A Jacobi-type Newton method for Nash equilibrium problems with descent guarantees

L. F. Bueno* G. Haeser† O. Kolossoski‡

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Abstract

A common strategy for solving an unconstrained two-player Nash equilibrium problem with continuous variables is applying Newton's method to the system obtained by the corresponding first-order necessary optimality conditions. However, when taking into account the game dynamics, it is not clear what is the goal of each player when considering they are taking their current decision following Newton's iterates. In this paper we provide an interpretation for Newton's iterate as follows: instead of minimizing the quadratic approximation of the objective functions parameterized by the other player current decision (the Jacobi-type strategy), we show that the Newton iterate follows this approach but with the objective function parameterized by a prediction of the other player action. This interpretation allows us to present a new Newtonian algorithm where a backtracking procedure is introduced in order to guarantee that the computed Newtonian directions, for each player, are descent directions for the corresponding parameterized functions. Thus, besides favoring global convergence, our algorithm also favors true minimizers instead of maximizers or saddle points, unlike the standard Newton method, which does not consider the minimization structure of the problem in the non-convex case. Thus, our method is more robust compared to other Jacobi-type strategies or the pure Newtonian approach, which is corroborated by our numerical experiments. We also present a proof of the well-definiteness of the algorithm under some standard assumptions, together with a preliminary analysis of its convergence properties taking into account the game dynamics.

Keywords: Nash equilibrium, Newtonian method, Jacobi-type methods, Non-convex game, Game dynamics.

1 Introduction

The Nash Equilibrium Problem (NEP), introduced in [20], models an economic game where each player aims at minimizing their own objective function, but the decision taken

*Institute of Science and Technology, Federal University of São Paulo, São José dos Campos, SP, Brazil. (lfelipebueno@gmail.com)

†Department of Applied Mathematics, University of São Paulo, São Paulo, SP, Brazil. (ghaeser@ime.usp.br)

‡Department of Applied Mathematics, University of São Paulo, São Paulo, SP, Brazil. (OliverKolossoski@gmail.com)

by each player influences the payoff of the other players. This is a fundamental problem in economics and social behavior theory that has been studied extensively. See, for instance, [1], [19] and [2]. For surveys on the subject, see [9] and [12]. In this paper we consider the unconstrained two-player NEP given by

$$\underset{x_1 \in \mathbb{R}^{n_1}}{\text{Minimize}} \quad f_1(x_1, x_2), \tag{1}$$

and

$$\underset{x_2 \in \mathbb{R}^{n_2}}{\text{Minimize}} \quad f_2(x_1, x_2), \tag{2}$$

where $f_1, f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ are (possibly non-convex) twice continuously differentiable functions that describe each player's objective to be minimized, parameterized by the other player's decision. A (local) solution to the NEP (1-2) is a point $(x_1^*, x_2^*) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that x_1^* is a (local) minimizer for (1) with $x_2 = x_2^*$, and x_2^* is a (local) minimizer for (2) with $x_1 = x_1^*$.

In order to solve the NEP (1-2), some studies deal with a reformulation of the problem, such as a variational inequality problem or as a complementarity problem; see for instance [18] and [11]. One problem with this approach is that several solutions of the reformulated problem may not be solutions of the associated equilibrium problem. Another possibility is to reformulate the NEP as an optimization problem via the Nikaido-Isoda function (see [21]), which transforms the NEP into a minimax problem. A penalty update scheme was also proposed in the works of [13] and [10]. Those indirect approaches are somewhat computationally demanding, as they require solving a nontrivial optimization problem at each step of the algorithm. In order to avoid such heavy computations, the method proposed in this paper aims at solving the system (1-2) without rewriting it as an indirect problem.

One may also consider partitioning methods, namely Jacobi- and Gauss-Seidel-type splitting methods (see [9] and [28]) that deal with problem (1-2) by iterating the idea of fixing the variables of one player and taking a step towards minimizing the problem of the other player. In particular, a standard Jacobi strategy would fix preliminary decisions for each player, and then iteratively update them simultaneously by solving a model for the corresponding optimization problem parameterized by the fixed decision of the other player. Alternatively, so-called best response methods fix the decision of each player as the best possible response, see [6] and [7]. The advantage of these approaches is that they are able to treat problem (1-2) in its original form, without losing information due to reformulations. However, these methods also require solving a general optimization problem at each step for finding the best response for the other player before solving the corresponding optimization problem for one player, which may also be computationally expensive, in addition to having very restrictive convergence conditions.

The minimization structure plays a key role when optimization problems are non-convex. Several important applications result in such problems, including NEPs (see [8]). Moreover, the dynamics of the algorithms, namely, the way that the iterates are computed, may capture some elements of the true dynamics of the game, reflecting the way a competitive game actually unfolds in practice. That is, we suspect that economic systems behave in practice as a dynamical system, where each player considers the current decision of the other players and a model for predicting their future behavior, before optimizing their strategy. We expect that Jacobi-type methods should capture this behavior, predicting accurately the final outcome of a game, even if no equilibrium is reached. In fact, differently from what is usual in an algorithm for solving a single optimization problem, where we expect and hope to build a sequence converging to a solution of the problem, in a NEP viewed as such dynamical system, the dynamics may converge to an equilibrium state, but other interesting phenomena may be observed such as orbiting, cycling or divergence. The reader should have this observation in mind when considering what type of theoretical property should be expected by an algorithm aiming to capture such dynamics.

In this paper we introduce a method based on a Jacobi-type approach within a Newtonian framework. The main novelty of our approach is that instead of taking a decision based on the current decision of the opponent, our Newton model is built around a prediction of the opponent step. This should reflect the reality of some equilibrium games in practice, where each player tries to make the best decision based on what is the expected response of the other player. The algorithm proposed in this paper is posed in such a way that each player aims at decreasing their objective function, with respect to the predicted decision of the other player, using quadratic approximations. This avoids the heavy computations of best response methods, while possibly still reflecting the dynamics of equilibrium games in practice.

Additionally, we note that our method considers the true minimization structure of the system (1-2) in a Newtonian framework, instead of considering only the system of equations given by the necessary optimality conditions associated with each problem. The idea is that one should favor minimizers in place of maximizers or saddle points when considering the solutions of this system of equations, which is usually not considered in other Newtonian algorithms for solving non-convex instances of (1-2).

In a related work, a Jacobi-type approach using quadratic approximations of players' objective functions was considered in [28]. The proposed algorithm consists in a trust region method which does not directly use the Hessian information on its iterative step; however, it relies on some strong assumptions on the true Hessians that we do not rely on in our work. Additionally, for each player, the method in [28] is based on the current response of the other players, rather than on a predicted one.

The paper is organized as follows: In Section 2 we introduce our Jacobi-type algorithm in a Newtonian framework. In Section 3 we establish some notations and summarize the overall assumptions to be considered throughout the paper, together with some preliminary results. In Section 4 we establish the well-definiteness of our algorithm. In Section 5 we state some theoretical properties of the algorithm, establishing convergence to stationarity under certain conditions. In section 6 we illustrate the behavior and effectiveness of the proposed method with some numerical experiments. Finally, some concluding remarks and future prospective works are given in Section 7.

Notation: Given a twice continuously differentiable function $f : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$, $(x_1, x_2) \mapsto f(x_1, x_2)$, for $i = 1, 2$ we denote by $\nabla_{x_i} f(\overline{x})$ its partial gradient with respect to variable x_i evaluated at $\overline{x}$. The partial Hessian with respect to variables x_i and x_j evaluated at $\overline{x}$ is denoted by $\nabla^2_{x_i x_j} f(\overline{x})$, for $i, j = 1, 2$. We use $\| \cdot \|$ to denote the 2-norm of vectors and matrices, while $u^T v$ denotes the canonical inner product of vectors u and v of the same dimensions. We denote by $\mathbb{N}$ the set of positive integers.

2 A Descent Newton Algorithm for the Two-Player Nash Equilibrium Problem

Recall that a solution (x_1, x_2) to the equilibrium problem (1-2) satisfies the following first-order necessary optimality condition:

$$\begin{pmatrix} \nabla_{x_1} f_1(x_1, x_2) \\ \nabla_{x_2} f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

Most algorithms for solving the NEP (1-2) rely on approaching directly the nonlinear system of equations (3), disregarding the minimization structure of problem (1-2). This poses no issues when f_1 and f_2 are convex with respect to their decision variables, however, in the presence of non-convexities, the algorithm may find local maximizers or saddle points as often as local minimizers. Our goal in this paper is to propose an algorithm that takes into account the minimization structure of the problem in order to find local minimizers more often, thus better reflecting the behavior of two-player games in practice.

Given a current iterate (x_1^k, x_2^k), the classical Newtonian step (d_1^k, d_2^k) for solving (3) is given by solving the following linear system:

$$\begin{pmatrix} \nabla^2_{x_1 x_1} f_1(x_1^k, x_2^k) & \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k) \\ \nabla^2_{x_1 x_2} f_2(x_1^k, x_2^k) & \nabla^2_{x_2 x_2} f_2(x_1^k, x_2^k) \end{pmatrix} \begin{pmatrix} d_1^k \\ d_2^k \end{pmatrix} = - \begin{pmatrix} \nabla_{x_1} f_1(x_1^k, x_2^k) \\ \nabla_{x_2} f_2(x_1^k, x_2^k) \end{pmatrix}. \tag{4}$$

It is well known that, for optimization problems, applying Newton's method to the corresponding first-order necessary optimality conditions can be better interpreted as minimizing a quadratic approximation of the objective function. With this interpretation, we can extend Newton's method to non-convex problems, and we can also provide elements for globalization schemes. Two well-stablished strategies for doing this are the use of trust regions and line search techniques, ensuring progress near the current point. The first one guarantees the existence of a minimizer for the quadratic approximation of the objective function by establishing a compact domain for the subproblem. In line search approaches, one should use some positive definite approximation for the Hessian of the objective function to obtain a coercive model and descent directions. In this case, one must carefully choose a convenient step-size along the Newton direction.

In order to exploit the minimization structure of the NEP (1-2), we shall consider an idea which resembles a standard line-search globalization described above. Given an approximate solution (x_1^k, x_2^k) at some iteration k, the most natural step to be made by the first player would be associated with minimizing, with respect to variable d_1, a simplified model for the function $f_1(x_1^k + d_1, x_2^k)$. Namely, the first player would select its response by computing

$$\underset{d_1}{\text{Minimize}} \quad \phi_1(x_1^k + d_1, x_2^k), \tag{5}$$

where ϕ_1 is an approximation of f_1.

Similarly, the second player would seek to minimize at the same time a model for the function $f_2(x_1^k, x_2^k + d_2)$ with respect to d_2, leading to

$$\underset{d_2}{\text{Minimize}} \quad \phi_2(x_1^k, x_2^k + d_2). \tag{6}$$

This approach is related to Jacobi's method and its theoretical investigation has been conducted in [28] under a trust-region globalization. The main concern that we want to address is to base the minimization model for each player on a prediction of the other player's action, rather than on its actual decision. Thus, the main contribution of this work is considering the case when the direction d_1^k, the decision of the first player, is obtained by a Newton step with respect to a predicted behavior of the second player. Namely, given $x_2^k + d_2^k$, a predicted decision for player two, the problem we aim to solve for player one is:

$$\underset{d_1}{\text{Minimize}} \quad \phi_1(x_1^k + d_1, x_2^k + d_2^k). \tag{7}$$

Inspired by a Newtonian appeal, let us consider the following convex quadratic approximation:

$$f_1(x_1^k + d_1, x_2^k + d_2^k) \approx f_1(x_1^k, x_2^k + d_2^k) + \nabla_{x_1} f_1(x_1^k, x_2^k + d_2^k)^T d_1 + \frac{1}{2} d_1^T H_1^k d_1, \tag{8}$$

where H_1^k is a positive definite approximation for the Hessian $\nabla^2_{x_1 x_1} f_1(x_1^k, x_2^k + d_2^k)$. Since d_2^k needs to be found together with d_1^k, suppose that the same approach is done simultaneously by the other player in order to make their decision d_2^k. Then, given (x_1^k, x_2^k), we would still have a Nash equilibrium problem in the variables (d_1, d_2). However, the resulting NEP is simpler to solve, since the objective functions are parameterized convex quadratics in the corresponding decision variables. But, since the term $\nabla_{x_1} f_1(x_1^k, x_2^k + d_2^k)^T d_1$ might still combine the variables in a non-linear way, the approximated problem might still be hard to solve. Thinking in eliminating this inconvenience with a typical idea from Newton's method, we use linear approximations of the gradients in the following way:

$$\nabla_{x_1} f_1(x_1^k, x_2^k + d_2^k) \approx \nabla_{x_1} f_1(x_1^k, x_2^k) + \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k) d_2^k. \tag{9}$$

In this way, our approximated problem (7) is now clearly defined by combining approximations (8) and (9), which has a simple structure and its solution is readily given by solving the linear system:

$$H_1^k d_1 + \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k) d_2^k + \nabla_{x_1} f_1(x_1^k, x_2^k) = 0.$$

Repeating the approach for the second player, with a positive definite $H_2^k \approx \nabla^2_{x_2 x_2} f_2(x_1^k + d_1^k, x_2^k)$, our iteration is based on the solution of the following linear system:

$$\begin{pmatrix} H_1^k & \nabla^2_{x_1x_2} f_1(x_1^k, x_2^k) \\ \nabla^2_{x_1x_2} f_2(x_1^k, x_2^k) & H_2^k \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \begin{pmatrix} \nabla_{x_1} f_1(x_1^k, x_2^k) \\ \nabla_{x_2} f_2(x_1^k, x_2^k) \end{pmatrix}. \tag{10}$$

Given the equivalence with the standard Newtonian system (4), it is natural to choose $H_1^k \approx \nabla^2_{x_1x_1} f_1(x_1^k, x_2^k)$ and $H_2^k \approx \nabla^2_{x_2x_2} f_2(x_1^k, x_2^k)$. Thus, we have associated the standard Newtonian system (4) with the original NEP, where every player makes its decision considering the predicted decision of the other player. It now seems natural to consider descent conditions associated with the predicted functions $f_1(x_1, x_2^k + d_2^k)$ and $f_2(x_1^k + d_1^k, x_2)$. However, we cannot guarantee that the resulting directions from (10) are descent directions for these functions, as is shown in the following example.

Example 2.1 *Consider the equilibrium problem where* $f_1(x_1, x_2) := \frac{1}{2}x_1^2 + (x_2^2 + 2x_2 + 1)x_1$ *and* $f_2(x_1, x_2) := \frac{1}{2}(x_2 + 2)^2$. *In this case, we have*

- $\nabla_{x_1} f_1(x_1, x_2) = x_1 + x_2^2 + 2x_2 + 1$, $\nabla^2_{x_1x_1} f_1(x_1, x_2) = 1$, $\nabla^2_{x_1x_2} f_1(x_1, x_2) = 2x_2 + 2$
- $\nabla_{x_2} f_2(x_1, x_2) = x_2 + 2$, $\nabla^2_{x_2x_2} f_2(x_2, x_2) = 1$ *and* $\nabla^2_{x_1x_2} f_2(x_2, x_2) = 0$.

Taking $(x_1^k, x_2^k) := (0, 0)$, *the Newton direction* (d_1^k, d_2^k) *obtained from* (10) *is the solution of:*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So $d_1^k = 3$ *and* $d_2^k = -2$. *Note that* $\nabla_{x_1} f_1(x_1^k, x_2^k) = \nabla_{x_1} f_1(x_1^k, x_2^k + d_2^k) = 1$ *and therefore* d_1^k *is not a descent direction for* $f_1(x_1, x_2^k + d_2^k)$ *(nor for* $f_1(x_1, x_2^k)$*) at* x_1^k.

This means that we do not expect the descent condition $f_1(x_1^k + td_1^k, x_2^k + d_2^k) < f_1(x_1^k, x_2^k + d_2^k)$ to hold even for arbitrarily small t. Therefore, an algorithm that requires an Armijo-like line search could be not well-defined. The reason for not obtaining a descent direction when minimizing the convex quadratic approximation of the function comes from the fact that we did not use its true gradient. That is, the approximation of $\nabla_{x_1} f_1(x_1^k, x_2^k + d_2^k)$ in (9) might not be accurate enough when $\|d_2^k\|$ is large. To improve this approximation, we should consider also taking small steps along the direction d_2^k. In this way, the line search should be done simultaneously for f_1 and f_2, considering a simultaneous backtracking in d_1^k and d_2^k. Thus, replacing d_2^k by td_2^k in the previous derivation of (10), we arrive at the following system of equations for computing (d_1, d_2), given a tentative step length t:

$$\begin{pmatrix} H_1^k & t\nabla^2_{x_1x_2} f_1(x_1^k, x_2^k) \\ t\nabla^2_{x_1x_2} f_2(x_1^k, x_2^k) & H_2^k \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \begin{pmatrix} \nabla_{x_1} f_1(x_1^k, x_2^k) \\ \nabla_{x_2} f_2(x_1^k, x_2^k) \end{pmatrix}.$$

Note that for $t > 0$ sufficiently small, we expect the solution (d_1, d_2) to be such that d_1 is a descent direction for problem (1) while d_2 is a descent direction for problem (2) considering the predicted decision for the other player. Hence, our method favors local minimizers for solving problem (1-2). Notice, however, that the solution (d_1, d_2) depends on the choice of the step length t, thus, once a step length is rejected by the descent condition, a new direction (d_1, d_2) must be recomputed, similarly to a trust region approach. Finally, once the descent condition is met for some t, the new iterate is defined as $(x_1^{k+1}, x_2^{k+1}) := (x_1^k, x_2^k) + t(d_1, d_2)$.

Our algorithm selects the stepsizes based on a standard Armijo-type condition, namely, checking whether the decrease along a component (with the other component fixed at the predicted point) is proportional to what is predicted by the first order approximation of the function. In mathematical terms, for some $\alpha \in (0,1)$, we perform the check

$$\begin{aligned} f_1(x_1^k + td_1, x_2^k + td_2) &\le f_1(x_1^k, x_2^k + td_2) + \alpha t \nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1, \\ f_2(x_1^k + td_1, x_2^k + td_2) &\le f_2(x_1^k + td_1, x_2^k) + \alpha t \nabla_{x_2} f_2(x_1^k + td_1, x_2^k)^T d_2, \end{aligned}$$

decreasing t and recomputing the directions whenever one of these inequalities do not hold.

Before formally presenting the algorithm, let us discuss some other conditions needed in the line search procedure. Differently from optimization problems, since the directions are not necessarily descent directions when t is large, we must ensure a negative slope over the search direction when computing the step-size t. In doing so, we are lead to also avoiding directions that are near orthogonal to the gradient of the predicted function. Additionally, in order to avoid stagnation when far from stationary points, we also demand that the direction is not too small compared with the corresponding gradient of the predicted function.

Mathematically, for $\theta \in (0,1)$ and $\gamma > 0$, our line search checks also if

$$\begin{aligned} \nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1 &\le -\theta \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \cdot \|d_1\|, \\ \|d_1\| &\ge \gamma \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\|, \end{aligned} \tag{11}$$

with a similar requirement for the second player. Notice that, in optimization, similar inequalities are standard requirements on the direction for guaranteeing global convergence of a descent method using an Armijo line search.

Guided by all these ideas, we formalize below the algorithm for computing a sequence $\{(x_1^k, x_2^k)\}_{k\in\mathbb{N}}$ for solving the NEP (1-2), where we note that the algorithm includes a safeguarding strategy that replaces the mixed hessians with zeroes whenever stationarity for the corresponding player has been reached and t is small enough, which will be explained later.

Algorithm 1 Jacobi-type descent Newton algorithm

Step 0. Given $(x_1^0, x_2^0) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, let $\alpha \in (0,1)$, $\tau \in (0,1]$, $\theta > 0, \gamma > 0$ and set $k := 0$.

Step 1. Compute $g_1^k := \nabla_{x_1} f_1(x_1^k, x_2^k)$ and $g_2^k := \nabla_{x_2} f_2(x_1^k, x_2^k)$, choose symmetric positive definite matrices H_1^k and H_2^k, and set $t := 1$.

Step 2. Set

$$\begin{aligned} M_1 &:= \begin{cases} \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k), & \text{if } \|g_1^k\| > 0 \text{ or } t > \tau, \\ 0, & \text{otherwise;} \end{cases} \\ M_2 &:= \begin{cases} \nabla^2_{x_1 x_2} f_2(x_1^k, x_2^k), & \text{if } \|g_2^k\| > 0 \text{ or } t > \tau, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{12}$$

Step 3. Check if the matrix

$$\begin{bmatrix} H_1^k & tM_1 \\ tM_2 & H_2^k \end{bmatrix}$$

is non-singular. If not, set $t := t/2$ and repeat Step 3.

Step 4. Find (d_1, d_2) by solving the linear system

$$\begin{pmatrix} H_1^k & tM_1 \\ tM_2 & H_2^k \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \begin{pmatrix} g_1^k \\ g_2^k \end{pmatrix}. \tag{13}$$

Step 5. Check the inequalities

$$f_1(x_1^k + td_1, x_2^k + td_2) \leq f_1(x_1^k, x_2^k + td_2) + \alpha t \nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1, \tag{14}$$

$$\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1 \leq -\theta \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \cdot \|d_1\|, \tag{15}$$

$$\gamma \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \cdot \|g_1^k\| \leq \|d_1\| \cdot \|g_1^k\|; \tag{16}$$

$$f_2(x_1^k + td_1, x_2^k + td_2) \leq f_2(x_1^k + td_1, x_2^k) + \alpha t \nabla_{x_2} f_2(x_1^k + td_1, x_2^k)^T d_2, \tag{17}$$

$$\nabla_{x_2} f_2(x_1^k + td_1, x_2^k)^T d_2 \leq -\theta \|\nabla_{x_2} f_2(x_1^k + td_1, x_2^k)\| \cdot \|d_2\|, \tag{18}$$

$$\gamma \|\nabla_{x_2} f_2(x_1^k + td_1, x_2^k)\| \cdot \|g_2^k\| \leq \|d_2\| \cdot \|g_2^k\|. \tag{19}$$

If one of the inequalities (14)-(19) do not hold, set $t := t/2$ and go to Step 2.

Step 6. Set $t^k := t, d_1^k := d_1, d_2^k := d_2$ and update $x_1^{k+1} := x_1^k + t^k d_1^k$, $x_2^{k+1} := x_2^k + t^k d_2^k$, and $k := k + 1$. Go to Step 1.

Notice that for finding the direction (d_1, d_2), we use the linear system (13), which for $i = 1, 2$ replaces the mixed Hessian matrix $t\nabla^2_{x_1 x_2} f_i(x_1^k, x_2^k)$, used in the previous discussion when deducing (10), with zeroes whenever $g_i^k = 0$ and $t \leq \tau$. This is done in order to force d_i to be zero if stationarity is already reached for player i and the step size is sufficiently small. As a consequence, a standard Newton step is set for the other player. The reason for doing this is that in these situations it may not be possible to obtain a function decrease for player i. We also replaced the aforementioned condition (11), which controls the ratio between gradient and direction, by (16). Both conditions are the same when $g_1^k \neq 0$, but using (16) ensures that when stationarity is reached for the first player, inequality (16) is automatically satisfied. A similar situation occurs for the second player with respect to condition (19).

On the other hand, taking $d_i^k = 0$ may not be the most adequate strategy, even if $g_i^k = 0$, since stationarity for player i may be lost after a step is made for the other player. Thus, this is done only as a safeguarding procedure after the stepsize is at least as small as a threshold value $\tau \in (0, 1]$. Notice that when $\tau := 1$, the safeguarding procedure is always activated whenever $g_i^k = 0$, while when $\tau < 1$, it is activated only after rejecting the Newtonian step at least once. This is done in order to try using the Newtonian direction as much as possible due to its fast local properties.

3 General Assumptions

The assumptions stated in this section are supposed to hold throughout the paper without specific mentioning. Before stating them, we establish some notation that shall be used in the sequel. We denote

$$g^k := \begin{bmatrix} g_1^k \\ g_2^k \end{bmatrix} = \begin{bmatrix} \nabla_{x_1} f_1(x_1^k, x_2^k) \\ \nabla_{x_2} f_2(x_1^k, x_2^k) \end{bmatrix}, \quad \mathcal{H}_{t,k} := \begin{bmatrix} H_1^k & tM_1 \\ tM_2 & H_2^k \end{bmatrix},$$
$$d^k := \begin{bmatrix} d_1^k \\ d_2^k \end{bmatrix}, \quad d := \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad \text{and} \quad x^k := \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix}.$$

Additionally, for each player $i = 1, 2$, we denote the index of the other player by $\neg i$. The general assumptions that we use in all our analysis are the following:
Assumption A There exists $L > 0$ such that $\nabla_{x_i} f_i(\cdot, x_{\neg i})$ are L-Lipschitz continuous for every fixed $x_{\neg i}$ and $i \in \{1, 2\}$. As a consequence, we have that

$$f_i(x_i + td_i, x_{\neg i}) \le f_i(x) + t\nabla_{x_i} f_i(x)^T d_i + \frac{t^2 L}{2}\|d_i\|^2$$

for every $x = (x_i, x_{\neg i}) \in \mathbb{R}^{n_i + n_{\neg i}}, d_i \in \mathbb{R}^{n_i}$, and $t > 0$.
Assumption B There exists $C_H > 0$ such that $\|\nabla^2_{x_1 x_2} f_i(x)\| \le C_H$ for all $x \in \mathbb{R}^{n_1+n_2}$ and $i \in \{1, 2\}$.
Assumption C There exists $C_R > 0$ and functions r_1 and r_2 such that for $i \in \{1, 2\}$ and for all $x = (x_i, x_{\neg i}) \in \mathbb{R}^{n_i + n_{\neg i}}$, all $d_{\neg i} \in \mathbb{R}^{n_{\neg i}}$ and $t > 0$ it holds

$$\nabla_{x_i} f_i(x_i, x_{\neg i} + td_{\neg i}) = \nabla_{x_i} f_i(x) + t\nabla^2_{x_1 x_2} f_i(x) d_{\neg i} + r_i(t, x, d_{\neg i}), \tag{20}$$

and
$$\|r_i(t, x, d_{\neg i})\| \le C_R t^2 \|d_{\neg i}\|^2.$$

Assumption D There exist $\lambda_{max} \ge \lambda_{min} > 0$ such that for all $i \in \{1, 2\}$ and $k \in \mathbb{N}$ the eigenvalues of H_i^k lie in $[\lambda_{min}, \lambda_{max}]$. As a consequence, for all $x_i \in \mathbb{R}^{n_i}$, we have that

$$\lambda_{min} \le \|H_i^k\| \le \lambda_{max}, \quad \frac{1}{\lambda_{max}} \le \|(H_i^k)^{-1}\| \le \frac{1}{\lambda_{min}}, \quad \text{and}$$
$$\lambda_{min}\|x_i\|^2 \le x_i^T H_i^k x_i \le \lambda_{max}\|x_i\|^2, \quad i \in \{1, 2\}.$$

We highlight that all Assumptions A to C are simple assumptions on the smoothness and boundedness of the high order derivatives of the functions f_1 and f_2, while Assumption D can be easily guaranteed when choosing the matrices H_i^k.

We end this section with a few results that arise from Assumptions A-D. The first result guarantees uniform boundedness of the matrices $\mathcal{H}_{t,k}$ and $\mathcal{H}_{t,k}^{-1}$ for small enough values of t.

Lemma 3.1 *For every $k \in \mathbb{N}$ and $t \in (0, 1]$, the matrix $\mathcal{H}_{t,k}$ satisfies*

$$\|\mathcal{H}_{t,k}\| \le \mu_{max} := \sqrt{\lambda_{max}^2 + 4\lambda_{max} C_H + C_H^2}. \tag{21}$$

Moreover, if $t \leq \frac{\lambda_{min}^2}{8\lambda_{max}C_H}$, *then* $\mathcal{H}_{t,k}$ *is non-singular and* $\|\mathcal{H}_{t,k}^{-1}\| \leq \mu_{min}$, *where the constant* μ_{min} *is given by*

$$\mu_{min} := \frac{\sqrt{2}}{\lambda_{min}}. \tag{22}$$

Proof: By the definition of $\mathcal{H}_{t,k}$, for a vector $z = (z_1, z_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we have that

$$\mathcal{H}_{t,k}z = \begin{bmatrix} H_1^k & tM_1 \\ tM_2 & H_2^k \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} H_1^k z_1 + tM_1 z_2 \\ tM_2 z_1 + H_2^k z_2 \end{bmatrix}.$$

Therefore, we can write

$$\begin{aligned}\|\mathcal{H}_{t,k}z\|^2 &= (H_1^k z_1 + tM_1 z_2)^T (H_1^k z_1 + tM_1 z_2) + (tM_2 z_1 + H_2^k z_2)^T (tM_2 z_1 + H_2^k z_2) \\ &= \|H_1^k z_1\|^2 + 2tz_1^T H_1^k M_1 z_2 + \|tM_1 z_2\|^2 + \|H_2^k z_2\|^2 + 2tz_2^T H_2^k M_2 z_1 \\ &+ \|tM_2 z_1\|^2. \end{aligned} \tag{23}$$

Now, by the definition of M_i on (12), we see that $\|M_i\| \leq \|\nabla^2_{x_i x_{\neg i}} f_i(x^k)\|, i = 1, 2$. Hence, by the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}&\|\mathcal{H}_{t,k}z\|^2 \leq \|H_1^k z_1\|^2 + 2t\|H_1^k\| \cdot \|\nabla^2_{x_1 x_2} f_1(x^k)\| \cdot \|z_1\| \cdot \|z_2\| \\ &+ t^2 \|\nabla^2_{x_1 x_2} f_1(x^k)\|^2 \cdot \|z_2\|^2 + \|H_2^k z_2\|^2 + 2t\|H_2^k\| \cdot \|\nabla^2_{x_2 x_1} f_2(x^k)\| \cdot \|z_1\| \cdot \|z_2\| \\ &+ t^2 \|\nabla^2_{x_2 x_1} f_2(x^k)\|^2 \cdot \|z_1\|^2.\end{aligned}$$

Using Assumptions B and D, and the fact that $t \leq 1$, we have that

$$\begin{aligned}&\|\mathcal{H}_{t,k}z\|^2 \leq \lambda_{max}^2 \|z_1\|^2 + 2t\lambda_{max}C_H \cdot \|z_1\| \cdot \|z_2\| + t^2 C_H^2 \cdot \|z_2\|^2 \\ &+ \lambda_{max}^2 \|z_2\|^2 + 2t\lambda_{max}C_H \cdot \|z_1\| \cdot \|z_2\| + t^2 C_H^2 \|z_1\|^2 \\ &\leq \lambda_{max}^2 \left(\|z_1\|^2 + \|z_2\|^2\right) + 4\lambda_{max}C_H \cdot \|z_1\| \cdot \|z_2\| + C_H^2 \left(\|z_1\|^2 + \|z_2\|^2\right) \\ &= \left(\lambda_{max}^2 + C_H^2\right)\left(\|z_1\|^2 + \|z_2\|^2\right) + 4\lambda_{max}C_H \|z_1\| \cdot \|z_2\|.\end{aligned}$$

Hence, observing that $\|z_i\| \leq \|z\|$ for $i = 1, 2$, we have that

$$\|\mathcal{H}_{t,k}z\|^2 \leq \left(\lambda_{max}^2 + 4\lambda_{max}C_H + C_H^2\right)\|z\|^2,$$

and thus (21) holds. To prove the bound on $\|\mathcal{H}_{t,k}\|$, recalling (23), we have that

$$\begin{aligned}\|\mathcal{H}_{t,k}z\|^2 &= \|H_1^k z_1\|^2 + 2tz_1^T H_1^k M_1 z_2 + \|tM_1 z_2\|^2 + \|H_2^k z_2\|^2 + 2tz_2^T H_2^k M_2 z_1 \\ &+ \|tM_2 z_1\|^2 \geq \lambda_{min}^2 \|z_1\|^2 - 2t\lambda_{max}\|M_1\| \cdot \|z_1\| \cdot \|z_2\| + \|tM_1 z_2\|^2 \\ &+ \lambda_{min}^2 \|z_2\|^2 - 2t\lambda_{max}\|M_2\| \cdot \|z_1\| \cdot \|z_2\| + \|tM_2 z_1\|^2.\end{aligned}$$

Since $\|tM_i z_{\neg i}\|^2 \geq 0$, $\|M_i\| \leq C_H$, $\|z\|^2 = \|z_1\|^2 + \|z_2\|^2$, and $\|z_i\| \leq \|z\|$, we have

$$\|\mathcal{H}_{t,k}z\|^2 \geq \left(\lambda_{min}^2 - 4t\lambda_{max}C_H\right)\|z\|^2.$$

Thus, for $t \leq \frac{\lambda_{min}^2}{8\lambda_{max}C_H}$, we have $\|\mathcal{H}_{t,k}z\|^2 \geq \frac{\lambda_{min}^2}{2}\|z\|^2$, so $\mathcal{H}_{t,k}$ is non-singular and $\|\mathcal{H}_{t,k}^{-1}\| \leq \frac{\sqrt{2}}{\lambda_{min}}$. □

Next we prove a technical result that shall be used later.

Lemma 3.2 *For every $k \in \mathbb{N}$, if $t \leq \frac{\lambda_{min}^2}{8\lambda_{max} C_H}$ and d is as in Step 4 of Algorithm 1, then $\|d\| \leq C_k := \mu_{min}(\|g_1^k\| + \|g_2^k\|)$, where μ_{min} is given by (22).*

Proof: By the system equation (13) and using Lemma 3.1, we have that

$$\|d\| = \|\mathcal{H}_{t,k}^{-1} g^k\| \leq \|\mathcal{H}_{t,k}^{-1}\| \cdot (\|g_1^k\| + \|g_2^k\|) \leq \mu_{min}(\|g_1^k\| + \|g_2^k\|) = C_k. \tag{24}$$

□

The next result shows that for each component $i = 1, 2$ and each step k, the gradients g_i^k are comparable with d_i provided that t is small enough.

Lemma 3.3 *For every $k \in \mathbb{N}$, $i = 1, 2$, if t satisfies*

$$t \leq \min\left\{\frac{\lambda_{min}^2}{8\lambda_{max} C_H}, \frac{\|g_i^k\|}{2C_H C_k}\right\}, \tag{25}$$

where C_k is given as in Lemma 3.2, then

$$\frac{1}{2}\|g_i^k\| \leq \|H_i^k d_i\| \leq \frac{3}{2}\|g_i^k\|. \tag{26}$$

Proof: We shall prove the result for $i = 1$, being the result analogous for $i = 2$. By the first equation of system (13), the definition of M_1, Assumption B and Lemma 3.2, we have that

$$\|H_1^k d_1\| = \| - tM_1 d_2 - g_1^k\| \leq tC_H C_k + \|g_1^k\|.$$

Thus, from (25), we conclude the right-hand side inequality of (26). Similarly, to prove the left-hand side inequality in (26), considering again the first equation of system (13), we have $\|g_1^k\| = \|-H_1^k d_1 - tM_1 d_2\| \leq \|H_1^k d_1\| + tC_H C_k$. Thus, the result follows from (25). □

4 Well Definiteness of the Algorithm

This section is devoted to proving well-definiteness of Algorithm 1. As previously mentioned, by the way Algorithm 1 is built, in cases where one of the gradients is zero, say $\|g_2^k\| = 0$, and the step-size t has sufficiently decreased ($t \leq \tau$), it turns out that the direction for the second player is null while for the first player the direction is computed as a standard Newton direction for optimization with Armijo line search, which is known to be well-defined and satisfy conditions (14)-(16) with $t \leq \frac{2(1-\alpha)\lambda_{min}^2}{L\lambda_{max}}$, $\theta = \frac{\lambda_{min}}{\lambda_{max}}$ and $\gamma = \frac{1}{\lambda_{max}}$ (see for instance the discussion in [3], pages 29-36, and Lemma 2.20 in [15]). Thus, we focus on the case where $\|g_i^k\| \neq 0, i = 1, 2$, and we show that the inequalities (14)-(19) are satisfied for sufficiently small t, meaning that the stepsize does not decrease indefinitely.

We begin with the gradient/direction ratio inequalities (16) and (19). However, we show a stronger version bounding the directions by the gradient on the points $(x_i^k, x_{\neg i}^k + td_{\neg i})$ from both above and below.

Lemma 4.1 *For every $k \in \mathbb{N}$ and $i = 1, 2$, if t satisfies*

$$t \leq \min\left\{\frac{\|g_i^k\|}{8C_H C_k}, \frac{\lambda_{min}^2}{8\lambda_{max} C_H}\right\}, \tag{27}$$

where C_k is given as in Lemma 3.2, then

$$\gamma \left\|\nabla_{x_i} f_i(x_i^k, x_{\neg i}^k + td_{\neg i})\right\| \leq \|d_i\| \leq \beta \left\|\nabla_{x_i} f_i(x_i^k, x_{\neg i}^k + td_{\neg i})\right\|, \tag{28}$$

for every γ and β such that

$$\gamma \leq \frac{2}{3\lambda_{max}}, \quad \beta \geq \frac{2}{\lambda_{min}}. \tag{29}$$

Proof: We prove the result only for $i = 1$, as the proof for $i = 2$ is similar. By (13), we have that

$$-g_1^k = H_1^k d_1 + tM_1 d_2. \tag{30}$$

Also, we have that

$$\nabla_{x_1} f_1(x_1^k, x_2^k + td_2) = g_1^k + t\left[\int_0^1 \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k + \xi td_2) d\xi\right] d_2. \tag{31}$$

Combining (30) and (31) we obtain

$$\nabla_{x_1} f_1(x_1^k, x_2^k + td_2) = -H_1^k d_1 + t\left[-M_1 + \int_0^1 \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k + \xi td_2) d\xi\right] d_2. \tag{32}$$

By the triangular inequality, we have that

$$\|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \geq \left|\|H_1^k d_1\| - t\left\|\left[-M_1 + \int_0^1 \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k + \xi td_2) d\xi\right] d_2\right\|\right| \tag{33}$$

and

$$\|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \leq \|H_1^k d_1\| + t\left\|\left[-M_1 + \int_0^1 \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k + \xi td_2) d\xi\right] d_2\right\|. \tag{34}$$

Next, we prove that for t sufficiently small it holds

$$\frac{1}{2}\|H_1^k d_1\| \leq \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \leq \frac{3}{2}\|H_1^k d_1\|. \tag{35}$$

Let us suppose that $\left[-M_1 + \int_0^1 \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k + \xi td_2) d\xi\right] d_2 \neq 0$, since otherwise this is trivial from (32). Hence, if we select t satisfying

$$t \leq \frac{\|H_1^k d_1\|}{2\left\|\left[-M_1 + \int_0^1 \nabla^2_{x_1 x_2} f_1(x_1^k, x_2^k + \xi td_2) d\xi\right] d_2\right\|}, \tag{36}$$

we see from (33) that the left-hand side inequality in (35) holds and, by (34), we obtain the right-hand side inequality. Now, using that $\lambda_{min}\|d_1\| \leq \|H_1^k d_1\| \leq \lambda_{max}\|d_1\|$, (35) implies that

$$\frac{\lambda_{min}}{2}\|d_1\| \leq \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \leq \frac{3\lambda_{max}}{2}\|d_1\|,$$

which implies the desired bounds in (28).

Finally, let us show that t satisfying (27) implies the bound (36). By Assumption B and Lemma 3.2, we see that

$$\begin{aligned}
&\left\| \left[-M_1 + \int_0^1 \nabla^2_{x_1x_2} f_1(x_1^k, x_2^k + \xi t d_2) d\xi \right] d_2 \right\| \\
&\leq \left[\|M_1\| + \left\| \int_0^1 \nabla^2_{x_1x_2} f_1(x_1^k, x_2^k + \xi t d_2) d\xi \right\| \right] \cdot \|d_2\| \\
&\leq \left[\|M_1\| + \int_0^1 \|\nabla^2_{x_1x_2} f_1(x_1^k, x_2^k + \xi t d_2)\| d\xi \right] \cdot \|d_2\| \leq 2C_H C_k,
\end{aligned}$$

therefore, we have that

$$\frac{\|H_1^k d_1\|}{4C_H C_k} \leq \frac{\|H_1^k d_1\|}{2\left\| \left[-M_1 + \int_0^1 \nabla^2_{x_1x_2} f_1(x_1^k, x_2^k + \xi t d_2) d\xi \right] d_2 \right\|},$$

thus, in order to attain (36), it suffices to select

$$t \leq \frac{\|H_1^k d_1\|}{4C_H C_k}. \tag{37}$$

Now, notice that inequalities (26) guaranteed by Lemma 3.3 ensure (37) holds with t satisfying the hypothesis (27). □

The next result ensures the angle inequalities (15) and (18) hold for t sufficiently small.

Lemma 4.2 *For every $k \in \mathbb{N}$ and $i = 1, 2$, if $g_i^k \neq 0$, t satisfies*

$$t \leq \min\left\{ \sqrt{\frac{\lambda_{min}\|g_i^k\|}{8\lambda_{max} C_R (C_k)^2}}, \frac{\|g_i^k\|}{8C_H C_k}, \frac{\lambda_{min}^2}{8\lambda_{max} C_H} \right\}, \tag{38}$$

where C_k is given as Lemma 3.2, and θ satisfies

$$\theta \leq \frac{\lambda_{min}}{4\lambda_{max}}, \tag{39}$$

then,

$$\nabla_{x_i} f_i(x_i^k, x_{\neg i}^k + td_{\neg i})^T d_i \leq -\theta \|\nabla f_i(x_i^k, x_{\neg i}^k + td_{\neg i})\| \cdot \|d_i\|. \tag{40}$$

Proof: Once again, we work the proof of the main claim for the first component. For the second one, the proof follows similarly. Since we are considering $g_1^k \neq 0$, we have that $M_1 = \nabla^2_{x_1 x_2} f_1(x^k)$. Hence, Algorithm 1 yields

$$H_1^k d_1 = -g_1^k - t\nabla^2_{x_1 x_2} f_1(x^k) d_2. \tag{41}$$

By Assumption C, we have that

$$\nabla_{x_1} f_1(x_1^k, x_2^k + td_2) = g_1^k + t\nabla^2_{x_1 x_2} f_1(x^k) d_2 + r_1(t, x^k, d_2), \tag{42}$$

where the remainder $r_1(t, x^k, d_2)$ is such that

$$\|r_1(t, x^k, d_2)\| \leq C_R t^2 \|d_2\|^2. \tag{43}$$

Therefore, adding (42) to (41), we have

$$H_1^k d_1 = -\nabla_{x_1} f_1(x_1^k, x_2^k + td_2) + r_1(t, x^k, d_2). \tag{44}$$

Hence, since $\|H_1^k d_1\| \leq \lambda_{max} \|d_1\|$, we have that

$$\|d_1\| \geq \frac{1}{\lambda_{max}} \| - \nabla_{x_1} f_1(x_1^k, x_2^k + td_2) + r_1(t, x^k, d_2)\|,$$

and using the triangular inequality at the right-hand side we obtain

$$\|d_1\| \geq \frac{1}{\lambda_{max}} \Big| \|r_1(t, x^k, d_2)\| - \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| \Big|. \tag{45}$$

On the other hand, multiplying (44) by d_1^T and using the Cauchy-Schwarz inequality, Assumptions C and Lemma 3.2, we have that

$$\begin{aligned} d_1^T \nabla_{x_1} f_1(x_1^k, x_2^k + td_2) &\leq -d_1^T H_1^k d_1 + \|r_1(t, x^k, d_2)\| \cdot \|d_1\| \\ &\leq -\lambda_{min} \|d_1\|^2 + C_R C_k^2 t^2 \|d_1\|. \end{aligned} \tag{46}$$

Therefore, if we choose t in (46) satisfying

$$t \leq \sqrt{\frac{\lambda_{min} \|d_1\|}{2 C_R C_k^2}},$$

we obtain

$$d_1^T \nabla_{x_1} f_1(x_1^k, x_2^k + td_2) \leq \frac{-\lambda_{min} \|d_1\|^2}{2}. \tag{47}$$

Thus, we can combine (45) and (47), to write

$$d_1^T \nabla_{x_1} f_1(x_1^k, x_2^k + td_2) \leq -\frac{\lambda_{min}}{2\lambda_{max}} \|d_1\| \Big| \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| - \|r_1(t, x^k, d_2)\| \Big|. \tag{48}$$

Next, to bound the right-hand side of (48), for t satisfying (38) we have by (26) that

$$t^2 \leq \frac{\lambda_{min}\|g_1^k\|}{8\lambda_{max}C_R C_k^2} \leq \frac{\lambda_{min}\|H_1^k d_1\|}{4\lambda_{max}C_R C_k^2} \leq \frac{\lambda_{min}\|d_1\|}{4C_R C_k^2}. \tag{49}$$

Hence, using (49) on (43), we have that

$$\|r_1(t,x^k,d_2)\| \leq C_R t^2\|d_2\|^2 \leq C_R C_k^2 t^2 \leq \frac{\lambda_{min}\|d_1\|}{4} \leq \frac{\|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\|}{2},$$

where we used Lemma 4.1 in the last inequality. Therefore, we have that

$$\begin{aligned}\Big|\|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| - \|r_1(t,x^k,d_2)\|\Big| &= \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\| - \|r_1(t,x^k,d_2)\| \\ &\geq \frac{\|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\|}{2}.\end{aligned}$$

Then, (48) gives us the desired inequality

$$d_1^T \nabla_{x_1} f_1(x_1^k, x_2^k + td_2) \leq -\frac{\lambda_{min}}{4\lambda_{max}}\|d_1\| \cdot \|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\|.$$

□

Now we are ready to finish establishing the well-definiteness of Algorithm 1, where we prove the remaining relations (14) and (17).

Proposition 4.1 *For $i = 1, 2$, suppose that $g_i^k \neq 0$ for some $k \in \mathbb{N}$, and assume that the stepsize t satisfies*

$$t \leq \min\left\{\sqrt{\frac{\lambda_{min}\|g_i^k\|}{8\lambda_{max}C_R C_k^2}}, \frac{\|g_i^k\|}{8C_H C_k}, \frac{\lambda_{min}^2}{8\lambda_{max}C_H}, \frac{(1-\alpha)2\theta}{\beta L}\right\}, \tag{50}$$

where β and θ are constants satisfying (29) and (39) respectively. Then all inequalities (14)-(19) are satisfied. As a consequence, iteration k of Algorithm 1 is well-defined.

Proof: By Lemmas 4.1 and 4.2, it remains to prove (14), since (17) is analogous. Fix $k \in \mathbb{N}$ and $i = 1$. By Assumption A, we have that

$$f_1(x_1^k + td_1, x_2^k + td_2) - f_1(x_1^k, x_2^k + td_2) \leq t\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1 + \frac{t^2 L_1}{2}\|d_1\|^2.$$

If $d_1 = 0$, then the result is immediate. If $d_1 \neq 0$ and $t > 0$, we have that

$$\begin{aligned}&t\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1 + \frac{t^2 L}{2}\|d_1\|^2 \leq \alpha t \nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1 \\ &\Leftrightarrow t \leq \frac{2(1-\alpha)\left[-\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)^T d_1\right]}{L\|d_1\|^2},\end{aligned}$$

which can be guaranteed by

$$t \leq \frac{2(1-\alpha)\theta\|\nabla_{x_1} f_1(x_1^k, x_2^k + td_2)\|}{L\|d_1\|}$$

due to (40). Also, by Lemma 4.1, we have that $\frac{\|\nabla_{x_1} f_1(x_1^k, x_2^k+td_2)\|}{\|d_1\|} \geq \frac{1}{\beta}$, and thus, it suffices to select t satisfying (50) to ensure (14). □

Notice that our well-definiteness results assume that t is smaller than a positive constant which depends on the gradients g_i^k at iteration k. This is the technical reason why we must switch to the standard Newton method for optimization whenever one player has already reached stationarity. We could not prove or find a counter-example for the well-definiteness of the method that never switches to standard Newton, that is, considering M_i always equals to $\nabla^2_{x_1 x_2} f_i(x^k), i = 1, 2$, in place of (12). With the use of this safeguarding procedure we have that any iteration of the algorithm is well defined.

We end this section with a particular result when the iterates all lie on a bounded set. The result states we can infer the boundedness of the stepsize sequence from the boundedness of both gradients from below. This guarantees that in this situation the line search criteria will not get increasingly hard to be satisfied, forcing small steps and stalling the computation of the iterates, whenever the decisions of both players are far from stationarity.

Proposition 4.2 *Suppose that the sequence $\{x^k\}$ generated by Algorithm 1 lie on a bounded set. If there exists a constant $c > 0$ and $k_0 \in \mathbb{N}$ such that $\|g_i^k\| > c$ for each component $i = 1, 2$ and all $k > k_0$, then the stepsize sequence $\{t^k\}$ is bounded away from zero.*

Proof: Let $0 < \bar{t} := \min\{t^k\}$ for $k \leq k_0$. For $k > k_0$, since $\{x^k\}$ is bounded, there exists an universal constant C_d such that $0 < C_k \leq C_d$, where C_k is defined in Lemma 3.2. Moreover, using that $\|g_i^k\| > c$, Proposition 4.1 ensures that if

$$t \leq t' := \min\left\{\sqrt{\frac{\lambda_{min} c}{8\lambda_{max} C_R C_d^2}}, \frac{c}{8C_H C_d}, \frac{\lambda_{min}^2}{8\lambda_{max} C_H}, \frac{(1-\alpha)2\theta}{\beta L}\right\},$$

then conditions (14)-(19) hold. Thus, by the stepsize update rule we have that, for all k,

$$t^k \geq \min\left\{1, \frac{t'}{2}, \bar{t}\right\} > 0.$$

□

5 Convergence Analysis

In this section we establish our convergence results for Algorithm 1. Before starting, we point out that equilibrium points are key elements in the study of a game, but they certainly do not capture everything one may need in order to understand the modeled problem. Unlike optimization problems, where the solution represents the goal for the decision maker, NEP solutions may not be desirable by the players. This happens, for example, in the famous Prisoner's Dilemma and in congestion problems (see [24] and [25]). In this way, an algorithm that captures the dynamics involved in the game, even if it does not obtain convergence, can sometimes be even more important than the identification of the solution itself. A more concrete example comes from the so-called evolutionary game theory (see [27]), which is used to explain the basis of evolutionary behavior. This field differs from standard game theory in the sense that it focus on how a change of strategy affects the dynamics of the game, rather than finding equilibrium points, since a change of strategies describes competition between populations. In arbitrary dynamical systems, one can obtain stable equilibrium points, where trajectories in their neighborhood converge to them. However, it is known that one can also observe trajectories that may orbit or cycle, or that may simply diverge from an unstable equilibrium point or even have a chaotic behavior, see [23]. Since our algorithm in some sense tries to emulate the game dynamics, we believe that all these situations may occur with the sequences it generates. Therefore, it does not seem reasonable to show convergence to equilibrium points in general problems.

On the other hand, Nash equilibrium situations are observed in many real-world applications. Therefore, it is natural to imagine that the dynamics involved in these applications favors the stability of the system. Moreover, when convergence occurs, the limit point must be an attractor and therefore it is expected to be an equilibrium point. In other words, we believe that the process of evolution of the generated sequence should privilege convergence to equilibrium points, even if they are not beneficial for the agents. However, in situations where conditions change very quickly it may be hard to achieve an equilibrium, because, in general, players tend to respond to outdated information. Traditional best-response algorithms suffer from this problem and therefore need very restrictive assumptions in order to achieve convergence. Our method is constructed on basing the iteration on a predicted response rather than on the current one, and therefore generates more stability in the process. This type of approach may be more related to the real dynamics of the modeled problem. Considering this, we expect to obtain equilibrium points more frequently than other Jacobi-type methods in the literature.

We also would like to mention that the proper mathematical study of such dynamical systems arising from realistic behavior of agents in a NEP together with iterative schemes meant to simulate their behavior is an interesting subject that is rarely discussed in the literature, with several nuances that must be addressed. In this aspect, our results are definitely incomplete, as we are mainly concerned with stationarity of a limit point of the sequence generated, as usual in the optimization practice. That is, since we believe that cycling or orbiting involving non-stationary points are reasonable outcomes of the dynamics, our assumptions are presented in order to force convergence of the algorithm, leaving the other possibilities to a later study. With all this in mind we start stating our main convergence results.

Proposition 5.1 *Consider the sequences* $\{x^k\}$ *and* $\{t^k\}$ *generated by Algorithm 1. Then, the following holds:*

i) If $\lim_{\mathbb{N}'} d^k = 0$ *on an infinite subset* $\mathbb{N}' \subset \mathbb{N}$*, then the sequence of gradients* $\{g_1^k\}$ *and* $\{g_2^k\}$ *converge to zero on the same subset.*

ii) If the whole sequence $\{x^k\}$ *converges to a point* x^**, then either* $\nabla_{x_1} f_1(x^*) = 0$ *or* $\nabla_{x_2} f_2(x^*) = 0$.

iii) If $\{x^k\}$ *converges to a point* x^* *and if in addition the sequence of step sizes* $\{t^k\}$ *is bounded away from zero, then* x^* *is stationary for the NEP (1-2), that is,* $\nabla_{x_1} f_1(x^*) = 0$ *and* $\nabla_{x_2} f_2(x^*) = 0$.

Proof: By (13) and (21), we have that $\|g^k\| = \|\mathcal{H}_{t,k} d^k\| \leq \|\mathcal{H}_{t,k}\| \cdot \|d^k\| \leq \mu_{max} \|d^k\|$. Therefore, if $\lim_{k \in \mathbb{N}'} d^k = 0$, then $\lim_{k \in \mathbb{N}'} g^k = 0$, which proves item (i).

To prove item (iii), assume that the sequence converges. Then, we have that

$$0 = \lim_{k \to \infty} x^{k+1} - x^k = \lim_{k \to \infty} t^k d^k.$$

Since $t^k \geq \bar{t}$ for some constant $\bar{t} > 0$, we have that d^k converges to zero. Thus, by item (i) the whole gradient sequence converges to zero implying that the limit point x^* is stationary.

Finally, to prove (ii), assume that it does not hold. Thus, there exists $\epsilon > 0$ such that $\|g_i^k\| \geq \epsilon$ for $i = 1, 2$ and k large enough, say $k > k_0 \in \mathbb{N}$. Additionally, since $\{x^k\}$ is convergent, the sequence lies on a bounded set. Hence, by Proposition 4.2, we have that $\{t^k\}$ is bounded away from zero, which by item (iii) implies that $g^k \to 0$, a contradiction. □

Differently from the case of optimization, when the sequence $\{x^k\}$ is not necessarily convergent, one can not expect to prove stationarity of its limit points. However, if there is some indication of asymptotic stationarity for one of the players, one can infer stationarity/optimality for the other player, in some sense. In the next proposition we show that when the sequence of iterates varies only with respect to the first player (that is, $d_2^k = 0$) and $\{t^k\}$ is bounded, then either $\{f_1(x^k)\}$ is unbounded from below, which would somehow characterize the ill-posedness of the problem, or all limit points of $\{x^k\}$ are stationary for the NEP (1-2). Clearly, the role of each player may be exchanged in the statement of this result.

Proposition 5.2 *Consider* $\{x^k\}$ *and* $\{t^k\}$ *given by Algorithm 1 and suppose that there exists a positive real number* $\bar{t}$ *such that* $t^k > \bar{t}$ *for all* $k \in \mathbb{N}$.

i) If there exists k_0 *such that* $d_2^k = 0$ *for all* $k \geq k_0$ *then either* $\{f_1(x^k)\}$ *is unbounded from below, or* $\lim_{k \to +\infty} g_1^k = 0$.

ii) If $\sum_{k=1}^{\infty} \|d_2^k\| = \bar{d}$ *for some* $\bar{d} \in \mathbb{R}$*, then either* $\{f_1(x^k)\}$ *is unbounded from below, or there exists an infinite subset* $\mathbb{N}' \subset \mathbb{N}$ *such that* $\lim_{k \in \mathbb{N}'} g_1^k = 0$.

As a consequence, considering that f_1 *is bounded, item i) states that every limit point of* $\{x^k\}$ *is stationary for the NEP (1-2) while item ii) states that there exists a limit point that is stationary.*

Proof: By conditions (14) and (15), for $k > k_0$ we have that

$$\begin{aligned} f_1(x_1^{k+1}, x_2^{k+1}) &\le f_1(x_1^k, x_2^{k+1}) + t^k\alpha\nabla_{x_1} f_1(x_1^k, x_2^{k+1})^T d_1^k \\ &\le f_1(x_1^k, x_2^{k+1}) - t^k\alpha\theta\|\nabla_{x_1} f_1(x_1^k, x_2^{k+1})\| \cdot \|d_1^k\|. \end{aligned}$$

Summing and subtracting $f(x^k)$, using that $t^k \ge \bar{t}$ and the mean value theorem, we have for some $\xi_k \in [0,1]$ that

$$\begin{aligned} f_1(x^{k+1}) - f_1(x^k) &\le f_1(x_1^k, x_2^{k+1}) - f_1(x_1^k, x_2^k) - t^k\alpha\theta\|\nabla_{x_1} f_1(x_1^k, x_2^{k+1})\| \cdot \|d_1^k\| \\ &\le t^k \nabla_{x_2} f_1(x_1^k, x_2^k + \xi_k t^k d_2^k)^T d_2^k - t^k\alpha\theta\|\nabla_{x_1} f_1(x_1^k, x_2^{k+1})\| \cdot \|d_1^k\| \\ &\le \|\nabla_{x_2} f_1(x_1^k, x_2^k + \xi_k t^k d_2^k)\|\|d_2^k\| - \bar{t}\alpha\theta\|\nabla_{x_1} f_1(x_1^k, x_2^{k+1})\| \cdot \|d_1^k\|. \quad (51) \end{aligned}$$

Using (16) and a telescoping sum on (51), if $d_2^k = 0$ for $k > k_0$ then

$$f_1(x^{k+1}) \le f_1(x^{k_0}) - \bar{t}\alpha\theta\gamma \sum_{j=k_0}^{n} \|g_1^k\|^2.$$

So, if $\{f_1(x^k)\}$ is bounded, we must have that $\lim_{k\to\infty} g_1^k = 0$. Since $d_2^k = 0$ implies that $\|d_1^k\| \le \frac{\|g_1^k\|}{\lambda_{min}}$, the result follows from Proposition 5.1.

Let us consider now the second case. Given $\epsilon > 0$, since $d_2^k \to 0$, Assumptions B and C ensure that there exists $k_1 \in \mathbb{N}$ such that if $\|g_1^k\| > \epsilon$ and $k > k_1$ then

$$\|\nabla_{x_2} f_1(x_1^k, x_2^k + \xi_k t^k d_2^k)\| \le 2\|g_1^k\|,$$

$$\frac{1}{2}\|g_1^k\| \le \|\nabla_{x_2} f_1(x_1^k, x_2^k + t^k d_2^k)\| \le 2\|g_1^k\|,$$

$$\|d_2^k\| \le \frac{\bar{t}\alpha\theta\gamma}{16}\|g_1^k\|.$$

So, in this situation we have by (51) and (16) that

$$\begin{aligned} f_1(x^{k+1}) - f_1(x^k) &\le 2\|g_1^k\| \cdot \|d_2^k\| - \frac{\bar{t}\alpha\theta}{2}\|g_1^k\| \cdot \|d_1^k\| \\ &\le -\frac{\bar{t}\alpha\theta\gamma}{8}\|g_1^k\|^2 \le -\frac{\bar{t}\alpha\theta\gamma\epsilon^2}{8}. \quad (52) \end{aligned}$$

If (52) holds for all k large enough we have that $\{f_1(x^k)\}$ is unbounded. Thus, assuming that it is bounded, there exist an infinite subset $\mathbb{N}' \subset \mathbb{N}$ such that $\lim_{k\in\mathbb{N}'} g_1^k = 0$. Using the first equation of (13), Assumptions B and D, and the fact that $d_2^k \to 0$ we conclude that $\lim_{k\in\mathbb{N}'} d_1^k = 0$ and so the result follows from Proposition 5.1.

□

The results of Proposition 5.2 are, in some sense, a generalization of what happens when Algorithm 1 is applied to a problem where optimality for one of the player is always satisfied, and only optimality of the other player is to be sought. In this case, t^k is always bounded away from zero and the Newton direction has a norm greater than a fraction of the gradient. Therefore, our result recovers the standard result from Newton's method that either the function is unbounded from below or all limit points are stationary. Note that, by Lemma 3.3, it would be possible to include in Algorithm 1 a test to ensure that $\gamma\|g_i^k\| \leq \|d_i^k\|$, and therefore $\lim g_2^k = 0$ would be a consequence of $\lim d_2^k = 0$. We chose not to do this explicitly since in some sense a similar requirement is already guaranteed by (16) and (19). Finally, we note that this type of analysis is fairly non-standard, and even a full analysis of the possible outcomes of the NEP viewed as a dynamical system is still lacking; this analysis would be very welcomed, as it would provide new elements for the analysis of algorithms of the type we propose.

We conclude our convergence results by showing that when the objective functions are strictly convex quadratic functions, the full step with $t = 1$ is always accepted by the line search and one iteration is enough to solve the problem. This is similar to what is known for the standard Newton method for optimization. This type of result corroborates the plausibility of the assumption on the boundedness away from zero of the stepsize, which was very important in the previous results.

Proposition 5.3 *Let $A_1 \in \mathbb{R}^{n_1\times n_1}$, $A_2 \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times n_2}$, $B_2 \in \mathbb{R}^{n_2\times n_1}$, $c_1 \in \mathbb{R}^{n_1}$, and $c_2 \in \mathbb{R}^{n_2}$ with A_1 and A_2 being positive definite matrices. Consider the NEP defined by the functions $f_1(x_1, x_2) = \frac{1}{2}(x_1)^T A_1 x_1 + (B_1 x_2 - c_1)^T x_1$ and $f_2(x_1, x_2) = \frac{1}{2}(x_2)^T A_2 x_2 + (B_2 x_1 - c_2)^T x_2$. If $\alpha \leq \frac{1}{2}$, $H_1^0 = A_1$, $H_2^0 = A_2$, and the matrix $\begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}$ is non-singular, then Algorithm 1 finds a solution in a single iteration without performing back-tracking; ie, the solution of system* (13) *with $t = 1$ satisfies all inequalities (14)-(19) with constants γ and θ satisfying* (29) *and* (39) *respectively and the new iterate x^1 is a solution of the problem.*

Proof: We prove the result for $i = 1$. For $i = 2$ the result is similar. Since $f_1(x)$ is quadratic with respect to x_1 and linear with respect to x_2, we have that $\nabla_{x_1} f_1(x)$ is linear. Thus, given an arbitrary x and d,

$$\nabla_{x_1} f_1(x_1 + d_1, x_2) = \nabla_{x_1} f_1(x) + A_1 d_1, \quad \text{and} \tag{53}$$

$$\nabla_{x_1} f_1(x_1, x_2 + d_2) = \nabla_{x_1} f_1(x) + B_1 d_2. \tag{54}$$

So, using (54) in (13) we obtain that

$$A_1 d_1 = -\nabla_{x_1} f_1(x_1^0, x_2^0 + d_2). \tag{55}$$

Then, by (53), $\nabla_{x_1} f_1(x^0 + d) = 0$. Since $\nabla^2_{x_1 x_1} f_1(x) = A_1 > 0$ for all x, we have that $x^0 + d$ is a solution of the NEP.

Next, we show that the inequalities (14)-(19) hold, which means that the point $x^1 := x^0 + d$ is accepted by the algorithm. By (55) we have that

$$\|\nabla_{x_1} f_1(x_1^0, x_2^0 + d_2)\| \leq \|A_1\|\|d_1\| \leq \lambda_{max}\|d_1\|.$$

So we obtain (16) and (19) for $\gamma \leq \frac{1}{\lambda_{max}}$. As a consequence of this and (55)

$$\begin{aligned} d_1^T \nabla_{x_1} f_1(x_1^0, x_2^0 + d_2) &= -d_1^T A_1 d_1 \leq -\lambda_{min} \|d_1\|^2 \\ &\leq -\lambda_{min} \gamma \|d_1\| \|\nabla_{x_1} f_1(x_1^0, x_2^0 + d_2)\|. \end{aligned}$$

Thus, (15) and (18) hold. Finally, to prove (14) and (17), we use that $f_1(\cdot, x_2^0 + d_2)$ is quadratic and (55) to obtain

$$\begin{aligned} f_1(x + d) &= f_1(x_1^0, x_2^0 + d_2) + \nabla_{x_1} f_1(x_1^0, x_2^0 + d_2)^T d_1 + \frac{1}{2} d_1^T A_1 d_1 \\ &= f_1(x_1^0, x_2^0 + d_2) + \frac{1}{2} d_1^T \nabla_{x_1} f_1(x_1^0, x_2^0 + d_2). \end{aligned}$$

□

Proposition 5.3 guarantees convergence, in one iteration, asking only that A_1 and A_2 are positive definite together with the non-singularity of the matrix $M := \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}$, without requiring its positive definiteness. In particular, without any special requirement on the matrices B_1 and B_2. In contrast, a standard Jacobi method for the system of equations (3) would require the norm of the iteration matrix to be less than 1 (which can be assured, for instance, when M is strictly diagonally dominant), while the Jacobi trust-region method proposed by [28] would require for convergence, among other things, the positive definiteness of M.

Additionally, the fact that Proposition 5.3 ensures convergence for quadratic problems in one iteration, hints at the behavior of the method close to the solution when applied to non-quadratic functions; namely, for sufficiently well-behaved functions, it is expected that the behavior of the algorithm would be similar to its behavior on quadratic functions. Thus, we would expect that close to the solution the method would not need to decrease the stepsize, and local convergence of the pure Newtonian strategy would be recovered, however the pursue of a result of this type is out of the scope of this paper.

6 Numerical Experiments

In this section we present some numerical experiments depicting the behavior of Algorithm 1. First we test it on five selected illustrative example problems and then we test the Algorithm on an application on facility location problems.

6.1 Illustrative Examples

For the first experiments, we chose the initial point as $x^0 := (-5, 1)^T$, an error tolerance of 10^{-4} for the norm of the gradients $\|g^k\|$ as stopping criterion, together with a maximum number of iterations of $k = 1000$. In Table 1 we show the convergence point, the error, and the number of iterations for the five selected equilibrium problems to be defined below. We compared Algorithm 1 (abbreviated by Alg1 in the tables) with the Jacobi-type trust-region method of [28] (abbreviated Yuan), an exact Jacobi-type method that deals with system (3) directly at each step (abbreviated Jacobi), and Newton's method for system (3) with unitary step, abbreviated Newton. We also display each problem's actual solution for comparison.

Jacobi's method for system (3) is done as follows: given (x_1^k, x_2^k) we algebrically compute the next iterate (x_1^{k+1}, x_2^{k+1}) by solving the equation $\nabla_{x_1} f_1(x_1, x_2^k) = 0$ for x_1 and $\nabla_{x_2} f_2(x_1^k, x_2) = 0$ for x_2, and then update $(x_1^{k+1}, x_2^{k+1}) := (x_1, x_2)$. As for Newton's method, given an iterate (x_1^k, x_2^k), we compute (d_1^k, d_2^k) by solving system (4) and then compute (x_1^{k+1}, x_2^{k+1}) as $(x_1^k, x_2^k) + (d_1^k, d_2^k)$.

Yuan's algorithm, a Jacobi-type method for NEPs that also uses Newtonian ideas, depends on several parameters, for details see [28]. In all our implementations we chose the parameters $\delta_v := 0.01, \tau_v := 1, t_{v,1} := 1$, $G_i = 1$, and $\beta_i := \frac{1}{2}, i = 1, 2$.

For our method, we chose $\alpha := 10^{-6}, \theta = 0.01, \gamma = 10^{-6}$, and $\tau = 0.99$. Recall that according to Lemmas 4.1 and 4.2, γ is proportional to $\frac{1}{\lambda_{max}}$ and θ to the ratio $\frac{\lambda_{min}}{\lambda_{max}}$. Selecting parameters γ and θ in this way, we enforce a large bounding interval for $\|H_i^k\|$, which guarantees the validity of Assumption D. In order to choose the matrices H_i^k, we computed the modified Cholesky decomposition of $\nabla^2_{x_i x_i} f_i(x^k)$ as described in [26], which resumes to this matrix whenever it is positive definite, and, when it is not, the procedure returns a positive definite matrix by adding a small diagonal perturbation to the matrix.

All tests were run in Matlab R2017a in an AMD Ryzen 5 2400G with 3.6Ghz graphics and 8Gb RAM processor. Our illustrative problem set is given by the following simple one-dimensional NEPs:

Example 6.1

$$f_1(x) := x_1^2 + x_1 x_2 - 5x_1, \quad f_2(x) := \frac{3x_2^2}{2} - x_1 x_2 - x_2.$$

This example defines positive definite quadratic functions such that solving its first-order necessary conditions is sufficient for solving the problem, in addition, the underlying linear system is strictly diagonally dominant; thus, we expect the problem to be solved properly even by the exact Jacobi-type method.

Example 6.2

$$f_1(x) := \frac{x_1^2}{4} + x_1 x_2 - 5x_1, \quad f_2(x) := \frac{x_2^2}{6} - x_1 x_2 - x_2.$$

This example is also a positive definite quadratic, however the spectral radius of the Jacobi iteration matrix is greater than 1, which may hinder the convergence of the exact Jacobi-type method.

Example 6.3

$$f_1(x) := x_1^2 + x_1 x_2 - 5x_1, \quad f_2(x) := -\frac{3x_2^2}{2} - x_1 x_2 - x_2.$$

In this example we considered a game without an equilibrium solution, given that f_2 has no local minimizer for all x_1 fixed. However, this problem has a non-equilibrium stationary point at $(3.2, -1.4)$.

Example 6.4

$$f_1 := -x_1(0.6 - x_2), \quad f_2 := x_2(0.7 - x_1).$$

This example comes from a mixed strategies formulation on a discrete equilibrium game, where the government needs to decide between the distribution of two different types of vaccines for a viral disease, for more details see [4]. This kind of problem has the special property that the Hessians are null, meaning that Jacobi's method is undefined.

Example 6.5

$$f_1(x) := \frac{x_1^3 x_2^2}{3} + \frac{x_1^2}{2}, \quad f_2(x) := \frac{x_1^2 x_2^3}{3} + \frac{x_2^2}{2}.$$

In the fifth example, we chose a cubic problem with an equilibrium point in $(0,0)$ *and a stationary point in* $(-1,-1)$*, which is not an equilibrium.*

The results are given in Table 1. For each method we report the last point obtained, the norm of g^k, and the number of iterations used by each method.

Table 1: Test results

	Example 6.1: Quadratic with matrix with spectral radius < 1 - Solution: $(2,1)$			
Method	Alg1	Yuan	Jacobi	Newton
Point	$(2,1)$	$(1.99997, 0.99999)$	$(2,1)$	$(2,1)$
$\|g^k\|$	0	$9.85431 \cdot 10^{-5}$	0	0
# iter	1	1	2	1
	Example 6.2: Quadratic with matrix with spectral radius > 1 - Solution: $(0.57142, 4.71428)$			
Method	Alg1	Yuan	Jacobi	Newton
Point	$(0.57142, 4.71428)$	$(0.57145, 4.71419)$	(∞, ∞)	$(0.57142, 4.71428)$
$\|g^k\|$	$2.22009 \cdot 10^{-16}$	$9.71940 \cdot 10^{-5}$	∞	$2.22009 \cdot 10^{-16}$
# iter	1	158	398	1
	Example 6.3: Quadratic problem without a solution.			
Method	Alg1	Yuan	Jacobi	Newton
Point	$(-\infty, \infty)$	$10^3 \cdot (-0.49401, 0.99900)$	$(3.19999, -1.39999)$	$(3.2, -1.4)$
$\|g^k\|$	∞	$2.50109 \cdot 10^3$	$4.28624 \cdot 10^{-5}$	$1.77615 \cdot 10^{-16}$
# iter	372	1000	8	1
	Example 6.4: Problem with null Hessians - Solution: $(0.7, 0.6)$			
Method	Alg1	Yuan	Jacobi	Newton
Point	$(0.7000, 0.60001)$	$10^9 \cdot (1.86823, -0.62812)$	-	$(0.7, 0.6)$
$\|g^k\|$	$7.00011 \cdot 10^{-5}$	$2.59973 \cdot 10^9$	-	$2.22009 \cdot 10^{-16}$
# iter	1	1000	-	1
	Example 6.5: Cubic problem - Solution: $(0,0)$			
Method	Alg1	Yuan	Jacobi	Newton
Point	$10^{-8} \cdot (0.27025, 0.27025)$	$10^{-4} \cdot (0.64551, 0.31008)$	$(-10^{-154} \cdot 0.74020, -\infty)$	$(-1,-1)$
$\|g^k\|$	$5.40980 \cdot 10^{-9}$	$8.50096 \cdot 10^{-5}$	∞	$5.04872 \cdot 10^{-16}$
# iter	9	25	6	7

In Examples 6.1 and 6.2 we obtained the expected results: Jacobi managed to converge for the first one but not for the second one, while all the other three methods found the solution. Algorithm 1 and Newton's method converged in one iteration since the problems are quadratic with positive definite Hessians.

In Example 6.3, which does not have a solution, the exact Jacobi and Newton's methods converged to the stationary point $(3.2, -1.4)$, meaning that they did not aim at solving the actual problem. Our algorithm and Yuan's algorithm, on the other hand, did not converge, as should be expected. Yuan's method performed the maximum number of iterations, while Algorithm 1 found $(-\infty, +\infty)$ in 372 iterations.

In Example 6.4 the exact Jacobi method is undefined, since the objective functions' Hessians $\nabla^2_{x_i x_i} f_i(x^k)$ are null, meaning that the component-wise systems in (3) are undefined. Since the problem is quadratic, Newton's method managed to solve it in one step, as expected. Our method does not use the Newton step here, as the Hessians are not positive definite, thus implying it had to compute a positive definite approximation. Even so, the method still managed to find a solution with the required precision in just one step. We highlight that Yuan's method could not find the solution in this example, due to the fact that Yuan's method rely on strong hypotheses on the system matrix that demand the positive definiteness of the jacobian of the function that defines system (3), which does not occur here. On the other hand, our method managed to reach the solution, since its theory is based on positive definite approximated Hessians instead.

As for Example 6.5, since the gradients are given by $x_i(x_i x^2_{\neg i} + 1)$ for each component i, the stationary points can be achieved either if $x_i = 0$ or if $x_i x^2_{\neg i} + 1 = 0$. In order to implement the exact Jacobi method for system (3), we need to choose how we are going to satisfy the system directly, either by selecting $x_i = 0$ (the equilibrium point) or selecting x_i satisfying $x_i x^2_{\neg i} + 1 = 0$. If we choose the former, Jacobi's method unsurprisingly takes one step. If instead we choose the latter, Jacobi's method behaves as follows: one of the components diverges decreasing the function value indefinitely and the other component manages to find a point that zeroes the respective gradient. On the other hand, Algorithm 1, Yuan's and Newton's method successfully managed to find a stationary point in this example. However, Newton's method converged to $(-1, -1)$, the closest stationary point, instead of to the actual solution. Algorithm 1 and Yuan's method, on the other hand, converged to the actual solution $(0, 0)$.

We highlight that Algorithm 1 never reduced the stepsize in order to achieve the conditions (14)-(19) in the quadratic Examples 6.1-6.4. This is expected and consistent with Proposition 5.3 for the strictly convex cases. In Example 6.5 the stepsize was reduced at some iterations, however, at most one reduction of the stepsize was needed for the direction computed to be accepted by the descent criteria (14)-(19). Even when the stepsize was reduced, this occurred only far from the solution, which is also consistent with a Newtonian approach and indicates that the hypothesis of the sequence $\{t^k\}$ being bounded away from zero is plausible.

6.2 Facility location problem

In the facility location problem, first studied in [14], one may choose the best place to open facilities. In general, servers seek to be close to their customers due to the various commercial and logistical advantages that this can entail. Here we consider the problem of two players who each want to open a facility in order to obtain the highest expected return on trading with N customers. A player's chances of serving a certain customer depends on the proximity of their facility with the position of the customer in relation to the competitor's proximity. Denoting by x_i the position of the facility i, $i \in \{1, 2\}$, and by z_j the position of the client j, $j \in \{1, \cdots, N\}$, let us consider here that the probability of player i serving client j is $1 - \frac{\|x_i - z_j\|^2}{\|x_i - z_j\|^2 + \|x_{\neg i} - z_j\|^2}$. Thus, if b_j^i is the profit of player i in serving customer j, the function to be minimized by each player to obtain the best expected revenue is

$$f_i(x_i, x_{\neg i}) = \sum_{j=1}^{N} \frac{b_j^i \|x_i - z_j\|^2}{\|x_i - z_j\|^2 + \|x_{\neg i} - z_j\|^2}, \quad i = 1, 2, \tag{56}$$

resulting in a NEP.

Notice that the functions are not defined when both facilities are located exactly at the position of one of the customers, a very pathological situation which was never encountered by any of the algorithms we run. Anyway, these problems are complex, since the functions are non-convex. In Figure 1 we illustrate the objective function of the first player in a one-dimensional problem, with the decision of the second player fixed at 0.915 and three clients located at $z_1 = 1, z_2 = -1$ and $z_3 = 3$, with a uniform profit of 1. For this problem an equilibrium solution is $(1.901, 0.915)$, which is found by Algorithm 1, with the same parameters used in Subsection 6.1, from the starting point $(2, 1)$.

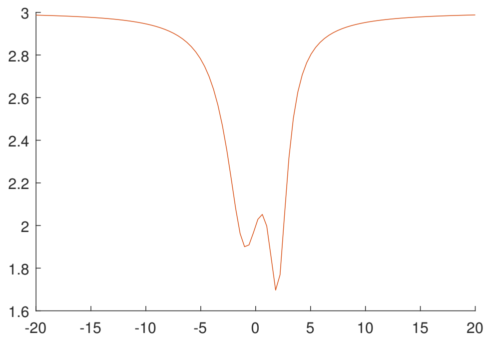

Figure 1: Unidimensional objective function for the facility location problem.

For a more extensive analysis, let us now consider a two-dimensional problem. For our test we selected a problem in the form (56) with the clients' coordinates $z_1 = (1, 0), z_2 = (0, 1), z_3 = (-1, 0)$, and $z_4 = (0, -1)$. We chose the profit vector for each player as $b^1 = (1, 2, 1, 1)$ and $b^2 = (1, 2, 2, 3)$. We run Algorithm 1 for 100 random initial points with coordinates generated in the interval $[-2, 2]$, comparing with Yuan's, Jacobi and Newton's method. Here, for the Jacobi method we solved problems (5) and (6) with $\phi_i = f_i$ using Matlab's *fminunc* solver instead of dealing with system (3). In Figure 2 we report the final outcome of each algorithm according to the stopping criterion $\|g^k\| \leq 10^{-6}$. We report simply whether the method converged to an equilibrium point, a non-equilibrium stationary point or if it diverged. We see that Jacobi and Yuan's method perform poorly due to the properties of the Jacobian of the correspondent first-order conditions. Jacobi diverged in 88 out of 100 runs while Yuan's method diverged in 97 runs, reaching the maximum number of iterations. Algorithm 1 converged to a true equilibrium point in all cases, taking on average 36 iterations and 1.08 seconds of CPU time, while Newton's method mostly converged to a non-equilibrium stationary point, having attained the true equilibrium point in only 13 of the 100 runs. Newton's method, on average, took 19 iterations and 0.76 seconds to converge, however, for the 13 initial points where both algorithms converged to the equilibrium, they were able to perform similarly as Algorithm 1 took 0.82 seconds to perform 19 iterations while Newton's method took 0.75 seconds to perform 16 iterations, on average.

In Figure 3 we see the behavior of Algorithm 1 for this problem for the initial point $x_1^0 = (2, 3)$ and $x_2^0 = (-3, 2)$. The filled circles in the plane represent the optimal facility locations found for the agents, while an empty circle represents the iterates. We see that Algorithm 1 forces the trajectory to the optimal placement. Running Newton's method with this same data yields a sequence that diverges, forcing the second facility to $(+\infty, -\infty)$, yet still stopping declaring success since the gradients tend to zero.

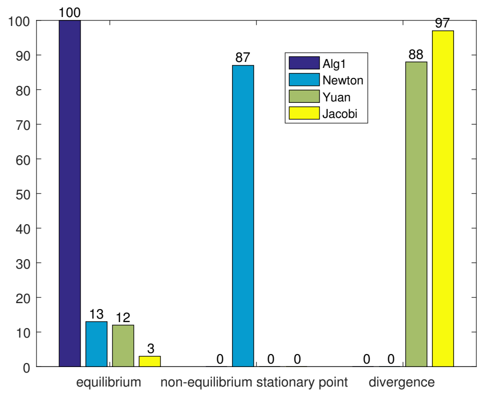

Figure 2: Facility allocation problem: quality of convergence for different methods.

In conclusion, our numerical experiments attest that our method may avoid converging to a non-equilibrium stationary point, differently from methods that deal only with the nonlinear system of equations obtained by the first order necessary optimality conditions as the standard Newton method. Also, differently from Jacobi and Yuan's method, our method is based on a predicted behavior of the other player, which seems to considerably speed up convergence. Moreover, Jacobi and Yuan's method seem to struggle when the Jacobian of the system of equations is not well-behaved, which does not occur with our method.

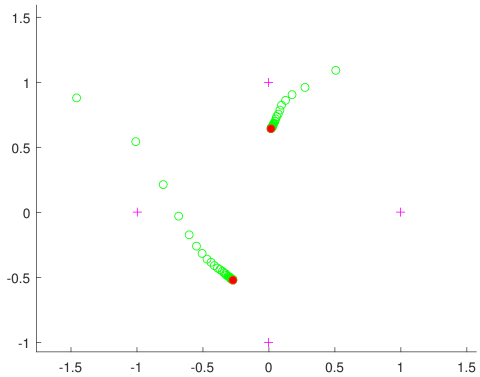

Figure 3: Bidimensional behavior of convergence of Algorithm 1.

7 Final remarks

The formulation of Nash Equilibrium Problems (NEPs) is one of the most important concepts in modeling the behavior of social and economic agents in practice. In particular, in this paper, we considered the unconstrained NEP with two players and continuous variables, where we focus on the possibility of non-convexities in the objective functions of each player. Usually, for this kind of problem in the optimization literature, one is interested only in computing a Nash equilibrium, or, more accurately, a point satisfying the system of nonlinear equations that are satisfied by all equilibrium points. This approach may be misleading as this system of equations may include also other undesired solutions.

The contribution of this paper is to propose an algorithm for this problem with two main characteristics: Firstly, our algorithm addresses the NEP in its entirety, considering its minimization structure and avoiding convergence to an undesired stationary point. Secondly, the iterates of our algorithm are computed in such a way that they can be interpreted as mimicking the behavior of the agents in practice, namely, the iterates computed by the method approximate in some sense the true decision made by the players in different time periods. That is, a new iterate is computed by player one by minimizing a convex quadratic model of their objective function parameterized by a prediction of the action that will be taken by the second player. The prediction is computed simultaneously, assuming that the second player is adopting the same strategy. This gives rise to a Jacobi-type Newton method, globalized by an Armijo-type linesearch, which guarantees that at the new iterate both players will decrease their predicted objectives; this avoids, as much as possible, convergence to undesired stationary points. On the other hand, the method can also be seen as a new interpretation of Newton's method, which is known to be intrinsically linked to fast local convergence.

Both characteristics of our algorithm are new and relevant for the literature of NEPs, especially when non-convexities are considered, favoring convergence to a true equilibrium rather than simply a stationary point. Moreover, our approach opens the path to new and interesting investigations of the behavior of Newtonian iterates built with the goal of mimicking the behavior of true economic agents. Clearly, this is a starting point for this type of investigation and many aspects still need to be elucidated. For example, it would be interesting to interpret the many parameters of the algorithm with the true agents' behavior. We believe that the synergy between the interpretation of the game's dynamics and the sequence generated by an algorithm can inspire both better techniques for solving NEPs and a better understanding of practical situations associated with models of this type.

In this paper, we focused on the analysis considering convergence of the iterative sequence when the step size sequence $\{t^k\}$ is bounded away from zero. An important open problem for a better understanding of the algorithm would be to prove whether or not it is possible to guarantee the boundedness away from zero of $\{t^k\}$. Additionally, it would be interesting to investigate whether or not the step $t = 1$ is always accepted when close enough to solutions that satisfy the second-order sufficient optimality condition. This would be associated with both the boundedness away from zero of $\{t^k\}$ and the quadratic convergence of the algorithm. However, more refined analysis should be carried out to also capture other types of phenomena that may occur to the sequence besides convergence. For instance, in a general dynamical system, the interest goes far beyond mere convergence to an equilibrium, since several other interesting phenomena may occur. We expect that subsequent research on this topic should focus on understanding and classifying such phenomena, with the analysis carried out to the algorithms built for mimicking the dynamics.

Taking into account that our algorithm converges more often to a true equilibrium, we envision several possible extensions of this work. One may consider $N > 2$ players, while also considering that each player's decision variable x_i, $i = 1, 2, \ldots, N$, is constrained to a feasible set $X_i \subset \mathbb{R}^{n_i}$, or, more generally, the feasible set may also be parameterized by the other players' decision $x_{\neg i}$, namely, $X_i := X_i(x_{\neg i})$, as in what is known as a *generalized NEP*. Our intention is to extend the ideas of this paper to consider such parameterized constrained problems, borrowing ideas from Sequential Quadratic Programming, where one considers a linearization of the constraints in a Newtonian framework. In addition, our unconstrained algorithm may be employed in an Augmented Lagrangian framework for the general constrained problem (see [16] and [5]). In this framework, the constraints are penalized and at each iteration of the algorithm an unconstrained NEP must be solved, and finding true solutions of the subproblems rather than mere stationary points should favor the algorithm in finding true solutions of the original problem.

Another interesting point to be investigated in the future is the connection with dynamic learning, reinforcing decisions that have better objective function values. Since the philosophy of using best response functions together with predicted decisions for the other player is similar to our approach, this may generate relevant ramifications when substituting the standard two step process of independent decision making and probability distribution updated considered in [22, 17] for ours. Furthermore, in stochastic versions of the algorithm, the notions of cycling and orbiting may be more naturally understood as generalized convergence, simplifying the analysis.

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