explicit characterization of $\text{conv}(X)$ is unlikely to be possible. In this section, we settle for a weaker structural result: in Theorem 1, we state an explicit description of $\text{conv}(X)$ that relies on the convex hulls of polyhedral sets. Naturally, describing these polyhedral sets is NP-hard as well; nonetheless, they are substantially easier to handle, thanks to the maturity of polyhedral theory.

We first define the polyhedral sets that are key to characterizing $\text{conv}(X)$.

Definition 1. Given $\boldsymbol{\alpha} \in \mathbb{R}^n$, define sets $P_0(\boldsymbol{\alpha}), P(\boldsymbol{\alpha}) \subseteq \mathbb{R}^{2n}$ as

$$\begin{aligned}
P_0(\boldsymbol{\alpha}) &\stackrel{\text{def}}{=} \left\{ (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times Z : \sum_{i=1}^n |\alpha_i x_i| \leq \sqrt{\sum_{i=1}^n \alpha_i^2 z_i} \right\}, \text{ and} \\
P(\boldsymbol{\alpha}) &\stackrel{\text{def}}{=} \text{conv}\Big(P_0(\boldsymbol{\alpha})\Big).
\end{aligned}$$

Note that set $P(\boldsymbol{\alpha})$ is the convex hull of a union of a finite number of polytopes, one for each $\boldsymbol{z} \in Z$. Thus, $P(\boldsymbol{\alpha})$ is a polytope itself. We defer to §4.1-4.2 the discussion on constructing relaxations of set $P(\boldsymbol{\alpha})$. As Proposition 2 below states, set $P(\boldsymbol{\alpha})$ is a relaxation of set X.

Proposition 2 (Validity)**.** *Set conv*$(X) \subseteq P(\boldsymbol{\alpha})$ *for all* $\boldsymbol{\alpha} \in \mathbb{R}^n$.

Proof. It suffices to show that $X \subseteq P_0(\boldsymbol{\alpha})$. Since $\boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0$ and $\boldsymbol{z} \in \{0,1\}^n$, we must have $x_i = x_i z_i = x_i \sqrt{z_i}$ for all $i = 1, \dots, n$. Hence, we find that

$$\sum_{i=1}^{n} |\alpha_i x_i| = \sum_{i=1}^{k} |(\alpha_i \sqrt{z_i}) x_i| \leq \sqrt{\sum_{i=1}^{n} \alpha_i^2 z_i} \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sqrt{\sum_{i=1}^{n} \alpha_i^2 z_i},$$

where the first inequality is due to Hölder's inequality, and the second one is because of $\sum_{i=1}^{n} x_i^2 \leq 1$. Hence, $(\boldsymbol{x}, \boldsymbol{z}) \in P_0(\boldsymbol{\alpha}) \subseteq P(\boldsymbol{\alpha})$, concluding the proof. □

Moreover, we now show how to use $P(\boldsymbol{\alpha})$ to construct an equivalent convex formulation of the NP-hard problem (8). Note that in Proposition 3 below, we set $\boldsymbol{\alpha} = \boldsymbol{a}$.

Proposition 3 (Optimality)**.** *Problem* (8) *is equivalent to*

$$\min_{\boldsymbol{x},\boldsymbol{z}} \ \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} \tag{13a}$$

$$\text{s.t. } (\boldsymbol{x}, \boldsymbol{z}) \in P(\boldsymbol{a}), \tag{13b}$$

that is, they both have the same optimal objective value and there exists an optimal solution of (13) *that is also optimal for* (8)*.*

Proof. It suffices to show that problem (8) is equivalent to

$$\min_{\boldsymbol{x},\boldsymbol{z}} \ \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} \tag{14a}$$

$$\text{s.t. } (\boldsymbol{x}, \boldsymbol{z}) \in P_0(\boldsymbol{a}). \tag{14b}$$

In any feasible solution of (14), we find that

$$\boldsymbol{a}'\boldsymbol{x} \geq -\sum_{i=1}^{n} |a_i x_i| \geq -\sqrt{\sum_{i=1}^{n} a_i^2 z_i}, \tag{15}$$

where the first inequality follows directly from the definition of the absolute value and the second inequality follows from constraints $(\boldsymbol{x}, \boldsymbol{z}) \in P_0(\boldsymbol{a})$. Moreover, both inequalities (15) hold at equality in an optimal solution, since otherwise, it is always possible to increase/decrease x_i for some index i without violating feasibility while improving the objective value. Thus, projecting out variables $\boldsymbol{x}$, problem (14) reduces to (10), which, as shown in the proof of Proposition 1, is equivalent to (8). □

Propositions 2 and 3 together come with an alternative representation of $\text{conv}(X)$ which is expressed as intersections of sets $P(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n$.

Theorem 1. *The convex hull of X can be described (with an infinite number of constraints, one for each $\boldsymbol{\alpha} \in \mathbb{R}^n$) as*

$$conv(X) = \bigcap_{\boldsymbol{\alpha} \in \mathbb{R}^n} P(\boldsymbol{\alpha}) \tag{16}$$

Proof. Note that both $\text{conv}(X)$ and $\bigcap_{\boldsymbol{\alpha}\in\mathbb{R}^n} P(\boldsymbol{\alpha})$ are convex and bounded. Thus, to show that they are equivalent, we show that their extreme points coincide. In other words, it is sufficient to show that the following two optimization problems

$$\min_{\boldsymbol{x},\boldsymbol{z}} \left\{ \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} : (\boldsymbol{x},\boldsymbol{z}) \in X \right\} \tag{17}$$

$$\min_{\boldsymbol{x},\boldsymbol{z}} \left\{ \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} : (\boldsymbol{x},\boldsymbol{z}) \in \bigcap_{\boldsymbol{\alpha}\in\mathbb{R}^n} P(\boldsymbol{\alpha}) \right\} \tag{18}$$

have the same solution for any $(\boldsymbol{a},\boldsymbol{c})$. First, due to Proposition 2, we find that (18) is a relaxation of (17). Second, the problem $\min_{\boldsymbol{x},\boldsymbol{z}} \{ \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} : (\boldsymbol{x},\boldsymbol{z}) \in P(\boldsymbol{a})\}$ is a further relaxation of (18), as it is obtained by dropping all the constraints but one. Third, due to Proposition 3, this further relaxation is exact, and thus (18) is exact as well. This concludes the proof. ☐

The description (16) of $\text{conv}(X)$ can be highly nonlinear, since it involves an infinite number of constraints. However, the significance of Theorem 1 is that to understand $\text{conv}(X)$ it suffices to study the polyhedral set $P(\boldsymbol{\alpha})$, which is arguably a simpler task due to advances in polyhedral theory, and since this set does not involve complementarity or other constraints linking the discrete and continuous variables. In §4.1 and §4.2 we discuss how to obtain strong relaxation of $P(\boldsymbol{\alpha})$ in general. However, an alternative approach to obtain valid inequalities is to restrict the values of $\boldsymbol{\alpha}$, as the examples below show.

Example 1. Let $\boldsymbol{\alpha} = \boldsymbol{e}_i$ for some $i \in \{1,\ldots,n\}$, where $\boldsymbol{e}_i$ is the standard i-th basis vector of $\mathbb{R}^n$. In this case,

$$\begin{aligned} P(\boldsymbol{e}_i) &= \text{conv}\left(\{(\boldsymbol{x},\boldsymbol{z}) \in \mathbb{R}^n \times Z : |x_i| \le \sqrt{z_i}\}\right) \\ &= \{(\boldsymbol{x},\boldsymbol{z}) \in \mathbb{R}^n \times \text{conv}(Z) : |x_i| \le z_i\}. \end{aligned}$$

Thus, we find that big-M constraints (1) are derived from the partial convexification $\bigcap_{i=1}^n P(\boldsymbol{e}_i) \supseteq \text{conv}(X)$. ☐

Example 2. Suppose that $Z = \{\boldsymbol{z} \in \{0,1\}^n : \|\boldsymbol{z}\|_1 = k\}$, and let $\boldsymbol{\alpha} = \boldsymbol{e}$. In this case

$$\begin{aligned} P(\boldsymbol{e}) &= \text{conv}\left(\left\{(\boldsymbol{x},\boldsymbol{z}) \in \mathbb{R}^n \times \{0,1\}^n : \|\boldsymbol{x}\|_1 \le \sqrt{\|\boldsymbol{z}\|_1},\ \|\boldsymbol{z}\|_1 = k\right\}\right) \\ &= \left\{(\boldsymbol{x},\boldsymbol{z}) \in \mathbb{R}^n \times [0,1]^n : \|\boldsymbol{x}\|_1 \le \sqrt{k},\ \|\boldsymbol{z}\|_1 = k\right\}. \end{aligned}$$

In particular we find that the inequality $\|\boldsymbol{x}\|_1 \le \sqrt{k}$, which was studied in [12] in the context of sparse PCA, is precisely the relaxation $P(\boldsymbol{e}) \supseteq \text{conv}(X)$. ☐

4 Relaxations

This section discusses how to describe or approximate $P(\boldsymbol{\alpha})$. Interestingly, this family of polyhedra has already been studied in the literature. In §4.1 we review existing results on the facial structure of $P(\boldsymbol{\alpha})$. In §4.2 we study the natural *nonlinear* relaxation of $P(\boldsymbol{\alpha})$, show that this relaxation is guaranteed to be strong, and establish links between this relaxation and the perspective relaxation R_{persp}.

4.1 Short review of relaxations via linear inequalities

We assume in this section that $Z = \{0,1\}^n$. Given $\boldsymbol{\alpha} \in \mathbb{R}^n$, the facial structure of polyhedron $P(\boldsymbol{\alpha})$ was first studied in [1], and the results were later refined in [19]. We now review these results, as they can be used to generate valid inequalities for $\text{conv}(X)$.

Define $N \overset{\text{def}}{=} \{1,\dots,n\}$, and define the set function $g : 2^N \to \mathbb{R}$ as $g(S) = \sqrt{\sum_{i\in S}\alpha_i^2}$. Since function g is submodular, the submodular inequalities of Nemhauser et al. [18] are valid for its hypograph. In particular, letting $\rho_i(S) = g(S\cup\{i\}) - g(S)$, the inequalities

$$\sum_{i=1}^n |\alpha_i x_i| \le g(S) - \sum_{i\in S}\rho_i(S\setminus\{i\})(1-z_i) + \sum_{i\in N\setminus S}\rho_i(\emptyset)z_i \qquad \forall S\subseteq N \tag{19a}$$

$$\sum_{i=1}^n |\alpha_i x_i| \le g(S) - \sum_{i\in S}\rho_i(N\setminus\{i\})(1-z_i) + \sum_{i\in N\setminus S}\rho_i(S)z_i \qquad \forall S\subseteq N \tag{19b}$$

are valid for $P(\boldsymbol{\alpha})$. However, coefficients $\rho_i(\emptyset)$ in (19a) and $\rho_i(N\setminus\{i\})$ in (19b) are not tight. Thus, inequalities (19) are, in general, weak, and better inequalities can be obtained via lifting. Specifically, given $S\subseteq N$, the base inequality

$$\sum_{i=1}^n |\alpha_i x_i| \le g(S) - \sum_{i\in S}\rho_i(S\setminus\{i\})(1-z_i) \tag{20}$$

is facet-defining for $\text{conv}\left(\left\{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{R}^n\times\{0,1\}^n : \sum_{i=1}^n|\alpha_i x_i| \le \sqrt{\sum_{i=1}^n \alpha_i^2 z_i},\ z_i = 0\ \forall i\in N\setminus S\right\}\right)$. Inequality (20) can then be lifted into a facet-defining inequality for $P(\boldsymbol{\alpha})$ through maximal lifting. In this case, lifting is sequence independent and the resulting inequality can be obtained in closed form, see [19, Theorem 4]. Similarly, inequality (19b) can be improved through lifting, see [19, Theorem 5]. Any of the resulting valid inequalities of the form $\sum_{i=1}^n |\alpha_i x_i| \le \rho_\pi + \boldsymbol{\pi}_{\boldsymbol{\alpha}}'\boldsymbol{z}$ –where $(\rho_\alpha, \boldsymbol{\pi}_{\boldsymbol{\alpha}})\in\mathbb{R}^{n+1}$ depends on $\boldsymbol{\alpha}$– are valid for $\text{conv}(X)$ for all values of $\boldsymbol{\alpha}$, and can thus be used to improve formulations. While the inequalities discussed here are facet-defining for the case $Z=\{0,1\}^n$, they may be weaker for the case with more general constraints. Nonetheless, we point out that strong valid inequalities have also been proposed for the case where Z is defined by a knapsack constraint, see [24].

On separation: Given a relaxation $R \supset \text{conv}(X)$ – for example, one may take $R = R_{\text{persp}}$–, we show that it is possible to separate extreme points of R from $\text{conv}(X)$ by calling a separation oracle for polyhedra $P(\boldsymbol{\alpha})$. Indeed, let $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})$ be an extreme point of R. Then there exists coefficients $(\bar{\boldsymbol{a}},\bar{\boldsymbol{c}})\in\mathbb{R}^{2n}$ such that $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})$ is the unique solution to $\min_{(\boldsymbol{x},\boldsymbol{z})\in R}\bar{\boldsymbol{a}}^\top\boldsymbol{x} + \bar{\boldsymbol{c}}^\top\boldsymbol{z}$. We now show that separation of $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})$ from $\text{conv}(X)$ is equivalent to separation from $P(\bar{\boldsymbol{a}})$.

Proposition 4. *If $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})\notin conv(X)$, then $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})\notin P(\bar{\boldsymbol{a}})$.*

Observe that reverse claim, $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})\notin P(\bar{\boldsymbol{a}}) \implies (\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})\notin \text{conv}(X)$ always holds from Theorem 1. Thus, Proposition 4 precisely characterizes which polytope is needed to separate an extreme point of R.

Proof of Proposition 4. Let $(\bar{\boldsymbol{x}},\bar{\boldsymbol{z}})\notin\text{conv}(X)$, and note that

$$\bar{\boldsymbol{a}}^\top\bar{\boldsymbol{x}} + \bar{\boldsymbol{c}}^\top\bar{\boldsymbol{z}} = \min_{(\boldsymbol{x},\boldsymbol{z})\in R}\bar{\boldsymbol{a}}^\top\boldsymbol{x} + \bar{\boldsymbol{c}}^\top\boldsymbol{x} < \min_{(\boldsymbol{x},\boldsymbol{z})\in\text{conv}(X)}\bar{\boldsymbol{a}}^\top\boldsymbol{x} + \bar{\boldsymbol{c}}^\top\boldsymbol{x}. \tag{21}$$

Indeed, any optimal solution to the right hand side problem in (21) is feasible for R (since R is a relaxation); since $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}})$ is the unique minimizer of this problem over R, the strict inequality follows. Thus, we find that

$$\bar{\boldsymbol{a}}^\top \bar{\boldsymbol{x}} + \bar{\boldsymbol{c}}^\top \bar{\boldsymbol{z}} < \min_{(\boldsymbol{x},\boldsymbol{z})\in \text{conv}(X)} \bar{\boldsymbol{a}}^\top \boldsymbol{x} + \bar{\boldsymbol{c}}^\top \boldsymbol{x} = \min_{(\boldsymbol{x},\boldsymbol{z})\in X} \bar{\boldsymbol{a}}^\top \boldsymbol{x} + \bar{\boldsymbol{c}}^\top \boldsymbol{z} = \min_{(\boldsymbol{x},\boldsymbol{z})\in P(\bar{\boldsymbol{a}})} \bar{\boldsymbol{a}}^\top \boldsymbol{x} + \bar{\boldsymbol{c}}^\top \boldsymbol{z},$$

where the last equality follows from Proposition 3. In particular we find that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}) \notin P(\bar{\boldsymbol{a}})$, concluding the proof. □

4.2 Nonlinear relaxation and approximation

Consider the natural nonlinear relaxation of $P(\boldsymbol{\alpha})$, obtained by simply dropping the integrality constraints on variables $\boldsymbol{z}$:

$$C(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \left\{ (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times \text{conv}(Z) : \sum_{i=1}^n |\alpha_i x_i| \leq \sqrt{\sum_{i=1}^n \alpha_i^2 z_i} \right\}.$$

While $C(\boldsymbol{\alpha})$ is hard to compute in general as it involves computing the convex hull of the feasible region Z, it can be obtained easily for example if $Z = \{0,1\}^n$, $Z = \{z \in \{0,1\}^n : \sum_{i=1}^n z_i \leq k\}$ for some $k \in \mathbb{Z}_+$, or more generally if the constraints defining Z are totally unimodular. Moreover, the nonlinear constraint defining $C(\boldsymbol{\alpha})$ is SOCP-representable. Thus, this continuous relaxation can be used with many off-the-shelf solvers.

Optimization over relaxation $C(\boldsymbol{\alpha})$ has also been studied in the literature [3]. Specifically, consider the convex relaxation of the problem (13) given by

$$\bar{\zeta} = \min_{\boldsymbol{x},\boldsymbol{z}} \boldsymbol{a}'\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{z} \tag{22a}$$

$$\text{s.t. } (\boldsymbol{x}, \boldsymbol{z}) \in C(\boldsymbol{a}), \tag{22b}$$

Proposition 5. *There exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}})$ of (22) where $\bar{\boldsymbol{z}}$ lies on an edge of conv(Z). Moreover, if $\mathbf{c}'\boldsymbol{z} \leq 0$ for all $\boldsymbol{z} \in Z$, then $(4/5)\bar{\zeta} \geq \zeta^* \geq (5/4)\zeta_r$, where ζ^* is the optimal objective value of problem (13) –equivalently, problem (8)–, and ζ_r is the objective value of the feasible solution obtained by rounding $\bar{\boldsymbol{z}}$ to the best of the two extreme points of conv(Z) defining the edge where it lies.*

In other words, Proposition 5 states that the solution of (22) is "close" to integral (e.g., if $Z = \{0,1\}^n$, then $\bar{\boldsymbol{z}}$ has at most one fractional coordinate), that its associated objective value is similar to the optimal objective value of the mixed-integer problem, and that rounding of this solution yields a constant factor approximation algorithm under mild conditions.

Proof of Proposition 5. Projecting out variables $\boldsymbol{x}$ exactly the same as the proof of Proposition 3, we find that problem (22) simplifies to

$$\min_{\boldsymbol{z}\in\text{conv}(Z)} \sum_{i=1}^n c_i z_i - \sqrt{\sum_{i=1}^n a_i^2 z_i}. \tag{23}$$

This particular continuous relaxation of the discrete problem with feasible region $\boldsymbol{z} \in Z$ was studied in [3]. The fact that there exists an optimal solution in an edge of conv(Z) follows from Proposition 5 in [3]: at a high level, the main idea of the argument is that any optimal solution $(\boldsymbol{x^*}, \boldsymbol{z^*})$ of (23) is also optimal for

$$\min_{\boldsymbol{z}} \sum_{i=1}^n c_i z_i \text{ s.t. } \sum_{i=1}^n a_i^2 z_i = \sum_{i=1}^n a_i^2 z_i^*, \; \boldsymbol{z} \in \text{conv}(Z),$$

and the extreme points of $\text{conv}(Z) \cap \left\{\boldsymbol{z} \in \mathbb{R}^n : \sum_{i=1}^n a_i^2 z_i = \sum_{i=1}^n a_i^2 z_i^*\right\}$ lies in edges of $\text{conv}(Z)$. The approximation ratio of (4/5) follows from Proposition 7 and Corollary 2 in the same paper: the main idea is that since an optimal solution $(\boldsymbol{x^*}, \boldsymbol{z^*})$ of (23) lies on an edge of $\text{conv}(z)$, there exists two points $(\boldsymbol{x^1}, \boldsymbol{z^1})$ and $(\boldsymbol{x^2}, \boldsymbol{z^2})$ in Z such that $(\boldsymbol{x^*}, \boldsymbol{z^*})$ is a convex combination of the two. Then, using properties of the square root function, it is possible to establish that the best point $(\boldsymbol{x^i}, \boldsymbol{z^i})$ has an objective value similar to $(\boldsymbol{x^*}, \boldsymbol{z^*})$. □

Now consider the relaxation of $\text{conv}(X)$, as defined in (16), obtained by replacing polyhedra $P(\boldsymbol{\alpha})$ with their nonlinear relaxations $C(\boldsymbol{\alpha})$:

$$\bar{C} \stackrel{\text{def}}{=} \bigcap_{\boldsymbol{\alpha} \in \mathbb{R}^n} C(\boldsymbol{\alpha}) = \left\{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^{2n} : (\boldsymbol{x}, \boldsymbol{z}) \in C(\boldsymbol{\alpha}), \; \forall \boldsymbol{\alpha} \in \mathbb{R}^n\right\}. \tag{24}$$

Proposition 6 below states that the relaxation $\bar{C}$ is in fact equivalent to the perspective relaxation.

Proposition 6. $\bar{C} = R_{persp}$.

Proof. Note that the set $\bar{C}$ can be described with constraint $\boldsymbol{z} \in \text{conv}(Z)$ and the single nonlinear constraint

$$0 \geq \max_{\boldsymbol{\alpha} \in \mathbb{R}^n} \sum_{i=1}^n |\alpha_i x_i| - \sqrt{\sum_{i=1}^n \alpha_i^2 z_i}. \tag{25}$$

Since the function in (25) is positively homogeneous in $\boldsymbol{\alpha}$, it follows that either the optimization problem in unbounded (and the constrained is violated), or the optimization problem is bounded (and the constraint is satisfied). Finally, a characterization on whether this problem is bounded or not can be found in [14, Proposition 2]: problem (25) is unbounded if and only if $\sum_{i=1}^n x_i^2/z_i > 1$. Thus, concluding the proof.

□

Remark 3. Observe that the big-M constraints (1) are not implied by relaxation $\bar{C} = R_{\text{persp}}$. Although these inequalities are not hugely beneficial (in light of Proposition 5), they should be still added to the relaxation due to their simplicity (see Example 1). □

5 Approximate robust counterpart

We now turn our attention to the robust optimization problem (5) with uncertainty set (6), discussed in §1.1. Instead of solving (5) directly, which is difficult due to the discrete uncertainty set, we propose to solve instead the perspective approximation

$$\xi = \min_{\boldsymbol{y} \in Y} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \max_{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times [0,1]^n} \left\{\boldsymbol{x}'\boldsymbol{y} : \sum_{i=1}^n (d_i x_i)^2 / z_i \leq b, \; \boldsymbol{z} \in \text{conv}(Z)\right\}. \tag{26}$$

Since we relaxed the inner maximization problem, it follows that (26) is a conservative approximation of (5). Moreover, since $\boldsymbol{z}$ does not appear in the objective of the inner maximization problem, the condition of Proposition 5 is satisfied: for any fixed $\boldsymbol{y}$ the objective value of the inner maximization problem in (26) is at most 5/4 times the corresponding objective value in (5). Thus, if $\tilde{\boldsymbol{a}}'\boldsymbol{y} \geq 0$ for all $\boldsymbol{y} \in Y$, then solving (26) results in a 1.25-approximation algorithm for (5). We now derive a conic-quadratic formulation of problem (26). The condition is necessary to have non-negative objective values and well-defined approximation ratios: otherwise, it would be possible to construct an instance where $\xi = 0$, and any suboptimal solution would have an infinite optimality gap.

Clearly, to solve (26), one needs to compute conv(Z), which is in principle a difficult task. However it can be naturally accomplished if Z is a simple set, e.g., Z is totally unimodular (where the natural relaxation describes the convex hull) or if $Z = \left\{\boldsymbol{z} \in \{0,1\}^n : \boldsymbol{c}^\top \boldsymbol{z} \leq k\right\}$ with all coefficients $c_i \in \mathbb{Z} \cap [0, u]$ (where conv(Z) can be expressed in an extended formulation using the path polytope with a network with $(O)(nu)$ nodes and arcs). We now assume that $\text{conv}(Z) = \{z \in [0,1]^n : \boldsymbol{A}\boldsymbol{z} \leq \boldsymbol{k}\}$ for a suitable matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and vector $\boldsymbol{k} \in \mathbb{R}^m$, and denote by $\boldsymbol{A_i}$ the i-th column of $\boldsymbol{A}$.

Proposition 7. *Problem* (26) *can be reformulated as the SOCP*

$$\min_{\boldsymbol{y},\boldsymbol{t},\lambda,\mu} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \lambda b + \mu k + \sum_{i=1}^{n} t_i \tag{27a}$$

$$s.t.\ (y_i/d_i)^2 \leq 4(t_i + \boldsymbol{\mu}^\top \boldsymbol{A_i})\lambda \qquad i = 1, \ldots, n \tag{27b}$$

$$\boldsymbol{y} \in Y \tag{27c}$$

$$\boldsymbol{t} \in \mathbb{R}^n_+,\ \lambda \in \mathbb{R}_+,\ \boldsymbol{\mu} \in \mathbb{R}^m_+. \tag{27d}$$

Observe that since both $\lambda \geq 0$ and $t_i + \mu \geq 0$, (27b) are rotated cone constraints and thus (27) is indeed SOCP-representable (provided that Y is). The derivation of Proposition 7 is based on the following Fenchel duality result used in [4].

Lemma 1 (Fenchel dual)**.** *For any $x \in \mathbb{R}$ and $0 \leq z \leq 1$,*

$$\frac{x^2}{z} = \max_{p\in\mathbb{R}} px - \frac{p^2}{4}z.$$

Proof. If $x = z = 0$, then both sides of the equality are 0. If $z = 0$ and $x \neq 0$, then both sides are equal to $+\infty$. Otherwise, an optimal solution of the maximization problem is $p^* = 2\frac{x}{z}$, and the corresponding objective value is $\frac{x^2}{z}$. □

Proof of Proposition 7. We find that

$$\xi = \min_{\substack{\boldsymbol{y}\in Y\\ \lambda\in\mathbb{R}_+,\boldsymbol{\mu}\in\mathbb{R}_+^m}} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \lambda b + \boldsymbol{\mu}^\top \boldsymbol{k} + \max_{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{R}^n\times[0,1]^n} \left\{ \boldsymbol{y}'\boldsymbol{x} - \lambda \sum_{i=1}^n (d_i x_i)^2/z_i - \boldsymbol{\mu}^\top \boldsymbol{A}\boldsymbol{z} \right\}$$

(∵ Slater condition holds and strong duality of Lagrangian relaxation)

$$= \min_{\substack{\boldsymbol{y}\in Y\\ \lambda\in\mathbb{R}_+,\boldsymbol{\mu}\in\mathbb{R}_+^m,\boldsymbol{p}\in\mathbb{R}^n}} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \lambda b + \boldsymbol{\mu}^\top \boldsymbol{k} + \max_{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{R}^n\times[0,1]^n} \left\{ \sum_{i=1}^n (y_i - \lambda d_i p_i) x_i + \sum_{i=1}^n \left(0.25\lambda p_i^2 - \boldsymbol{\mu}^\top \boldsymbol{A_i}\right) z_i \right\}$$

(∵ Lemma 1:$(d_i x_i)^2/z_i \to p_i d_i x_i - p_i^2 z_i/4$; and Sion's minimax theorem)

$$= \min_{\substack{\boldsymbol{y}\in Y\\ \lambda\in\mathbb{R}_+,\boldsymbol{\mu}\in\mathbb{R}_+^m}} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \lambda b + \boldsymbol{\mu}^\top \boldsymbol{k} + \max_{\boldsymbol{z}\in[0,1]^n} \left\{ \sum_{i=1}^n \left(0.25\frac{(y_i/d_i)^2}{\lambda} - \boldsymbol{\mu}^\top \boldsymbol{A_i}\right) z_i \right\} \qquad (\because p_i^* = y_i/(\lambda d_i))$$

$$= \min_{\substack{\boldsymbol{y}\in Y\\ \lambda\in\mathbb{R}_+,\boldsymbol{\mu}\in\mathbb{R}_+^m}} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \lambda b + \boldsymbol{\mu}^\top \boldsymbol{k} + \sum_{i=1}^n \max\left\{0, 0.25\frac{(y_i/d_i)^2}{\lambda} - \boldsymbol{\mu}^\top \boldsymbol{A_i}\right\}.$$

$$(\because z_i^* = \mathbb{1}_{\{0.25(y_i/d_i)^2 > \lambda(\boldsymbol{\mu}^\top \boldsymbol{A_i})\}})$$

The formulation above corresponds directly to the SOCP formulation (27). □

6 Computations

According to the results of §4.2, the perspective is a simple relaxation that is guaranteed to be strong (Proposition 5). Thus, we suggest its use in practice. Note that if set X appears directly in an optimization problem (e.g., the first two applications discussed in §1.1), the perspective is arguably already the state-of-the-art relaxation – thus we omit computations for those cases. However, we illustrate its application to the robust optimization problem (5) with uncertainty set (6). In particular, we consider a simple portfolio optimization problem with $Y = \{\boldsymbol{y} \in \mathbb{R}^n : \sum_{i=1}^n y_i = 1,\ \boldsymbol{y} \geq 0\}$ and $Z = \{\boldsymbol{z} \in \{0,1\}^n : \sum_{i=1}^n z_i \leq k\}$. Since Z is totally unimodular, $\text{conv}(Z)$ is simply its natural continuous relaxation. Moreover, since $\boldsymbol{z}$ does not appear in the objective of the inner maximization problem (5), Z is permutation-invariant and thus the perspective relaxation guarantees exact solutions (Proposition 8 in the appendix), the formulation is exact in this case.

6.1 Methods

We compare three conservative approximations of (5) – the first two are based on commonly used methods in the literature.

Budgeted uncertainty This approach, inspired by [8], replaces the ellipsoidal constraint with simple bound constraints and solves instead

$$\min_{\boldsymbol{y}\in Y} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \max_{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{R}^n\times\{0,1\}^n} \left\{ \boldsymbol{x}'\boldsymbol{y} : |x_i| \leq \sqrt{b}/d_i,\ \sum_{i=1}^n z_i \leq k,\ \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0 \right\}.$$

This optimization problem can be reformulated as the linear optimization [8]

$$\begin{aligned}
\min_{\boldsymbol{y},\boldsymbol{t},\mu} \; & \tilde{\boldsymbol{a}}'\boldsymbol{y} + b\mu + \sum_{i=1}^{n} t_i \\
\text{s.t.} \; & (\sqrt{b}/d_i)|y_i| \leq \mu + t_i \qquad i = 1, \ldots, n \\
& \boldsymbol{y} \in Y, \; \boldsymbol{t} \in \mathbb{R}^n_+, \; \mu \in \mathbb{R}_+.
\end{aligned}$$

Note that $\boldsymbol{y} \geq 0$ in our experiments. Thus, we replace $|y_i|$ with y_i in all constraints.

Ellipsoidal uncertainty This approach, inspired by [7], ignores the cardinality constraint and solves instead

$$\min_{\boldsymbol{y}\in Y} \tilde{\boldsymbol{a}}'\boldsymbol{y} + \max_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \boldsymbol{x}'\boldsymbol{y} : \sum_{i=1}^{n} (d_i x_i)^2 \leq b \right\}.$$

This optimization problem can be reformulated as the SOCP [7]

$$\begin{aligned}
\min_{\boldsymbol{y},\boldsymbol{t},\mu} \; & \tilde{\boldsymbol{a}}'\boldsymbol{y} + \sqrt{b} \cdot \sqrt{\sum_{i=1}^{n} (y_i/d_i)^2} \\
\text{s.t.} \; & \boldsymbol{y} \in Y, \; \boldsymbol{t} \in \mathbb{R}^n_+, \; \mu \in \mathbb{R}_+.
\end{aligned}$$

Perspective approximation The approach we propose, described in §5.

6.2 Results

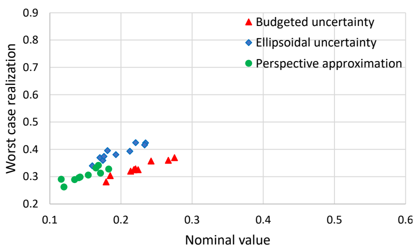
We set $n = 200$ in our computations, and we set $k \in \{5, 10, 20\}$ and $b \in \{5, 10, 20\}$ in our computations. Each entry of $\boldsymbol{a}$ and $\boldsymbol{d}$ is drawn from an uniform distribution on the interval $[0, 1]$. All optimization problems are solved using CPLEX 12.8 with the default settings, in a laptop with Intel Core i7-8550U CPU and 16 GB RAM. Solution times for all methods are less than 0.1 seconds in all cases.

For each combination of parameters (b, k), we generate 10 instances and record for each method: the nominal objective value $\tilde{\boldsymbol{a}}'\boldsymbol{y}^*$, where $\boldsymbol{y}^*$ is the solution produced; and the worst-case realization given by

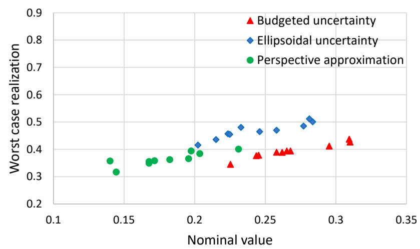
$$\tilde{\boldsymbol{a}}'\boldsymbol{y}^* + \max_{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{R}^n\times\{0,1\}^n} \left\{ \boldsymbol{x}'\boldsymbol{y}^* : \sum_{i=1}^{n} (d_i x_i)^2/z_i \leq b, \; \sum_{i=1}^{n} z_i \leq k, \; \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0 \right\}. \tag{28}$$

Note that computing the worst-case realization requires solving a mixed-integer optimization problem. However, since the perspective reformulation results in a strong relaxation and $n = 200$ is not too large, problem (28) can be comfortably solved to optimality using CPLEX. Figure 1 presents the results, showing the nominal objective value and worst-case realization for each combination of parameters and each instance.

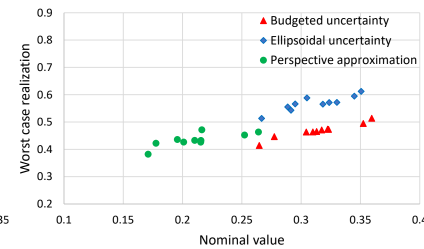
We observe that the budgeted uncertainty approach consistently has the worst nominal performance, although it tends to be better in terms of robustness than the ellipsoidal uncertainty. The perspective approximation results in the "best" worse-case realizations for all the combinations of parameters (as expected, since it delivers optimal solutions with respect to this metric). It also results in the best solutions in terms of the nominal values, except for the case with $k = 20$ and $b = 5$ (where the ellipsoidal uncertainty has slightly better nominal performance). *Thus, in our experiments, we can conclude that the perspective approximation is the best approach, delivering the most reliable solutions without affecting (and in most cases improving) the nominal performance.*



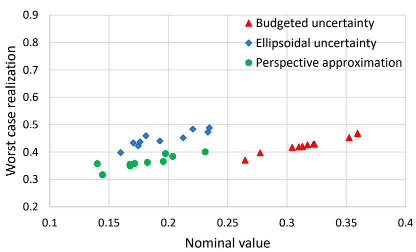
(a) $k = 5, b = 5$



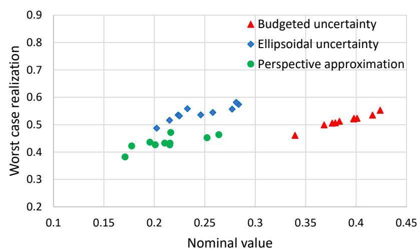
(b) $k = 5, b = 10$



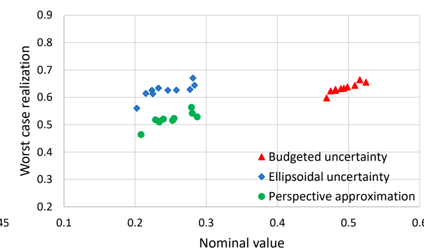
(c) $k = 5, b = 20$



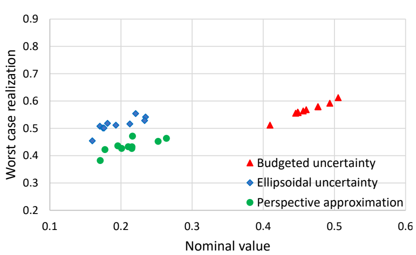
(d) $k = 10, b = 5$



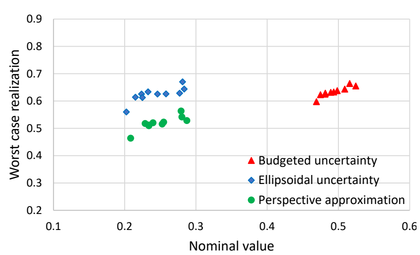
(e) $k = 10, b = 10$



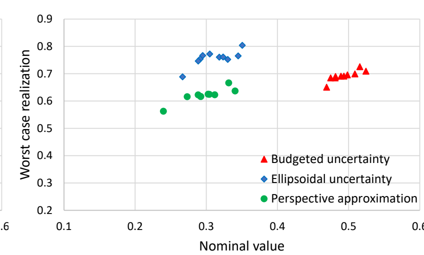
(f) $k = 10, b = 20$



(g) $k = 20, b = 5$



(h) $k = 20, b = 10$



(i) $k = 20, b = 20$

Figure 1: Nominal value versus worst-case realization for different cardinality and budget parameters. The **budgeted uncertainty** approach (red triangles) typically yields solutions with large nominal values, particularly for large values of k. The **ellipsoidal uncertainty** approach (blue rhombuses) often results in good nominal values (particularly for large k), but the worst-case realizations are large. The **perspective approximation** (red circles) always results in the best worst-case realizations, and often in the best nominal values.

7 Conclusion

We gave an implicit description of $\text{conv}(X)$, established links between the convexification of this set and convexification of polyhedral sets, and studied the strength of the perspective relaxation R_{persp}. On the one hand, we showed in this paper that the perspective reformulation is insufficient to describe $\text{conv}(X)$, and that it can be interpreted as a nonlinear relaxation of the polyhedral sets. On the other hand, we showed that while the perspective reformulation can be strengthened using polyhedral theory as discussed in §4.1, it is already quite strong. Our experiments on robust optimization suggest that explicitly accounting for discrete nonlinear uncertainty sets can deliver better quality solutions than existing approaches while preserving tractability.

8 Acknowledgments

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References

[1] Shabbir Ahmed and Alper Atamtürk. Maximizing a class of submodular utility functions. *Mathematical Programming*, 128(1-2):149–169, 2011.

[2] M Selim Aktürk, Alper Atamtürk, and Sinan Gürel. A strong conic quadratic reformulation for machine-job assignment with controllable processing times. *Operations Research Letters*, 37:187–191, 2009.

[3] Alper Atamtürk and Andrés Gómez. Maximizing a class of utility functions over the vertices of a polytope. *Operations Research*, 65:433–445, 2017.

[4] Alper Atamtürk and Andrés Gómez. Safe screening rules for L0-regression from perspective relaxations. In *International Conference on Machine Learning*, pages 421–430. PMLR, 2020.

[5] Alper Atamtürk and Vishnu Narayanan. Polymatroids and mean-risk minimization in discrete optimization. *Operations Research Letters*, 36(5):618–622, 2008.

[6] Tiziano Bacci, Antonio Frangioni, Claudio Gentile, and Kostas Tavlaridis-Gyparakis. New MINLP formulations for the unit commitment problems with ramping constraints. *Optimization*, 2019.

[7] Aharon Ben-Tal and Arkadi Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, 88(3):411–424, 2000.

[8] Dimitris Bertsimas and Melvyn Sim. The price of robustness. *Operations Research*, 52(1): 35–53, 2004.

[9] Juan S Borrero and Leonardo Lozano. Modeling defender-attacker problems as robust linear programs with mixed-integer uncertainty sets. *INFORMS Journal on Computing*, 33(4):1570–1589, 2021.

[10] Sebastián Ceria and João Soares. Convex programming for disjunctive convex optimization. *Mathematical Programming*, 86(3):595–614, 1999.

[11] Alexandre d'Aspremont, Francis Bach, and Laurent El Ghaoui. Optimal solutions for sparse principal component analysis. *Journal of Machine Learning Research*, 9(7), 2008.

[12] Santanu S Dey, Rahul Mazumder, and Guanyi Wang. Using ℓ_1-relaxation and integer programming to obtain dual bounds for sparse pca. *Operations Research*, 2021.

[13] Antonio Frangioni and Claudio Gentile. Perspective cuts for a class of convex 0–1 mixed integer programs. *Mathematical Programming*, 106(2):225–236, 2006.

[14] Andrés Gómez. Strong formulations for conic quadratic optimization with indicator variables. *Mathematical Programming*, 188(1):193–226, 2021.

[15] Oktay Günlük and Jeff Linderoth. Perspective reformulations of mixed integer nonlinear programs with indicator variables. *Mathematical Programming*, 124:183–205, 2010.

[16] Jinhak Kim, Mohit Tawarmalani, and Jean-Philippe P Richard. Convexification of permutation-invariant sets and an application to sparse principal component analysis. *Mathematics of Operations Research*, 2021.

[17] Yongchun Li and Weijun Xie. Exact and approximation algorithms for sparse PCA. *arXiv preprint arXiv:2008.12438*, 2020.

[18] George L Nemhauser, Laurence A Wolsey, and Marshall L Fisher. An analysis of approximations for maximizing submodular set functions—I. *Mathematical Programming*, 14(1):265–294, 1978.

[19] Xueyu Shi, Oleg A Prokopyev, and Bo Zeng. Sequence independent lifting for a set of submodular maximization problems. *Mathematical Programming*, pages 1–46, 2022.

[20] Linchuan Wei, Andrés Gómez, and Simge Küçükyavuz. On the convexification of constrained quadratic optimization problems with indicator variables. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 433–447. Springer, 2020.

[21] Linchuan Wei, Andrés Gómez, and Simge Küçükyavuz. Ideal formulations for constrained convex optimization problems with indicator variables. *Mathematical Programming*, 2021.

[22] Linchuan Wei, Alper Atamtürk, Andrés Gómez, and Simge Küçükyavuz. On the convex hull of convex quadratic optimization problems with indicators. *arXiv preprint arXiv:2201.00387*, 2022.

[23] Weijun Xie and Xinwei Deng. Scalable algorithms for the sparse ridge regression. *SIAM Journal on Optimization*, 30(4):3359–3386, 2020.

[24] Jiajin Yu and Shabbir Ahmed. Maximizing a class of submodular utility functions with constraints. *Mathematical Programming*, 162(1-2):145–164, 2017.

[25] Hui Zou, Trevor Hastie, and Robert Tibshirani. Sparse principal component analysis. *Journal of Computational and Graphical Statistics*, 15(2):265–286, 2006.

A The Equivalence in Remark 1

Let us consider

$$Y = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_0 \le k,\ \|\boldsymbol{x}\|_2^2 \le 1 \right\}, \tag{29}$$

and its corresponding perspective relaxation

$$R_{\text{persp}} \stackrel{\text{def}}{=} \left\{ (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n \times [0,1]^n : \sum_{i=1}^n x_i^2/z_i \le 1, \sum_{i=1}^n z_i \le k \right\}. \tag{30}$$

Proposition 8. *We have that conv$(Y) :=$ proj$_{\boldsymbol{x}}(R_{persp})$.*

Proof. By definition, we have set $Y \subseteq \text{proj}_{\boldsymbol{x}}(R_{persp})$. To show the equivalence, since both sets Y and R_{persp} are compact. It is sufficient to show that for any $\boldsymbol{a} \in \mathbb{R}^n$, we must have

$$v_1 \stackrel{\text{def}}{=} \min_{\boldsymbol{x} \in Y} \boldsymbol{a}^\top \boldsymbol{x} = v_2 \stackrel{\text{def}}{=} \min_{\boldsymbol{x} \in \text{proj}_{\boldsymbol{x}}(R_{persp})} \boldsymbol{a}^\top \boldsymbol{x}.$$

First of all, let $z_i = 1$ if $x_i \neq 0$ for each $i = 1, \ldots, n$. Then $v_1 = \min_{\boldsymbol{x} \in Y}$ is equivalent to

$$\begin{aligned} v_1 := \min_{\boldsymbol{x},\boldsymbol{z}}\ & \boldsymbol{a}'\boldsymbol{x} \\ \text{s.t. } & \|\boldsymbol{x}\|_2^2 \le 1 \\ & \boldsymbol{x} \circ (\boldsymbol{e} - \boldsymbol{z}) = 0 \\ & \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{z} \in \{0,1\}^n\ \sum_{i=1}^n z_i \le k, . \end{aligned}$$

According to the proof of Proposition 1, without loss of generality, suppose that $a_1 \ge a_2 \ge \ldots \ge a_n$, we have $v_1 = \sqrt{\sum_{i=1}^k a_i^2}$.

On the other hand, $v_2 := \min_{\boldsymbol{x} \in \text{proj}_{\boldsymbol{x}}(R_{persp})} \boldsymbol{a}^\top \boldsymbol{x}$ is equivalent to

$$\begin{aligned} v_2 := \min_{\boldsymbol{x},\boldsymbol{z}}\ & \boldsymbol{a}'\boldsymbol{x} \\ \text{s.t. } & \sum_{i=1}^n \frac{x_i^2}{z_i} \le 1 \\ & \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{z} \in [0,1]^n\ \sum_{i=1}^n z_i \le k, \end{aligned}$$

which is equivalent to

$$\begin{aligned} v_2 := \min_{\boldsymbol{x},\boldsymbol{z}}\ & \sum_{i=1}^n a_i \sqrt{z_i} \frac{x_i}{\sqrt{z_i}} \\ \text{s.t. } & \sum_{i=1}^n \frac{x_i^2}{z_i} \le 1 \\ & \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{z} \in [0,1]^n\ \sum_{i=1}^n z_i \le k. \end{aligned}$$

For any given $\boldsymbol{z} \in [0,1]^n$, according to the Cauchy-schwartz inequality, we have

$$v_2 \geq \min_{\boldsymbol{z}} \; -\sqrt{\sum_{i=1}^n a_i^2 z_i}$$
$$\text{s.t. } \boldsymbol{z} \in [0,1]^n \; \sum_{i=1}^n z_i \leq k.$$

Next, optimizing over $\boldsymbol{z}$ on the right-hand side of the inequality, we obtain that $v_2 \geq \sqrt{\sum_{i=1}^k a_i^2} = v_1$. Note that the the equality is obtainable by letting $x_i = a_i / \sqrt{\sum_{\ell=1}^k a_\ell^2}$ for each $i = 1, \ldots, k$ and 0, otherwise, and $z_i = 1$ for each $i = 1, \ldots, k$ and 0, otherwise. Thus, we have $v_2 = \sqrt{\sum_{i=1}^k a_i^2} = v_1$. This completes the proof. $\square$

B Detailed example

Consider the optimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^2, \boldsymbol{z} \in \{0,1\}^2} \; x_1^2 + 3x_2^2 + (2x_1 - x_2)^2 - 6x_1 - 7x_2 \tag{31a}$$
$$\text{s.t. } z_1 + z_2 \leq 1 \tag{31b}$$
$$x_1^2 + x_2^2 \leq 1 \tag{31c}$$
$$x_1(1 - z_1) = 0, \; x_2(1 - z_2) = 0. \tag{31d}$$

Observe that variables $\boldsymbol{z}$ do not appear directly in the objective. We now discuss several relaxations of (31).

B.1 Reformulation of the norm constraint

Consider three possible formulations of (31c).

basic The constraint is directly formulated as (31c).

p. inv. Reformulating the constraint based on the permutation-invariance of set Y given in Remark 1 [16]. This formulation calls for the introduction of five additional variables u_1, u_2, r, t_1, t_2 and replaces constraint (31c) with the system

$$u_1^2 + u_2^2 \leq 1, \; u_1 \geq u_2, \; u_1 + u_2 = x_1 + x_2, \; u_1 \geq r + t_1, \; x_1 \leq t_1 + r, \; x_2 \leq t_2 + r$$
$$u_1 \geq 0, \; u_2 = 0, \; t_1 \geq 0, \; t_2 \geq 0, \; r \text{ free}.$$

persp. Using the perspective reformulation, as advocated in the paper, and replacing (31c) with $x_1^2/z_1 + x_2^2/z_2 \leq 1$.

B.2 Reformulation of the objective

Consider two reformulations of the objective.

basic The objective is directly formulated as (31a).

persp. Using the perspective reformulation, as commonly advocated in the literature, replacing (31a) with $x_1^2/z_1 + 3x_2^2/z_2 + (2x_1 - x_2)^2 - 6x_1 - 7x_2$

B.3 Relaxation quality

Table 1 presents solutions and objective value of several combinations of the reformulations of (31). Note that the ideal convex reformulation of (31) (which in general cannot be obtained in practice) can be obtained in this case by exploiting the fact that $z_1 + z_2 \leq 1$ implies $x_1 x_2 = 0$, and is given by

$$\min_{\boldsymbol{x},\boldsymbol{z}} 5x_1^2/z_1 + 4x_2^2 - 6x_1 - 7x_2 \text{ s.t. } z_1 + z_2 \leq 1,\ |x_i| \leq z_i \text{ for } i \in \{1,2\}.$$

Table 1: Solution and objective values of convex relaxations of (31).

Norm constraint			Objective		Solution				
basic	**p. inv.**	**persp.**	**basic**	**persp.**	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	**obj.**
x			x		0.63	0.77	0.48	0.50	-6.78
	x		x		0.42	0.58	0.42	0.57	-5.33
		x	x		0.42	0.58	0.42	0.57	-5.33
	x			x	0.40	0.60	0.27	0.73	-4.49
		x		x	0.39	0.60	0.32	0.68	-4.44
—— ideal ——					0.00	0.88	0.00	1.00	-3.06

The key observations are: • if the reformulation of the objective is "basic", e.g., does not involve the discrete variables, then the permutation invariant relaxation of set Y and the perspective are equivalent; • however, if the objective is improved by using variables $\boldsymbol{z}$, then better approximations of conv(X) than conv(Y) yield stronger formulations (rows 4 and 5 of the table).