

On the strength of recursive McCormick relaxations for binary polynomial optimization

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Abstract

Recursive McCormick relaxations have been among the most popular convexification techniques for binary polynomial optimization problems. It is well-understood that both the quality and the size of these relaxations depend on the recursive sequence, and finding an optimal recursive sequence amounts to solving a difficult combinatorial optimization problem. In this paper, we prove that any recursive McCormick relaxation is implied by the extended flower relaxation, a linear programming relaxation that is a natural generalization of the flower relaxation introduced by Del Pia and Khajavirad [8], which for binary polynomial optimization problems with fixed degree can be solved in strongly polynomial time.

Key words: Binary polynomial optimization, Recursive McCormick relaxations, Multilinear polytope, Extended flower relaxation, Cutting planes.

1 Binary polynomial optimization

We consider a binary polynomial optimization problem of the form:

$$\begin{aligned} \max \quad & \sum_{I \in \bar{\mathcal{I}}_0} c_I^0 \prod_{i \in I} x_i \\ \text{s.t.} \quad & \sum_{I \in \bar{\mathcal{I}}_j} c_I^j \prod_{i \in I} x_i \leq b_j, \quad \forall j \in \{1, \dots, m\} \\ & x \in \{0, 1\}^n, \end{aligned} \tag{BP}$$

where for each $j \in \{0, 1, \dots, m\}$, we denote by $\bar{\mathcal{I}}_j$ a family of nonempty subsets of $\{1, \dots, n\}$, and c_I^j , $I \in \bar{\mathcal{I}}_j$ are nonzero real-valued coefficients. We refer to each product term $\prod_{i \in I} x_i$ with $|I| \geq 2$ as a *multilinear term*, and we refer to $r = \max\{|I| : I \in \cup_{j=0}^m \bar{\mathcal{I}}_j\}$ as the degree of Problem (BP). Throughout this paper, we assume that $r \geq 3$. Problem (BP) is \mathcal{NP} -hard even when $m = 0$, and it has been the subject of extensive research by the mixed-integer nonlinear optimization (MINLP) community over the past three decades [5, 21, 19, 3, 22, 15, 2, 7, 9, 6, 8, 10, 11, 4].

In this paper, we are interested in the quality of linear programming (LP) relaxations for Problem (BP). To convexify Problem (BP), it is common practice to first linearize the objective function and all constraints by introducing a new variable y_I for every multilinear term $\prod_{i \in I} x_i$ and obtain an equivalent optimization

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problem in a lifted space:

$$\begin{aligned}
\max \quad & \sum_{I \in \tilde{\mathcal{I}}_0 \setminus \mathcal{I}_0} c_I^0 x_I + \sum_{I \in \mathcal{I}_0} c_I^0 y_I \\
\text{s.t.} \quad & \sum_{I \in \tilde{\mathcal{I}}_j \setminus \mathcal{I}_j} c_I^j x_I + \sum_{I \in \mathcal{I}_j} c_I^j y_I \leq b_j, \quad \forall j \in \{1, \dots, m\} \\
& y_I = \prod_{i \in I} x_i, \quad \forall I \in \bigcup_{j=0}^m \mathcal{I}_j \\
& x \in \{0, 1\}^n, \quad \forall i \in \{1, \dots, n\},
\end{aligned} \tag{EBP}$$

where \mathcal{I}_j , $j \in \{0, 1, \dots, m\}$ consists of all elements in $\tilde{\mathcal{I}}_j$ of cardinality at least two. Subsequently, a convex relaxation of the feasible region of Problem (EBP) is constructed and the resulting problem is solved to obtain an upper bound on the optimal value of Problem (BP). The Reformulation Linearization technique (RLT) [20] and the sum-of-squares hierarchy [17] are two general schemes for constructing LP and semi-definite programming (SDP) relaxations for polynomial optimization problems, respectively. While these methods often lead to strong bounds, they are too expensive to solve. In the following, we focus on a number of convexification techniques that generate cheap LP relaxations for Problem (EBP) by building polyhedral relaxations for the set

$$\mathcal{S} = \left\{ (x, y) : y_I = \prod_{i \in I} x_i, \forall I \in \tilde{\mathcal{I}} := \bigcup_{j=0}^m \mathcal{I}_j, x \in \{0, 1\}^n \right\}. \tag{1}$$

Throughout this paper, we refer to set \mathcal{S} defined by (1) as the *multilinear set* and we refer to its convex hull as the *multilinear polytope*. Without loss of generality, we assume that each x_i , $i \in \{1, \dots, n\}$ appears in at least one multilinear term. Moreover, we refer to $r = \max\{|I| : I \in \tilde{\mathcal{I}}\}$ as the *rank* of the multilinear set. Notice that the rank of a multilinear set equals to the degree of the corresponding binary polynomial optimization problem.

1.1 Recursive McCormick relaxations

Motivated by factorable programming techniques [16], a polyhedral relaxation of the multilinear set defined by (1) can be constructed by a recursive application of bilinear envelopes [1]. In the following we give a brief overview of this method in its full generality:

- *Initialization.* Initialize the sets $\mathcal{R}_I = \{I\}$ for all $I \in \tilde{\mathcal{I}}$, and $\tilde{\mathcal{K}} = \emptyset$. Mark each $L \in \mathcal{R}_I$ for all $I \in \tilde{\mathcal{I}}$ as *unchecked*.
- *Recursive Decomposition.* For each $I \in \tilde{\mathcal{I}}$, and for each $L \in \mathcal{R}_I$ such that $L \in \{I\} \cup \tilde{\mathcal{K}}$, consider the following cases:
 - If L is unchecked, then denote by $P_L = \{J, K\}$ a partition of L ; that is, J, K are nonempty, $J \cup K = L$ and $J \cap K = \emptyset$. If $|J| > 1$ (resp. $|K| > 1$), let $\mathcal{R}_I = \mathcal{R}_I \cup \{J\}$ (resp. $\mathcal{R}_I = \mathcal{R}_I \cup \{K\}$). Moreover, if $|J| > 1$ and $J \notin \tilde{\mathcal{I}} \cup \tilde{\mathcal{K}}$ (resp. $|K| > 1$ and $K \notin \tilde{\mathcal{I}} \cup \tilde{\mathcal{K}}$), then introduce the new *artificial variable* $y_J = \prod_{i \in J} x_i$, update $\tilde{\mathcal{K}} = \tilde{\mathcal{K}} \cup \{J\}$, and mark J as unchecked (resp. $y_K = \prod_{i \in K} x_i$, update $\tilde{\mathcal{K}} = \tilde{\mathcal{K}} \cup \{K\}$, and mark K as unchecked). Finally, mark L as *checked*.
 - If L is checked, then consider its partition $P_L = \{J, K\}$, constructed above. If $|J| > 1$ (resp. $|K| > 1$), let $\mathcal{R}_I = \mathcal{R}_I \cup \{J\}$ (resp. $\mathcal{R}_I = \mathcal{R}_I \cup \{K\}$).

It then follows that for each $L \in \mathcal{R}_I$ with $P_L = \{J, K\}$, we have $y_L = y_J y_K$, where we define $y_{\{i\}} := x_i$ for all $i \in \{1, \dots, n\}$. Therefore, the multilinear set \mathcal{S} can be written, in a lifted space of variables, as a collection of bilinear equations of the form $y_{J_t \cup K_t} = y_{J_t} y_{K_t}$, $t \in \{1, \dots, T\}$. Subsequently a polyhedral relaxation of this set is obtained by replacing each bilinear equation by its convex hull over the unit hypercube:

$$\begin{aligned}
y_{J_t \cup K_t} &\geq 0, & y_{J_t \cup K_t} &\geq y_{J_t} + y_{K_t} - 1, & \forall t \in \{1, \dots, T\}. \\
y_{J_t \cup K_t} &\leq y_{J_t}, & y_{J_t \cup K_t} &\leq y_{K_t},
\end{aligned} \tag{2}$$

This simple relaxation technique is used extensively in the literature and is often referred to as the *recursive McCormick relaxation* (RMC). In the remainder of this paper, given an RMC of the multilinear set \mathcal{S} , we refer to \mathcal{R}_I as the *recursive sequence* of the multilinear term $y_I = \prod_{i \in I} x_i$, $I \in \tilde{\mathcal{I}}$ and we refer to \mathcal{R}_I , $I \in \tilde{\mathcal{I}}$ as the recursive sequence of \mathcal{S} . Moreover, by the *size* of an RMC, we imply the number of its artificial variables; i.e., $|\tilde{\mathcal{K}}|$.

For instance, the global solver BARON [14] uses an RMC scheme in which for each $y_{J_t \cup K_t} = y_{J_t} y_{K_t}$, it chooses J_t, K_t such that $|J_t| = 1$ (see [23] for further details). For example, consider the multilinear set $\{(x, y) : y_{1234} = x_1 x_2 x_3 x_4, x \in \{0, 1\}^4\}$; then the factorable decomposition module in BARON constructs the following bilinear equations $y_{1234} = x_1 y_{234}$, $y_{234} = x_2 y_{34}$, $y_{34} = x_3 x_4$; in this case, the recursive sequence of y_{1234} is given by $\mathcal{R}_{1234} = \{1234, 234, 34\}$ and artificial variables of this RMC are y_{234}, y_{34} .

For a multilinear set, there exist many different recursive sequences, in general. It is well-understood that the choice of the recursive sequence impacts both the size and the quality of the resulting RMC. In [18], the authors study the impact of the recursive sequence on the size of and the quality of the resulting RMC. First, the authors show that even for multilinear sets of rank three, the problem of finding a minimum-size RMC is NP-hard. Moreover, they propose a mixed-integer programming (MIP) formulation to solve this problem. Subsequently, they propose a MIP formulation to identify RMCs of a given size with the best possible upper bound. Their numerical experiments suggest that their proposed RMCs often outperform the simpler RMCs implemented in global solvers. To better illustrate this point, we use the running example of [18] which we will later use to explain our results as well:

Example 1. Consider the problem of maximizing $f(x) = -x_1 x_2 x_3 + x_2 x_3 x_4 + x_1 x_3 x_4$ over $x \in \{0, 1\}^4$. It can be checked that if we solve an RMC with the recursive sequence

$$\mathcal{R}_{123}^1 = \{123, 12\}, \mathcal{R}_{234}^1 = \{234, 23\}, \mathcal{R}_{134}^1 = \{134, 13\},$$

we obtain the upper bound $U = \frac{4}{3}$, while if we solve an RMC with the recursive sequence

$$\mathcal{R}_{123}^2 = \{123, 13\}, \mathcal{R}_{234}^2 = \{234, 34\}, \mathcal{R}_{134}^2 = \{134, 34\},$$

we obtain the upper bound $U = 1.0$, which is sharp. The recursive sequence \mathcal{R}^2 outperforms \mathcal{R}^1 since the artificial variable y_{34} introduced in \mathcal{R}^2 appears in the recursive sequence of two multilinear terms $y_{234} = x_2 x_3 x_4$, $y_{134} = x_1 x_3 x_4$, while all artificial variables of \mathcal{R}^1 appear in the recursive sequence of one multilinear term.

1.2 Multilinear polytopes and hypergraph acyclicity

In [7], we introduce a hypergraph representation of multilinear sets which in turn enables us to characterize multilinear polytopes of hypergraphs with various degrees of acyclicity [12] in the space of original variables [8, 10]. As a result, we obtain strong polyhedral relaxations for general multilinear sets [11]. In the following, we briefly review two polyhedral relaxations of multilinear sets and introduce a new stronger polyhedral relaxation; in the next section, we compare these relaxations with RMCs.

Before proceeding further, we introduce our hypergraph terminology. Recall that a hypergraph G is a pair (V, E) where $V = V(G)$ is the set of nodes of G , and $E = E(G)$ is a set of subsets of V of cardinality at least two, called the edges of G . With any hypergraph G , we associate a multilinear set \mathcal{S}_G defined as follows:

$$\mathcal{S}_G = \left\{ z \in \{0, 1\}^d : z_e = \prod_{v \in e} z_v, e \in E \right\}, \quad (3)$$

where $d = |V| + |E|$. We denote by MP_G the convex hull of \mathcal{S}_G . Note that variables z_v , $v \in V$ in (3) correspond to variables x_i , $i \in \{1, \dots, n\}$ in (1) and variables z_e , $e \in E$ in (3) correspond to variables y_I , $I \in \tilde{\mathcal{I}}$ in (1).

The standard linearization. A simple polyhedral relaxation of \mathcal{S}_G can be obtained by replacing each multilinear term $z_e = \prod_{v \in e} z_v$, by its convex hull over the unit hypercube:

$$\begin{aligned} \text{MP}_G^{\text{LP}} = \left\{ z : \right. & z_v \leq 1, \forall v \in V, \\ & z_e \geq 0, z_e \geq \sum_{v \in e} z_v - |e| + 1, \forall e \in E, \\ & \left. z_e \leq z_v, \forall v \in e, \forall e \in E \right\}. \end{aligned} \quad (4)$$

The above relaxation has been used extensively in the literature and is often referred to as the *standard linearization* of the multilinear set (cf. [13, 5]). In [8], we prove that $\text{MP}_G^{\text{LP}} = \text{MP}_G$ if and only if G is a Berge-acyclic hypergraph; i.e., the most restrictive type of hypergraph acyclicity [12]. For general hypergraphs however, the standard linearization often provides a weak relaxation of multilinear sets. As we show in the next section, the standard linearization corresponds to the projection of an RMC, if the common product terms across various multilinear terms are not exploited. The following example demonstrates this result:

Example 2. Consider the binary polynomial optimization problem considered in Example 1. It can be checked that using the standard linearization of the corresponding multilinear set, we obtain an upper bound $U = \frac{4}{3}$, which is equal to the bound obtained using the RMC with no artificial variable appearing in the recursive sequence of more than one multilinear term.

The flower relaxation and the extended flower relaxation. In [8], we introduce the flower relaxation of multilinear sets, a polyhedral relaxation that is often significantly stronger than the standard linearization. To formally define this relaxation, we first need to introduce flower inequalities [8]. Let e_0 be an edge of G and let $e_k, k \in K$, be the set of all edges adjacent to e_0 with

$$|e_0 \cap e_k| \geq 2, \quad \forall k \in K. \quad (5)$$

Let T be a nonempty subset of K such that

$$e_0 \cap e_i \cap e_j = \emptyset, \quad \forall i \neq j \in T. \quad (6)$$

Then the *flower inequality* centered at e_0 with neighbors $e_k, k \in T$, is given by:

$$\sum_{v \in e_0 \setminus \cup_{k \in T} e_k} z_v + \sum_{k \in T} z_{e_k} - z_{e_0} \leq |e_0 \setminus \cup_{k \in T} e_k| + |T| - 1. \quad (7)$$

We refer to inequalities of the form (7), for all nonempty $T \subseteq K$ satisfying conditions (5) and (6), as the system of flower inequalities centered at e_0 . We define the *flower relaxation* MP_G^F as the polytope obtained by adding the system of flower inequalities centered at each edge of G to MP_G^{LP} . In [8], we prove that $\text{MP}_G^F = \text{MP}_G$ if and only if G is a γ -acyclic hypergraph. We should remark that γ -acyclic hypergraphs represent a significant generalization of Berge-acyclic hypergraphs [12]. As we detail in [11], while the separation problem over the flower relaxation is NP-hard for general hypergraphs, it can be solved in $O(|E|^2)$ time for hypergraphs with fixed rank.

Example 3. Consider the binary polynomial optimization problem considered in Example 1. It can be checked that using the flower relaxation of the corresponding multilinear set, we obtain an upper bound $U = 1.0$, which coincides with the best RMC bound.

While in Example 3, the flower relaxation is as good as the best RMC, this is not always the case. Next, we introduce a stronger polyhedral relaxation of multilinear sets that dominates all RMC relaxations. To this end, we first present a class of valid inequalities for multilinear sets that can be considered as a generalization of flower inequalities. Let e_0 be an edge of G and as before, denote by $e_k, k \in K$, the set of all edges adjacent to e_0 . Let T be a nonempty subset of K such that

$$\gamma_i := \left| (e_0 \cap e_i) \setminus \bigcup_{j \in T \setminus \{i\}} (e_0 \cap e_j) \right| \geq 2, \quad \forall i \in T. \quad (8)$$

Then the *extended flower inequality* centered at e_0 with neighbors e_k , $k \in T$, is given by (7). Similarly, we define the *extended flower relaxation* MP_G^{EF} as the polytope obtained by adding the system of extended flower inequalities centered at each edge of G to MP_G^{LP} . Notice that conditions (5) and (6) imply condition (8); namely, flower inequalities are a special case of extended flower inequalities. Indeed, extended flower inequalities are a significant generalization of flower inequalities as they subsume cases in which $e_0 \cap e_i \cap e_j \neq \emptyset$ for some $i \neq j \in T$.

We should remark that in the special case where the neighbours e_k , $k \in T$ satisfy the so-called *running intersection property*, it can be checked that *running intersection inequalities* [10] imply extended flower inequalities. However, in general, extended flower inequalities are not implied by running intersection inequalities. The following example illustrates these facts:

Example 4. Consider the hypergraph $G = (V, E)$ with $V = \{v_1, \dots, v_9\}$ and $E = \{e_0, e_1, e_2, e_3\}$, where $e_0 = V$, $e_1 = \{v_1, \dots, v_4\}$, $e_2 = \{v_4, \dots, v_7\}$, and $e_3 = \{v_1, v_7, v_8, v_9\}$. Then all flower inequalities of \mathcal{S}_G are given by:

$$\begin{aligned} \sum_{v \in e_0 \setminus e_i} z_v + z_{e_i} - z_{e_0} &\leq 5, & \forall i \in \{1, 2, 3\}, \\ z_{e_0} - z_{e_i} &\leq 0, \end{aligned} \quad (9)$$

and the remaining running intersection inequalities of \mathcal{S}_G are given by

$$-z_{e_i \cap e_j} + \sum_{v \in e_0 \setminus (e_i \cup e_j)} z_v + z_{e_i} + z_{e_j} - z_{e_0} \leq 2, \quad \forall i \neq j \in \{1, 2, 3\}. \quad (10)$$

Moreover, in addition to inequalities (9), we have the following extended flower inequalities:

$$\sum_{v \in e_0 \setminus (e_i \cup e_j)} z_v + z_{e_i} + z_{e_j} - z_{e_0} \leq 3, \quad \forall i \neq j \in \{1, 2, 3\} \quad (11)$$

$$z_{e_1} + z_{e_2} + z_{e_3} - z_{e_0} \leq 2. \quad (12)$$

First notice that using inequalities $z_v \leq 1$ for all $v \in V$, which are present in the standard linearization of \mathcal{S}_G , we deduce that extended flower inequalities (11) are implied by the running intersection inequalities (10). However, the extended flower inequality (12) is not implied by any other inequalities. To see this consider the point \tilde{z}_p , $p \in V \cup E$ defined as:

$$\begin{aligned} \tilde{z}_v &= 1, & \forall v \in \bar{V} := \{v_1, v_4, v_7\} \\ \tilde{z}_v &= \tilde{z}_e = \frac{3}{4}, & \forall v \in V \setminus \bar{V}, \forall e \in \{e_1, e_2, e_3\} \\ \tilde{z}_{e_0} &= 0. \end{aligned} \quad (13)$$

Clearly, \tilde{z} violates the extended flower inequality (12), since $3(\frac{3}{4}) \not\leq 2$. However, it can be checked that \tilde{z} satisfies all inequalities present in the standard linearization of \mathcal{S}_G , as well as flower inequalities (9) and running intersection inequalities (10). This is due to the fact that while the set $\{e_i, e_j\}$, has the running intersection property for all $i \neq j \in \{1, 2, 3\}$, the set $\{e_1, e_2, e_3\}$ do not have the running intersection property and hence no running intersection inequality can be generated for the latter. See [10] for details regarding running intersection inequalities.

Remark 1. It is important to note that assumption (8) is without loss of generality. To see this, consider an edge e_0 and a subset of adjacent edges e_k , $k \in T$ some of which do not satisfy assumption (8). The following cases arise:

- $\gamma_i = 0$ for some $i \in T$. In this case, the extended flower inequality is implied by two inequalities: (i) $z_{e_i} \leq 1$, and (ii) the extended flower inequality centered at e_0 with neighbours e_k , $k \in T \setminus \{i\}$.
- $\gamma_i = 1$ for some $i \in T$. In this case, the extended flower inequality is implied by two inequalities: (i) $z_{e_i} \leq z_{\bar{v}}$, where $\{\bar{v}\} = e_0 \cap e_i$, and (ii) the extended flower inequality centered at e_0 with neighbours e_k , $k \in T \setminus \{i\}$.

Hence we conclude that to construct the extended flower relaxation, it suffices to consider neighbours e_k , $k \in T$ satisfying assumption (8).

We conclude this section by commenting on complexity of optimizing over the extended flower relaxation.

Separating over extended flower inequalities. In the following, we present a separation algorithm over extended flower inequalities that runs in strongly polynomial-time for hypergraphs $G = (V, E)$ of fixed degree r ; i.e., in a number of iterations bounded by a polynomial in $|V|$ and $|E|$. By equivalence of separation and optimization, this in turn implies that optimizing over the extended flower relaxation can be done in polynomial time. The fixed degree assumption is reasonable as for binary polynomial optimization problems, $|V|$ and $|E|$ represent the number of variables and multilinear terms, respectively, while r is the degree of the polynomial. Indeed, for all problems appearing in applications, while we have hundreds or thousands of variables, the degree of the polynomial is a small number; namely, $r \ll 10$. In [10], we present an efficient separation algorithm for running-intersection inequalities. While the extended flower inequalities do not imply and are not implied by running-intersection inequalities, they can be separated using similar ideas. Hence, in the following, we provide a sketch of our separation algorithm and refer the reader to [10] for further details.

Let us start by formally defining the separation problem: Given a hypergraph $G = (V, E)$, and a vector $\bar{z} \in [0, 1]^{V+E}$, decide whether \bar{z} satisfies all extended flower inequalities or not, and in the latter case, find an extended flower inequality that is violated by \bar{z} .

Consider an edge $e_0 \in E$, and denote by E_{e_0} the subset of E containing, for every $f \subseteq e_0$ with $|f| \geq 2$, among all the edges $e \in E \setminus \{e_0\}$ with $e_0 \cap e = f$, only one that maximizes \bar{z}_e . It then follows that in order to solve the separation problem over all extended flower inequalities in G centered at e_0 , it suffices to do the separation over the (much smaller) set of extended flower inequalities centered e_0 , with neighbors e_k contained in E_{e_0} . In the following, we assume that a rank- r hypergraph $G = (V, E)$ is represented by an incidence-list in which every edge stores the vertices it contains. We assume the vertex list for each edge is sorted, otherwise, such a representation can be constructed in $O(r|E|)$ operations using some integer sorting algorithm. We are now ready to present our separation algorithm:

Proposition 1. *Given a rank- r hypergraph $G = (V, E)$ and a vector $\bar{z} \in [0, 1]^{V+E}$, there exists an algorithm that solves the separation problem in $O(|E|(r2^r|E| + r^22^{r^2/2}))$ operations.*

Proof. Let $e_0 \in E$; we present the separation algorithm over all extended flower inequalities centered at e_0 . By applying the algorithm $|E|$ times, we can solve the separation problem over all extended flower inequalities in G .

Observe that by assumption (8), the number of neighbours t in an extended flower inequality does not exceed $\frac{r}{2}$. First we construct the set E_{e_0} defined above. For a rank- r hypergraph G , the number of elements in E_{e_0} is at most $2^r - r - 1$. It can be checked that the set E_{e_0} can be constructed in $O(r2^r|E|)$ operations. It then follows that the total number of possible sets of neighbors of cardinality at most $\frac{r}{2}$ is given by $N = \sum_{i=1}^{r/2} \binom{2^r - r - 1}{i} \leq (2^r - r)^{r/2}$. For each possible set of t neighbors, we can check the validity of condition (8) in $O(tr)$ operations. Finally, for each valid set of t neighbors, we generate the corresponding unique extended flower inequality in $O(t+r)$ operations. Therefore, the total running time of the separation algorithm is given by $O(|E|(r2^r|E| + r^22^{r^2/2}))$. \square

Hence by Proposition 1, for a fixed-rank hypergraph $G = (V, E)$, the separation problem over all extended flower inequalities can be solved in $O(|E|^2)$ operations. As we show in the next section, the extended flower relaxation implies *all* RMCs of a multilinear set. This is an important result as it indicates instead of solving a MIP to identify an optimal RMC and subsequently solving an LP in a lifted space (due to the addition of artificial variables), one can solve the extended flower relaxation efficiently and obtain no worse (in fact, often better) bounds.

2 A theoretical comparison of the relaxations

In this section, we present a theoretical comparison of RMCs, the standard linearization, and the extended flower relaxation of multilinear sets. Let $G = (V, E)$ be a hypergraph, and consider an RMC for the

multilinear set \mathcal{S}_G . We denote by $z_e, e \in \bar{E}$ the set of all artificial variables corresponding to the RMC. From the definition of RMCs it follows that $E \cap \bar{E} = \emptyset$, and $|e| \geq 2$ for all $e \in \bar{E}$. We denote by MP_G^{RMC} the projection of the RMC onto the original space; i.e., the space of variables $z_v, v \in V$ and $z_e, e \in E$.

2.1 Recursive McCormik relaxations vs the standard linearization

Given an RMC of a multilinear set \mathcal{S}_G , in the following we characterize the conditions under which optimizing over this RMC is equivalent to optimizing over the standard linearization MP_G^{LP} defined by (4). We say that an RMC for \mathcal{S}_G with the recursive sequence $\mathcal{R}_e, e \in E$ is *non-overlapping*, if $\mathcal{R}_e \cap \mathcal{R}_{e'} = \emptyset$ for all $e \neq e' \in E$. For example, the RMC of Example 1 with the recursive sequence \mathcal{R}^1 is non-overlapping since $\mathcal{R}_{123}^1 \cap \mathcal{R}_{234}^1 = \mathcal{R}_{123}^1 \cap \mathcal{R}_{134}^1 = \mathcal{R}_{134}^1 \cap \mathcal{R}_{234}^1 = \emptyset$, whereas, the recursive sequence \mathcal{R}^2 is not non-overlapping, since $\mathcal{R}_{234}^2 \cap \mathcal{R}_{134}^2 = \{34\}$. The following proposition states the relationship between RMCs and the standard linearization of multilinear sets.

Proposition 2. *Let $G = (V, E)$ be a hypergraph and consider an RMC for the multilinear set \mathcal{S}_G . Then $\text{MP}_G^{\text{RMC}} = \text{MP}_G^{\text{LP}}$, if and only if the RMC is non-overlapping.*

Proof. First suppose that the RMC is non-overlapping with the recursive sequence $\mathcal{R}_e, e \in E$. Consider an edge $\tilde{e} \in E$. By the non-overlapping assumption, we have the following two properties:

- There exists no $e \in \mathcal{R}_{\tilde{e}}$ such that $e \neq \tilde{e}$ and $e \in E$, as otherwise we have $\mathcal{R}_{\tilde{e}} \cap \mathcal{R}_e = e$, which contradicts the non-overlapping assumption.
- For any $e' \in \mathcal{R}_{\tilde{e}}$, we have $e' \notin \mathcal{R}_e$ for all $e \in E \setminus \{\tilde{e}\}$.

By above observations, for each $\tilde{e} \in E$, all variables $z_{\tilde{e}}, \tilde{e} \in \mathcal{R}_{\tilde{e}} \setminus \{\tilde{e}\}$ are artificial variables and these variables do not appear in any other \mathcal{R}_e with $e \neq \tilde{e}$. Hence, for each $e \in E$, our task is to consider inequalities containing only variables $z_e, e \in \mathcal{R}_e$ and to project out all $z_{\tilde{e}}$ with $\tilde{e} \subset e$ from these inequalities. Since the standard linearization is obtained by replacing each $z_e = \prod_{v \in e} z_v, z_v \in \{0, 1\}, e \in E$, by its convex hull, we deduce that $\text{MP}_G^{\text{LP}} \subseteq \text{MP}_G^{\text{RMC}}$. Hence it suffices to show that $\text{MP}_G^{\text{RMC}} \subseteq \text{MP}_G^{\text{LP}}$. Consider a point $\tilde{z}_p, p \in V \cup E$ in MP_G^{RMC} . Recall that $z_e, e \in \bar{E}$ denotes the artificial variables of the RMC. By definition of MP_G^{RMC} there exists $\tilde{z}_e, e \in \bar{E}$ such that $\tilde{z}_p, p \in V \cup E \cup \bar{E}$ satisfies the RMC. First, from all inequalities of the form $z_{p_1 \cup p_2} \leq z_{p_1}, z_{p_1 \cup p_2} \leq z_{p_2}$ in this RMC, we deduce that $\tilde{z}_e \leq \tilde{z}_v$ for all $v \in e$ and for all $e \in E$. Second, from all inequalities of the form $z_{p_1 \cup p_2} \geq z_{p_1} + z_{p_2} - 1$ and $z_{p_1 \cup p_2} \leq z_{p_1}$, we deduce that $\tilde{z}_v \leq 1$ for all $v \in V$. Finally, from all inequalities of the form $z_{p_1 \cup p_2} \geq z_{p_1} + z_{p_2} - 1$ in this RMC we deduce that $\tilde{z}_e \geq \sum_{v \in e} \tilde{z}_v - |e| + 1$ for all $e \in E$. Notice that the RMC includes inequalities $z_e \geq 0$ for all $e \in E$. We conclude that $\tilde{z}_p, p \in V \cup E$ is present in MP_G^{LP} . Therefore, $\text{MP}_G^{\text{RMC}} \subseteq \text{MP}_G^{\text{LP}}$, implying if the RMC is non-overlapping, we have $\text{MP}_G^{\text{RMC}} = \text{MP}_G^{\text{LP}}$.

Next, suppose that the RMC is not non-overlapping; i.e., $\mathcal{R}_{e'} \cap \mathcal{R}_{e''} = \tilde{e}$ for some $e', e'' \in E$ and some $\tilde{e} \in E \cup \bar{E}$. Two cases arise:

- If $\tilde{e} \in E$, then without loss of generality, let $\tilde{e} = e''$. It then follows that the RMC implies the inequality $z_{e'} \leq z_{e''}$, while this inequality is not implied by the standard linearization. To see this, consider the point $\tilde{z}_v = \tilde{z}_e = \frac{1}{2}$ for all $v \in V, e \in E \setminus \{e''\}$, and $\tilde{z}_{e''} = 0$. It can be checked that $\tilde{z} \in \text{MP}_G^{\text{LP}}$, while $\tilde{z}_{e'} > \tilde{z}_{e''}$.
- If $\tilde{e} \in \bar{E}$, then the RMC implies the following inequalities $z_{e'} \leq z_{\tilde{e}}$ and $z_{e''} \geq z_{\tilde{e}} + \sum_{v \in e'' \setminus \tilde{e}} z_v - |e'' \setminus \tilde{e}|$. Projecting out $z_{\tilde{e}}$ from these two inequalities, we conclude that the inequality

$$\sum_{v \in e'' \setminus \tilde{e}} z_v + z_{e'} - z_{e''} \leq |e'' \setminus \tilde{e}|, \quad (14)$$

is implied by MP_G^{RMC} . We argue that the above inequality is not implied by MP_G^{LP} . To see this, consider the point \tilde{z} defined as: $\tilde{z}_v = 1$ for all $v \in e'' \setminus \tilde{e}$, $\tilde{z}_v = \frac{1}{2}$ for all $v \in (e' \setminus e'') \cup \tilde{e}$, $\tilde{z}_v = 0$ for the remaining nodes in G , $\tilde{z}_{e'} = \frac{1}{2}$, $\tilde{z}_{e''} = 0$, $\tilde{z}_e = 1$ for all $e \subseteq e'' \setminus \tilde{e}$, $\tilde{z}_e = 0$ for all $e \not\subseteq e' \cup e''$ and $\tilde{z}_e = \frac{1}{2}$ for all remaining edges in G . This point does not satisfy inequality (14), as $|e'' \setminus \tilde{e}| + \frac{1}{2} - 0 \not\leq |e'' \setminus \tilde{e}|$. However, it can be checked that \tilde{z} belongs to MP_G^{LP} , provided that $|\tilde{e} \cap e''| \geq 2$, which always holds since $\tilde{e} \subset e''$ and $|\tilde{e}| \geq 2$.

By (i) and (ii) above, it follows that if $MP_G^{\text{RMC}} = MP_G^{\text{LP}}$, then the RMC is non-overlapping, and this completes the proof. \square

Roughly speaking, Proposition 2 indicates that the standard linearization is equivalent to the *weakest* type of RMCs; i.e., RMCs not exploiting common product terms in various multilinear terms $z_e = \prod_{v \in e} z_v$. Namely, when such common expressions exist, one can always construct RMCs that are stronger than the standard linearization (see Examples 1 and 2).

2.2 Recursive McCormik relaxation vs the extended flower relaxation

We now present the main the result of this paper. Namely, the extended flower relaxation implies all RMCs of a multilinear set. In fact, as we show via an example, the extended flower relaxation often provides a stronger relaxation than the best RMC.

Proposition 3. *Let $G = (V, E)$ be a hypergraph, and consider the multilinear set \mathcal{S}_G . Then $MP_G^{\text{EF}} \subseteq MP_G^{\text{RMC}}$ for all RMCs of \mathcal{S}_G .*

Proof. Consider an RMC of \mathcal{S}_G ; we assume that this RMC does not satisfy the non-overlapping property as otherwise the result follows from Proposition 2. For each $e \in E$ and each artificial edge $\bar{e} \in \mathcal{R}_{\bar{e}} \cap \bar{E}$, define

$$\mathcal{P}_{\bar{e}, \bar{e}} := \{e \in E \setminus \{\bar{e}\} : \bar{e} \in \mathcal{R}_e\}. \quad (15)$$

For each $e \in E$, denote by T_e a partition of e such that each $p \in T_e$ satisfies $p \in e \cup \mathcal{R}_e$ and for $p \in T_e \cap \bar{E}$, we have $\mathcal{P}_{e,p} \neq \emptyset$. Denote by $w_{e,p}$ be an element of $\mathcal{P}_{e,p}$. Recall that by definition of partition we have $p \cap p' = \emptyset$ for all $p, p' \in T_e$ and $\cup_{p \in T_e} p = e$. We refer to any such partition T_e of e as a *flower partition* of e . Denote by \mathcal{T}_e the set consisting of all flower partitions of e . Let \bar{T}_e be a subset of T_e consisting of all edges corresponding to artificial variables. To complete the proof it suffices to prove the following claim:

Claim 1. *The polytope MP_G^{RMC} is defined by the following inequalities:*

$$z_v \leq 1, \quad \forall v \in V, \quad z_e \geq 0, \quad \forall e \in E \quad (16)$$

$$z_e \leq z_p, \quad \forall p \in e \cup (\mathcal{R}_e \cap E), \quad \forall e \in E \quad (17)$$

$$\sum_{p \in T_e \setminus \bar{T}_e} z_p + \sum_{p \in \bar{T}_e} z_{w_{e,p}} - z_e \leq |T_e| - 1, \quad \forall T_e \in \mathcal{T}_e, \quad \forall e \in E. \quad (18)$$

All above inequalities are implied by MP_G^{EF} . That is, inequalities (16) are present in the standard linearization; inequalities (17) are flower inequalities centered at p , if $p \in E$, and are present in the standard linearization, if $p \in e$. Inequalities (18) are extended flower inequalities (or are implied by extended flower inequalities, see Remark 1) centered at e , if there exists some $p \in T_e$ with $|p| \geq 2$, and are present in the standard linearization otherwise, as in the latter case they simplify to $\sum_{v \in e} z_v - z_e \leq |e| - 1$.

To prove Claim 1 we need to introduce some notation. Consider an artificial variable $z_{\bar{e}}$, $\bar{e} \in \bar{E}$. We define the *level* of $z_{\bar{e}}$ as

$$t_{\bar{e}} := \max_{e \in E: \bar{e} \in \mathcal{R}_e} \left| \{e' \in \mathcal{R}_e \cap \bar{E} : \bar{e} \supseteq e'\} \right|.$$

Note that from the definition of t_e it follows that $1 \leq t_e \leq r - 2$ for all $e \in \bar{E}$, where as before r denotes the rank of \mathcal{S}_G . Moreover, we define *the level of an RMC* as

$$t := \max_{e \in \bar{E}} t_e,$$

and we define $t := 0$, if the RMC contains no artificial variables.

Proof of Claim 1. We prove by induction on level t of the RMC:

The base case. Consider an RMC of level zero; that is, the RMC does not contain any artificial variables. Then all inequalities defining the RMC are of the following forms:

$$\begin{aligned} z_{p_1 \cup p_2} &\geq 0, \\ z_{p_1 \cup p_2} - z_{p_1} &\leq 0, \\ z_{p_1 \cup p_2} - z_{p_2} &\leq 0, \\ z_{p_1} + z_{p_2} - z_{p_1 \cup p_2} &\leq 1, \end{aligned}$$

where $p_1, p_2 \in V \cup E$ and $p_1 \cup p_2 \in E$. All above inequalities are present in System (16)-(18). The first inequality is of the form (16), the second and third inequalities are of the form (17) and, the fourth inequality is of the form (18) with $\tilde{T}_e = \emptyset$ and $|T_e| = 2$ for all $e \in E$.

The inductive step. Now consider an RMC of level k for the multilinear set \mathcal{S}_G , $G = (V, E)$ with the artificial variables z_e , $e \in \tilde{E}$. Let z_e , $e \in \tilde{E}$ denote all level-1 artificial variables in this RMC. If $\tilde{E} = \emptyset$, then the proof follows from the base case. Henceforth, suppose that $\tilde{E} \neq \emptyset$. It can be checked that this RMC is equivalent to an RMC of level $k - 1$ for the multilinear set $\mathcal{S}_{G'}$, $G' = (V, E')$, where $E' = E \cup \tilde{E}$ and with the artificial variables z_e , $e \in \tilde{E} \setminus \tilde{E}$. Therefore, by the induction hypothesis, the polytope $\text{MP}_{G'}^{\text{RMC}}$ is defined by the following inequalities:

$$\begin{aligned} z_v &\leq 1, \quad \forall v \in V \\ z_e &\geq 0, \quad \forall e \in E \cup \tilde{E} \\ z_e &\leq z_p, \quad \forall p \in e \cup (\mathcal{R}_e \cap (E \cup \tilde{E})), \quad \forall e \in E \cup \tilde{E} \\ \sum_{p \in T_e \setminus \tilde{T}_e} z_p + \sum_{p \in \tilde{T}_e} z_{w_{e,p}} - z_e &\leq |T_e| - 1, \quad \forall T_e \in \mathcal{T}_e, \quad \forall e \in E \cup \tilde{E}, \end{aligned} \tag{19}$$

where as before $w_{e,p}$ is an element of $\mathcal{P}_{e,p}$ defined by (15). It can be checked that any inequality in System (19) not containing any artificial variable z_e , $e \in \tilde{E}$ is also present in System (16)-(18). Hence to construct MP_G^{RMC} , it suffices to project out the remaining artificial variables z_e , $e \in \tilde{E}$ from System (19). Notice that all artificial edges $e \in \tilde{E}$ are level-1 edges; i.e., for each $e \in \tilde{E}$, we have $e = p_1 \cup p_2$ for some $p_1, p_2 \in V \cup E$. As a result, there exists no $e \neq e' \in \tilde{E}$ such that $e' \in \mathcal{R}_e$. Now consider the RMC of $\mathcal{S}_{G'}$ defined above and for each $e \in \tilde{E}$, consider a flower partition T_e of e for which there exists some $\tilde{e} \in \tilde{E}$ satisfying $\tilde{e} \in T_e$. Define $\tilde{T}_e := T_e \cap \tilde{E}$. It can be checked that T_e is also a flower partition of e in the RMC for \mathcal{S}_G with the only difference that the subset of T_e containing edges of the artificial variables are $\tilde{T}_e \cup \tilde{T}_e$. It then follows that any artificial variable $z_{\tilde{e}}$, $\tilde{e} \in \tilde{E}$ appears in the following inequalities of System (19):

$$z_{\tilde{e}} \geq 0 \tag{20}$$

$$z_{\tilde{e}} \leq z_{p_1}, \quad z_{\tilde{e}} \leq z_{p_2} \tag{21}$$

$$z_{p_1} + z_{p_2} - z_{\tilde{e}} \leq 1 \tag{22}$$

$$z_e \leq z_{\tilde{e}}, \quad \forall e \in E : \tilde{e} \in \mathcal{R}_e \tag{23}$$

$$\sum_{p \in \tilde{T}_e} z_p + \sum_{p \in T_e \setminus \tilde{T}_e} z_p + \sum_{p \in \tilde{T}_e \setminus \tilde{T}_e} z_{w_{e,p}} - z_e \leq |T_e| - 1, \quad \forall T_e : \tilde{e} \in \tilde{T}_e, \quad \forall e \in E : \tilde{e} \in \mathcal{R}_e \tag{24}$$

Next, using Fourier-Motzkin elimination, we project out $z_{\tilde{e}}$, $\tilde{e} \in \tilde{E}$ from System (20)-(24). To this end, it suffices to consider the following cases:

Inequalities (20)-(23): first consider inequalities (20) and (21); projecting out $z_{\tilde{e}}$ we obtain $z_{p_1} \geq 0$ and $z_{p_2} \geq 0$, both of which are implied by inequalities (16) and (17). Next, consider inequalities (21) and (22); projecting out $z_{\tilde{e}}$ we obtain $z_{p_1} \leq 1$ and $z_{p_2} \leq 1$; again these inequalities are implied by inequalities (16) and (17). Finally, we consider inequalities (21) and (23). Projecting out $z_{\tilde{e}}$ we obtain

$$z_e \leq z_{p_1}, \quad z_e \leq z_{p_2}, \quad \forall e \in E : \tilde{e} \in \mathcal{R}_e.$$

Recall that $\tilde{e} = p_1 \cup p_2$ for some $p_1, p_2 \in V \cup E$. Since $\tilde{e} \in \mathcal{R}_e$, from the definition of \mathcal{R}_e it follows that, if $|p_1| > 1$ (resp. $|p_2| > 1$), then $p_1 \in \mathcal{R}_e$ (resp. $p_2 \in \mathcal{R}_e$). Hence, the above inequalities are present among inequalities (17).

Inequalities (22)- (24): We now project out $z_{\bar{e}}$, $\bar{e} \in \tilde{T}_e$ from inequalities (24). First note that if we use inequality (20) to project some $z_{\bar{e}}$ from (24), we obtain a redundant inequality, since the left-hand side of the resulting inequality cannot exceed the right-hand side. Next, consider inequalities (23) for some $\bar{e} \in \tilde{T}_e$; If $\{e \in E : \bar{e} \in \mathcal{R}_e\} = \{\hat{e}\}$, then projecting $z_{\bar{e}}$ from (23) and (24) gives the redundant inequality:

$$\sum_{p \in \tilde{T}_e \setminus \{\hat{e}\}} z_p + \sum_{p \in T_e \setminus \tilde{T}_e} z_p + \sum_{p \in \tilde{T}_e \setminus \tilde{T}_e} z_{w_{e,p}} \leq |T_e| - 1.$$

Hence, we use inequalities (23) to project out an artificial variable $z_{\bar{e}}$ only if $|e \in E : \bar{e} \in \mathcal{R}_e| \geq 2$.

Now let $\tilde{T}_e = \tilde{T}_e^1 \cup \tilde{T}_e^2$, for any (possibly empty) \tilde{T}_e^1 and \tilde{T}_e^2 . Let $\bar{e} = p_1^{\bar{e}} \cup p_2^{\bar{e}}$ for any $\bar{e} \in \tilde{T}_e^1$, where $p_1^{\bar{e}}, p_2^{\bar{e}} \in V \cup E$. We now project out artificial variables $z_{\bar{e}}$, $\bar{e} \in \tilde{T}_e$ from inequalities (24), using inequalities (22) for all $\bar{e} \in \tilde{T}_e^1$, and using inequalities (23) for all $\bar{e} \in \tilde{T}_e^2$, to obtain:

$$\sum_{\bar{e} \in \tilde{T}_e^1} (z_{p_1^{\bar{e}}} + z_{p_2^{\bar{e}}}) + \sum_{\bar{e} \in \tilde{T}_e^2} z_{f_{\bar{e}}} + \sum_{p \in T_e \setminus \tilde{T}_e} z_p + \sum_{p \in \tilde{T}_e \setminus \tilde{T}_e} z_{w_{e,p}} - z_e \leq |T_e| + |\tilde{T}_e^1| - 1, \quad (25)$$

for all T_e such that $\tilde{T}_e \neq \emptyset$ and for all $e \in E$ such that $\mathcal{R}_e \cap \tilde{E} \neq \emptyset$, where we define $f_{\bar{e}}$ an edge in E different from e such that $\bar{e} \in \mathcal{R}_{f_{\bar{e}}}$. Now consider the flower partition T_e of e and define a new flower partition T'_e of e , where each $\bar{e} \in \tilde{T}_e^1$ is replaced by $p_1^{\bar{e}}, p_2^{\bar{e}}$. Since by definition $\bar{e} = p_1^{\bar{e}} \cup p_2^{\bar{e}}$, it follows that $|T'_e| = |T_e| + |\tilde{T}_e^1|$. Hence, any inequality of the form (25) is present among inequalities (18), or by Remark 1 is implied by these inequalities and this completes the proof. \square

We conclude this paper by showing via a simple example that the extended flower relaxation (even the flower relaxation) often provides a stronger relaxation than any RMC.

Example 5. Consider the hypergraph $G = (V, E)$ with $V = \{v_1, \dots, v_8\}$ and $E = \{e_0, e_1, e_2, e_3, e_4\}$, where $e_0 = \{v_1, \dots, v_4\}$, $e_1 = \{v_1, v_2, v_5\}$, $e_2 = \{v_2, v_3, v_6\}$, $e_3 = \{v_3, v_4, v_7\}$, and $e_4 = \{v_1, v_4, v_8\}$. Then the flower relaxation of \mathcal{S}_G contains the following six flower inequalities centered at e_0 :

$$\sum_{v \in e_0 \setminus e_i} z_v + z_{e_i} - z_{e_0} \leq 2, \quad i \in \{1, \dots, 4\} \quad (26)$$

$$z_{e_i} + z_{e_j} - z_{e_0} \leq 1, \quad (i, j) \in \{(1, 3), (2, 4)\}. \quad (27)$$

Now consider an RMC of \mathcal{S}_G . Then the above inequalities are implied by this RMC, only if $\mathcal{R}_{e_0} \supseteq \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}\}$. However this is not possible as the set of all possible recursive sequences of z_{e_0} are given by

- (i) $\{\{v_1, v_2, v_3, v_4\}, \{v_i, v_j, v_k\}, \{v_i, v_j\}\}$ for all $i \neq j \neq k \in \{1, \dots, 4\}$, which implies only one of the flower inequalities of (26), centered at e_0 , and
- (ii) $\{\{v_1, v_2, v_3, v_4\}, \{v_i, v_j\}, \{v_k, v_l\}\}$ for all $i \neq j \neq k \neq l \in \{1, \dots, 4\}$, which implies two flower inequalities of (26) and one flower inequality of (27).

References

- [1] F.A. Al-Khayyal and J.E. Falk. Jointly constrained biconvex programming. *Mathematics of Operations Research*, 8:273–286, 1983.
- [2] X. Bao, A. Khajavirad, N.V. Sahinidis, and M. Tawarmalani. Global optimization of nonconvex problems with multilinear intermediates. *Mathematical Programming Computation*, pages 1–37, 2014.
- [3] E. Boros and P.L. Hammer. Pseudo-boolean optimization. *Discrete Applied Mathematics*, 123(1):155–225, 2002.
- [4] R. Chen, S. Dash, and O. Günlük. Cardinality constrained multilinear sets. In *International Symposium on Combinatorial Optimization*, pages 54–65. Springer, 2020.

- [5] Y. Crama. Concave extensions for non-linear 0–1 maximization problems. *Mathematical Programming*, 61:53–60, 1993.
- [6] Y. Crama and E. Rodríguez-Heck. A class of valid inequalities for multilinear 0 – 1 optimization problems. *Discrete Optimization*, 25:28–47, 2017.
- [7] A. Del Pia and A. Khajavirad. A polyhedral study of binary polynomial programs. *Mathematics of Operations Research*, 42(2):389–410, 2017.
- [8] A. Del Pia and A. Khajavirad. The multilinear polytope for acyclic hypergraphs. *SIAM Journal on Optimization*, 28:1049–1076, 2018.
- [9] A. Del Pia and A. Khajavirad. On decomposability of multilinear sets. *Mathematical Programming*, 170:387–415, 2018.
- [10] A. Del Pia and A. Khajavirad. The running intersection relaxation of the multilinear polytope. *Mathematics of Operations Research*, 46:1008–1037, 2018.
- [11] A. Del Pia, A. Khajavirad, and N. Sahinidis. On the impact of running intersection inequalities for globally solving polynomial optimization problems. *Mathematical programming computation*, 12(2):165–191, 2020.
- [12] Ronald Fagin. Degrees of acyclicity for hypergraphs and relational database schemes. *Journal of the ACM (JACM)*, 30(3):514–550, 1983.
- [13] F. Glover and E. Woolsey. Converting the 0-1 polynomial programming problem to a 0-1 linear program. *Operations research*, 22(1):180–182, 1974.
- [14] A. Khajavirad and N. V. Sahinidis. A hybrid LP/NLP paradigm for global optimization relaxations. *Mathematical Programming Computation*, 10(3):383–421, 2018.
- [15] J. Luedtke, M. Namazifar, and J.T. Linderoth. Some results on the strength of relaxations of multilinear functions. *Mathematical Programming*, 136:325–351, 2012.
- [16] G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part i—convex underestimating problems. *Mathematical programming*, 10(1):147–175, 1976.
- [17] P. A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical programming*, 96(2):293–320, 2003.
- [18] A. U. Raghunathan, C. Cardonha, D. Bergman, and C. J. Nohra. Recursive mccormick linearization of multilinear programs. *arXiv preprint arXiv:2207.08955*, 2022.
- [19] H.S. Ryou and N.V. Sahinidis. Analysis of bounds for multilinear functions. *Journal of Global Optimization*, 19:403–424, 2001.
- [20] H. D. Sherali and C. H. Tuncbilek. A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique. *Journal of Global Optimization*, 2:101–112, 1992.
- [21] H.D. Sherali. Convex envelopes of multilinear functions over a unit hypercube and over special discrete sets. *Acta Mathematica Vietnamica*, 22:245–270, 1997.
- [22] M. Tawarmalani, J-P. Richard, and C. Xiong. Explicit convex and concave envelopes through polyhedral subdivisions. *Mathematical Programming*, 138:531–577, 2013.
- [23] M. Tawarmalani and N. V. Sahinidis. Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Mathematical programming*, 99(3):563–591, 2004.