Evaluating Mixed-Integer Programming Models over Multiple Right-hand Sides

Rachael M. Alfant\textsuperscript{a}, Temitayo Ajayi\textsuperscript{b}, Andrew J. Schaefer\textsuperscript{a,\ast}

\textsuperscript{a}Department of Computational Applied Mathematics and Operations Research, Rice University, Houston, TX 77005
\textsuperscript{b}Nature Source Improved Plants, Ithaca, NY 14850

Abstract

A critical measure of model quality for a mixed-integer program (MIP) is the difference, or gap, between its optimal objective value and that of its linear programming relaxation. Although in many contexts, only an approximation of the right-hand side(s) is available, there is no consensus on appropriate measures for MIP model quality over multiple right-hand sides. In this paper, we provide model formulations for the expectation and extrema of absolute and relative MIP gap functions over finite discrete sets.

Keywords: Mixed-integer programming, superadditive duality, value function

1. Introduction

Given a (maximization) mixed-integer program (MIP), the gap is the difference between the optimal objective value of its linear programming (LP) relaxation and that of the MIP. The MIP gap is a critical measure of model quality for MIPs with fixed data. Some theoretical implications include improving solution algorithms, such as branch and bound (Le Bodic and Nemhauser (2017)). Practical implications include the interpretation of the dual objective (price) function, which tells us how much extra resources are worth (Wolsey (1981)). A classic example for a fixed right-hand side is the Uncapacitated Facility Location Problem, for which there exist “Weak” and “Strong” models. The Strong model yields tighter bounds in a branch-and-cut algorithm (Cornuéjols et al. (1983), Conforti et al. (2014)), and thus performs better than the Weak model.

In practice, the right-hand side may not be known exactly or it may vary. Thus, evaluative metrics must be developed in order to assess a MIP model’s quality over multiple right-hand sides. Such metrics may have applications in sensitivity analysis (e.g., Wolsey (1981), Gülcelsoy (2009))

\ast Corresponding author

Email addresses: rma10@rice.edu (Rachael M. Alfant), tayo.ajayi25@gmail.com (Temitayo Ajayi), andrew.schaefer@rice.edu (Andrew J. Schaefer)
and stochastic programming (e.g., Sen (2005), Kong et al. (2006), Özaltın et al. (2012)). Blair and Jeroslow (1977) provide properties of the MIP value function and use superadditivity to conduct sensitivity analyses on the optimal value as the right-hand side varies. Blair and Jeroslow (1979) subsequently bound the MIP gap as the right-hand side varies. Blair (1995) subsequently identifies a class of polynomially computable formulas such that for every rational MIP, and for all feasible right-hand sides, there is a formula equal to its value function.

Value functions and superadditive duality play central roles in this paper, and have various applications in optimization. Johnson (1974) extends integer programming (IP) duality theory (see Güzelsoy and Ralphs (2010) for an extensive survey on IP duality) to MIPs through examining the group problem. Ralphs and Hassanzadeh (2014) characterize MIP value functions and present a cutting-plane algorithm for their construction. Schaefer (2009) provides a polyhedral description of the inverse-feasible objectives for an IP using superadditive duality; Lamperski and Schaefer (2015) use superadditive duality to formulate the inverse MIP problem (the problem of determining the values of unobserved parameters of a MIP such that a provided solution is optimal).

Gomory (1965) proves that absolute IP gap functions are periodic with respect to the columns of the constraint matrix, while Ajayi et al. (2021) reproduce this result using superadditivity. To our knowledge, our paper is the first to study the periodicity of MIP gap functions. Furthermore, Ajayi et al. (2021) optimize IP gap functions over multiple right-hand sides; this paper expands the framework presented in Ajayi et al. (2021) by formulating optimization problems that serve as measures of MIP model quality over finite, discrete sets. These formulations compute the expectation and extrema of absolute and relative MIP gap functions over multiple right-hand sides.

2. Preliminaries

Let \( A \in \mathbb{Z}^{m \times n}_+ \), \( G \in \mathbb{Q}^{m \times p}_+ \), \( b \in \mathbb{R}^m_+ \), \( c \in \mathbb{R}^n_+ \), and \( h \in \mathbb{R}^p_+ \). Let \( a_j \) be the \( j \)th column of \( A \) and \( g_k \) the \( k \)th column of \( G \). Consider the mixed-integer programming problem:

\[
 z_{MIP}(b) = \max_{x \in \mathbb{Z}^n_+, y \in \mathbb{R}^p_+} \{ c^\top x + h^\top y \mid Ax + Gy \leq b \}. \tag{1}
\]

Let \( z_{LPR}(b) \) be the optimal objective value of the LP relaxation of (1) with right-hand side \( b \).

**Definition 2.1.** The absolute gap for a MIP is defined as: \( z_{LPR}(b) - z_{MIP}(b) \).

**Definition 2.2.** The relative gap for a MIP is defined as: \( \frac{z_{MIP}(b)}{z_{LPR}(b)} \).
Note that the relative gap for MIPs is undefined for any $b$ such that $z_{LP}(b) = 0$. Define $\mathcal{B}(0, b) := \prod_{i=0}^{m} [0, b_i]$, i.e., the Cartesian product of the intervals $[0, b_1], \ldots, [0, b_m]$, and $\tilde{\mathcal{B}}(0, b) := \mathcal{B}(0, b) \cap \mathbb{Z}_+^m$. We study the MIP gap over $\mathcal{B}(0, b)$. Throughout this paper, we assume the right-hand side parameter $\beta$ is in $\mathcal{B}(0, b)$ and $\beta \in \beta \mathcal{B}(0, b)$ is such that $\beta \leq \beta$.

2.1. IP Value Functions and Duality

Our approach to gap functions for MIPs is very closely related to MIP value functions and MIP duality. Thus, to study gap functions for MIPs, we first characterize superadditive duality for pure IPs and define value functions of pure IPs and LPs. For any $\beta \in \mathcal{B}(0, b)$ and $\beta \in \beta \mathcal{B}(0, b)$, the parametrized LP, $LP(\beta - \beta)$, with value function $z_{LP}(\beta - \beta)$, is defined as:

$$z_{LP}(\beta - \beta) := \max_{y \in \mathbb{R}^p_+} \{h^\top y \mid Gy \leq \beta - \beta\}. \quad (2)$$

Let $\pi$ be the dual variable corresponding to the constraint in (2). The dual of $LP(\beta - \beta)$, $LPD(\beta - \beta)$, with value function $z_{LPD}(\beta - \beta)$, is defined as follows:

$$z_{LPD}(\beta - \beta) := \min_{\pi \in \mathbb{R}^m_+} \{\pi^\top (\beta - \beta) \mid \pi^\top G \geq h^\top\}. \quad (3)$$

Let $\mathcal{Y} := \{\pi \in \mathbb{R}^m_+ \mid \pi^\top G \geq h^\top\}$. Denote $\Omega := \{\pi^r \mid r \in \mathcal{R}\}$ the set of extreme points of $\mathcal{Y}$. Because $\mathcal{Y}$ is bounded, $\min_{\pi \in \Omega} \pi^\top (\beta - \beta) = z_{LPD}(\beta - \beta)$. Furthermore, because there exist a finite number of constraints and variables for $LPD(\beta - \beta)$, the set of extreme points of $\mathcal{Y}$ is finite, i.e., $|\mathcal{R}| < +\infty$, as a consequence of Weyl’s Theorem (Charnes and Cooper, 1958). We use the variable, $\pi_{\beta - \beta}$, to model the value of $z_{LPD}$ on the lattice $\beta \mathcal{B}(0, \beta)$.

Assumption 2.1. $A$ and $G$ have no zero columns.

Assumption 2.1 implies finite optima for $LP(\beta - \beta)$; weak duality then yields finite optimum for $LPD(\beta - \beta)$. For $\beta \in \beta \mathcal{B}(0, \beta)$, the parametrized IP, $IP(\beta)$, with value function $z_{IP}$, is:

$$z_{IP}(\beta) := \max_{x \in \mathbb{Z}^n_+} \{c^\top x \mid Ax \leq \beta\}. \quad (4)$$

Remark 2.1. For all $\beta \in \beta \mathcal{B}(0, \beta)$, $z_{IP}(\beta) < +\infty$ and $IP(\beta)$ is feasible.

Because we assume the data are nonnegative, Remark 2.1 is a direct result of Assumption 2.1.

Definition 2.3. (Blair and Jeroslow, 1982) Chvátal functions are the smallest class of functions constructed recursively from taking sums, nonnegative multiples, and ceilings of linear functions.
**Definition 2.4.** (Blair and Jeroslow (1982)) *Gomory functions* are an extension of Chvátal functions, which include functions constructed recursively from taking maximums of linear functions.

**Proposition 2.1.** (Blair and Jeroslow (1982)) The value function, \( z_{IP} \), belongs to the class of *Gomory functions*, which can be written as the maximum of finitely many Chvátal functions.

There are various formulations for the dual of \( \text{IP}(\beta) \), though some are less adaptable for developing measures of model quality over multiple right-hand sides. Recall \( A \in \mathbb{Z}_+^{m \times n} \) and \( \beta \in \widehat{B}[0, \widehat{\beta}] \), and consider the following dual formulation.

**Theorem 2.1.** (Johnson (1974)) A strong dual to the IP (4) is:

\[
\min \{ F(\beta) \mid F(a_i) \geq c_j \ \forall j \in \{1, 2, \ldots, n\}, \ F \text{ nondecreasing and superadditive, } F(0) = 0 \}. \tag{5}
\]

The superadditive dual of \( \text{IP}(\beta) \) is as follows (Wolsey (1981), Ajayi et al. (2021)):

\[
z_{\text{SIP}}(\beta) := \min \phi(\beta)
\]

\[
\text{s.t. } \phi(a_j) \geq c_j \ \forall j \in 1, 2, \ldots, n,
\]

\[
\phi \text{ nondecreasing and superadditive,}
\]

\[
\phi(0) = 0,
\]

\[
\phi(\beta_1) \in \mathbb{R}_+ \ \forall \beta_1 \in \widehat{B}[0, \widehat{\beta}].
\]

Note that we use the variable, \( \phi(\beta) \), to model the value of \( z_{\text{SIP}} \) on the lattice \( \widehat{B}[0, \widehat{\beta}] \). Denote \( \Phi(\widehat{\beta}) := \{ \phi \in \mathbb{R}^{\lceil \widehat{\beta} \rceil} \mid (\text{6b}) - (\text{6e}) \} \). Ajayi et al. (2021) use (6) to model IP gap functions parametrized over a set of right-hand sides as LPs (albeit of exponentially large size) with at most one SOS1 constraint. In addition, Wolsey (1981) proves that (6) is a strong dual to \( \text{IP}(\beta) \) for a single \( \beta \), and Ajayi et al. (2021) extend this result to all \( \beta \in \widehat{B}[0, \widehat{\beta}] \).

**Theorem 2.2.** (Ajayi et al. (2021)) For all \( \beta \in \widehat{B}[0, \widehat{\beta}] \), \( \text{SIP}(\beta) \) is a strong dual to \( \text{IP}(\beta) \).

**Proposition 2.2.** (Wolsey (1981), Ajayi et al. (2021)) For any \( \phi \in \Phi(\widehat{\beta}) \), \( \phi(\beta) \geq z_{IP}(\beta) \), for all \( \beta \in \widehat{B}[0, \widehat{\beta}] \).

### 2.2. MIP Value Functions and Superadditive Duality

Let \( S(\widehat{\beta}) := \{ (x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p \mid Ax + Gy \leq \widehat{\beta} \} \). We define the parametrized mixed-integer program, \( \text{MIP}(\widehat{\beta}) \), with value function \( z_{\text{MIP}} \), as:

\[
z_{\text{MIP}}(\widehat{\beta}) := \max_{x,y} \{ c^\top x + h^\top y \mid (x, y) \in S(\widehat{\beta}) \}. \tag{7}
\]
Proposition 2.3. (Blair and Jeroslow (1977)) The value function, \(z_{\text{MIP}}\), is superadditive, i.e., for any \(\hat{\beta}_1, \hat{\beta}_2 \in B[0, b]\) with \(\hat{\beta}_1 + \hat{\beta}_2 \in B[0, b]\), \(z_{\text{MIP}}(\hat{\beta}_1) + z_{\text{MIP}}(\hat{\beta}_2) \leq z_{\text{MIP}}(\hat{\beta}_1 + \hat{\beta}_2)\).

Duality for MIPs is more complex than that of IPs and LPs. One approach is to use a directional derivative formulation. Let \(d \in \mathbb{R}^m_+\), and define the directional derivative:

\[
F : \mathbb{R}^m_+ \to \mathbb{R}, \quad F(d) := \lim_{\lambda \downarrow 0^+} \frac{F(\lambda d)}{\lambda}.
\]

Theorem 2.3. (Johnson (1974), Blair and Jeroslow (1977), Nemhauser and Wolsey (1999)) Let \(S(\bar{\beta}) := \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ \mid Ax + Gy \leq \bar{\beta}\}\) and \(z_{\text{MIP}}(\bar{\beta}) := \max \{c^\top x + h^\top y \mid (x, y) \in S(\bar{\beta})\}\).

A strong dual to the mixed-integer programming problem is:

\[
w := \min \; F(\bar{\beta}) \quad (8a)
\]

s.t. \(F(a_j) \geq c_j \; \forall j \in \{1, 2, \ldots, n\}\), \(F(g_k) \geq h_k \; \forall k \in \{1, 2, \ldots, p\}\), \(F\) nondecreasing and superadditive, \(F(0) = 0\). \(8c\)

Note that \(z_{\text{MIP}}(\bar{\beta})\) is an optimal solution to (8). Observe that unlike (5), (8) contains a constraint with a directional derivative, (8c), which may present modeling complications and makes a formulation similar to (6) problematic. For this reason, we instead formulate the superadditive dual of MIP by exploiting the structure of the MIP value function presented in Proposition 2.4.

Proposition 2.4. (Lamperski and Schaefer (2015)) For any \(\bar{\beta} \in B[0, b]\),

\[
z_{\text{MIP}}(\bar{\beta}) := \max_{\beta \in [0, \bar{\beta}]} \{z_{\text{IP}}(\beta) + z_{\text{LP}}(\bar{\beta} - \beta)\}. \tag{9}
\]

Proposition 2.5 extends the dual formulation presented in Wolsey (1981) to MIPs.
Proposition 2.5. [Lamperski and Schaefer (2015)] Let $A \in \mathbb{Z}_{+}^{m \times n}$, $G \in \mathbb{Q}_{+}^{m \times p}$, and $\beta \in \mathbb{B}[0, b]$. Then, $SDMIP(\hat{\beta})$ is equivalent to:

\[
\begin{align*}
\text{(11a)} & \quad z_{SDMIP}(\hat{\beta}) := \min_{\phi, \pi} \phi(\hat{\beta}) + \pi_{\hat{\beta} - \beta}(\hat{\beta} - \beta) \\
\text{s.t.} & \quad \phi(\hat{\beta}) + \pi_{\hat{\beta} - \beta}(\hat{\beta} - \beta) \geq \phi(\beta) + \pi_{\beta - \beta}(\beta - \beta) \quad \forall \beta \in \hat{B}[0, \hat{\beta}], \\
\text{(11b)} & \quad \phi(a_j) \geq c_j \quad \forall j \in \{1, 2, \ldots, n\}, \\
\text{(11c)} & \quad \phi(0) = 0, \\
\text{(11d)} & \quad \phi(\beta_1) + \phi(\beta_2) \leq \phi(\beta_1 + \beta_2) \quad \forall \beta_1, \beta_2, \beta_1 + \beta_2 \in \hat{B}[0, \hat{\beta}], \\
\text{(11e)} & \quad \phi(\beta_1) \leq \phi(\beta_2) \quad \forall \beta_1 \leq \beta_2 \in \hat{B}[0, \hat{\beta}], \\
\text{(11f)} & \quad \phi(\beta) \in \mathbb{R}_+ \quad \forall \beta \in \hat{B}[0, \hat{\beta}], \\
\text{(11g)} & \quad \pi_{\beta - \beta} \in \Omega \quad \forall \beta \in \hat{B}[0, \hat{\beta}].
\end{align*}
\]

Formulation (11) avoids the use of directional derivatives, but at the expense of an even larger LP. Vector $\phi$ is indexed by $\beta$; vector $\pi$ is indexed by $(\hat{\beta} - \beta)$ and dot-multiplied with $(\hat{\beta} - \beta)$ in (11a) and (11b). We show $SDMIP(\hat{\beta})$ is a strong dual to $MIP(\hat{\beta})$.

Theorem 2.4. Let $\hat{\beta} \in \mathbb{B}[0, b]$, and let $\beta^* \in \arg \max_{\beta \in \mathbb{B}[0, b]} z_{IP}(\beta) + z_{LP}(\hat{\beta} - \beta)$. Then, for $\phi \in \Phi(\hat{\beta})$ and $\pi_{\hat{\beta} - \beta^*} \in \Omega$, we have that $\phi(\beta^*) + \pi_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*) \geq z_{MIP}(\hat{\beta})$.

Proof. Let $\hat{\beta} \in \mathbb{B}[0, b]$. Choose $\beta^* \in \hat{B}[0, \hat{\beta}]$ such that $z_{MIP}(\hat{\beta}) = z_{IP}(\beta^*) + z_{LP}(\hat{\beta} - \beta^*)$. Let $\phi \in \Phi(\hat{\beta})$ and $\pi_{\hat{\beta} - \beta^*} \in \Omega$ be such that $(\phi, \pi_{\hat{\beta} - \beta^*})$ is feasible for $SDMIP(\hat{\beta})$. Then, $z_{SDMIP}(\hat{\beta}) \leq \phi(\beta^*) + \pi_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*)$. By Proposition 2.2, $z_{IP}(\beta^*) \leq \phi(\beta^*)$, and by LP weak duality, $z_{LP}(\hat{\beta} - \beta^*) \leq \pi_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*)$. Thus, $z_{MIP}(\hat{\beta}) = z_{IP}(\beta^*) + z_{LP}(\hat{\beta} - \beta^*) \leq \phi(\beta^*) + \pi_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*)$. □

Theorem 2.5. Let $\hat{\beta} \in \mathbb{B}[0, b]$. Then, $SDMIP(\hat{\beta})$ is a strong dual to $MIP(\hat{\beta})$.

Proof. Let $\hat{\beta} \in \mathbb{B}[0, b]$. Choose $\beta^* \in \hat{B}[0, \hat{\beta}]$ such that $z_{MIP}(\hat{\beta}) = z_{IP}(\beta^*) + z_{LP}(\hat{\beta} - \beta^*)$. Let $x^* \in \text{opt}_{IP}(\beta^*)$ and $y^* \in \text{opt}_{LP}(\hat{\beta} - \beta^*)$. By Theorem 2.2, $z_{IP}(\beta^*) = c^T x^* = \phi(\beta^*) = z_{SIP}(\beta^*)$ for some $\phi^* \in \Phi(\hat{\beta})$. Also, by LP duality, $z_{LP}(\hat{\beta} - \beta^*) = h^T y^* = \pi_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*) = z_{LPD}(\hat{\beta} - \beta^*)$ for $\pi_{\hat{\beta} - \beta^*} \in \Omega$ where $\pi_{\hat{\beta} - \beta^*} \in \arg \min_{\pi \in \Omega} \pi^T (\hat{\beta} - \beta^*)$. Now, consider the tuple $(\phi^*, \pi_{\hat{\beta} - \beta^*})$. Note that this tuple is feasible for $SDMIP(\hat{\beta})$: $\phi^*$ satisfies (11c)-(11g), $\pi_{\hat{\beta} - \beta^*}$ satisfies (11b), and because of the way $\beta^*$ was chosen, i.e., because $\beta^* \in \arg \max_{\beta \in \mathbb{B}[0, b]} z_{IP}(\beta) + z_{LP}(\hat{\beta} - \beta)$, $\phi(\beta^*) + \pi_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*) \geq \phi^*(\beta^*) + \pi^*_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*)$ for all $\beta \in \hat{B}[0, \hat{\beta}]$, thus satisfying (11b). Furthermore: $\phi^*(\beta^*) + \pi^*_{\hat{\beta} - \beta^*}(\hat{\beta} - \beta^*) = z_{SIP}(\beta^*) + z_{LPD}(\hat{\beta} - \beta^*) = z_{IP}(\beta^*) + z_{LP}(\hat{\beta} - \beta^*) = z_{MIP}(\hat{\beta})$. By
Theorem 2.4. \( \phi^*(\beta^*) + \pi^T_{\beta-\beta^*}(\beta - \beta^*) = z_{MIP}(\beta) \leq \phi(\beta^*) + \pi^T_{\beta-\beta^*}(\beta - \beta^*) \) for all \( \phi \in \Phi(\beta) \) and \( \pi_{\beta-\beta^*} \in \Omega \). Thus, \( z_{SDMIP}(\beta) = \phi^*(\beta^*) + \pi^T_{\beta-\beta^*}(\beta - \beta^*) \), and \( z_{MIP}(\beta) = z_{SDMIP}(\beta) \). \( \square \)

Now, consider the LP relaxation of (7):

\[
z_{LPR}(\beta) := \max_{x \in \mathbb{R}_+^n, y \in \mathbb{R}_+^p} \{ c^T x + h^T y \mid Ax + Gy \leq \beta \}. \tag{12}
\]

Define \( \mathcal{P}(\beta) := \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^p \mid Ax + Gy \leq \beta \} \). Note that Proposition 2.3 applies to the LP relaxation value function, i.e., we have that \( z_{LPR}(\beta_1) + z_{LPR}(\beta_2) \leq z_{LPR}(\beta_1 + \beta_2) \) for any \( \beta_1, \beta_2 \in \mathcal{B}[0, b] \) with \( \beta_1 + \beta_2 \in \mathcal{B}[0, b] \).

Remark 2.2. For all \( \beta \in \mathcal{B}[0, b] \), \( z_{LPR}(\beta) < +\infty \) and \( \text{LPR}(\beta) \) is feasible.

Because we assume the data are nonnegative, Remark 2.2 follows from Assumption 2.1. Let \( \mathcal{Q} := \{ u \in \mathbb{R}_+^m \mid A^T u \geq c, G^T u \geq h \} \), and let \( \{ u^q \mid q \in \kappa \} \) be the set of extreme points of \( \mathcal{Q} \). The dual of \( z_{LPR}(\beta) \) may be formulated as follows:

\[
z_{DLPR}(\beta) = \min_{q \in \kappa} \beta^T u^q. \tag{13}
\]

Because \( c \) and \( h \) are strictly positive, as a result of Assumption 2.1, Remark 2.2, and Weyl's Theorem (Charnes and Cooper [1958]), \( \text{DLPR}(\beta) \) is feasible and \( z_{DLPR}(\beta) \geq 0 \) for all \( \beta \in \mathcal{B}[0, b] \). Furthermore, there are a finite number of extreme points, i.e., \( |\kappa| < +\infty \).

Remark 2.3. There always exists an extreme point of \( \mathcal{Q} \) that is an optimal solution to \( \text{DLPR}(\beta) \).

Remark 2.3 allows for one to encode the objective function of \( \text{DLPR}(\beta) \) as a function of the extreme points of \( \mathcal{Q} \). We exploit this in Sections 4 and 5, where we optimize the expectation and extrema of absolute and relative MIP gap functions over finite discrete sets.

2.3. MIP Gap Functions

Definition 2.5. Given a set of right-hand sides, \( \mathcal{B}[0, b] \), the absolute gap function for MIPs is defined as: \( \Gamma : \mathcal{B}[0, b] \to \mathbb{R}_+ \cup \{\infty\} \), \( \Gamma(\beta) := z_{LPR}(\beta) - z_{MIP}(\beta) \).

An absolute gap that is close to zero indicates that the LP relaxation provides a high-quality approximation for the optimal objective value of the corresponding MIP. In addition, because \( S(\beta) \subseteq \mathcal{P}(\beta) \), we have that for all \( \beta \in \mathcal{B}[0, b] \), \( \Gamma(\beta) \geq 0 \). Define \( \mathcal{B}^+[0, b] := \{ \beta \in \mathcal{B}[0, b] \mid z_{MIP}(\beta) > 0 \} \).

Definition 2.6. Given a set of right-hand sides, \( \mathcal{B}^+[0, b] \), the relative gap function for MIPs is defined as: \( \gamma : \mathcal{B}^+[0, b] \to \mathbb{R}_+ \), \( \gamma(\beta) := \frac{z_{MIP}(\beta)}{z_{LPR}(\beta)} \).
A relative gap that is close to 1 indicates that the LP relaxation provides a high-quality approximation for the optimal objective value of the corresponding MIP. The domain of $\gamma$ is restricted to $B^+[0, b]$ in order to avoid division by zero. Thus, $\gamma(\tilde{\beta}) \in [0, 1]$ for all $\tilde{\beta} \in B^+[0, b]$.

3. Properties of Absolute MIP Gap Functions

[Ajayi et al. (2021)] prove that the absolute IP gap function defined over rational vectors is the minimum of finitely many Gomory functions. We extend this result to absolute MIP gap functions.

**Proposition 3.1.** The absolute MIP gap function defined over $B[0, b]$ is the minimum of finitely many Gomory functions.

The proof of Proposition 3.1 (and other results with omitted proofs) is in the E.C. The remainder of this section presents results for absolute MIP gap function periodicity. Gomory (1965) proves that the absolute IP gap function defined over rational vectors is the minimum of finitely many Gomory functions. We extend this result to absolute MIP gap functions. We provide analogous results for absolute IP gap functions.

**Proposition 3.2.** If $(x^*, y^*) \in \text{opt}_{MIP}(\tilde{\beta})$, then for all $(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p$ such that $0 \leq x \leq x^*$ and $0 \leq y \leq y^*$, $(x, y) \in \text{opt}_{MIP}(Ax + Gy)$.

**Proposition 3.3.** Let $\tilde{\beta} \in B[0, b]$ and $(x^*, y^*) \in \text{opt}_{MIP}(\tilde{\beta})$. Given $\eta \in \mathbb{N}$, let $J_\eta = \{1, \ldots, n\}$ denote the set of indices such that $x^*_j \geq \eta$ for $j \in J_\eta$. Denote $K := \{1, \ldots, p\}$, and let $\lambda^* = \min_{k \in K} \{y^*_k\}$. Then, for any $j \in J_\eta$, $k \in K$, and $\lambda \in [0, \lambda^*]$, $z_{MIP}(\tilde{\beta} - \eta a_j - \lambda g_k) = z_{MIP}(\tilde{\beta} - \eta c_j - \lambda h_k)$.

**Remark 3.1.** If $J_\eta = \emptyset$, Proposition 3.3 implies LP complementary slackness. If $K = \emptyset$, Proposition 3.3 implies IP complementary slackness (Nemhauser and Wolsey (1999)).

**Proposition 3.4.** Let $\tilde{\beta} \in B[0, b]$ and $(x^*, y^*) \in \text{opt}_{LPR}(\tilde{\beta})$. Given $\eta \in \mathbb{N}$, let $J_\eta = \{1, \ldots, n\}$ denote the set of indices such that $x^*_j \geq \eta$ for $j \in J_\eta$. Denote $K := \{1, \ldots, p\}$, and let $\lambda^* = \min_{k \in K} \{y^*_k\}$. Then, for any $j \in J_\eta$, $k \in K$, and $\lambda \in [0, \lambda^*]$, $z_{LPR}(\tilde{\beta} - \eta a_j - \lambda g_k) = z_{LPR}(\tilde{\beta} - \eta c_j - \lambda h_k)$.

**Theorem 3.1.** Let $\tilde{\beta} \in B[0, b]$, $(\tilde{x}^M, \tilde{y}^M) \in \text{opt}_{MIP}(\tilde{\beta})$, and $(\tilde{x}^L, \tilde{y}^L) \in \text{opt}_{LPR}(\tilde{\beta})$. Given $\eta \in \mathbb{N}$, let $J_\eta = \{1, \ldots, n\}$ denote the set of indices such that $\tilde{x}^M_j, \tilde{x}^L_j \geq \eta$ for $j \in J_\eta$. Denote $K := \{1, \ldots, p\}$, and let $\lambda^* = \min_{j \in J_\eta} \{y^M_j, y^L_j\}$. Then, for $j \in J_\eta$, $k \in K$, and $\lambda \in [0, \lambda^*]$, we have that $\Gamma(\tilde{\beta} - \eta a_j - \lambda g_k) = \Gamma(\tilde{\beta})$. If, in addition, $z_{LPR}(\tilde{\beta}) > \eta c_j + \lambda h_k$ and $z_{MIP}(\tilde{\beta}) \geq \eta c_j + \lambda h_k$, then $\gamma(\tilde{\beta} - \eta a_j - \lambda g_k) = z_{MIP}(\tilde{\beta} - \eta c_j - \lambda h_k)$.
Proof. Let \( \hat{\beta} \in \mathcal{B}[0, b] \). Note that the relative gap function result follows directly from Propositions 3.3 and 3.4. Let \((\hat{x}^M, \hat{y}^M) \in \text{opt}_{MIP}(\hat{\beta}) \) and \((\tilde{x}^L, \tilde{y}^L) \in \text{opt}_{LP R}(\hat{\beta}) \). By definition, \( \Gamma(\hat{\beta} - \eta a_j - \lambda g_k) = z_{LP R}(\hat{\beta} - \eta a_j - \lambda g_k) - z_{MIP}(\hat{\beta} - \eta a_j - \lambda g_k) \). By hypothesis, we consider a pair of indices, \((j, k)\), with \( j \in J_n \) and \( k \in K \), such that \( \tilde{x}^M_j, \tilde{x}^L_j \geq \eta \) and \( 0 \leq \lambda^* \leq \tilde{y}^M_k, \tilde{y}^L_k \). Then, by Propositions 3.3 and 3.4, \( z_{LP R}(\hat{\beta} - \eta a_j - \lambda g_k) = z_{LP R}(\hat{\beta}) - \eta c_j - \lambda h_k \) and \( z_{MIP}(\hat{\beta} - \eta a_j - \lambda g_k) = z_{LP R}(\hat{\beta}) - \eta c_j - \lambda h_k \) for all \( \lambda \in [0, \lambda^*] \). Thus, \( \Gamma(\hat{\beta} - \eta a_j - \lambda g_k) = z_{LP R}(\hat{\beta}) - \eta c_j - \lambda h_k - (z_{MIP}(\hat{\beta}) - \eta c_j - \lambda h_k) = z_{LP R}(\hat{\beta}) - z_{MIP}(\hat{\beta}) = \Gamma(\hat{\beta}) \).

4. Absolute Gap Functions over a Discrete Set

In this section, we present formulations for optimizing the expectation, infimum, and supremum of the absolute gap function, \( \Gamma \), over finite discrete sets. The following results extend those of Ajayi et al. (2021). Thus, following Ajayi et al. (2021), each formulation is associated with three letters: the first letter indicates the quality measure (expectation, infimum, or supremum), the second letter designates the gap function (absolute or relative), and the third letter, \( D \), indicates that the gap is measured over a discrete set. For all of the absolute and relative gap function formulations, it is important to note that as a consequence of Remark 2.3, the optimal objective value of DLPR(\( \hat{\beta} \)) can be written solely in terms of the extreme points of the feasible region \( Q \) for all \( \hat{\beta} \).

For each formulation in this section, let \( D \) be a finite, discrete subset of \( \mathcal{B}[0, b] \). The expectation of the absolute gap function can be used to determine the expected performance of the LP relaxation as an approximation for the MIP, with a gap close to zero indicating a high-quality approximation for the MIP in expectation. The infimum can be used to determine the best-case performance, with a gap of zero indicating a perfect formulation for at least one right-hand side in \( D \). Finally, the supremum can be used to determine the worst-case performance, with a gap close to zero indicating a consistently high-quality approximation for the MIP.

Note that while the formulations presented in this section bear similarities to those presented in Ajayi et al. (2021), there are a number of nontrivial differences. These include: the domain over which the formulations are defined, the inclusion of the dual variables corresponding to the LP embedded in the MIP, and the substantially greater number of constraints for each formulation. Furthermore, unlike the formulations presented in Ajayi et al. (2021), there are two right-hand sides for us to consider: the right-hand side corresponding to the MIP, \( \hat{\beta} \in \mathcal{B}[0, b] \), and the portion of \( \hat{\beta} \) allocated to the IP embedded in the MIP, \( \beta \in \hat{\mathcal{B}}[0, \hat{\beta}] \).
4.1. Expectation of the Absolute Gap Function over a Discrete Set

Denote $\xi$ a discrete random variable with event space $D$ and discretization size 1. Let $\mathbb{P}\{\xi = \hat{\beta}\} = \mu(\hat{\beta})$. The expectation of the absolute gap function over $D$ is: $E_\xi[\Gamma(\xi)] := \sum_{\hat{\beta} \in D} \mu(\hat{\beta})\Gamma(\hat{\beta})$. Consider the formulation:

$$\delta_{EAD} = \max_{\hat{\beta} \in D} \sum_{\beta \in D} \mu(\hat{\beta})\psi(\beta)$$  \hspace{1cm} (14a)

s.t. $\psi(\hat{\beta}) \leq \hat{\beta}^T u^q - (\phi(\beta) + \pi_{\hat{\beta} - \beta}(\hat{\beta} - \beta)) \forall q \in \kappa$, $\beta \in \hat{B}[0, \hat{\beta}], \hat{\beta} \in D$, \hspace{1cm} (14b)

$\phi \in \Phi(b)$, \hspace{1cm} (14c)

$\pi_{\hat{\beta} - \beta} \in \Omega \forall \beta \in \hat{B}[0, \hat{\beta}], \hat{\beta} \in D$, \hspace{1cm} (14d)

$\psi \in \mathbb{R}^{|D|}$. \hspace{1cm} (14e)

**Theorem 4.1.** The optimal objective value of (14) is $\delta_{EAD} = E_\xi[\Gamma(\xi)]$.

**Proof.** Let $\bar{\psi}(\hat{\beta}) = \Gamma(\hat{\beta})$ for all $\hat{\beta} \in D$, and let $\beta^* \in \arg \max_{\beta \in \hat{B}[0, \hat{\beta}]} z_{LP}(\beta) + z_{MP}(\beta - \hat{\beta})$. For each $\hat{\beta} \in D$, let $\phi(\beta^*) = z_{LP}(\beta^*)$ and $\pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta^*) = z_{MP}(\beta^* - \hat{\beta})$ such that $\pi_{\hat{\beta} - \beta}^* \in \Omega$ (note that this is guaranteed to exist by Remark 2.1). We show that the triple $(\phi, \pi_{\hat{\beta} - \beta}^*, \bar{\psi})$ is feasible for (14).

Note that by Theorem 2.2, $\bar{\phi}$ satisfies (11c)-(11g). Furthermore, by strong duality, $z_{LP}(\beta - \beta^*) = z_{MP}(\beta - \beta^*)$. So, $z_{LP}(\beta - \beta^*) = \hat{\beta}^T u^q - (\phi(\beta) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta^*))$ where $\pi_{\hat{\beta} - \beta}^* \in \Omega$. Thus, $\pi_{\hat{\beta} - \beta}^*$ satisfies (11h). In addition, by construction, $\bar{\phi}(\beta^*) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta^*) \geq \phi(\beta) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta) \forall \beta \in \hat{B}[0, \hat{\beta}]$, thus, satisfying (11b). By strong duality, $z_{LP}(\beta) = z_{MP}(\beta)$ for all $\beta \in D$. Thus,

$$\bar{\psi}(\hat{\beta}) = \Gamma(\hat{\beta}) = z_{MP}(\hat{\beta}) - z_{MP}(\hat{\beta}) = \min_{q \in \kappa} \hat{\beta}^T u^q - (\phi(\beta^*) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta^*))$$

$$\leq \hat{\beta}^T u^q - (\phi(\beta^*) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta^*)) \forall q \in \kappa, \hat{\beta} \in D.$$ 

Therefore, the triple $(\phi, \pi_{\hat{\beta} - \beta}^*, \bar{\psi})$ is feasible for (14).

Suppose $(\phi^*, \pi_{\hat{\beta} - \beta}^*, \psi^*)$ is feasible for (14). By Theorem 2.4, $\phi^*(\beta^*) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta^*) \geq z_{MP}(\hat{\beta})$ for all $\hat{\beta} \in D$. By feasibility, $\psi^*(\hat{\beta}) \leq \hat{\beta}^T u^q - (\phi^*(\beta) + \pi_{\hat{\beta} - \beta}^*(\hat{\beta} - \beta)) \forall q \in \kappa, \beta \in \hat{B}[0, \hat{\beta}]$, and $\hat{\beta} \in D$. It follows that $\psi^*(\hat{\beta}) \leq z_{LP}(\hat{\beta}) - z_{MP}(\hat{\beta}) = \bar{\psi}(\hat{\beta})$. Hence, $\sum_{\hat{\beta} \in D} \mu(\hat{\beta})\psi^*(\hat{\beta}) \leq \sum_{\hat{\beta} \in D} \mu(\hat{\beta})\bar{\psi}(\hat{\beta}) = E_\xi[\Gamma(\xi)]$, i.e., the optimal objective value of (14) is $E_\xi[\Gamma(\xi)]$. \qed

4.2. Infimum of the Absolute Gap Function over a Discrete Set

Because, trivially, $\Gamma(0) = 0 = \min_{\beta \in \hat{B}[0, \hat{\beta}]} \Gamma(\beta)$, we exclude $\{0\}$ from consideration. Denote $D^+ = D \setminus \{0\}$. The infimum of the absolute gap function over $D^+$ is: $\Delta_{IAD} := \inf_{\hat{\beta} \in D^+} \Gamma(\hat{\beta}) = \min_{\beta \in D^+} \Gamma(\hat{\beta}).$
Consider the formulation:

\[ \delta_{IAD} = \max \psi \]
\[ \text{s.t. } \psi \leq \beta^\top u^q - (\phi(\beta) + \pi^\top_{\tilde{\beta} - \beta}(\tilde{\beta} - \beta)) \quad \forall q \in \kappa, \ \beta \in \tilde{B}[0, \tilde{\beta}] \setminus \{0\}, \ \tilde{\beta} \in \mathcal{D}^+, \]  
\[ \phi \in \Phi(b), \]  
\[ \pi_{\tilde{\beta} - \beta} \in \Omega \quad \forall \beta \in \tilde{B}[0, \tilde{\beta}] \setminus \{0\}, \ \tilde{\beta} \in \mathcal{D}^+, \]  
\[ \psi \in \mathbb{R}^+. \]  

(16a)
(16b)
(16c)
(16d)
(16e)

Theorem 4.2. The optimal objective value of (16) is \( \Delta_{IAD} \). That is, \( \delta_{IAD} = \Delta_{IAD} \).

4.3. Supremum of the Absolute Gap Function over a Discrete Set

Let \( \text{SOS1}(\{w(\tilde{\beta})\})_{\tilde{\beta} \in \mathcal{D}} \) denote a Special Ordered Set constraint of Type 1 on the decision variable \( w \in \mathbb{R}^{|D|}_+ \), so that \( |\{\tilde{\beta} \in \mathcal{D} | w(\tilde{\beta}) > 0\}| \leq 1 \) (Beale and Tomlin [1970]). The supremum of the absolute gap function over \( \mathcal{D} \) is: \( \Delta_{SAD} := \sup_{\tilde{\beta} \in \mathcal{D}} \Gamma(\tilde{\beta}) = \max_{\tilde{\beta} \in \mathcal{D}} \Gamma(\tilde{\beta}) \). Consider the formulation:

\[ \delta_{SAD} = \max \sum_{\tilde{\beta} \in \mathcal{D}} \psi(\tilde{\beta}) \]
\[ \text{s.t. } \psi(\tilde{\beta}) \leq \beta^\top u^q - (\phi(\beta) + \pi^\top_{\tilde{\beta} - \beta}(\tilde{\beta} - \beta)) \quad \forall q \in \kappa, \ \beta \in \tilde{B}[0, \tilde{\beta}], \ \tilde{\beta} \in \mathcal{D}, \]  
\[ \text{SOS1}(\{\psi(\tilde{\beta})\})_{\tilde{\beta} \in \mathcal{D}}, \]  
\[ \phi \in \Phi(b), \]  
\[ \pi_{\tilde{\beta} - \beta} \in \Omega \quad \forall \beta \in \tilde{B}[0, \tilde{\beta}], \ \tilde{\beta} \in \mathcal{D}, \]  
\[ \psi \in \mathbb{R}^{|D|}_+. \]  

(17a)
(17b)
(17c)
(17d)
(17e)
(17f)

Theorem 4.3. The optimal objective value of (17) is \( \Delta_{SAD} \). That is, \( \delta_{SAD} = \Delta_{SAD} \).

5. Relative Gap Functions over a Discrete Set

In this section, we optimize the expectation, infimum, and supremum of the relative gap function, \( \gamma \), over finite discrete sets. The following results extend those of Ajayi et al. [2021]. We maintain the same notation from Section 4.

Recall \( \mathcal{B}^+[0, b] = \{\tilde{\beta} \in \mathcal{B}[0, b] | z_{MIP}(\tilde{\beta}) > 0\} \). Let \( \mathcal{S}^+ \) be a finite subset of \( \mathcal{B}^+[0, b] \), and let \( \tilde{\mathcal{B}}^+[0, b] = \mathcal{B}^+[0, b] \cap \mathbb{Z}^m_+ \). The expectation of the relative gap function can be used to determine the expected performance of the LP relaxation as an approximation for the MIP, with a gap close to
Thus, for all $q \in S$. Hence, the triple $(\tilde{\beta}, b, \psi)$. Consider the formulation:

$$
\delta_{ERD} = \min_{\beta \in S^+} \sum_{\beta \in S^+} \mu(\beta)\psi(\beta)
$$

s.t. $\psi(\beta)\tilde{\beta}^\top u^q \geq \phi(\beta) + \pi^\top_{\beta - \beta}(\tilde{\beta} - \beta) \quad \forall q \in \kappa, \beta \in \tilde{B}^+[0, \tilde{\beta}], \tilde{\beta} \in S^+$, (18b)

$\phi \in \Phi(b)$, (18c)

$\pi_{\tilde{\beta} - \beta} \in \Omega \quad \forall \beta \in \tilde{B}^+[0, \tilde{\beta}], \tilde{\beta} \in S^+$, (18d)

$\psi \in \mathbb{R}^{[S^+]_+}$. (18e)

**Theorem 5.1.** The optimal objective value of (18) is $\delta_{ERD} = \mathbb{E}_\xi[\gamma(\xi)]$.

**Proof.** Let $\tilde{\psi}(\tilde{\beta}) = \gamma(\tilde{\beta})$ for all $\tilde{\beta} \in S^+$, and let $\beta^* \in \arg \max_{\beta \in \tilde{B}^+[0, \tilde{\beta}]} z_{IP}(\beta) + z_{LP}(\tilde{\beta} - \beta)$. For each $\tilde{\beta} \in S^+$, let $\tilde{\phi}(\beta^*) = z_{IP}(\beta^*)$ and $\tilde{\pi}^\top_{\beta - \beta}(\tilde{\beta} - \beta^*) = z_{LP}(\tilde{\beta} - \beta^*)$ such that $\tilde{\pi}_{\beta - \beta^*} \in \Omega$. By arguments similar to those in the proof of Theorem 4.1, the triple satisfies (11b)-(11h).

By strong duality, $z_{LPR}(\tilde{\beta}) = z_{DLPR}(\tilde{\beta}) = \min_{q \in \kappa} \tilde{\beta}^\top u^q$. In addition, $z_{LPR}(\tilde{\beta}) > 0$ for all $\tilde{\beta} \in S^+$. Thus, for all $q \in \kappa, \tilde{\beta} \in S^+$, and $\beta \in \tilde{B}^+[0, \tilde{\beta}]$:

$$
\tilde{\psi}(\tilde{\beta})\tilde{\beta}^\top u^q = \gamma(\tilde{\beta})\tilde{\beta}^\top u^q \geq \gamma(\tilde{\beta})z_{LPR}(\tilde{\beta}) = z_{MIP}(\tilde{\beta}) = z_{IP}(\beta^*) + z_{LP}(\tilde{\beta} - \beta^*)
$$

$$
= \tilde{\phi}(\beta^*) + \tilde{\pi}^\top_{\beta - \beta^*}(\tilde{\beta} - \beta^*) \geq \tilde{\phi}(\beta) + \tilde{\pi}^\top_{\beta - \beta^*}(\tilde{\beta} - \beta).
$$

Hence, the triple $(\tilde{\phi}, \tilde{\pi}_{\beta - \beta^*}, \tilde{\psi})$ is feasible for (18).

Suppose $(\phi^*, \pi^*_{\beta - \beta^*}, \psi^*)$ is feasible for (18). By Theorem 2.4, $\phi^*(\beta^*) + \pi^*_{\beta - \beta^*}(\tilde{\beta} - \beta^*) \geq z_{MIP}(\tilde{\beta})$ for all $\tilde{\beta} \in S^+$. By feasibility, $\psi^*(\tilde{\beta}) \geq \phi^*(\tilde{\beta}) + \pi^*_{\beta - \beta^*}(\tilde{\beta} - \beta^*) \geq \frac{\phi^*(\beta^*) + \pi^*_{\beta - \beta^*}(\tilde{\beta} - \beta^*)}{\tilde{\beta}^\top u^q} \quad \forall q \in \kappa, \beta \in \tilde{B}^+[0, \tilde{\beta}]$, and $\tilde{\beta} \in S^+$. Hence, $\psi^*(\tilde{\beta}) \geq \frac{\phi^*(\beta^*) + \pi^*_{\beta - \beta^*}(\tilde{\beta} - \beta^*)}{z_{LPR}(\tilde{\beta})} \geq \frac{z_{MIP}(\tilde{\beta})}{z_{LPR}(\tilde{\beta})} = \gamma(\tilde{\beta}) = \tilde{\psi}(\tilde{\beta})$. Thus, $\mathbb{E}_\xi[\Gamma(\xi)] = \sum_{\tilde{\beta} \in S^+} \mu(\tilde{\beta})\gamma(\tilde{\beta}) = \sum_{\tilde{\beta} \in S^+} \mu(\tilde{\beta})\tilde{\psi}(\tilde{\beta}) \leq \sum_{\tilde{\beta} \in S^+} \mu(\tilde{\beta})\psi^*(\tilde{\beta})$, i.e., $\delta_{ERD} = \mathbb{E}_\xi[\Gamma(\xi)]$. \qed
5.2. Infimum of the Relative Gap Function over a Discrete Set

The infimum of the relative gap function over $S^+$ is: $\Delta_{IRD} := \min_{\beta \in S^+} \gamma(\hat{\beta})$. Consider the formulation:

$$\delta_{IRD} = \max_{\beta \in S^+} \sum_{\beta \in S^+} \psi(\hat{\beta})$$

s.t. $(1 - \psi(\hat{\beta})) \hat{\beta}^T u^q \geq \phi(\beta) + \pi_{\hat{\beta} - \beta}(\hat{\beta} - \beta)$ $\forall q \in \kappa$, $\beta \in \hat{B}^+[0, \hat{\beta}]$, $\hat{\beta} \in S^+$, (20b)

SOS1($\{\psi(\hat{\beta})\}_{\hat{\beta} \in S^+}$), (20c)

$\phi \in \Phi(b)$, (20d)

$\pi_{\hat{\beta} - \beta} \in \Omega$ $\forall \beta \in \hat{B}^+[0, \hat{\beta}]$, $\hat{\beta} \in S^+$, (20e)

$\psi \in \mathbb{R}^{|S^+|}$, (20f)

Theorem 5.2. The optimal objective value of (20) is $1 - \Delta_{IRD}$. That is, $\delta_{IRD} = 1 - \Delta_{IRD}$.

Proof. The proof follows similarly from Theorem 4.3 and is therefore omitted.

5.3. Supremum of the Relative Gap Function over a Discrete Set

The supremum of the relative gap function over $S^+$ is: $\Delta_{SRD} := \max_{\beta \in S^+} \gamma(\hat{\beta})$. Consider the formulation:

$$\delta_{SRD} = \min_{\psi} \psi$$

s.t. $\psi \cdot \hat{\beta}^T u^q \geq \phi(\beta) + \pi_{\hat{\beta} - \beta}(\hat{\beta} - \beta)$ $\forall q \in \kappa$, $\beta \in \hat{B}^+[0, \hat{\beta}]$, $\hat{\beta} \in S^+$, (21b)

$\phi \in \Phi(b)$, (21c)

$\pi_{\hat{\beta} - \beta} \in \Omega$ $\forall \beta \in \hat{B}^+[0, \hat{\beta}]$, $\hat{\beta} \in S^+$, (21d)

$\psi \in \mathbb{R}^+$. (21e)

Theorem 5.3. The optimal objective value of (21) is $\Delta_{SRD}$. That is, $\delta_{SRD} = \Delta_{SRD}$.

Proof. The proof follows similarly from Theorem 4.2 and is therefore omitted.

Acknowledgments

The authors thank Seth Brown and Dr. Mustafa Can Camur of Rice University for their helpful comments. This research was supported by National Science Foundation grant CMMI-1933373.
References


E.C. Electronic Companion

E.C.1. Section 3 Results

Proposition 3.1. The absolute MIP gap function defined over $\mathcal{B}[0, b]$ is the minimum of finitely many Gomory functions.

Proof. Let $\tilde{\beta} \in \mathcal{B}[0, b]$. Consider the negative of the value function:

$$-z_{MIP}(\tilde{\beta}) = \min_{x \in \mathbb{Z}_+^n, y \in \mathbb{R}_+^p} \{-c^\top x - h^\top y \mid Ax + Gy \leq \tilde{\beta}\}.$$

Blair and Jeroslow (1984) prove that for any $\tilde{\beta}$ such that MIP($\tilde{\beta}$) is feasible, which includes $\mathcal{B}[0, b]$, $-z_{MIP}$ is the minimum of finitely many Gomory functions. So, let $-z_{MIP}(\tilde{\beta}) = \min\{G_1(\tilde{\beta}), \ldots, G_L(\tilde{\beta})\}$, where $\{G_i(\tilde{\beta}) \mid i = 1, \ldots, L\}$ are Gomory functions. Recall that $z_{DLP}(\beta) = \min_{q \in \kappa} \beta^\top u^q$.

By strong duality, $z_{LP}(\beta) = z_{DLP}(\beta)$. Thus, $z_{LP}(\beta) = \min_{q \in \kappa} \beta^\top u^q$, where $|\kappa| < +\infty$. Let $q(\tilde{\beta})^* \in \kappa$ be such that $q(\tilde{\beta})^* \in \{q \in \kappa \mid \arg \min_{q \in \kappa} \beta^\top u^q \}$. Recall that as a consequence of Assumption 2.1, Remark 2.2, Weyl’s Theorem (Charnes and Cooper (1958)), and the assumption that $c, h > 0$, $\beta^\top u^q(\tilde{\beta})^* = z_{DLP}(\beta) \geq 0$. Thus, for $\tilde{\beta} \in \mathcal{B}[0, b]$, $\Gamma(\beta) = z_{LP}(\beta) = z_{MIP}(\tilde{\beta}) = \beta^\top u^q(\tilde{\beta})^* + \min_{i = 1, \ldots, L} G_i(\beta) = \min_{i = 1, \ldots, L} (\beta^\top u^q(\tilde{\beta})^* + G_i(\beta))$. Notice for each $i$, $\beta^\top u^q(\tilde{\beta})^* + G_i(\beta)$ is a Gomory function, as it is the sum of two Gomory functions. Thus, $\Gamma(\beta)$ is the minimum of finitely many Gomory functions for all $\tilde{\beta} \in \mathcal{B}[0, b]$.

Proposition 3.2. If $(x^*, y^*) \in \text{opt}_{MIP}(\beta)$, then for all $(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p$ such that $0 \leq x \leq x^*$ and $0 \leq y \leq y^*$, $(x, y) \in \text{opt}_{MIP}(Ax + Gy)$.

Proof. Let $(x^*, y^*) \in \text{opt}_{MIP}(\beta)$. Suppose for the sake of contradiction that there exists $(\tilde{x}, \tilde{y}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p$ such that $0 \leq \tilde{x} \leq x^*$, $0 \leq \tilde{y} \leq y^*$, and $(\tilde{x}, \tilde{y}) \not\in \text{opt}_{MIP}(Ax + Gy)$. Then, there exists $(x_0, y_0) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p$ such that $Ax_0 + Gy_0 \leq Ax + Gy$ and $c^\top \tilde{x} + h^\top \tilde{y} \leq c^\top x_0 + h^\top y_0 = z_{MIP}(Ax + Gy)$.

Now, let $\tilde{x} = x^* - \tilde{x}$ and $\tilde{y} = y^* - \tilde{y}$. Note that since $0 \leq \tilde{x} \leq x^*$ and $0 \leq \tilde{y} \leq y^*$, $\tilde{x}$ and $\tilde{y}$ are nonnegative. Then, $Ax^* + Gy^* = A(\tilde{x} + \tilde{x}) + G(\tilde{y} + \tilde{y}) \leq \beta$, i.e., $(\tilde{x} + \tilde{x}, \tilde{y} + \tilde{y})$ is feasible for MIP($\beta$). However, we also have that $z_{MIP}(\beta) = c^\top x^* + h^\top y^* = c^\top (\tilde{x} + \tilde{x}) + h^\top (\tilde{y} + \tilde{y}) < c^\top x_0 + h^\top y_0 + c^\top \tilde{x} + h^\top \tilde{y}$, which contradicts the fact that $(x^*, y^*) \in \text{opt}_{MIP}(\beta)$. Thus, $(\tilde{x}, \tilde{y}) \not\in \text{opt}_{MIP}(Ax + Gy)$.

Proposition 3.3. Let $\tilde{\beta} \in \mathcal{B}[0, b]$ and $(x^*, y^*) \in \text{opt}_{MIP}(\tilde{\beta})$. Given $\eta \in \mathbb{N}$, let $J_\eta \subseteq \{1, \ldots, n\}$ denote the set of indices such that $x^*_j \geq \eta$ for $j \in J_\eta$. Denote $K := \{1, \ldots, p\}$, and let $\lambda^* = \min_{k \in K} \{y^*_k\}$.

Then, for any $j \in J_\eta$, $k \in K$, and $\lambda \in [0, \lambda^*]$, $z_{MIP}(\beta - \eta a_j - \lambda g_k) = z_{MIP}(\beta) - \eta c_j - \lambda h_k$. 

17
Proof. Denote $e_j$ the $j^{th}$ unit vector in $\mathbb{R}_+^n$ and $\epsilon_k$ the $k^{th}$ unit vector in $\mathbb{R}_+^p$. Let $(x^*, y^*) \in \text{opt}_{MIP}(\tilde{\beta})$, so $z_{MIP}(\tilde{\beta}) = c^T x^* + h^T y^*$. Let $\lambda \in [0, \lambda^*]$ where $\lambda^* = \min_{k \in K} \{y_k^*\}$. Suppose $J_\eta \neq \emptyset$, and choose $\eta \in \mathbb{N}$ is such that $x_j^* \geq \eta$ for all $j \in J_\eta$. Note that $A(x^* - \eta e_j) + G(y^* - \lambda \epsilon_k) = Ax^* + Gy^* - \eta a_j - \lambda g_k \leq \tilde{\beta} - \lambda a_j - \lambda g_k \leq \tilde{\beta}$, with $x^* - \eta e_j \in \mathbb{Z}_+^n$ and $y^* - \lambda \epsilon_k \in \mathbb{R}_+^p$. Therefore, $(x^* - \eta e_j, y^* - \lambda \epsilon_k)$ is feasible for $MIP(\tilde{\beta} - \eta a_j - \lambda g_k)$ and $MIP(\tilde{\beta})$.

Now, suppose for the sake of contradiction that $(x^* - \eta e_j, y^* - \lambda \epsilon_k) \notin \text{opt}_{MIP}(\tilde{\beta} - \eta a_j - \lambda g_k)$. Note that $\text{opt}_{MIP}(\tilde{\beta} - \eta a_j - \lambda g_k) \neq \emptyset$ due to Assumption 2.1 and the data being nonnegative. Thus, let $(\tilde{x}, \tilde{y}) \in \text{opt}_{MIP}(\tilde{\beta} - \eta a_j - \lambda g_k)$. Then, $A \tilde{x} + G \tilde{y} \leq \tilde{\beta} - \eta a_j - \lambda g_k \leq \tilde{\beta}$, with $\tilde{x} \in \mathbb{Z}_+^n$ and $\tilde{y} \in \mathbb{R}_+^p$, so $(\tilde{x}, \tilde{y})$ is feasible for $MIP(\tilde{\beta})$. Also, $c^T \tilde{x} + h^T \tilde{y} > c^T (x^* - \eta e_j) + h^T (y^* - \lambda \epsilon_k)$. Now, let $\tilde{x} = \tilde{x} + \eta e_j$ and $\tilde{y} = \tilde{y} + \lambda \epsilon_k$. Note that $A \tilde{x} + G \tilde{y} = A(x^* + \eta e_j) + G(y^* + \lambda \epsilon_k) \leq \tilde{\beta} - \eta a_j - \lambda g_k + \eta a_j + \lambda g_k = \tilde{\beta}$, with $\tilde{x} \in \mathbb{Z}_+^n$ and $\tilde{y} \in \mathbb{R}_+^p$. Thus, $(\tilde{x}, \tilde{y})$ is feasible for $MIP(\tilde{\beta})$. Moreover, $c^T \tilde{x} + h^T \tilde{y} = c^T (x^* + \eta e_j) + h^T (y^* + \lambda \epsilon_k) + \eta c_j + \lambda h_k = c^T x^* + h^T y^*$, which contradicts the fact that $(x^*, y^*) \in \text{opt}_{MIP}(\tilde{\beta})$. Thus, $(x^* - \eta e_j, y^* - \lambda \epsilon_k) \in \text{opt}_{MIP}(\tilde{\beta} - \eta a_j - \lambda g_k)$.

Therefore, $z_{MIP}(\tilde{\beta} - \eta a_j - \lambda g_k) = c^T(x^* - \eta e_j) + h^T(y^* - \lambda \epsilon_k) = c^T x^* + h^T y^* - \eta c_j - \lambda h_k = z_{MIP}(\tilde{\beta}) - \eta c_j - \lambda h_k$. \hfill \Box

**Proposition 3.4** Let $\tilde{\beta} \in \mathcal{B}[0, b]$ and $(x^*, y^*) \in \text{opt}_{LP}(\tilde{\beta})$. Given $\eta \in \mathbb{N}$, let $J_\eta \subseteq \{1, \ldots, n\}$ denote the set of indices such that $x_j^* \geq \eta$ for $j \in J_\eta$. Denote $K := \{1, \ldots, p\}$, and let $\lambda^* = \min_{k \in K} \{y_k^*\}$. Then, for any $j \in J_\eta$, $k \in K$, and $\lambda \in [0, \lambda^*]$, $z_{LP}(\tilde{\beta} - \eta a_j - \lambda g_k) = z_{LP}(\tilde{\beta}) - \eta c_j - \lambda h_k$.

**Proof.** This follows from LP complementary slackness, and can be proven also by contradiction analogously to Proposition 3.3. \hfill \Box

**E.C.2. Section 4 Results**

**Theorem 4.2** The optimal objective value of (16) is $\Delta_{IAD}$. That is, $\delta_{IAD} = \Delta_{IAD}$.

**Proof.** Let $\phi = \Delta_{IAD}$, and let $\beta^* = \arg \max_{\beta \in \mathcal{B}[0, \bar{b}]} z_{LP}(\beta)$. For each $\tilde{\beta} \in \mathcal{D}^+$, let $\tilde{\phi}(\beta^*) = z_{LP}(\beta^*)$ and $\tilde{\pi}_{\beta^*}^\top(\tilde{\beta} - \beta^*) = z_{LP}(\tilde{\beta} - \beta^*)$ such that $\tilde{\pi}_{\beta^*}^\top \in \Omega$ (note that this is guaranteed to exist by Remark 2.1). We show that the triple $(\tilde{\phi}, \tilde{\pi}_{\beta^*}^\top, \psi)$ is feasible for (16).

By arguments similar to those in the proof of Theorem 4.1, the triple satisfies (11b)-(11h), and $\psi = \Delta_{IAD} = \min_{\tilde{\beta} \in \mathcal{D}^+} \tilde{\Gamma}(\tilde{\beta}) \leq z_{LP}(\tilde{\beta}) - z_{LP}(\tilde{\beta}) \leq \tilde{\beta}^\top u^q - (\phi^*(\beta^*) + \tilde{\pi}_{\beta^*}^\top(\tilde{\beta} - \beta^*))$ for all $q \in \kappa$ and $\tilde{\beta} \in \mathcal{D}^+$. Hence, the triple $(\tilde{\phi}, \tilde{\pi}_{\beta^*}^\top, \psi)$ is feasible for (16).

Now, suppose $(\phi^*, \tilde{\pi}_{\beta^*}^\top, \psi^*)$ is feasible for (16). By Theorem 2.4, $\phi^*(\beta^*) + \pi_{\beta^*}^\top(\tilde{\beta} - \beta^*) \geq z_{MIP}(\tilde{\beta})$ for all $\tilde{\beta} \in \mathcal{D}^+$. Furthermore, by feasibility, $\psi^* \leq \tilde{\beta}^\top u^q - (\phi^*(\beta) + \pi_{\beta}^\top(\tilde{\beta} - \beta))$ for all $\tilde{\beta} \in \mathcal{D}^+$.
all $q \in \kappa$, $\beta \in \mathbb{B}[0, \beta] \setminus \{0\}$, and $\hat{\beta} \in \mathcal{D}^+$. It follows that $\psi^* \leq \Delta_{IAD} = \tilde{\psi}$. Thus, $\tilde{\psi} = \delta_{IAD}$, and the optimal objective value of (16) is $\Delta_{IAD}$.

**Theorem 4.3.** The optimal objective value of (17) is $\Delta_{SAD}$. That is, $\delta_{SAD} = \Delta_{SAD}$.

**Proof.** Let $\hat{\beta}_{\text{max}} \in \arg \max_{\beta \in \mathcal{D}} \{\Gamma(\hat{\beta})\}$. Let $\hat{\psi}(\hat{\beta}_{\text{max}}) = \Gamma(\hat{\beta}_{\text{max}})$ and $\tilde{\psi}(\hat{\beta}) = 0$ for all $\hat{\beta} \in \mathcal{D} \setminus \hat{\beta}_{\text{max}}$. Note that by construction, $\tilde{\psi}$ satisfies (17c). Let $\hat{\psi}(\hat{\beta}) = \Gamma(\hat{\beta})$ for all $\hat{\beta} \in \mathcal{D}$, and let $\beta^* \in \arg \max_{\beta \in \mathbb{B}[0, \beta]} z_{LP}(\beta) + z_{MIP}(\beta)$. For each $\beta \in \mathcal{D}$, let $\tilde{\phi}(\beta^*) = z_{LP}(\beta^*)$ and $\tilde{\pi}_{\beta-\beta^*}(\hat{\beta} - \beta^*) = z_{LP}(\hat{\beta} - \beta^*)$ such that $\tilde{\pi}_{\beta-\beta^*} \in \Omega$ (note that this is guaranteed to exist by Remark 2.1). By arguments similar to those in the proof of Theorem 4.1, the triple satisfies (11b)-(11h).

Now, note that $\psi^*(\hat{\beta}) = 0 \leq \Gamma(\hat{\beta})$ for all $\hat{\beta} \in \mathcal{D} \setminus \hat{\beta}_{\text{max}}$. Therefore for $\hat{\beta} \in \mathcal{D} \setminus \hat{\beta}_{\text{max}},$

$$\psi^*(\hat{\beta}) \leq \Gamma(\hat{\beta}) = z_{LP}(\hat{\beta}) - z_{MIP}(\hat{\beta}) \leq \hat{\beta}^\top u^q - (\tilde{\phi}(\beta^*) + \tilde{\pi}_{\beta-\beta^*}(\hat{\beta} - \beta^*)) \quad \forall q \in \kappa, \hat{\beta} \in \mathcal{D}^+.$$

Recall that by strong duality, $z_{LP}(\hat{\beta}_{\text{max}}) = z_{MIP}(\hat{\beta}_{\text{max}})$. Thus, for $\hat{\beta} = \hat{\beta}_{\text{max}},$

$$\tilde{\psi}(\hat{\beta}_{\text{max}}) = \Gamma(\hat{\beta}_{\text{max}}) = z_{LP}(\hat{\beta}_{\text{max}}) - z_{MIP}(\hat{\beta}_{\text{max}}) = \min_{q \in \kappa} \hat{\beta}_{\text{max}}^\top u^q - (\tilde{\phi}(\beta^*) + \tilde{\pi}_{\beta-\beta^*}(\hat{\beta}_{\text{max}} - \beta^*)) \leq \hat{\beta}_{\text{max}}^\top u^q - (\tilde{\phi}(\beta^*) + \tilde{\pi}_{\beta-\beta^*}(\hat{\beta}_{\text{max}} - \beta^*)) \quad \forall q \in \kappa.$$

Hence, the triple $(\tilde{\phi}, \tilde{\pi}_{\beta-\beta^*}, \tilde{\psi})$ is feasible for (17).

Now, let $(\phi^*, \pi^*_{\beta-\beta^*}, \psi^*)$ be feasible for (17) such that there exists some $\beta^* \in \mathcal{D}$ for which $\psi^*(\hat{\beta}) = 0$ for all $\hat{\beta} \in \mathcal{D} \setminus \hat{\beta^*}$. By Theorem 2.4, $\phi^*(\beta^*) + \pi^*_{\beta-\beta^*}(\hat{\beta} - \beta^*) \geq z_{MIP}(\hat{\beta})$ for all $\hat{\beta} \in \mathcal{D}$. Furthermore, by feasibility, $\psi^*(\hat{\beta}) \leq \hat{\beta}^\top u^q - (\phi^*(\beta) + \pi^*_{\beta-\beta^*}(\hat{\beta} - \beta))$ for all $q \in \kappa$, $\beta \in \mathbb{B}[0, \beta]$, and $\hat{\beta} \in \mathcal{D}$. It follows that $\psi^*(\hat{\beta^*}) \leq z_{LP}(\hat{\beta^*}) - z_{MIP}(\hat{\beta^*}) = \Gamma(\hat{\beta^*})$. Now, recall that $\hat{\beta}_{\text{max}} \in \arg \max_{\beta \in \mathcal{D}} \{\Gamma(\hat{\beta})\}$. Then, $\psi^*(\hat{\beta^*}) \leq \Gamma(\hat{\beta^*}) \leq \Gamma(\hat{\beta}_{\text{max}}) = \tilde{\psi}(\hat{\beta}_{\text{max}})$. Thus, $\sum_{\beta \in \mathcal{D}} \psi^*(\hat{\beta}) = \sum_{\beta \in \mathcal{D}} \tilde{\psi}(\hat{\beta}) = \Delta_{SAD}$, i.e., the optimal objective value of (17) is $\Delta_{SAD}$. \qed