
WATER FINDS ITS LEVEL: A LOCALIZED METHOD FOR MULTICOMMODITY FLOW PROBLEM

A PREPRINT

Pengfei Liu

Building No.1, Zhonguancun Road
Haidian District
Beijing, China
pengfeil89@foxmail.com

September 14, 2022

ABSTRACT

This paper describes a local-control method for multicommodity flow problem. Both the capacity constraints and the flow conservation constraints are relaxed. If the flow exceeds the capacity on an edge, the edge would have a congestion cost. If the flow into a vertex is not equal to that out of the vertex, the vertex would have a height. Then a new conception, stable pseudo-flow, is introduced. Potential difference reduction algorithms, which don't rely on any shortest path or augmenting path, are designed to obtain stable pseudo-flow. If the stable pseudo-flow is a nonzero-stable pseudo-flow, there exists no feasible solution for multicommodity flow problem; if the stable pseudo-flow is a zero-stable pseudo-flow, there exists a feasible solution and the zero-stable pseudo-flow is the feasible solution. Additionally, the algorithms work in a localized manner and can be efficiently implemented in parallel, which would further improve the performance.

Keywords local-control · optimality condition · exact algorithm · potential difference · multicommodity flow

1 Introduction

The *multicommodity flow problem* is the problem of designing flow of several different commodities through a common network with edge capacities. Given a directed graph $G(V, E)$ which has n vertexes and m edges, a capacity function $u : E \rightarrow Q^+$, K origin-destination pairs of nodes defined by (s_k, t_k, d_k) where s_k and t_k are the origin and destination of commodity k , and d_k is the demand, the objective is to obtain an assignment of flow which satisfies the demand for each commodity without violating the capacity constraints. The constraints can be summarized as follows:

$$\begin{aligned} \sum_{k \in \mathcal{K}} f_{ij,k} &\leq u_{ij}, \forall (i, j) \in E \\ \sum_{j \in \delta^+(i)} f_{ij,k} - \sum_{j \in \delta^-(i)} f_{ji,k} &= \begin{cases} d_k, & \text{if } i = s_k \\ -d_k, & \text{if } i = t_k \\ 0, & \text{if } i \in V - \{s_k, t_k\} \end{cases} \\ f_{ij,k} &\geq 0, \forall k \in \mathcal{K}, (i, j) \in E \end{aligned} \tag{1}$$

where $\delta^+(i) = \{j | (i, j) \in E\}$, $\delta^-(i) = \{j | (j, i) \in E\}$ and $\mathcal{K} = \{1, 2, \dots, K\}$. In this paper, we assume $s_k \neq t_k$. The first expressions are capacity constraints. The second are flow conservation constraints and the last are non-negative constraints.

Multicommodity flow problems have attracted great attention since the publication of the works of [3] and [4]. Wang [5, 6] and Salimifard et al.[7] gives a comprehensive survey about this problem. Many special solution methods based on

linear programming have been suggested to exploit in some way the special block-angular structure of multicommodity flow problem [18, 19, 20, 21, 23, 35, 22, 17, 24]. There are also many approximation methods for multicommodity flow problem [36, 27, 26, 25]. Awerbuch and Leighton [28, 29] propose an approximation algorithm which uses local-control techniques similar to the preflow-push algorithm of Goldberg and Tarjan [30, 11]. Unlike previous approximation algorithms that attempt to find shortest paths or augmenting paths to push flow towards sinks, their algorithm uses an "edge-balancing" technique that attempts to send a commodity across an edge (i, j) if there is more of the commodity quened at i than that quened at j .

Recent improvements compute the max-flow by a sequence of electric flows [40, 41, 42, 43, 44, 45, 46]. There are also some pseudo-flow approaches [37, 38, 39] for solving max-flow problem. Pyakurel et al. [8, 9, 10] propose flow problems with intermediate storage. Besides, network equilibrium [12, 13, 14] is proposed to solve multicommodity flow problem.

In this paper, not only the capacity constraints but also the flow conservation constraints are relaxed. We introduce congestion function for each edge and height function for each vertex and commodity. Then a simple rule is given, that is, the commodity k should flow from vertex i to vertex j if the height difference between vertex i and vertex j for commodity k is greater than the congestion function of edge (i, j) . If the feasible region of Expression (1) is not empty, the feasible solution would be obtained by this simple rule. This is consistent with our common sense: water finds its level.

The network notation introduced here is summarized in Table 1. Further notation is introduced as needed.

1.1 Our contribution

Our contributions can be summarized as follows.

- (i) There has been much less progress on exact algorithms for multicommodity flow problem, perhaps because it is much more difficult to handle K commodities than to handle one. By introducing height and congestion functions, we give a very simple and intuitive optimality condition, which may inspire the research on exact algorithms for multicommodity flow problem.
- (ii) We give an exact algorithm with complexity $O(K|E| \frac{\lambda_{max}}{\alpha} \ln \frac{\Lambda \sum_{k \in \mathcal{K}} d_k}{\epsilon \alpha})$ for multicommodity flow problem¹, where Λ is the maximum degree of graph $G(V, E)$ and λ_{max} the largest eigenvalue of $Q^T Q$ (see Expression 13).
- (iii) Unlike other exact methods based on solving linear programming, our method just needs to compute the congestion of every edge and the height of every vertex and commodity, and then update the flow by *potential difference*. The convex programming in this paper is mainly used to prove the optimality condition and to analyze algorithmic complexity.
- (iv) We also give a practical potential difference reduction algorithm by an inexact line search method.
- (v) Unlike previously exact algorithms for multicommodity flow problem, our new algorithm could be executed in a localized manner, which are especially suited to environments where global control is not possible. Besides, the algorithm can work in parallel, which would greatly improve the computational efficiency.

2 Optimality Condition

2.1 Stable pseudo-flow

Unlike other methods, our method *does not maintain the capacity constraints and flow conservation constraints*. The method, however, maintains a *pseudo-flow*, which is a function $\mathbf{f} : K \times E \rightarrow \mathbb{R}^+$. Let Δ_{ik} be the amount by which the flow of commodity k into the vertex i exceeds that out of the vertex, i.e.,

$$\Delta_{ik} = \sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} + \hat{\Delta}_{ik}, \forall i \in V, k \in \mathcal{K} \quad (2)$$

where $\delta^+(i) = \{j | (i, j) \in E\}$, $\delta^-(i) = \{j | (j, i) \in E\}$ and $\hat{\Delta}_{ik}$ is defined as:

$$\hat{\Delta}_{ik} = \begin{cases} 0 & \text{if } i \in V - \{s_k, t_k\} \\ d_k & \text{if } i = s_k \\ -d_k & \text{if } i = t_k. \end{cases} \quad (3)$$

¹For the definition of α , see the proof of Lemma 6. However, the estimate of the lower bound of α is still an open problem.

Table 1: Basic Network Notation

$G(V, E)$	a directed graph
V	node (index) set
E	edge (index) set
\mathcal{K}	set of commodities, i.e., $\mathcal{K} = \{1, \dots, k, \dots, K\}$
$f_{ij,k}$	flow of commodity k on edge (i, j) , $\mathbf{f} = (\dots, f_{ij,k}, \dots)$
f_{ij}	flow on edge (i, j) , i.e., $f_{ij} = \sum_{k \in \mathcal{K}} f_{ij,k}$
(s_k, t_k, d_k)	s_k and t_k are the origin and destination of commodity k , and d_k is the demand
u_{ij}	the capacity of edge (i, j)
$\delta^+(i)$	$\{j (i, j) \in E\}$
$\delta^-(i)$	$\{j (j, i) \in E\}$
ψ_{ij}	congestion function of edge (i, j)
h_{ik}	height function of vertex i for commodity k
Δ_{ik}	the amount by which the k th flow into the vertex i exceeds that out of the
r_{ij}	the unused capacity for edge (i, j)
\mathbf{x}	the concat of \mathbf{f} and \mathbf{r} , i.e., $\mathbf{x} = \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}$
\mathbf{x}_j	the j th component of \mathbf{x}
m	the number of edges of graph $G(V, E)$
n	the number of vertexes of graph $G(V, E)$
λ_{max}	the largest eigenvalue of $Q^T Q$
Λ	the maximum degree of graph $G(V, E)$
$\phi_{ij,k}$	the potential difference of edge (i, j) and commodity k

For each commodity k at each vertex i , a height function $h_{ik} = h_{ik}(\Delta_{ik})$ is introduced, where $h_{ik}(\cdot)$ represents the relationship between Δ_{ik} and the height of vertex i for commodity k . Specifically, h_{ik} is defined as

$$h_{ik}(\Delta_{ik}) = \Delta_{ik}. \quad (4)$$

That is, the height of vertex i for commodity k is the amount by which the flow of commodity k into the vertex exceeds that out of that vertex.

Additionally, a *congestion function* $\psi_{ij} = \psi_{ij}(f_{ij})$ for each edge is introduced, where $\psi_{ij}(\cdot)$ represents the relationship between the flow and the degree of congestion for edge (i, j) . Specifically, ψ_{ij} is defined as

$$\psi_{ij}(f_{ij}) = \begin{cases} 0 & \text{if } f_{ij} \leq u_{ij} \\ f_{ij} - u_{ij} & \text{if } f_{ij} > u_{ij} \end{cases} \quad (5)$$

where u_{ij} is the capacity of edge (i, j) and $f_{ij} = \sum_{k \in \mathcal{K}} f_{ij,k}$. If the flow on edge (i, j) is less than the capacity, the congestion of edge (i, j) is zero. Otherwise, the congestion of edge (i, j) is the amount by which the flow exceeds the capacity.

The greater the h_{ik} is, the higher the vertex i for commodity k is. The greater the ψ_{ij} is, the more 'congested' the edge (i, j) is. As is often said, water finds its level. Intuitively, if the height difference between vertex i and vertex j for commodity k is greater than the congestion of edge (i, j) , the flow $f_{ij,k}$ should be increased; if the height difference is less than the congestion, the flow $f_{ij,k}$ should be decreased. In the following we would give a strict proof to demonstrate that the feasible solution for multicommodity flow problem may be obtained by this intuitive idea.

Firstly, we introduce the conception of *potential difference* and *stable pseudo-flow* here.

Definition 1 For every edge (i, j) and commodity k , define the potential difference $\phi_{ij,k}$ as the amount that the height difference between vertex i and vertex j minus the congestion of edge (i, j) , i.e.,

$$\phi_{ij,k} = h_{ik} - h_{jk} - \psi_{ij} \quad (6)$$

Definition 2 By using $\{\psi_{ij}, \forall (i, j) \in E\}$ as congestion function and $\{h_{ik}, \forall i \in V, \forall k \in \mathcal{K}\}$ as height function, a pseudo-flow \mathbf{f} is called a *stable pseudo-flow* if it satisfies the following conditions:

- (i) for any used edge of commodity k , the height difference between vertex i and vertex j for commodity k is equal to the congestion of edge (i, j) , i.e., the potential difference $\phi_{ij,k}$ is zero;

(ii) for any unused edge of commodity k , the height difference between vertex i and vertex j is less than or equal to the congestion of edge (i, j) , i.e., the potential difference $\phi_{ij,k}$ is less than or equal to zero;

where an edge (i, j) is called *used* by commodity k if there exists $s_k - t_k$ flow on edge (i, j) , otherwise it is called *unused*.

Definition 3 A stable pseudo-flow \mathbf{f} is called zero-stable pseudo-flow if $\psi_{ij} = 0$, $h_{ik} = 0$, $\forall (i, j) \in E, i \in V, k \in \mathcal{K}$. Otherwise, it is called nonzero-stable pseudo-flow.

Obviously, by the definitions above, a zero-stable pseudo-flow is a feasible flow that satisfies Expression (1). Therefore, we have the following theorem:

Theorem 1 Given $\{(s_k, t_k, d_k) : k \in \mathcal{K}\}$ and capacity reservation $\{u_{ij} : (i, j) \in E\}$, the feasible region of Expression (1) is not empty if and only if there exists zero-stable pseudo-flow.

In fact, if there exists a nonzero-stable pseudo-flow, there is no feasible solution for Expression (1). Before proving this conclusion, we would give a non-linear programming formulation of multicommodity flow problem, whose solution is stable pseudo-flow.

2.2 The basic formulation

Let f_{ij} be the sum of the flow of all pairs on edge (i, j) and Δ_{ik} defined as Expression (2). Define the following programming:

$$\begin{aligned} \min \quad z &= \sum_{(i,j) \in E} \int_0^{f_{ij}} \psi_{ij}(\omega) d\omega + \sum_{i,k} \int_0^{\Delta_{ik}} h_{ik}(\omega) d\omega \\ \text{s.t.} \quad & f_{ij,k} \geq 0, \forall k \in \mathcal{K}, (i, j) \in E \end{aligned} \quad (7)$$

where ψ_{ij} is the congestion function and h_{ik} the height function.

In the programming above, the objective function is the sum of the integrals of the edge congestion functions and the integrals of the vertex height functions. The only constraints are non-negative constraints. Note that there is no capacity constraint or flow conservation constraint here. According to the definitions of the congestion function ψ and the height function h , we have the following lemma:

Lemma 1 The feasible region of Expression (1) is not empty if and only if the minimum value of the objective function of Expression (7) is zero; the feasible region of Expression (1) is empty if and only if the minimum value of the objective function of Expression (7) is greater than zero.

2.3 Equivalence

To demonstrate the equivalence between the stable pseudo-flow and the optimal solution of Programming (7), it has to be shown that any flow pattern that solves Programming (7) satisfies the stable conditions. This equivalency is demonstrated in this part by proving that the Karush-Kuhn-Tucker conditions for Programming (7) are identical to the stable conditions.

Lemma 2 Let \mathbf{f}^* be a solution of Programming (7). \mathbf{f}^* is the optimal solution of Programming (7) if and only if \mathbf{f}^* satisfies the Karush-Kuhn-Tucker conditions of Programming (7).

Proof: Firstly, the objective function of Programming (7) is convex. Secondly, the inequality constraints of Programming (7) are continuously differentiable concave functions. So Karush-Kuhn-Tucker conditions are necessary and sufficient for the optimality of Programming (7) (see [15]). \square

Since Programming (7) is a minimization problem with non-negativity constraints, the Karush-Kuhn-Tucker conditions of such formulation are as following:

Stationarity

$$-\frac{\partial z}{\partial f_{ij,k}} = -\mu_{ij,k}, \forall k \in \mathcal{K}, (i, j) \in E$$

Primal feasibility

$$-f_{ij,k} \leq 0, \forall k \in \mathcal{K}, (i, j) \in E \quad (8)$$

Dual feasibility

$$\mu_{ij,k} \geq 0, \forall k \in \mathcal{K}, (i, j) \in E$$

Complementary slackness

$$\mu_{ij,k} f_{ij,k} = 0, \forall k \in \mathcal{K}, (i, j) \in E.$$

Obviously,

$$\begin{aligned} \frac{\partial z}{\partial f_{ij,k}} &= \psi_{ij}(f_{ij}) \frac{\partial f_{ij}}{\partial f_{ij,k}} + h_{ik}(\Delta_{ik}) \frac{\partial \Delta_{ik}}{\partial f_{ij,k}} + h_{jk}(\Delta_{jk}) \frac{\partial \Delta_{jk}}{\partial f_{ij,k}} \\ &= \psi_{ij}(f_{ij}) + (-h_{ik}(\Delta_{ik})) + (h_{jk}(\Delta_{jk})) \\ &= \psi_{ij} + h_{jk} - h_{ik} \end{aligned} \quad (9)$$

Substituting the expression above into Stationarity expression in KKT conditions,

$$h_{ik} - h_{jk} - \psi_{ij} = -\mu_{ij,k}, \forall k \in \mathcal{K}, (i, j) \in E.$$

For any used edge of commodity k , i.e. $f_{ij,k} > 0$, by complementary slackness $\mu_{ij,k} f_{ij,k} = 0$ in KKT conditions, we have $\mu_{ij,k} = 0$. Therefore,

$$h_{ik} - h_{jk} - \psi_{ij} = 0, \forall k \in \mathcal{K}, (i, j) \in E.$$

That is, the potential difference between vertex i and vertex j for commodity k is equal to zero for any used edge of commodity k .

For any unused edge of commodity k , due to $\mu_{ij,k} \geq 0$, we have

$$h_{ik} - h_{jk} - \psi_{ij} = -\mu_{ij,k} \leq 0, \forall k \in \mathcal{K}, (i, j) \in E.$$

That is, the potential difference between vertex i and vertex j for commodity k is less than or equal to zero for any unused edge of commodity k .

With the interpretation above, it is now clear that:

- (i) for any used edge of commodity k , the potential difference between vertex i and vertex j for commodity k is zero, i.e. $\phi_{ij,k} = 0$;
- (ii) for any unused edge of commodity k , the potential difference between vertex i and vertex j for commodity k is less than or equal to zero, i.e. $\phi_{ij,k} \leq 0$.

So we have the following lemma:

Lemma 3 *The optimal solution of Programming (7) is a stable pseudo-flow.*

Obviously, a stable pseudo-flow also satisfies the KKT conditions. By Lemma 2, we have

Lemma 4 *The stable pseudo-flow is the optimal solution of Programming (7).*

By Lemma 1, Lemma 3, Lemma 4, the following theorem holds.

Theorem 2 *Given $\{(s_k, t_k, d_k) : k \in \mathcal{K}\}$ and capacity reservation $\{u_{ij} : (i, j) \in E\}$, the feasible region of Expression (1) is empty if and only if there exists a nonzero-stable pseudo-flow.*

Remark 1 *In fact, both Theorem 1 and Theorem 2 are true if the height function and the congestion function satisfy the following conditions:*

- (i) *the height function $h_{ik}(\Delta_{ik})$ is a strictly monotone increasing function and $h_{ik}(0) = 0$;*

(ii) the congestion function ψ_{ij} satisfies the following definition:

$$\psi(f_{ij}) = \begin{cases} 0 & \text{if } f_{ij} \leq u_{ij} \\ g(f_{ij} - u_{ij}) & \text{if } f_{ij} > u_{ij} \end{cases}$$

where $g(0) = 0$ and $g(\cdot)$ is a strictly monotone increasing function.

Remark 2 Expression (9) shows that the potential difference is exactly the same as the negative gradient of Programming (7), which is the key to prove the convergence of the following algorithms.

3 Potential Difference Reduction Algorithm

Theorem 1 and Theorem 2 above suggest a simple algorithmic approach for solving the multicommodity flow problem. That is, design an algorithm to obtain the stable pseudo-flow, which may be achieved by adjusting the commodity flow $f_{ij,k}$ until the potential difference $\phi_{ij,k} = 0$ or $f_{ij,k} = 0$. In other words, if the potential difference between vertex i and vertex j for commodity k is greater than zero, the flow $f_{ij,k}$ should be increased; if the potential difference is less than zero and $f_{ij,k} > 0$, the flow $f_{ij,k}$ should be decreased. In this paper we call the algorithm potential difference reduction algorithm. Before describing the algorithm, we need to rewrite Programming (7) as a quadratic programming problem with nonnegative constraints.

3.1 Quadratic programming

Firstly,

$$\int_0^{\Delta_{ik}} h_{ik}(\omega) d\omega = \frac{1}{2} \Delta_{ik}^2 = \frac{1}{2} \left(\sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} + \hat{\Delta}_{ik} \right)^2 \quad (10)$$

By the definition of congestion function ψ_{ij} , we have

$$\int_0^{f_{ij}} \psi_{ij}(\omega) d\omega = \begin{cases} 0 & \text{if } f_{ij} \leq u_{ij} \\ \frac{1}{2} (f_{ij} - u_{ij})^2 & \text{if } f_{ij} > u_{ij} \end{cases} \quad (11)$$

By introducing an auxiliary variable $\{r_{ij} : r_{ij} \geq 0\}$ which is the unused capacity for edge (i, j) , Programming (7) could be rewritten as

$$\begin{aligned} \min \quad & z = \frac{1}{2} \sum_{(i,j) \in E} \left(\sum_{k \in \mathcal{K}} f_{ij,k} + r_{ij} - u_{ij} \right)^2 + \frac{1}{2} \sum_{i,k} \left(\sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} + \hat{\Delta}_{ik} \right)^2 \\ \text{s.t.} \quad & f_{ij,k} \geq 0, \forall k \in \mathcal{K}, (i, j) \in E \\ & r_{ij} \geq 0, \forall (i, j) \in E \end{aligned} \quad (12)$$

Programming (12) is a quadratic programming problem with nonnegative constraints, which could be rewritten to the matrix form.

To write the matrix form, we define a $n \times m$ matrix A of $Graph(V, E)$ with element A_{ve} for the vertex v and the edge e (connecting vertexes v_i and v_j) defined by

$$A_{ve} = \begin{cases} -1 & \text{if } v = v_i \\ 1 & \text{if } v = v_j \\ 0 & \text{otherwise} \end{cases}$$

Now define P as a $nK \times m(K+1)$ matrix composed of $K \times (K+1)$ blocks of size $n \times m$ each:

$$P = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ 0 & A & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & A & 0 \end{bmatrix}.$$

Define C as a $m \times m(K+1)$ matrix constructed from $K+1$ blocks of identity matrices of dimension $m \times m$:

$$C = (I_{m \times m} I_{m \times m} \cdots I_{m \times m})$$

For simplicity, define

$$Q = \begin{pmatrix} P \\ C \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \quad (13)$$

Then the objective function of Programming (12) could be rewritten as:

$$\begin{aligned} z &= \frac{1}{2} \sum_{(i,j) \in E} \left(\sum_{k \in \mathcal{K}} f_{ij,k} + r_{ij} - u_{ij} \right)^2 + \frac{1}{2} \sum_{i,k} \left(\sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} + \hat{\Delta}_{ik} \right)^2 \\ &= \frac{1}{2} (\mathbf{Q}\mathbf{x} - \mathbf{b})^T (\mathbf{Q}\mathbf{x} - \mathbf{b}) \\ &= \frac{1}{2} \mathbf{x}^T \mathbf{Q}^T \mathbf{Q} \mathbf{x} - \mathbf{b}^T \mathbf{Q} \mathbf{x} + \frac{1}{2} \mathbf{b}^T \mathbf{b}, \end{aligned} \quad (14)$$

where $b_{ik} = -\hat{\Delta}_{ik}$ and $b_{ij} = u_{ij}$.

Therefore, Programming (12) may be rewritten as:

$$\begin{aligned} \min \quad & z = \frac{1}{2} \mathbf{x}^T \mathbf{Q}^T \mathbf{Q} \mathbf{x} - \mathbf{b}^T \mathbf{Q} \mathbf{x} + \frac{1}{2} \mathbf{b}^T \mathbf{b} \\ \text{s.t.} \quad & \mathbf{x} \geq \mathbf{0} \end{aligned} \quad (15)$$

The gradient projection methods [16, 1, 2] are efficient for the above problem. Bertsekas [16] shows a linear rate of convergence of the gradient projection method under the assumption that the smallest eigenvalue of the matrix $\mathbf{Q}^T \mathbf{Q}$ is larger than zero. In this paper we would show that the gradient projection method for Programming (15) is linearly convergent, though $\mathbf{Q}^T \mathbf{Q}$ is positive semi-definite.

3.2 Algorithm of linear convergence

Let λ_{max} be the largest eigenvalue of $\mathbf{Q}^T \mathbf{Q}$ and $\mathbf{x} \geq \mathbf{0}$ means that every element of \mathbf{x} is greater than or equal to zero. The potential difference reduction algorithm for multicommodity flow problem is as following:

Algorithm 1 Potential difference reduction algorithm

- 1: Given $\mathbf{x}^0 \geq \mathbf{0}$; $\sigma \in (0, 1)$ and set the step size $\beta \leq \frac{2\sigma}{\lambda_{max}}$
- 2: **for** $l = 0, 1, \dots$, **do**
- 3: Calculate the height $h_{ik}^l, \forall i \in V, k \in \mathcal{K}$:

$$h_{ik}^l = \sum_{j \in \delta^-(i)} f_{ji,k}^l - \sum_{j \in \delta^+(i)} f_{ij,k}^l + \hat{\Delta}_{ik}.$$

- 4: Calculate the congestion $\psi_{ij}^l, \forall (i, j) \in E$:

$$\psi_{ij}^l = \sum_{k \in \mathcal{K}} f_{ij,k}^l + r_{ij}^l - u_{ij}.$$

- 5: Calculate the potential difference $\phi_{ij,k}^l, \forall (i, j) \in E, k \in \mathcal{K}$:

$$\phi_{ij,k}^l = h_{ik}^l - h_{jk}^l - \psi_{ij}^l.$$

- 6: Update $f_{ij,k}^{l+1}, r_{ij}^{l+1}, \forall (i, j) \in E, k \in \mathcal{K}$:

$$\begin{aligned} f_{ij,k}^{l+1} &= \max\{f_{ij,k}^l + \beta \phi_{ij,k}^l, 0\} \\ r_{ij}^{l+1} &= \max\{r_{ij}^l - \beta \psi_{ij}^l, 0\}. \end{aligned}$$

- 7: **end for**
-

Note that the flow $f_{ij,k}$ is updated by adding the potential difference, but r_{ij} is updated by minus its gradient. As is shown by Equation (9), the negative gradient of the objective function of Programming (7) is exactly the same as the potential difference. Additionally, the gradient $\frac{\partial z}{\partial \mathbf{x}} = Q^T Q \mathbf{x} - Q^T b$. For simplicity, Algorithm 1 could be reduced to the much more condensed matrix form.

Algorithm 2 Potential difference reduction algorithm in matrix form

Initialization: Given $\mathbf{x}^0 \geq 0$, $\sigma \in (0, 1)$. Set the step size $\beta \leq \frac{2\sigma}{\lambda_{max}}$ and $\mathbf{y}^0 := \mathbf{x}^0$.

Iterative step: Given \mathbf{x}^l , calculate:

$$\begin{cases} \mathbf{y}^{l+1} = \mathbf{x}^l - \beta(Q^T Q \mathbf{x}^l - Q^T b) \\ \mathbf{x}^{l+1} = \max(\mathbf{y}^{l+1}, 0) \end{cases} \quad (16)$$

Remark 3 Obviously, Algorithm 1 and Algorithm 2 are exactly equivalent.

Lemma 5 $\{\mathbf{x}^l\}_{l \geq 0}$, generated by Algorithm 2, is convergent, and $\{\mathbf{x}^l\}_{l \geq 0}$ converges to a optimal solution of Programming (15).

Remark 4 Lemma 5 can be deduced directly, as a special case, from Theorem 6 in [34]. We refer the readers to [34] for more details.

Lemma 6 Let $\{\mathbf{x}^l\}_{l \geq 0}$ and $\{\mathbf{y}^l\}_{l \geq 0}$ be the sequence generated by Algorithm 2. Then there exists a real $q \in (0, 1)$ such that, for all $l \geq 0$,

$$\|\mathbf{y}^{l+2} - \mathbf{y}^{l+1}\|_2 \leq (1 - q)\|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2$$

Proof: Set $B = \beta Q^T Q$, which is a symmetric positive semi-definite matrix. Let λ_{max}^B the largest eigenvalue of B . Obviously, $\lambda_{max}^B = \beta \lambda_{max} \leq 2\sigma < 2$. So we have

$$\|(I - B)x\|_2 \leq \|x\|_2. \quad (17)$$

We denote by $\mathcal{N}(Q)$ the null space of the matrix Q and define the real number η

$$\eta = 1 - \max \left\{ \frac{\|P_{\mathcal{N}(Q)}(\mathbf{x})\|_2}{\|\mathbf{x}\|_2} \mid \mathbf{x} \geq 0, \mathbf{x} \neq 0 \right\} \quad (18)$$

where $P_{\mathcal{N}(Q)}(\mathbf{x})$ is the projection of \mathbf{x} onto $\mathcal{N}(Q)$.

Since

$$\begin{aligned} \mathbf{x}^T B \mathbf{x} &= \beta \left(\sum_{(i,j) \in E} \left(\sum_{k \in \mathcal{K}} f_{ij,k} + r_{ij} \right)^2 + \sum_{i,k} \left(\sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} \right)^2 \right) \\ &\geq \beta \left(\sum_{(i,j) \in E} \sum_{k \in \mathcal{K}} f_{ij,k}^2 + \sum_{(i,j) \in E} r_{ij}^2 \right) \\ &= \beta \mathbf{x}^T \mathbf{x}, \forall \mathbf{x} \geq 0, \end{aligned} \quad (19)$$

we have $\mathbf{x}^T B \mathbf{x} \geq \beta \mathbf{x}^T \mathbf{x} = \beta \|\mathbf{x}\|_2^2$. Let $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$ where $\mathbf{x}_0 \in \mathcal{N}(Q)$ and \mathbf{x}_1 is orthogonal to $\mathcal{N}(Q)$. Since $\mathbf{x}^T B \mathbf{x} = \mathbf{x}_1^T B \mathbf{x}_1 \leq \lambda_{max}^B \|\mathbf{x}_1\|_2^2$, we have $\beta \|\mathbf{x}\|_2^2 \leq \lambda_{max}^B \|\mathbf{x}_1\|_2^2$. That is,

$$\begin{aligned} \|\mathbf{x}_0\|_2^2 &= \|\mathbf{x}\|_2^2 - \|\mathbf{x}_1\|_2^2 \\ &\leq \left(1 - \frac{\beta}{\lambda_{max}^B}\right) \|\mathbf{x}\|_2^2 \\ &\leq \left(1 - \frac{\beta}{2}\right) \|\mathbf{x}\|_2^2 \\ &\leq \left(1 - \frac{\beta}{4}\right)^2 \|\mathbf{x}\|_2^2 \end{aligned} \quad (20)$$

By Inequality (20) and Equation (18), we have

$$\frac{\beta}{4} \leq \eta \leq 1. \quad (21)$$

Let J be a nonempty subset of $\{1, 2, \dots, N\}$ where N is the dimension of \mathbf{x} . Denote the family of all such subsets by \mathcal{J} . For each $J \in \mathcal{J}$, introduce the subspace $X_J \in R^N$,

$$X_J = \{\mathbf{x} \in R^N | \mathbf{x}_j = 0, \forall j \notin J\}. \quad (22)$$

Define $H_J = X_J \cap \mathcal{N}(Q)$ and denote by Y_J the subspace of X_J which is orthogonal to H_J . Given any $J \in \mathcal{J}$, let α_J be the largest non-negative number for which

$$\|(I - B)\mathbf{y}\|_2 \leq (1 - \alpha_J)\|\mathbf{y}\|_2, \forall \mathbf{y} \in Y_J \quad (23)$$

where I denotes the unit matrix. Clearly, α_J is actually positive.

The first line of Equation (16) could be rewritten as:

$$\mathbf{y}^{l+1} = \mathbf{x}^l - (B\mathbf{x}^l - \beta Q^T \mathbf{b}). \quad (24)$$

Therefore,

$$\mathbf{y}^{l+1} - \mathbf{y}^l = (I - B)(\mathbf{x}^l - \mathbf{x}^{l-1}). \quad (25)$$

Furthermore, if $\mathbf{y}_j^{l+1} \geq 0$ or $\mathbf{y}_j^l \geq 0$, using $\mathbf{x}^l = \max(\mathbf{y}^l, 0)$, we have

$$|\mathbf{x}_j^{l+1} - \mathbf{x}_j^l| + |\mathbf{y}_j^{l+1} - \mathbf{x}_j^{l+1}| + |\mathbf{y}_j^l - \mathbf{x}_j^l| = |\mathbf{y}_j^{l+1} - \mathbf{y}_j^l| \quad (26)$$

Since

$$\begin{aligned} \mathbf{h}^T (B\mathbf{x} - \beta Q^T \mathbf{b}) &= \mathbf{h}^T B\mathbf{x} - \mathbf{h}^T \beta Q^T \mathbf{b} \\ &= \mathbf{h}^T (\beta Q^T Q)\mathbf{x} - \mathbf{h}^T (\beta Q^T \mathbf{b}) \\ &= \beta (Q\mathbf{h})^T (Q\mathbf{x} - \mathbf{b}) \\ &= 0, \forall \mathbf{h} \in \mathcal{N}(Q), \end{aligned}$$

it follows that

$$\mathbf{h}^T (\mathbf{y}^{l+1} - \mathbf{x}^l) = 0, \forall \mathbf{h} \in \mathcal{N}(Q) \quad (27)$$

Now let $\mathbf{w}^l = \mathbf{x}^l - \mathbf{x}^{l-1}$ and define

$$J_l = \{j | \mathbf{y}_j^l \geq 0 \text{ or } \mathbf{y}_j^{l-1} \geq 0\} \quad (28)$$

Decompose \mathbf{w}^l , which clearly belongs to X_{J_l} , as $\mathbf{w}^l = \mathbf{w}^{l,0} + \mathbf{w}^{l,1}$ where $\mathbf{w}^{l,0} \in H_{J_l}$ and $\mathbf{w}^{l,1} \in Y_{J_l}$, and consider the following two cases.

Case I. $\|\mathbf{w}^{l,0}\|_2 \leq (1 - \varepsilon)\|\mathbf{w}^l\|_2$ for some $\varepsilon \in (0, 1)$.

$$\begin{aligned} \|(I - B)\mathbf{w}^l\|_2^2 &= \|(I - B)\mathbf{w}^{l,0}\|_2^2 + \|(I - B)\mathbf{w}^{l,1}\|_2^2 \\ &\leq \|\mathbf{w}^{l,0}\|_2^2 + (1 - \alpha_J)^2 \|\mathbf{w}^{l,1}\|_2^2 \\ &= (2\alpha_J - \alpha_J^2) \|\mathbf{w}^{l,0}\|_2^2 + (1 - \alpha_J)^2 \|\mathbf{w}^l\|_2^2 \\ &\leq (1 - \varepsilon)^2 (2\alpha_J - \alpha_J^2) \|\mathbf{w}^l\|_2^2 + (1 - \alpha_J)^2 \|\mathbf{w}^l\|_2^2 \\ &= (1 - \varepsilon\alpha_J(2 - \varepsilon)(2 - \alpha_J)) \|\mathbf{w}^l\|_2^2 \end{aligned} \quad (29)$$

Hence,

$$\|(I - B)\mathbf{w}^l\|_2^2 \leq (1 - \varepsilon\alpha_J) \|\mathbf{w}^l\|_2^2 \quad (30)$$

By Equation (25) and Equation (26),

$$\begin{aligned} \|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2 &= \|(I - B)(\mathbf{x}^l - \mathbf{x}^{l-1})\|_2 \\ &\leq \sqrt{(1 - \varepsilon\alpha_J)} \|\mathbf{x}^l - \mathbf{x}^{l-1}\|_2 \\ &\leq \sqrt{(1 - \varepsilon\alpha_J)} \|\mathbf{y}^l - \mathbf{y}^{l-1}\|_2 \end{aligned} \quad (31)$$

Case II. $\|\mathbf{w}^{l,0}\|_2 \geq (1 - \varepsilon)\|\mathbf{w}^l\|_2$ for some $\varepsilon \in (0, 1)$.

Rewrite $\mathbf{w}^l = \mathbf{y}^l - \mathbf{x}^{l-1} + \mathbf{x}^l - \mathbf{y}^l$. Define $\bar{\mathbf{x}}$ as following:

$$\bar{\mathbf{x}}_j = \begin{cases} \mathbf{x}_j^l - \mathbf{y}_j^l, \forall j \in J_l \\ 0, \forall j \notin J_l \end{cases}$$

By Equation (27), the vector $\mathbf{y}^l - \mathbf{x}^{l-1}$ is orthogonal to H_{J_l} . Therefore, $\mathbf{w}^{l,0} = P_{H_{J_l}}(\mathbf{x}^l - \mathbf{y}^l) = P_{H_{J_l}}(\bar{\mathbf{x}})$ is the projection of $\bar{\mathbf{x}}$ onto H_{J_l} , from which, by Equation (18), we have

$$\begin{aligned} (1 - \varepsilon)\|\mathbf{w}^l\|_2 &\leq \|\mathbf{w}^{l,0}\|_2 = \|P_{H_{J_l}}(\bar{\mathbf{x}})\|_2 \\ &\leq (1 - \eta)\|\bar{\mathbf{x}}\|_2. \end{aligned}$$

Therefore,

$$\|\mathbf{w}^l\|_2 \leq \frac{(1 - \eta)}{(1 - \varepsilon)}\|\bar{\mathbf{x}}\|_2,$$

which, together with Equation (26), implies

$$\|\mathbf{w}^l\|_2 \leq \frac{(1 - \eta)}{(1 - \varepsilon)}\|\mathbf{y}^l - \mathbf{y}^{l-1}\|_2. \quad (32)$$

By Equation (25) and Inequality (17), we have the following the inequality

$$\begin{aligned} \|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2 &= \|(I - B)\mathbf{w}^l\|_2 \\ &\leq \|\mathbf{w}^l\|_2 \leq \frac{(1 - \eta)}{(1 - \varepsilon)}\|\mathbf{y}^l - \mathbf{y}^{l-1}\|_2. \end{aligned} \quad (33)$$

Taking $\varepsilon = \frac{\eta}{1 + \alpha_J}$, we can summarize Inequalities (31) and (33) as

$$\begin{aligned} \|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2 &\leq \left(1 - \frac{\eta\alpha_J}{2(1 + \alpha_J)}\right)\|\mathbf{y}^l - \mathbf{y}^{l-1}\|_2 \\ &\leq (1 - q)\|\mathbf{y}^l - \mathbf{y}^{l-1}\|_2, \end{aligned} \quad (34)$$

where

$$q = \min \left\{ \frac{\eta\alpha_J}{2(1 + \alpha_J)} \mid \forall J \in \mathcal{J} \right\}.$$

Set $\alpha = \min\{\alpha_J \mid \forall J \in \mathcal{J}\}$, then we have

$$q = \frac{\eta\alpha}{2(1 + \alpha)} \quad (35)$$

□

Remark 5 The proof above is similar to that of Theorem 10 in [34]. The main difference, in comparison with the result in [34], is that we do not suppose here that every element of matrix Q is greater than or equal to zero.

Remark 6 Although $Q^T Q$ is positive semi-definite, by Lemma 6, Algorithm 2 is of linear convergence rate.

Theorem 3 Let $\{\mathbf{x}^l\}_{l \geq 0}$ and $\{\mathbf{y}^l\}_{l \geq 0}$ be the sequence generated by Algorithm 2. \mathbf{x}^* and \mathbf{y}^* are the limits of the two sequences, respectively. Set the step size $\beta = \frac{2\sigma}{\lambda_{max}}$. Then the following statements hold:

- (i) we can find an index L in $O\left(\frac{\lambda_{max}}{\alpha} \ln \frac{\Lambda \sum_{k \in \mathcal{K}} d_k}{\varepsilon \alpha}\right)$ iterations such that $\|\mathbf{x}^l - \mathbf{x}^*\|_2 \leq \varepsilon \forall l \geq L$;
- (ii) each iteration can be done in $O(K|E|)$ time.

Proof:

By Formulation (35) and Inequality (21),

$$q = \frac{\eta\alpha}{2(1 + \alpha)} \geq \frac{\alpha\beta}{8(1 + \alpha)} = \frac{\alpha\sigma}{4(1 + \alpha)\lambda_{max}} \quad (36)$$

By Lemma 6,

$$\|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2 \leq (1 - q)^l \|\mathbf{y}^1 - \mathbf{y}^0\|_2. \quad (37)$$

Therefore,

$$\begin{aligned}
\|\mathbf{y}^L - \mathbf{y}^*\|_2 &\leq \sum_{l=L}^{\infty} \|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2 \\
&\leq \|\mathbf{y}^1 - \mathbf{y}^0\|_2 \sum_{l=L}^{\infty} (1-q)^l \\
&= \|\mathbf{y}^1 - \mathbf{y}^0\|_2 \frac{(1-q)^L}{q}
\end{aligned} \tag{38}$$

Since $\mathbf{x}^l = \max\{\mathbf{y}^l, 0\}$, the following inequality holds,

$$\|\mathbf{x}^{l+1} - \mathbf{x}^l\|_2 \leq \|\mathbf{y}^{l+1} - \mathbf{y}^l\|_2, \tag{39}$$

together with Inequality (38), we have

$$\|\mathbf{x}^L - \mathbf{x}^*\|_2 \leq \|\mathbf{y}^1 - \mathbf{y}^0\|_2 \frac{(1-q)^L}{q} \tag{40}$$

Define the starting point as $\{f_{ij,k}^0 = 0, r_{ij}^0 = u_{ij}, \forall (i, j) \in E, k \in \mathcal{K}\}$, i.e., $\mathbf{x}^0 = \mathbf{y}^0 = \begin{pmatrix} \mathbf{f}^0 \\ \mathbf{r}^0 \end{pmatrix}$. So

$$\begin{aligned}
\|\mathbf{y}^1 - \mathbf{y}^0\|_2 &= \|\beta(Q^T Q \mathbf{x}^0 - Q^T b)\|_2 \\
&= \beta \sqrt{\sum_{k \in \mathcal{K}} \sum_{(i,j) \in E} \left(\frac{\partial z}{\partial f_{ij,k}^0}\right)^2 + \sum_{(i,j) \in E} \left(\frac{\partial z}{\partial r_{ij}^0}\right)^2} \\
&= \beta \sqrt{\sum_{k \in \mathcal{K}} \sum_{(i,j) \in E} (\psi_{ij}^0 + h_{jk}^0 - h_{ik}^0)^2 + \sum_{(i,j) \in E} (\psi_{ij}^0)^2} \\
&= \beta \sqrt{\sum_{k \in \mathcal{K}} \sum_{(i,j) \in E} (0 + h_{jk}^0 - h_{ik}^0)^2 + \sum_{(i,j) \in E} 0^2} \\
&= \beta \sqrt{\sum_{k \in \mathcal{K}} \sum_{(i,j) \in E} (\hat{\Delta}_{jk} - \hat{\Delta}_{ik})^2} \\
&\leq \beta \sum_{k \in \mathcal{K}} \sum_{(i,j) \in E} |\hat{\Delta}_{jk} - \hat{\Delta}_{ik}| \\
&\leq \beta \sum_{k \in \mathcal{K}} (2\Lambda d_k) \\
&= 2\beta\Lambda \sum_{k \in \mathcal{K}} d_k \\
&= \frac{4\sigma\Lambda \sum_{k \in \mathcal{K}} d_k}{\lambda_{max}},
\end{aligned} \tag{41}$$

where Λ is the maximum degree of Graph $G(V, E)$.

$$\frac{(1-q)^L}{q} \|\mathbf{y}^1 - \mathbf{y}^0\|_2 \leq \epsilon \tag{42}$$

is satisfied, provide that

$$L * \ln(1-q) + \ln \|\mathbf{y}^1 - \mathbf{y}^0\|_2 \leq \ln(\epsilon q). \tag{43}$$

Since $\ln(1+q) \leq q, \forall q > -1$, Inequality (43) is true if

$$L \geq \frac{1}{q} \ln \frac{\|\mathbf{y}^1 - \mathbf{y}^0\|_2}{\epsilon q}. \tag{44}$$

By Inequality (36) and (41),

$$\frac{1}{q} \ln \frac{\|\mathbf{y}^1 - \mathbf{y}^0\|_2}{\epsilon q} \leq \frac{4(1+\alpha)\lambda_{max}}{\alpha\sigma} \ln \frac{16(1+\alpha)\Lambda \sum_{k \in \mathcal{K}} d_k}{\epsilon\alpha}. \tag{45}$$

Therefore, if

$$\begin{aligned} l \geq L &= \frac{4(1+\alpha)\lambda_{max}}{\alpha\sigma} \ln \frac{16(1+\alpha)\Lambda \sum_{k \in \mathcal{K}} d_k}{\epsilon\alpha} \\ &= O\left(\frac{\lambda_{max}}{\alpha} \ln \frac{\Lambda \sum_{k \in \mathcal{K}} d_k}{\epsilon\alpha}\right), \end{aligned} \quad (46)$$

Inequality (42) holds. By Inequality (40), the first statement of this theorem holds.

Obviously, the height $\{h_{ik}, \forall i \in V, k \in \mathcal{K}\}$ can be obtained in $2K|E|$ time; the congestion function $\{\psi_{ij}, \forall (i, j) \in E\}$ can be obtained in $O(K|E|)$ time. Therefore, each iteration of Algorithm 1 can be done in $O(K|E|)$ time. So the proof is complete. \square

Remark 7 Lemma 7 would give an estimate of the upper bound of λ_{max} . However, the estimate of the lower bound of α is still an open problem.

Remark 8 Since the largest eigenvalue λ_{max} is usually unavailable, it is not known a priori how small one should take the step size β . A possible way is to substitute the λ_{max} by its upper bound in Lemma 7.

Lemma 7 Let λ_{max} be the largest eigenvalue of $Q^T Q$, then

$$\lambda_{max} \leq K + 1 + 2\Lambda, \quad (47)$$

where Λ is the maximum degree of graph $G(V, E)$.

Proof:

$$\begin{aligned} \mathbf{x}^T Q^T Q \mathbf{x} &= \sum_{(i,j) \in E} \left(\sum_{k \in \mathcal{K}} f_{ij,k} + r_{ij} \right)^2 + \sum_{i,k} \left(\sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} \right)^2 \\ &\leq \sum_{(i,j) \in E} \left((K+1) \left(\sum_{k \in \mathcal{K}} f_{ij,k}^2 + r_{ij}^2 \right) \right) + \sum_{i,k} \left(\Lambda \left(\sum_{j \in \delta^+(i)} f_{ij,k}^2 + \sum_{j \in \delta^-(i)} f_{ji,k}^2 \right) \right) \\ &= (K+1) \sum_{(i,j) \in E} \left(\sum_{k \in \mathcal{K}} f_{ij,k}^2 + r_{ij}^2 \right) + 2\Lambda \sum_{k \in \mathcal{K}} \left(\sum_{(i,j) \in E} f_{ij,k}^2 \right) \\ &\leq (K+1+2\Lambda) \left(\sum_{(i,j) \in E} \sum_{k \in \mathcal{K}} f_{ij,k}^2 + \sum_{(i,j) \in E} r_{ij}^2 \right) \\ &= (K+1+2\Lambda) \mathbf{x}^T \mathbf{x}, \forall \mathbf{x}. \end{aligned}$$

Since $\mathbf{x}^T Q^T Q \mathbf{x} \leq (k+1+2\Lambda) \mathbf{x}^T \mathbf{x}$, $\lambda_{max} \leq k+1+2\Lambda$. \square

3.3 A practical algorithm

When λ_{max} is large, the step size β is small, which would lead to slow convergence. In this section we would give a practical potential difference reduction algorithm whose step size is determined by an inexact line search method.

Let $\phi(\mathbf{f})$ be the vector of potential difference when the flow is \mathbf{f} , i.e., $\phi(\mathbf{f}) = (\dots, \phi_{ij,k}, \dots)$. And $\max(\mathbf{u}, \mathbf{v})$ denotes the component-wise maximum of \mathbf{u}, \mathbf{v} . Algorithm 3 describes the adaptive potential difference reduction algorithm.

Algorithm 3 Adaptive potential difference reduction algorithm

```

1: Set  $\beta_0 = 1, \mu = 0.5, v = 0.9$ , given  $\mathbf{f}^0$ 
2: for  $l = 0, 1, \dots$ , if the stopping criterion is not satisfied, do
3:   calculate  $\phi(\mathbf{f}^l)$  by Algorithm 4;
4:    $\hat{\mathbf{f}}^l = \max\{\mathbf{f}^l + \beta^l \phi(\mathbf{f}^l), \mathbf{0}\}$ ;
5:   calculate  $\phi(\hat{\mathbf{f}}^l)$  by Algorithm 4;
6:    $\omega^l = \beta_l \|\phi(\mathbf{f}^l) - \phi(\hat{\mathbf{f}}^l)\|_2 / \|\mathbf{f}^l - \hat{\mathbf{f}}^l\|_2$ .
7:   while  $\omega^l > v$  do
8:      $\beta^l := \beta^l * 0.8 / \omega^l$ ;
9:      $\hat{\mathbf{f}}^l = \max\{\mathbf{f}^l + \beta^l \phi(\mathbf{f}^l), \mathbf{0}\}$ ;
10:    Calculate  $\phi(\hat{\mathbf{f}}^l)$  by Algorithm 4;
11:     $\omega^l = \beta_l \|\phi(\mathbf{f}^l) - \phi(\hat{\mathbf{f}}^l)\|_2 / \|\mathbf{f}^l - \hat{\mathbf{f}}^l\|_2$ .
12:   end while
13:    $\mathbf{f}^{l+1} = \hat{\mathbf{f}}^l$ ;
14:    $\phi(\mathbf{f}^{l+1}) = \phi(\hat{\mathbf{f}}^l)$ .
15:   if  $\omega^l \leq \mu$  then
16:      $\beta_l := \beta_l * 1.5$ .
17:   end if
18: end for

```

Algorithm 4 Given $\mathbf{f} \geq \mathbf{0}$, compute potential difference

```

1: Calculate the height  $h_{ik}, \forall i \in V, k \in \mathcal{K}$ :

```

$$h_{ik} = \sum_{j \in \delta^-(i)} f_{ji,k} - \sum_{j \in \delta^+(i)} f_{ij,k} + \hat{\Delta}_{ik}. \quad (48)$$

```

2: Calculate the congestion function  $\psi_{ij}, \forall (i, j) \in E$ :

```

$$\psi_{ij} = \begin{cases} 0 & \text{if } \sum_{k \in \mathcal{K}} f_{ij,k} \leq u_{ij} \\ \sum_{k \in \mathcal{K}} f_{ij,k} - u_{ij} & \text{if } \sum_{k \in \mathcal{K}} f_{ij,k} > u_{ij}. \end{cases} \quad (49)$$

```

3: Calculate the potential difference  $\phi_{ij,k}, \forall (i, j) \in E, k \in \mathcal{K}$ :

```

$$\phi_{ij,k} = h_{ik} - h_{jk} - \psi_{ij}. \quad (50)$$

Remark 9 By Equation (9), the negative gradient of the objective function of Programming (7) is exactly the same as the potential difference. Therefore, Algorithm 3 is a gradient projection method for Programming (7). For the convergence of gradient projection method, see [16, 32].

Remark 10 The step-size rule in Algorithm 3 is to start with an initial guess for the step size and then, if some condition is not satisfied, reduced the step size successively by multiplication with a constant scalar until it meets the condition. The reducing rule for β_l is of Armijo type [33, 16].

Remark 11 Too small step size β_l will lead to slow convergence. If $\omega^l \leq \mu$, we would enlarge β_l by $\beta_l := \beta_l * 1.5$ (see [31]).

4 Further Discussion on optimality Condition

As commonly understood, directed edges correspond to pipes of the physical network and vertices are pipe junctions. In practice, for any physical network, it is impossible that the flow in a pipe exceeds its capacity. In this section we

would discuss the optimality condition for multicommodity flow problem when only relaxing the flow conservation constraints.

The basic formulation is as following:

$$\begin{aligned}
\min \quad & z = \sum_{i,k} \int_0^{\Delta_{ik}} h_{ik}(\omega) d\omega \\
\text{s.t.} \quad & \sum_{k \in \mathcal{K}} f_{ij,k} \leq u_{ij}, \forall (i,j) \in E \\
& f_{ij,k} \geq 0, \forall k \in \mathcal{K}, (i,j) \in E
\end{aligned} \tag{51}$$

where h_{ik} is the height function and Δ_{ik} defined as Expression (2).

In the programming above, the objective function is the sum of the integrals of the vertex height function. The flow conservation constraints are relaxed but the capacity constraints are maintained. Obviously, the feasible region of Expression (1) is not empty if and only if the minimum value of the objective function of Programming (51) is zero; the feasible region of Expression (1) is empty if and only if the minimum value of the objective function of Programming (51) is greater than zero. Since Programming (51) is convex, the solution f^* of Programming (51) is optimal if and only if it satisfies the following Karush-Kuhn-Tucker conditions:

Stationarity

$$-\frac{\partial z}{\partial f_{ij,k}} = -\mu_{ij,k} + \gamma_{ij}, \forall k \in \mathcal{K}, (i,j) \in E$$

Primal feasibility

$$\begin{aligned}
& \sum_{k \in \mathcal{K}} f_{ij,k} \leq u_{ij}, \forall (i,j) \in E \\
& -f_{ij,k} \leq 0, \forall k \in \mathcal{K}, (i,j) \in E
\end{aligned} \tag{52}$$

Dual feasibility

$$\begin{aligned}
& \mu_{ij,k} \geq 0, \forall k \in \mathcal{K}, (i,j) \in E \\
& \gamma_{ij} \geq 0, \forall (i,j) \in E
\end{aligned}$$

Complementary slackness

$$\begin{aligned}
& \mu_{ij,k} f_{ij,k} = 0, \forall k \in \mathcal{K}, (i,j) \in E \\
& \gamma_{ij} \left(\sum_{k \in \mathcal{K}} f_{ij,k} - u_{ij} \right) = 0, \forall (i,j) \in E.
\end{aligned}$$

Obviously,

$$\begin{aligned}
\frac{\partial z}{\partial f_{ij,k}} &= h_{ik}(\Delta_{ik}) \frac{\partial \Delta_{ik}}{\partial f_{ij,k}} + h_{jk}(\Delta_{jk}) \frac{\partial \Delta_{jk}}{\partial f_{ij,k}} \\
&= (-h_{ik}(\Delta_{ik})) + (h_{jk}(\Delta_{jk})) \\
&= h_{jk} - h_{ik}.
\end{aligned} \tag{53}$$

Substituting the expression above into Stationarity expression in KKT conditions,

$$\gamma_{ij} = \mu_{ij,k} + h_{ik} - h_{jk}, \forall k \in \mathcal{K}, (i,j) \in E.$$

If edge (i,j) is not saturated, by complementary slackness, $\gamma_{ij} = 0$. If edge (i,j) is saturated, $\gamma_{ij} \geq 0$. Intuitively, γ_{ij} could be interpreted as the *pressure* on pipes generated by the fluid. If edge (i,j) is not saturated, the *pipe pressure* is zero, otherwise it is greater than or equal to zero.

For any used edge of commodity k , i.e. $f_{ij,k} > 0$, by complementary slackness $\mu_{ij,k} f_{ij,k} = 0$ in KKT conditions, we have $\mu_{ij,k} = 0$. Therefore,

$$\gamma_{ij} = h_{ik} - h_{jk}, \forall k \in \mathcal{K}, (i,j) \in E$$

That is, the height difference between vertex i and vertex j for commodity k is equal to the pipe pressure γ_{ij} of edge (i,j) .

For any unused edge of commodity k , due to $\mu_{ij,k} \geq 0$, we have

$$\gamma_{ij} = \mu_{ij,k} + h_{ik} - h_{jk} \geq h_{ik} - h_{jk}, \forall k \in \mathcal{K}, (i, j) \in E$$

That is, the height difference between vertex i and vertex j for commodity k is less than or equal to the pipe pressure γ_{ij} .

Note that the pipe pressure γ_{ij} is zero when the edge is unsaturated. Therefore, the optimality conditions can be concluded as following:

- (i) for any used edge of commodity k , if it is unsaturated, the height difference between vertex i and vertex j for commodity k is zero;
- (ii) for any used edge of commodity k , if it is saturated, the height difference between vertex i and vertex j for commodity k is equal to the pipe pressure γ_{ij} ;
- (iii) for any unused edge of commodity k , the height difference between vertex i and vertex j for commodity k is less than or equal to the pipe pressure γ_{ij} .

5 Conclusion

This paper gives a localized method for the multicommodity flow problem. We relax both the capacity constraints and flow conservation constraints, and introduce *congestion function* ψ for each edge and *height function* h for each vertex and commodity. Based on the congestion function ψ and the height function h , a very simple and intuitive optimal condition is described for multicommodity flow problem. By the optimal condition, we give an exact algorithm based on potential difference reduction approach, which works in a localized manner and doesn't rely on any shortest path or augmenting path. Generally, we give a new idea to solve multicommodity flow problem, which would help people better understand flow problems and may inspire more research in this area.

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