STEINER CUT DOMINANTS

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Abstract. For a subset T of nodes of an undirected graph G, a *T-Steiner cut* is a cut $\delta(S)$ with $T \cap S \neq \varnothing$ and $T \setminus S \neq \varnothing$. The *$T$-Steiner cut dominant* of G is the dominant $\mathrm{CUT}_+(G,T)$ of the convex hull of the incidence vectors of the T-Steiner cuts of G. For $T = \{s,t\}$, this is the well-understood s-t-cut dominant. Choosing T as the set of all nodes of G, we obtain the *cut dominant*, for which an outer description in the space of the original variables is still not known. We prove that, for each integer τ, there is a finite set of inequalities such that for every pair (G,T) with $|T| \leq \tau$ the non-trivial facet-defining inequalities of $\mathrm{CUT}_+(G,T)$ are the inequalities that can be obtained via iterated applications of two simple operations, starting from that set. In particular, the absolute values of the coefficients and of the right-hand-sides in a description of $\mathrm{CUT}_+(G,T)$ by integral inequalities can be bounded from above by a function of $|T|$. For all $|T| \leq 5$ we provide descriptions of $\mathrm{CUT}_+(G,T)$ by facet defining inequalities, extending the known descriptions of s-t-cut dominants.

1. Introduction

Let (G,T) be a *Steiner graph*, i.e., G is a connected graph with node set $V(G)$ and edge set $E(G)$, and $T \subseteq V(G)$ with $|T| \geq 2$ is a subset of at least two *terminals*. For $S \subseteq V(G)$ we denote by $\delta_G(S)$ or $\delta(S)$ the *cut* in G defined by S, i.e., the set of all edges with one endnode in S and the other one in $V \setminus S$, where we use $\delta(v) := \delta(\{v\})$ to denote the *star* of a node $v \in V(G)$. A cut $\delta(S)$ in T is called a *T-Steiner cut* if both $T \cap S \neq \varnothing$ and $T \setminus S \neq \varnothing$ are non-empty. The T-Steiner cut polytope $\mathrm{CUT}(G,T)$ is the convex hull of the incidence vectors $\chi(\delta(S)) \in \{0,1\}^{E(G)}$ of the T-Steiner cuts in G (with $\chi(\delta(S))_e = 1$ if and only if $e \in \delta(S)$ holds). The *T-Steiner cut dominant* is defined to be the polyhedron

$$\mathrm{CUT}_+(G,T) = \mathrm{CUT}(G,T) + \mathbb{R}^{E(G)}_{\geq 0} ,$$

i.e. the polyhedron formed by all points y that dominate some point $x \in \mathrm{CUT}(G,T)$ in the sense of $y \geq x$. The dominant of a polyhedron P is the polyhedron that is essential for minimizing linear functions with nonnegative coefficients over P.

In case of $T = \{s,t\}$ for some $s,t \in V(G)$, the polyhedron $\mathrm{CUT}_+(G,\{s,t\})$ is the well understood *s-t-cut dominant of* G, for which

(1) $$\mathrm{CUT}_+(G,\{s,t\}) = \left\{x \in \mathbb{R}^{E(G)}_{\geq \mathbb{O}} : x(P) \geq 1 \text{ for all } s\text{-}t\text{-paths } P \subseteq E(G)\right\}$$

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Date: October 5, 2022.

can be deduced via the max-flow min-cut theorem and flow decomposition (for a closer investigation of s-t-cut dominants see Skutella and Weber [14]). For more than two terminals, however, inequality descriptions of the Steiner cut dominants have not been known so far. In particular, for $T = V(G)$ the polyhedron

$$\mathrm{CUT}_+(G) = \mathrm{CUT}_+(G, V(G))$$

is the much less understood *cut dominant of* G (see, e.g., Conforti, Fiorini, and Pashkovich [5]), whose facets are still unknown. Despite the fact that this maybe makes up the most prominent example of a polynomial time solvable combinatorial optimization problem (see the remarks below) for whose associated polyhedron no inequality description is known, understanding the facets of cut dominants seems desirable also because of the following relation. By blocking duality the coefficient vectors of the non-trivial facet-defining inequalities of $\mathrm{CUT}_+(G)$ are the vertices of the *graphical subtour relaxation polyhedron* associated with G (see, e.g., [5]). Among those inequalities, the ones with the property that—when scaled such that the right-hand-side equals two—the components sum up to two over each star are the vertices of the *subtour relaxation polytope* of G, which (usually considered for the complete graph K_n on n nodes) makes up the most important relaxation of the *traveling salesman polytope*. Interest in the vertices of the latter relaxation (thus, in the facets of $\mathrm{CUT}_+(K_n)$) has been raised, e.g., by the *4/3-conjecture* on the quality of the *subtour relaxation bound* for the metric traveling-salesman problem (see, e.g., Goemans [7]).

The *minimum Steiner cut problem*, i.e. the task to find for nonnegative edge weights a T-Steiner cut of minimal weight, can obviously be solved in polynomial time by finding (in polynomial time) minimum s-t cuts, e.g., by computing maximum s-t-flows, for $t = t_1, \dots, t_{|T|-1}$. That optimization problem arises, for instance, as the separation problem associated with the inequalities in standard integer programming formulations for the *Steiner tree problem* (see, e.g., Letchford, Saeideh, and Theis [11] or Fleischmann [6]). In fact, there has been great progress recently both w.r.t. reducing the number of max-flow computations in the approach mentioned above (by Li and Panigrahi [13]) as well as in solving the max-flow problem (by Chen, Kyng, Liu, Peng, Gutenberg, and Sachdeva [4]), resulting in a randomized algorithm for computing minimum Steiner cuts in graphs G that runs in time $|E(G)|^{1+\mathrm{o}(1)}$ with high probability (where the weights are assumed to be integral numbers whose sizes are bounded by a polynomial in $|E(G)|$, see also the remarks in the updated version of [12]). For the minimum Steiner cut problem in *planar* graphs, Jue and Klein [9] designed a deterministic algorithm whose running time can be bounded by $\mathrm{O}(|V(G)| \cdot \log|V(G)| \cdot \log|T|)$.

Using the fact that the set of T-Steiner cuts is the union of the sets of s-t_i-cuts for $|T| - 1$ pairs (s, t_i) and following Balas' disjunctive programming paradigm [1] one can easily come up with an *extended formulation* for $\mathrm{CUT}_+(G, T)$ based on (1). When using flow-based extended formulations for the s-t-cut dominants instead of (1) one even obtains an extended formulation for $\mathrm{CUT}_+(G, T)$ with both $\mathrm{O}(|T| \cdot |E(G)|)$ variables and constraints. In fact, striving for further improving the size of the representation (at least for dense graphs), Carr, Konjevod, Little, Natarajan, and Parekh [3] introduced a polyhedron in $\mathrm{O}(|V|^2)$-dimensional space described by $\mathrm{O}(|T| \cdot |V|^2)$ many inequalities for which the dominant of its projection to the ambient space of $\mathrm{CUT}_+(G, T)$ equals $\mathrm{CUT}_+(G, T)$. Consequently, finding a minimum T-Steiner cut w.r.t. nonnegative weights can be done by solving a linear program over that polyhedron. (Actually, they provide a construction for $T = V(G)$ only, but that one can be readily generalized to arbitrary terminal sets.)

In this paper, however, we are not concerned with deriving algorithms for computing minmum weight Steiner cuts or in designing extended formulations, but rather in the facets of Steiner cut dominants, i.e., we search for descriptions of

Steiner cut dominants by means of inequalities for their ambient spaces $\mathbb{R}^{E(G)}$. The difficulty of deriving such inequality descriptions may become apparent from the fact that even for the cut dominants of planar graphs such descriptions are not yet known, in contrast to the situation for the cut *polytopes* of planar graphs which are described by the *cycle inqualities* (see Barahona and Mahjoub [2]).

One approach that has been taken in order to understand better the inequalities needed in descriptions of cut dominants is to classify them according to their right-hand-sides when scaled to be in *minimum integer form*, i.e., such that their non-zero coefficients are relatively coprime positive integers. Conforti, Fiorini, and Pashkovich [5] derived a forbidden-minor characterization of the graphs G for which $\text{CUT}_+(G)$ has a description by inequalities with right-hand-sides at most two. In fact, we will use that characterization later (see Theorem 7).

Besides appearing to be of independent interest, the concept of Steiner cut dominants offers another approach to classify facet defining inequalities for cut dominants. In view of the fact that every inequality that is valid for $\text{CUT}_+(G)$ is obviously valid for $\text{CUT}_+(G,T)$ for every choice $T \subseteq V(G)$ of terminals (in general, for $T_1 \subseteq T_2$ we obviously have $\text{CUT}_+(G,T_1) \subseteq \text{CUT}_+(G,T_2)$), we define the *Steiner degree* of an inequality defining a facet of $\text{CUT}_+(G)$ to be the minimal τ for which there is some $T \subseteq V(G)$ with $|T| = \tau$ such that the inequality defines a facet of $\text{CUT}_+(G,T)$. Of course, this notion of *Steiner degree* can be transfered readily to the vertices of subtour relaxation polyhedra and polytopes via the above mentioned blocking duality.

If an inequality defining a facet of $\text{CUT}_+(G,T)$ is valid for $\text{CUT}_+(G)$, then it clearly is a facet defining inequality for $\text{CUT}_+(G)$ of Steiner degree at most $|T|$ (as both polyhedra are full-dimensional). For instance, the inequalities in (1) that arise from *Hamiltonian* s-t-paths are facet defining inequalities for $\text{CUT}_+(G)$ of Steiner degree two. A consequence (see Theorem 5) of our first main result is that the right-hand-side (and the coefficients) of any facet defining inequality in minimum integer form for a cut dominant is bounded from above by a function depending only on its Steiner degree. Note that the reverse of that statement does not hold, as, e.g., for every spanning tree $Q \subseteq E(G)$ of G the inequality $x(Q) \geq 1$ defines a facet of $\text{CUT}_+(G)$ whose Steiner degree equals the number of leaves of Q (which follows from Part (2) of Remark 3 and the remarks at the beginning of Section 7). A consequence of the second main result of our paper will be a classification of the facet defining inequalities for cut dominants of Steiner degree at most five (see Corollary 2).

Before we develop our results precisely in the following sections, we provide informal descriptions of them.

- **Main Result I.:** We discuss two operations (*subdividing* and *gluing*) that produce facet defining inequalities from facet defining inequalities and show that for each τ there is a finite set of inequalities from which, for every Steiner cut dominant with τ terminals, a description by a system of facet defining inequalities can be obtained by repeated applications of those two operations (see Remark 8 and Theorem 4).
- **Main Result II.:** We introduce two classes of inequalities (*Steiner tree inequalities* and *Steiner cactus inequalities*) for which we prove that each Steiner cut dominant with at most five terminals is described by the inequalities from those two classes (see Remark 2 and Theorem 6).

The paper is structured in the following way. We start in Section 2 by defining some notions and stating some basic results on Steiner cut dominants, most of which are well-known in the more general context of *dominant* polyhedra (i.e., polyhedra whose recession cones are non-negative orthants). In Section 3 we establish a crucial structural result on every (non-trivial) facet of a Steiner cut dominant: It contains a subset of vertices whose associated Steiner cuts are induced by a laminar family of node sets with as many members as an inequality describing the facet has nonzero coefficients. This result generalizes a corresponding one for cut dominants, but is somewhat more delicate to prove. The most important consequence of this laminarity result in our context is that the number of nonzero coefficients in inequalities defining facets of Steiner cut dominants is bounded from above by the number of nodes of the graph plus the number of terminals minus three. In Section 5 we then introduce the two operations *subdividing* and *gluing* referred to above, before we state and prove the two main results in Section 6 and 8. We conclude with some open questions in Sect 9.

2. Facet inducing Steiner graphs

As the recession cone of $\mathrm{CUT}_+(G,T)$ is the nonnegative orthant, if $cx \geq \gamma$ is an inequality which is valid for $\mathrm{CUT}_+(G,T)$, then c is nonnegative. We refer to c_e or $c(e)$ as the *c-weight* of edge e, and denote by $\gamma_c(G,T) \geq \gamma$ the minimum c-weight of any T-Steiner cut in (G,T). We say that the vector c is in *minimum integer form* if its components are nonnegative integers whose greatest common divisor equals one. The *minimum integer form* of a nonnegative rational non-zero vector is its unique scalar multiple that is in minimum integer form. We can (and always will) assume that $cx \geq \gamma$, i.e. the coefficient vector c, is in minimum integer form. We furthermore will always assume that $\gamma = \gamma_c(G,T) \geq 1$ holds (thus excluding nonnegativity constraints from our considerations). The T-Steiner cuts with c-weight equal to $\gamma_c(G,T)$ are called the *roots* of c and of the inequality. We sometimes refer to $\gamma_c(G,T)$ as the *right-hand side* of c.

The graph $G_c = (V_c, E_c)$ is the subgraph of G where E_c is the set of all edges with non-zero (thus positive) c-weights, and V_c is the set of all endpoints of edges in E_c. We clearly have $\gamma_c(G,T) = \gamma_c(G_c,T)$. The following statements follows from the fact that $\mathrm{CUT}_+(G,T)$ is a dominant polyhedron in the nonnegative orthant that is full-dimensional. Whenever we make statements about Steiner cuts in a linear algebra context those statements refer to the corresponding incidence vectors.

Remark 1. *For each inequality $cx \geq \gamma = \gamma_c(G,T)$ that is valid for $\mathrm{CUT}_+(G,T)$ the following hold:*

1. *If $\gamma = 0$, then the inequality defines a facet of $\mathrm{CUT}_+(G,T)$ if and only if it is a nonnegativity constraint.*
2. *If $\gamma > 0$, then the inequality defines a facet of $\mathrm{CUT}_+(G,T)$ if and only if (G_c,T) has $|E_c|$ many c-minimum T-Steiner cuts that are linearly independent.*

If $cx \geq \gamma$ in minimum integer form defines a facet of $\mathrm{CUT}_+(G,T)$ with $G_c = G$ and $\gamma = \gamma(G,T) > 0$, then (G,T) is called a *facet inducing Steiner graph*, and we refer to c as *facet weights* for (G,T). Due to Remark 1, a vector c provides facet weights for (G,T) if and only if c has minimum integer form and there is a *root basis* of c, i.e., a basis of $\mathbb{R}^{E(G)}$ consisting of roots of c.

A *Steiner subgraph* of a Steiner graph (G,T) is a Steiner graph (G',T') with the same terminal set $T' = T$ and G' being a subgraph of G.

Remark 2. *In order to determine an inequality description of $\mathrm{CUT}_+(G,T)$ for some Steiner graph (G,T) it suffices due to Remark 1 to find all facet weights for all facet inducing Steiner subgraphs of (G,T).*

Therefore, we will concentrate on identifying facet inducing Steiner graphs and their facet weights subsequently. It will be one of the consequences of our results that for $|T| \leq 5$ the facet weights of a facet inducing Steiner graph (G,T) are uniquely determined. However, for larger values of $|T|$ we conjecture that this does not hold in general (see also Section 9).

In the following remarks we collect a few useful observations on facet inducing Steiner graphs, where for a node $v \in V(G)$ of a graph G we denote by $G \setminus v$ the graph $(V(G) \setminus \{v\}, E(G) \setminus \delta(v))$.

Remark 3. *For every facet inducing Steiner graph (G, T) with facet weights c the following hold:*

(1) *G is connected.*
(2) *Every node in $V(G) \setminus T$ has degree at least two.*
(3) *The facet of $\mathrm{CUT}_+(G, T)$ defined by $cx \geq \gamma_c(G, T)$ is bounded.*
(4) *If $\delta(S)$ is a root of c, then both S and $V(G) \setminus S$ induce connected subgraphs of G.*
(5) *Every edge of G is contained in at least one root of c.*
(6) *For every $v \in V(G)$, each component of $G \setminus v$ contains a terminal from T.*
(7) *For every $e \in E(G)$, we have $c(e) \leq \gamma_c(G, T)$ with equality if and only if e is a bridge (i.e., removing the edge e from G results in a disconnected graph).*

3. Laminar families

For a subset $A \subseteq V$ of a ground set V we denote by $\bar{A} := V \setminus A$ the *complement* of A. Two sets A and B *intersect* if $A \cap B$, $A \setminus B$, and $B \setminus A$ are all nonempty.

A family $\mathcal{L}$ of subsets of V (i.e., a set of pairwise distinct subsets of V) is *laminar* if no two sets in $\mathcal{L}$ intersect. That is, $\mathcal{L}$ is laminar if and only if each pair of sets is either disjoint or comparable w.r.t. inclusion. We define $\mathcal{L}_{\min}$ to be the subfamily consisting of the inclusionwise minimal members of $\mathcal{L}$. If $\mathcal{L}$ is a laminar family, then the members of $\mathcal{L}_{\min}$ have pairwise empty intersections.

Lemma 1. *Every laminar family $\mathcal{L}$ of distinct nonempty subsets of $V \neq \varnothing$ satisfies $|\mathcal{L}| \leq |V| + |\mathcal{L}_{\min}| - 1$.*

Proof. Since the inequality in the lemma is equivalent to $|\mathcal{L}| - |\mathcal{L}_{\min}| \leq |V| - 1$, it suffices to establish it for a laminar family $\mathcal{L}$ of distinct nonempty subsets of V that maximizes $|\mathcal{L}| - |\mathcal{L}_{\min}|$ and that has $|\mathcal{L}|$ largest possible among all maximizers.

It is easy to see that for such a family $|\mathcal{L}_{\min}|$ consists of all singletons of V. Hence we have $|\mathcal{L}_{\min}| = |V|$, and the claim follows from the well-known (and easy to prove) fact that a laminar family of non-empty pairwise distinct subsets of V cannot have more than $2|V| - 1$ members. □

For the proof of Theorem 1 below (which follows arguments outlined in [10]) the following two observations are very useful.

Lemma 2. *If the sets A and B intersect, and the set C intersects at least one of the sets*

$$A \cap B, \quad A \cup B, \quad A \setminus B, \quad B \setminus A,$$

then C and A intersect or C and B intersect.

Proof. If C and $A \cap B$ intersect, then we have $C \cap A \neq \varnothing$ and $C \cap B \neq \varnothing$, neither $A \subseteq C$ nor $B \subseteq C$, and not both $C \subseteq A$ and $C \subseteq B$. Hence C and A or C and B intersect.

If C and $A \cup B$ intersect, then neither $C \subseteq A$ nor $C \subseteq B$ holds. Due to the symmetry of the statement in the lemma, we may assume $C \cap A \neq \varnothing$. Thus, if $A \not\subseteq C$ is true, C and A intersect. Otherwise (i.e., $A \subseteq C$ holds), $C \cap B \neq \varnothing$ is non-empty as well (since then, as A and B intersect, we have $\varnothing \neq A \cap B \subseteq C$) and $B \not\subseteq C$ holds (due to $A \cup B \not\subseteq C$), thus C and B intersect.

As the statement of the lemma is symmetric in A and B, it remains to consider the case that C and $A \setminus B$ intersect. In this case we have $C \cap A \neq \varnothing$ and $A \not\subseteq C$. Hence, if C and A do not intersect, $C \subseteq A$ holds, which, however, implies that C and B intersect since C and $A \setminus B$ intersect. □

For a family $\mathcal{L}$ of subsets of V and some $S \subseteq V$, we define

$$I(S, \mathcal{L}) = \{L \in \mathcal{L} : S \text{ and } L \text{ intersect}\}.$$

Lemma 3. *If $L \in \mathcal{L}$ is a member of a laminar family $\mathcal{L}$ of subsets of V, and $S \subseteq V$ is some set such that S and L intersect, then we have*

$$I(S \cap L, \mathcal{L}), \quad I(S \cup L, \mathcal{L}), \quad I(S \setminus L, \mathcal{L}), \quad I(L \setminus S, \mathcal{L}) \quad \subsetneq \quad I(S, \mathcal{L})\,.$$

Proof. If $L' \in \mathcal{L}$ is a set which intersects one of the sets

$$S \cap L, \quad S \cup L, \quad S \setminus L, \quad L \setminus S\,,$$

then, by Lemma 2, L' intersects S or L, where the latter is impossible as both L and L' belong to the laminar family $\mathcal{L}$. This argument establishes the above inclusions, which obviously are proper as $L \in I(S, \mathcal{L})$ does not intersect any of the four sets above. □

4. Laminar root bases

The aim of this section is to establish the following result.

Theorem 1. *If c provides facet weights for the facet inducing Steiner graph (G, T) then there is a* laminar root basis *for c, i.e., a root basis $\delta(S)$ ($S \in \mathcal{L}$) for c with a laminar family $\mathcal{L}$ of subsets of $V(G)$.*

Before we start to prove Theorem 1, we have a closer look at laminar root bases.

Remark 4. *Let $\delta(S)$ ($S \in \mathcal{L}$) be a laminar root basis for facet weights c of a facet inducing Steiner graph (G, T) with a laminar family $\mathcal{L}$ of subsets of $V(G)$. Then $\mathcal{L}_{\min}$ consists of singleton elements that are in T.*

Proof. Each set $S \in \mathcal{L}_{\min}$ must contain some node from T, because $\delta(S)$ is a T-Steiner cut. If S contains more than one node, then due to Part (4) of Remark 3 there must be an edge with both end nodes in S. However, due to the laminarity of $\mathcal{L}$ such an edge is not contained in any of the cuts $\delta(S)$ ($S \in \mathcal{L}$), contradicting the fact that those cuts form a basis of $\mathbb{R}^{E(G)}$. □

In the following, in order to simplify reading, we denote by $\delta(S)$ also the incidence vector of the subset $\delta(S)$ of the edge set of a graph.

Lemma 4. *If $\delta(S_1)$ and $\delta(S_2)$ are roots of the facet weights c for the facet inducing Steiner graph (G, T) then at least one of the following holds:*

$\delta(S_1 \cap S_2)$ and $\delta(S_1 \cup S_2)$ are both roots and

$$\delta(S_1) + \delta(S_2) = \delta(S_1 \cap S_2) + \delta(S_1 \cup S_2) \tag{2}$$

or

$\delta(S_1 \setminus S_2)$ and $\delta(S_2 \setminus S_1)$ are both roots and

$$\delta(S_1) + \delta(S_2) = \delta(S_1 \setminus S_2) + \delta(S_2 \setminus S_1)\,. \tag{3}$$

Proof. We define

$$S_{12} := S_1 \cap S_2, \quad S_{\bar{1}\bar{2}} := V \setminus (S_1 \cup S_2), \quad S_{1\bar{2}} := S_1 \setminus S_2, \quad S_{\bar{1}2} := S_2 \setminus S_1$$

(see Fig. 1). As both $\delta(S_1)$ and $\delta(S_2)$ are T-Steiner cuts, the set T is not contained in any row or column of Fig 1. Hence at least one of the following holds:

$$\delta(S_{12}),\ \delta(S_{\bar{1}\bar{2}}) \text{ are both } T\text{-Steiner cuts} \tag{4}$$

or

$$\delta(S_{1\bar{2}}),\ \delta(S_{\bar{1}2}) \text{ are both } T\text{-Steiner cuts}\,. \tag{5}$$

If (4) holds, for the incidence vectors we use the relation

$$\delta(S_1) + \delta(S_2) = \delta(S_{12}) + \delta(S_{\bar{1}\bar{2}}) + 2 \cdot \delta(S_{1\bar{2}}, S_{2\bar{1}}) \tag{6}$$

where $\delta(S_{1\bar{2}}, S_{2\bar{1}})$ is the set of edges with one endnode in $S_{1\bar{2}}$ and the other one in $S_{2\bar{1}}$. Since both $\delta(S_1)$ and $\delta(S_2)$ are roots for the weights $c > \mathbb{O}$ and (4) holds, this implies $\delta(S_{1\bar{2}}, S_{2\bar{1}}) = \emptyset$ and thus (2).

If (5) holds, a similar argument implies (3). □

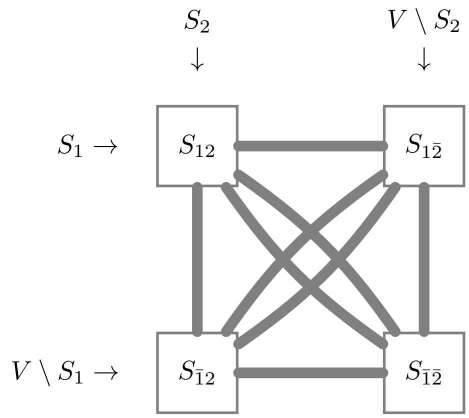


FIGURE 1. Notations used in Lemma 4 with $\chi(\delta(S_1)) + \chi(\delta(S_2))$ indicated.

Proof of Theorem 1. Let (G, T) be a facet inducing Steiner graph with facet weights c. Let $\delta(S)$ $(S \in \mathcal{L})$ be a laminar family of linearly independent roots for c that has a maximal number of members.

We claim that $\mathcal{L}$ is a root basis. In order to establish this statement, assume that it does not hold. Then there are roots that are not contained in the subspace spanned by the members of $\mathcal{L}$. Among those roots, we choose $\delta(S)$ to be one that minimizes $|I(S, \mathcal{L})|$. By maximality of $\mathcal{L}$ we have $I(S, \mathcal{L}) \neq \varnothing$.

Let $L \in I(S, \mathcal{L})$. By Lemma 4, one of the following two cases applies.

Case 1: : $\delta(S \cap L)$, $\delta(S \cup L)$ are roots and

$$\delta(S) + \delta(L) = \delta(S \cap L) + \delta(S \cup L) .$$

As $\delta(S)$ is not contained in the subspace spanned by $\mathcal{L}$, this equation shows that at least one of $\delta(S \cap L)$ and $\delta(S \cup L)$ is not contained in that subspace. Because of Lemma 3, this, however, contradicts the choice of S (minimizing $|I(S, \mathcal{L})|$).

Case 2: : $\delta(S \setminus L)$, $\delta(L \setminus S)$ are roots and

$$\delta(S) + \delta(L) = \delta(S \setminus L) + \delta(L \setminus S) ,$$

which implies a contradiction that is similar to the one we encountered in Case 1.

This completes the proof of Theorem 1.

For a laminar family $\mathcal{L}$ we define the *width* of $\mathcal{L}$ to be $|\mathcal{L}_{\min}|$ unless $\mathcal{L}$ has only one maximal member w.r.t. inclusion, in which case the width is $|\mathcal{L}_{\min}| + 1$. Theorem 1 then in particular implies that the Steiner rank of a facet defining inequality for a cut dominant is the smallest width of any laminar root basis for that inequality.

5. Subdivision and gluing

In this section we investigate two operations that construct facet inducing Steiner graphs from smaller ones. We also characterize operations that are inverse to those operations.

We say that the Steiner graph (G,T) has been obtained by a *subdivision* from the Steiner graph (G',T') if G arises from G' by replacing an edge uv with the edges uw and vw, where $w \notin V(T')$ is a newly added node, and $T = T'$ holds (i.e., the new node $w \notin T$ is a non-terminal node). If c' is a vector of edge weights of G' then the accordingly subdivided vector c of edge weights of G has $c(uw) = c(vw) = c'(uv)$ and $c(e) = c'(e)$ for all other edges. The following fact is straight forward.

Remark 5. *If the Steiner graph (G,T) is obained by subdivision from a facet inducing Steiner graph (G',T') with facet weights c', then also (G,T) is a facet inducing Steiner graph with facet weights c obtained by subdividing c' accordingly.*

In order to investigate the operation that is reverse to subdivision, let $w \in V(G)$ be a node of degree two in the graph G whose two neighbors u and v are not adjacent to each other. Then *reducing* w means to remove w and to replace its two incident edges uw and vw by the edge uv.

Analyzing the effect of reductions on facet inducing Steiner graphs is a bit more involved than the reasoning behind Remark 5 is. The following simple observation will be helpful for that purpose.

Remark 6. *If $\delta(S)$ is a root of the weight vector c for a Steiner graph (G,T) and $w \in V(G) \setminus T$ is a non-terminal node, then $c(\delta(w) \cap \delta(S)) \leq c(\delta(w) \setminus \delta(S))$ holds.*

Theorem 2. *Let (G,T) be a facet inducing Steiner graph with facet weights c. Let $w \in V(G)\setminus T$ be a non-terminal node with exactly two neighbors, say u and v. Then the following hold:*

1. *We have $c(uw) = c(vw)$ and u and v are not adjacent.*
2. *If G' is obtained from G by reducing node w, then (G',T') with $T' = T$ is a facet inducing Steiner graph with facet weights c' obtained from c by setting $c'(uv) = c(uw) = c(vw)$ and $c'(e) = c(e)$ for all other edges.*

Note that (G,T) can be obtained from (G',T') by subdivision of the edge uv, and c is the weight vector obtained by subdividing c' accordingly.

Proof. Due to $w \notin T$ (and $c > \mathbb{O}$) no root of c contains both uw and vw. A first consequence is that (since each of those two edges is contained in at least one root of c) Remark 6 implies $c(uw) = c(vw)$. As a second consequence, u and v are not adjacent to each other, because otherwise (as every cut intersects a triangle in none or two edges) all roots of c were contained in the hyperplane defined by $x_{uw} + x_{vw} = x_{uv}$, a contradiction to the existence of a root basis for c.

For the proof of the second statement we first observe that if $\delta_{G'}(S')$ with $S' \cap \{u,v\} \neq \varnothing$ is a T-Steiner cut in G', then $\delta_G(S' \cup \{w\})$ is a T-Steiner cut in G with $c'(\delta_{G'}(S')) = c(\delta_G(S' \cup \{w\}))$. This implies $\gamma_{c'}(G',T') \geq \gamma_c(G,T)$. In fact, we have $\gamma_{c'}(G',T') = \gamma_c(G,T)$, since for each root $\delta_G(S)$ of c with $w \in S$ we have $S \cap \{u,v\} \neq \varnothing$, thus $c'(\delta_{G'}(S')) = c(\delta_G(S))$ holds for the T-Steiner cut $\delta_{G'}(S')$ in G' with $S' = S \setminus \{w\}$, which hence is a root of c'.

In order to show that there is a root basis for c' let $\delta_G(S)$ $(S \in \mathcal{S})$ be a root basis for c with $w \in S$ for all $S \in \mathcal{S}$. As we just argued above, for each $S \in \mathcal{S}$ the T-Steiner cut $\delta_{G'}(S')$ with $S' = S \setminus \{w\}$ is a root for c'. Since those roots are the images of the root basis $\delta_G(S)$ $(S \in \mathcal{S})$ under the surjective linear map $\mathbb{R}^{E(G)} \to \mathbb{R}^{E(G')}$ that maps x to x' with $x'_{uv} = x_{uw} + x_{vw}$ and $x'_e = x_e$ for all other edges, they contain a root basis for c'. □

The second operation we consider is to *glue* two Steiner graphs (G_1,T_1) and (G_2,T_2) with $V(G_1) \cap V(G_2) = \{w\}$ and $w \in T_1 \cap T_2$ (possibly after replacing the graphs by isomorphic copies) at a common terminal node w in order to obtain a Steiner graph (G,T) with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ as well as $T = T_1 \cup T_2$ or $T = (T_1 \cup T_2) \setminus \{w\}$, i.e. w may or may not belong to T.

For two vectors c_1 and c_2 of edge weights of G_1 and G_2 in minimum integer form, respectively, the vector obtained by gluing c_1 and c_2 is the vector c of edge weights of G that is the minimum integer form of

$$(\gamma_{c_2}(G_2, T_2) \cdot c_1, \gamma_{c_1}(G_1, T_1) \cdot c_2) .$$

The set of roots of c is the union of the set of roots of c_1 and the set of roots of c_2, and $\gamma_c(G,T)$ devides the least common multiple of $\gamma_{c_1}(G_1,T_1)$ and $\gamma_{c_2}(G_2,T_2)$. Consequently, the following holds.

Remark 7. *If the Steiner graph (G,T) is obtained by gluing two facet inducing Steiner graphs (G_1,T_1) and (G_2,T_2) with facet weigts c_1 and c_2, respectively, then (G,T) is a facet inducing Steiner graph as well and the weights c obtained from gluing c_1 and c_2 accordingly are facet weights for (G,T).*

For the analysis of the reverse operation to gluing recall that a *cut node* in a connected graph G is a node w for which $G \setminus w$ is not connected.

Theorem 3. *Let (G,T) be a facet inducing Steiner graph with facet weights c, and let $w \in V(G)$ be a cut node of G. Let $V_1 \subseteq V(G)$ be the node set of one of the components of $G \setminus w$ and $V_2 := V(G \setminus w) \setminus V_1$. We denote by G_1 and G_2 the subgraphs of G induced by $V_1 \cup \{w\}$ and $V_2 \cup \{w\}$, respectively, and define $T_1 = (T \cap V_1) \cup \{w\}$ as well as $T_2 = (T \cap V_2) \cup \{w\}$.*

Then we have $|T_1|, |T_2| \geq 2$, and both (G_1,T_1) and (G_2,T_2) are facet inducing Steiner graphs with facet weights c_1 and c_2, respectively, that are the minimum integer forms of the restrictions of c to $E(G_1)$ and to $E(G_2)$, respectively.

Note that (G,T) can be obtained by gluing (G_1,T_1) and (G_2,T_2) at their common terminal node w, and c is the weight vector obtained by gluing c_1 and c_2 accordingly.

Proof. The fact that we have $|T_1|, |T_2| \geq 2$ is due to Part 6 of Remark 3.

Let $\delta_G(S)$ $(S \in \mathcal{S})$ be a root basis for c with $w \notin S$ for all $S \in \mathcal{S}$. Part 4 of Remark 3 implies that for every $S \in \mathcal{S}$ we have $S \subseteq V_1$ or $S \subseteq V_2$. From this we conclude that $\delta_{G_1}(S)$ $(S \in \mathcal{S}, S \subseteq V_1)$ form a root basis of c_1, and $\delta_{G_2}(S)$ $(S \in \mathcal{S}, S \subseteq V_2)$ form a root basis of c_2. □

6. Irreducible facet inducing Steiner graphs

We define a (weighted) Steiner graph (G,T) to be *constructable* from a family $\mathcal{F}$ of (weighted) Steiner graphs if there is a sequence $(G_0,T_0), \ldots, (G_r,T_r) = (G,T)$ with $(G_0,T_0) \in \mathcal{F}$ such that, for each $i \in [r]$, the Steiner graph (G_i,T_i) can be obtained as a subdivision of (G_{i-1},T_{i-1}) or from gluing (G_{i-1},T_{i-1}) with some member of $\mathcal{F}$. Due to Remarks 5 and 7 every (weighted) Steiner graph that is constructable from a family of facet inducing (weighted) Steiner graphs is facet inducing.

We call a facet inducing Steiner graph (G,T) *irreducible* if G has no cut node and no node in $V(G) \setminus T$ has degree two. The results from Section 5 imply the following.

Remark 8. *A (weighted) Steiner graph (G,T) is facet inducing if and only if it is constructable from irreducible facet inducing (weighted) Steiner graphs with at most $|T|$ terminals.*

Therefore, it suffices to determine the (weighted) irreducible facet inducing Steiner graphs. Due to (1) the only irreducible facet inducing Steiner graph with two terminals is the *Steiner edge* $(K_2, V(K_2))$ and the facet inducing Steiner graphs with two terminals are the *Steiner paths* $(G, \{s,t\})$, where G is a path with end nodes s and t.

Remark 9. *Each irreducible facet inducing Steiner graph (G,T) with $|T| \geq 3$ has $\deg(v) \geq 2$ for all $v \in T$ and $\deg(v) \geq 3$ for all $v \in V(G) \setminus T$.*

We are now prepared to establish a bound that in particular implies the first main result stated in the introduction.

Lemma 5. *Every irreducible facet inducing Steiner graph (G,T) with $|T| \geq 3$ has*

$$|E(G)| \leq |V(G)| + |T| - 3 \quad \textit{and} \quad |V(G)| \leq 3|T| - 6\,.$$

If equality holds in any of those two inequalities, then, for every choice of facet weights, $\delta(t)$ is a root for every $t \in T$.

Proof. Due to Remark 9 we have

$$2|E(G)| \geq 2|T| + 3|V(G) \setminus T| = 3|V(G)| - |T|\,.$$

Let c be facet weights of (G,T) and let $\delta(S)$ $(S \in \mathcal{L})$ be a laminar root basis for c as guaranteed to exist by Theorem 1. Due to Remark 4 there is some terminal $t^\star \in T$ with $\{t^\star\} \in \mathcal{L}_{\min}$. Replacing every set $S \in \mathcal{L}$ with $t^\star \in S$ by its complement $V(G) \setminus S$, we may assume that $\mathcal{L}$ is a laminar family (note that those sets S form a maximal chain in $\mathcal{L}$, thus laminarity is preserved) of pairwise distinct non-empty subsets of $V(G) \setminus \{t^\star\}$. Again by Remark 4, we have $|\mathcal{L}_{\min}| \leq |T| - 1$. Therefore, the bound from Lemma 1 yields

$$|E(G)| = |\mathcal{L}| \leq |V(G) \setminus \{t^\star\}| + |\mathcal{L}_{\min}| - 1 \leq |V(G)| + |T| - 3\,,$$

where equality between the left-hand and the right-hand side implies $|\mathcal{L}_{\min}| = |T| - 1$, i.e. that $\mathcal{L}_{\min}$ contains each terminal-singleton from $T \setminus \{t^\star\}$.

Combining the above two inequalities we obtain $|V(G)| \leq 3|T| - 6$ as claimed. □

The above lemma has the following immediate consequence.

Theorem 4. *There is a function f such that the number of irreducible facet inducing Steiner graphs (G,T) is bounded by $f(|T|)$ (up to isomorphisms).*

Of course, each irreducible facet inducing Steiner graph admits only finitely many facet weights (since a polyhedron has only finitely many facets). In particular, Theorem 4 yields the next result.

Theorem 5. *There is a function g such that the facet defining inequalities (in minimum integer form) for every Steiner cut dominant $\mathrm{CUT}_+(G,T)$ have right-hand sides (and coefficients) that are bounded by $g(|T|)$.*

Proof. This follows from Theorem 4 via Remark 2 and Remark 8, since the right hand-side of every inequality that is constructable from a finite set of inequalities (in minimum integer form) devides the least common multiple of their right-hand-sides. □

7. Steiner trees and Steiner cacti

We call a Steiner graph (G,T) a *Steiner tree* if G is a tree whose degree-one nodes are all in T (see Figure 2). The Steiner trees are the Steiner graphs that are constructible from Steiner edges. In particular they are facet inducing, where assigning a weight of one to each edge yields the unique facet weight vector (in minimum integer form). We call the corresponding inequalities (with right-hand-side one) *Steiner tree inequalities.*

Since for every Steiner graph (G,T), a subset of $E(G)$ contains a T-Steiner cut if and only if it intersects $E(G')$ for each Steiner subtree (G',T) of (G,T) the Steiner tree inequalities associated with the Steiner subtrees of (G,T) (together with the nonnegativity constraints) provide an integer programming formulation for $\mathrm{CUT}_+(G,T) \cap \mathbb{Z}^{E(G)}$.

Conversely, if for a Steiner subgraph (G',T) of the Steiner graph (G,T) the edge set $E(G')$ intersects every T-Steiner cut in (G,T) then (G',T) contains a Steiner subtree. This implies the following.

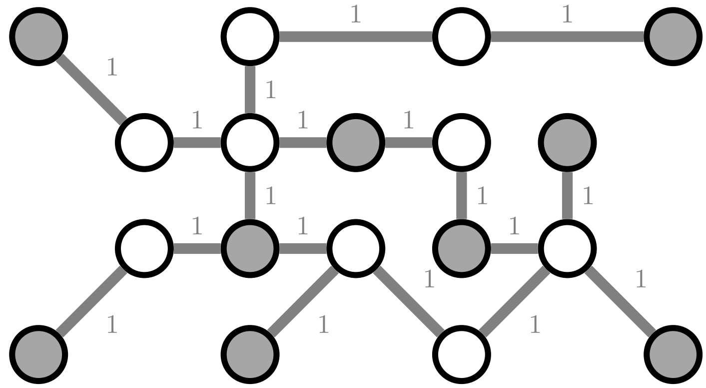


Figure 2. A Steiner tree with its facet weights (the dark nodes are the terminals).

Remark 10. *The only facet defining inequalities (in minimum integer form) with right-hand-side one for Steiner cut dominants are the Steiner tree inequalities.*

A *cactus* is a connected graph (we only consider graphs without loops or multiple edges) that has at meast one cycle, but in which every edge is contained in at most one cycle. We call the Steiner graph (G, T) a *Steiner cactus* if G is a cactus whose degree-one nodes are in T, and in which every cycle contains at least three nodes that are cut nodes of G or terminals (this in particular implies $|T| \geq 3$) (see Figure 3).

Steiner cacti are the Steiner graphs that are constructible from Steiner edges and Steiner cycles. In particular, they are facet inducing, where assigning to each edge weight one or two depending on whether the edge is contained in some cycle or not, respectively, yields the unique facet weight vector (in minimum integer form). We call the corresponding inequalities with right-hand-side two *Steiner cacti inequalities.*

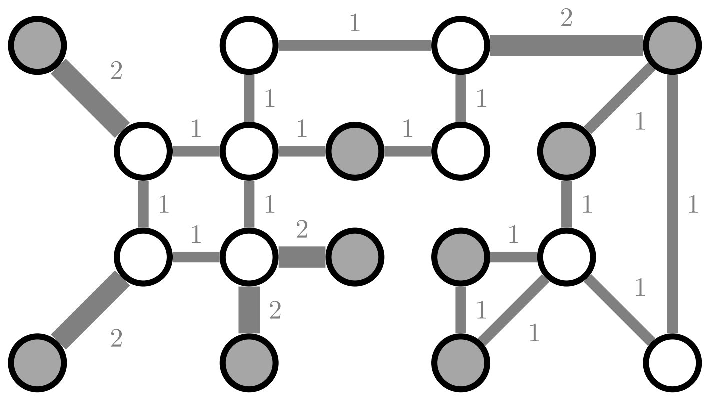


Figure 3. A Steiner cactus with its facet weights (the dark nodes are the terminals).

It seems worth to note that in contrast to Remark 10 there are facet defining inequalities (in minimum integer form) with right-hand-side two for Steiner cut dominants that are not Steiner cacti inequalities. Indeed, whenever G arises from some two-edge conected graph by replacing each edge by a path consisting of three edges, then (G, T) with $T \subseteq V(G)$ including all nodes of G with degree two is an irreducible facet inducing Steiner graph with facet weights provided by the all-ones vector and right-hand side two.

As it simplifies later arguments, we state the following fact (that follows easily from Theorems 2 and 3).

Remark 11. *If (G,T) is a facet inducing Steiner graph where G is a tree or a cactus, then (G,T) is a Steiner tree or a Steiner cactus, respectively.*

Since the only connected subgraphs of trees are trees, we find that if (G,T) is a Steiner graph with a tree G, then $\mathrm{CUT}_+(G,T)$ is described by the nonnegativity constraints and the Steiner tree inequality defined by the unique Steiner subtree of (G,T) with terminal set T.

Similarly, the only connected subgraphs of cacti are trees and cacti. Hence if (G,T) is a Steiner graph with a cactus G, then $\mathrm{CUT}_+(G,T)$ is described by the nonnegativity constraints and the Steiner tree and Steiner cactus inequalities defined by the Steiner subtrees and by the Steiner subcacti of (G,T) with terminal set T, respectively.

8. At most five terminals

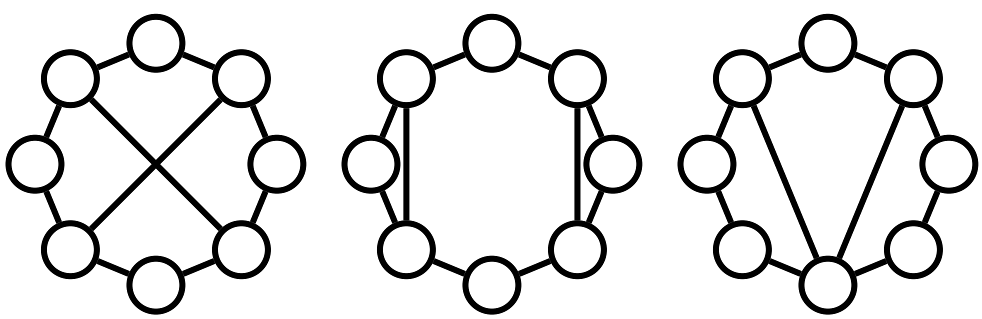
The facet inducing Steiner graphs with two terminals are the Steiner paths, among which the irreducible ones are the Steiner edges. The purpose of this section is to prove the following result.

Theorem 6. *The facet inducing Steiner graphs with at most five terminals are Steiner trees and Steiner cacti.*

Theorem 6 implies that, for $\tau \in \{3,4,5\}$, the irreducible facet inducing Steiner graphs with τ terminals are cycles of length τ with all nodes being terminals.

In order to prove Theorem 6 it remains to show that every facet inducing Steiner graph (G,T) with $|T| \in \{3,4,5\}$ is a Steiner tree or a Steiner cactus. We make some preparations for proving this.

Lemma 6. *If (G,T) is a facet inducing Steiner graph and G properly contains a Hamiltonian cycle $C \subsetneq E(G)$, then $E(G) \setminus C$ contains at least two edges that cross each other w.r.t. C (see. Figure 4).*



(a) (b) (c)

Figure 4. Crossing (a) and non-crossing (b,c) pairs of edges.

Proof. Let c be some facet weights for (G,T). If $\delta(S)$ is any root of c, then $|C \cap \delta(S)|$ is positive and even as C is Hamiltonian. According to Remark 3, both S and $V(G) \setminus S$ induce connected subgraphs of G. In case of $|C \cap \delta(S)| \geq 4$ this would require to have two edges in $E(G) \setminus C$ that cross each other w.r.t. C. As G does not contain such a pair of edges, we conclude that $|C \cap \delta(S)| = 2$ holds for every root S of c.

But this implies that the facet of $\mathrm{CUT}_+(G,T)$ defined by $cx \geq \gamma_c(G,T)$ is contained in the hyperplane defined by $x(C) = 2$, which yields a contradiction due to $C \neq E(G)$ and $c > 0$. □

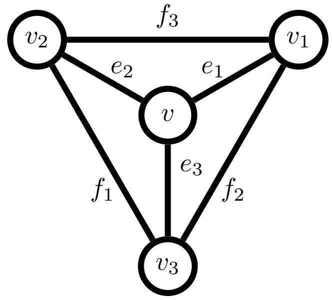


FIGURE 5. Notations used for YΔ-reductions.

The following operation turns out to be very helpful in reducing the sizes of graphs that we need to consider.

Lemma 7. *Let (G, T) be a facet inducing Steiner graph with facet weights c and $v \in V(G) \setminus T$ a non-terminal node of degree three with neighbors v_1, v_2, and v_3. With $e_i := vv_i$ for all $i \in \{1, 2, 3\}$ the following statements hold:*

(1) *No root contains $\{e_1, e_2, e_3\}$.*
(2) *We have*

$$c(e_1) \le c(e_2) + c(e_3)$$
$$c(e_2) \le c(e_3) + c(e_1)$$
$$c(e_3) \le c(e_1) + c(e_2)\,,$$

where at most one of the inequalities is satisfied at equality.

(3) *If $c(e_i) < c(e_j) + c(e_k)$ holds for i, j, k with $\{i, j, k\} = \{1, 2, 3\}$ then no root contains both e_j and e_k, and $v_j v_k$ is not an edge of G.*

Proof. The first statement follows from the fact that $v \in V(G) \setminus T$, which also implies the inequalities in the second statement (as every edge is contained in at least one root, see Remark 3). Clearly, because of $c > \mathbb{O}$, no two of those inequalities can be satisfied at equality.

To prove the third statement, observe that in case of $c(e_i) < c(e_j) + c(e_k)$ (for $\{i, j, k\} = \{1, 2, 3\}$) no root contains both e_j and e_k because of $v \in V(G) \setminus T$. Consequently, $v_j v_k \in E(G)$ in this case would imply that every root x satisfies $x_{v_j v_k} = x_{e_j} + x_{e_k}$, which, however, contradicts the fact that the roots span $\mathbb{R}^{E(G)}$. □

Definition 1. *Let (G, T) be a facet inducing Steiner graph with facet weights c and $v \in V(G) \setminus T$ a non-terminal node of degree three with neighbors v_1, v_2, and v_3. For each $i \in \{1, 2, 3\}$, we denote by $e_i := vv_i \in E(G)$ the edge in G connecting v to v_i and by f_i the edge (possibly in $E(G)$ or not) connecting the two neighbors of v different from v_i, and define $\zeta_i := c(\delta_G(v)) - 2c(e_i)$. Let G' be the graph with $V(G') = V(G) \setminus \{v\}$ and*

$$E(G') = E(G) \setminus \{e_1, e_2, e_3\} \cup \{f_i \,:\, \zeta_i > 0\}\,.$$

Let $c' \in \mathbb{R}^{E(G')}$ be the vector obtained from c by removing the components indexed by e_1, e_2, and e_3, and setting

$$c'(f_i) := c(f_i) + \frac{\zeta_i}{2}$$

for all $i \in \{1, 2, 3\}$ with $f_i \in E(G')$ (with $c(f_i) := 0$ in case of $f_i \notin E(G)$).

Then we say that the Steiner graph (G', T') with $T' = T$ and c' have been obtained from (G, T) and c by applying the YΔ-reduction *at v (see Figure 5).*

Note that applying a Y∇-reduction as in Definition 1 to a simple graph (recall that we restrict our considerations to simple graphs throughout the paper) results in a simple graph again. Lemma 7 now implies the following.

Remark 12. *In Definition 1, we have*

$$\zeta_i \geq 0 \quad \text{for all } i \in \{1,2,3\} \tag{7}$$

with

$$\zeta_i = 0 \quad \text{for at most one } i \in \{1,2,3\}\,. \tag{8}$$

Furthermore, if $\zeta_i > 0$ then $f_i \notin E(G)$ holds.

Lemma 8. *Applying a Y∇-reduction to a facet inducing Steiner graph (G,T) with facet weights c results in a facet inducing Steiner graph (G',T') with $T' = T$ and facet weights c'.*

Proof. We first observe that $c'(f) > 0$ holds for all $f \in E(G')$ (see (7)) .

With $\mathcal{S}'$ denoting the family of all $S \subseteq V(G) \setminus \{v\}$ with $S \cap T \neq \varnothing$, $T \not\subseteq S$, and $|S \cap \{v_1,v_2,v_3\}| \leq 1$, we have

$$c'(\delta_{G'}(S)) = c(\delta_G(S) \quad \text{for all } S \in \mathcal{S}'\,. \tag{9}$$

Indeed, this clearly holds for all $S \in \mathcal{S}'$ with $S \cap \{v_1,v_2,v_3\} = \varnothing$, and if $S \in \mathcal{S}'$ satisfies, say, $S \cap \{v_1,v_2,v_3\} = \{v_1\}$ we find

$$c'(\delta_{G'}(S)) = c(\delta_G(S)) - c(e_1) + \frac{\zeta_2}{2} + \frac{\zeta_3}{2} = c_G(\delta_G(S)$$

as well.

As the family $\delta_{G'}(S)$ $(S \in \mathcal{S}')$ is the family of all T'-Steiner cuts in G', and since $\delta_G(S)$ is a T-Steiner cut in G for each T'-Steiner cut $\delta_{G'}(S)$ in G' with $S \in \mathcal{S}'$ (recall $T' = T$), the equations in (9) imply $\gamma_{c'}(G',T') \geq \gamma_c(G,T)$.

In order to prove that c' provide facet weights, let the cuts $\delta_G(S)$ for $S \in \mathcal{S}$ form a root basis of c with $v \notin S$ for all $S \in \mathcal{S}$. For each $S \in \mathcal{S}$, the cut $\delta_{G'}(S)$ clearly is a T'-Steiner cut in G', for which we claim

$$c'_{G'}(\delta_{G'}(S)) = c_G(\delta(S) = \gamma_c(G,T)\,, \tag{10}$$

where this in particular implies $\gamma_{c'}(G',T') = \gamma_c(G,T)$. Indeed, due to Lemma 7 each $S \in \mathcal{S}$ satisfies $|S \cap \{v_1,v_2,v_3\}| \leq 2$. If that intersection has less than two elements, then we have $S \in \mathcal{S}'$ (with $\mathcal{S}'$ defined above), thus (10) is one of the equations (9). Otherwise, say if $S \cap \{v_1,v_2,v_3\} = \{v_2,v_3\}$ holds, we have $\zeta_1 = 0$ due to Lemma 7, and hence

$$\begin{aligned} c'(\delta_{G'}(S)) &= c(\delta_G(S)) - c(e_2) - c(e_3) + \frac{\zeta_2}{2} + \frac{\zeta_3}{2} \\ &= c_G(\delta_G(S)) - \zeta_1 \\ &= c_G(\delta_G(S)) \end{aligned}$$

as well.

We thus have proved that the cuts $\delta_{G'}(S)$ $(S \in \mathcal{S})$ are roots of c', and therefore it suffices to show that they span $\mathbb{R}^{E(G')}$. Towards this end, let $\varphi : \mathbb{R}^{E(G)} \to \mathbb{R}^{E(G')}$ be the linear map with $\varphi(x)_e = x_e$ for all $e \in E(G') \cap E(G)$ and $\varphi(x)_{f_i} = x(\delta_G(v)) - x_{e_i}$ for all $i \in \{1,2,3\}$ with $f_i \in E(G') \setminus E(G)$. As the image of φ is the entire space $\mathbb{R}^{E(G')}$, it hence is enough to argue that for each $S \in \mathcal{S}$ we have $\varphi(\delta_G(S)) = \delta_{G'}(S)$. As we clearly have $\varphi(\delta_G(S))_e = \delta_{G'}(S)_e$ for every $e \in E(G') \cap E(G)$, we thus only have to establish $\varphi(\delta_G(S))_{f_i} = \delta_{G'}(S)_{f_i}$ for each $i \in \{1,2,3\}$ with $f_i \in E(G') \setminus E(G)$. For such an index i and $\{1,2,3\} = \{i,j,k\}$ we have $\varphi(\delta_G(S))_{f_i} = |\delta_G(S) \cap \{e_j,e_k\}|$ and $\zeta_i > 0$, where according to Lemma 7 the latter inequality implies $|\delta_G(S) \cap \{e_j,e_k\}| \leq 1$. Due to $|S \cap \{v_j,v_k\}| = |\delta_G(S) \cap \{e_j,e_k\}|$, and as we have $\delta_{G'}(S)_{f_i} = 1$ if and only if $|S \cap \{v_j,v_k\}| = 1$ holds, this completes the proof. □

It is easy to see that applying a Y∇-reduction (according to Definition 1) to a Steiner tree or to a Steiner cactus yields a Steiner cactus. We will, however, need the following weakened version of the reverse statement that requires some arguments to be established.

Lemma 9. *If (G', T') is a Steiner tree or a Steiner cactus that has been obtained from a facet inducing Steiner graph (G, T) by applying a Y∇-reduction (according to Definition 1), then (G, T) is a Steiner tree or a Steiner cactus.*

Proof. We continue to use the notations from Definition 1 and furthermore denote by G_0 the subgraph of G induced by $\{v, v_1, v_2, v_3\}$.

Due to (8), after possibly renumbering the nodes, we have $\zeta_1, \zeta_2 > 0$. This readily implies $f_1, f_2 \in E(G')$ and (see Remark 12) $f_1, f_2 \notin E(G)$.

Let us first consider the case $f_3 \in E(G')$, i.e., the entire triangle $\{f_1, f_2, f_3\}$ is contained in $E(G')$. Then G' (which is a tree or a cactus) is a cactus, and the graph obtained from G' by removing the three edges f_1, f_2, f_3 has three connected components G_1, G_2, G_3, where each G_i intersects the triangle exactly in node v_i. As G' is a cactus, every graph G_i is a tree (possibly consisting of the single node v_i) or a cactus. Due to $f_1, f_2 \notin E(G)$ the graph G_0 is either the star with edge set $E(G_0) = \{e_1, e_2, e_3\}$ or the cactus with edge set $E(G_0) = \{e_1, e_2, e_3, f_3\}$. Therefore, $G = G_0 \cup G_1 \cup G_2 \cup G_3$ is a tree or a cactus.

It remains to consider the case $f_3 \notin E(G')$. In this case, by definition of G', we also have $f_3 \notin E(G)$. Thus G is the graph that arises from G' by removing f_1 and f_2 and adding the star G_0 with edge set $E(G_0) = \{e_1, e_2, e_3\}$.

If at most one of the edges f_1, f_2 is contained in some cycle of G' then G is connected with no edge being contained in more than one cycle, hence G is a tree or a cactus.

Therefore it remains to consider the case that f_1 is contained in some cycle $C_1 \subseteq E(G')$ and f_2 is contained in some cycle $C_2 \subseteq E(G')$ of the cactus G'. As G' is a cactus, the two cycles are either identical or the intersection of their node sets is $\{v_3\}$.

If $C_1 = C_2$ holds, then G is a cactus (arising from the cactus G' by inserting the star G_0 "into that cycle" and removing f_1, f_2).

Otherwise (i.e., the two cycles C_1 and C_2 with $f_1 \in C_1$ and $f_2 \in C_2$ intersect only in the common node v_3 of f_1 and f_2) let H be the cactus that arises from the cactus G' by removing f_1, f_2 and adding the node v as well as the two edges e_1, e_2. The cactus H contains the cycle $C := C_1 \cup C_2 \setminus \{f_2, f_1\} \cup \{e_1, e_2\}$ with node set U. The graph G arises from adding the chord e_3 of C to the cactus H. Denoting by G'' the subgraph $(U, C \cup \{e_3\})$ of G and by T'' the union of $U \cap T$ and the set of all nodes in U that are cut nodes of G, we conclude from Theorem 3 that (G'', T'') is a facet inducing Steiner graph. However, G'' is a Hamiltonian cycle (with edge set C) plus the one edge e_3, which contradicts Lemma 6. $\square$

Finally, we will make use of the main result of Conforti, Fiorini, and Pashkovich [5], which translated into our terminology is the following.

Theorem 7 (Thm. 5 and Lem. 4 in [5])**.** *If $cx \geq \gamma$ (in minimum integer form) defines a facet of* $\mathrm{CUT}_+(G)$ *with $\gamma > 2$ then G has a minor that is a prism or a pyramid (see Figure 8).*

Corollary 1. *If (G, T) is a facet inducing Steiner graph with facet weights c that satisfy $\gamma := \gamma_c(G, T) > 2$ and $cx \geq \gamma$ is valid for* $\mathrm{CUT}_+(G)$*, then G has a minor that is a prism or a pyramid.*

Proof. This follows readily from Theorem 7 as due to

$$\dim(\mathrm{CUT}_+(G)) = |E(G)| = \dim(\mathrm{CUT}_+(G, T))$$

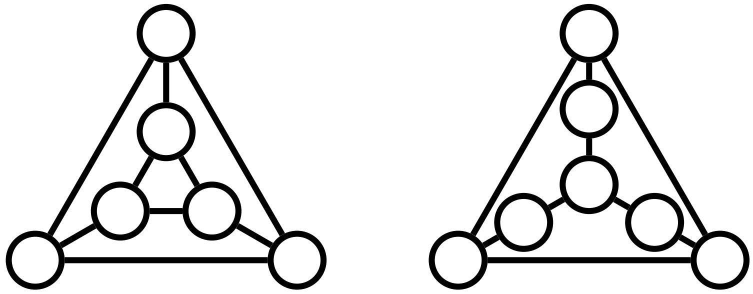


FIGURE 6. The *prism* (left) and the *pyramid* (right).

every inequality that is facet defining for $\mathrm{CUT}_+(G,T)$ and valid for $\mathrm{CUT}_+(G)$ is also facet defining for $\mathrm{CUT}_+(G)$ (which contains $\mathrm{CUT}_+(G,T)$). □

In order to establish Theorem 6, it suffices to prove the following result (see Remark 11).

Proposition 1. *For every irreducible facet inducing Steiner graph (G,T) with $|T| \in \{3,4,5\}$ the graph G is a tree or a cactus (in fact, G is an edge or a cycle).*

Proof. Supposing that the statement does not hold, we choose (G,T) to be an irreducible facet inducing Steiner graph with $\tau := |T| \in \{3,4,5\}$ such that G is neither a tree nor a cactus, where we make our choice such that $n := |V(G)|$ is minimal. Let c be facet weights for (G,T) and $\gamma := \gamma_c(G,T)$.

Lemma 5 provides the following upper bound on the number $m := |E(G)|$ of edges of G:

$$m \leq n + \tau - 3 \tag{11}$$

On the other hand, as G is two-node connected, but not a cycle, we have the lower bound

$$m \geq n + 1 \tag{12}$$

on the number of edges (see the remarks on ear decompositions below). Note that those two bounds already imply $\tau \geq 4$.

Clearly, due to the irreducibility, all terminal nodes have degree at least two and all non-terminal nodes have degree at least three. In fact, by the minimality of n we even deduce

$$|\delta(v)| \geq 4 \quad \text{for all } v \in V(G) \setminus T \tag{13}$$

from Lemma 8 and Lemma 9. This in particular implies

$$2m = \sum_{v \in V(G)} |\delta(v)| \geq 2n + 2(n - \tau)\,,$$

hence

$$m \geq 2n - \tau\,. \tag{14}$$

Note that (11) and (14) imply $n \leq 2\tau - 3$.

The system (11), (12), (14) has the following seven integral solutions with $3 \leq \tau \leq 5$ and $n \geq \tau$:

	τ	n	m
(1)	4	4	5
(2)	4	5	6
(3)	5	5	6
(4)	5	5	7
(5)	5	6	7
(6)	5	6	8
(7)	5	7	9

The cases (1), (3), and (4) (i.e., the ones with $n = \tau$) can be ruled out immediately by the result of Conforti, Fiorini, and Pashkovich cited above. Indeed, in those cases we have $T = V(G)$, and due to $m \geq n + 1$ (see (12)) there must be a node (thus, a terminal) of degree at least three, which then implies $\gamma \geq 3$. Hence, according to Theorem 7, G would have a prism or a pyramid as a minor in each of those cases, which, however, is not true as they all satisfy $n \leq 5$.

In order to enumerate the possible graphs for each of the remaining four parameter combinations, we exploit the well-known fact that the two-node connected graph G can be constructed by means of an *ear decomposition* (Whitney [15]): Starting from an arbitrary *initial cycle* in G, we repeatedly add *ears*, i.e., paths with at least one edge whose end nodes are disjoint nodes in the part of G that has already been constructed, and whose inner nodes have not yet appeared in the construction so far. It is possible to arrange the construction such that ears without inner nodes appear only after all nodes have shown up. The number of ears in an ear decomposition equals $m - n$.

A first consequence of the existence of an ear-decomposition of G is the following. As we have already ruled out the case $n = \tau$, the graph G has at least one non-terminal node. Thus, according to (13) it has a node with degree at least four, hence any ear decomposition has at least two ears. From this we conclude $m - n \geq 2$, which rules out cases (2) and (5).

Therefore, we are left with the task to show that cases (6) and (7) cannot occur. As in both of those cases we have $m - n = 2$, every ear decomposition of G has exactly two ears. In particular, no node degree exceeds four, thus all non-terminal nodes have degree equal to four (again, due to (13)). Furthermore, both remaining cases satisfy the equation

$$m = n + \tau - 3\,, \tag{15}$$

hence we know

$$c(\delta(t)) = \gamma \quad \text{for all } t \in T \tag{16}$$

according to Lemma 5, i.e., every terminal singleton defines a root. We conclude the proof by enumerating the graphs that remain to be considered and derive a specific contradiction for each of them.

Case (6): $\tau = 5$, $n = 6$, $m = 8$

Among the cycles that contain the non-terminal node $v \in V(G) \setminus T$, we chose one, say C, of maximal length as the initial cycle of an ear decomposition of G. Again, C is not a Hamiltonian cycle (since the two edges in $E(G) \setminus C$ would both be incident to the non-terminal node of degree four, thus they would not cross w.r.t. C, contradicting Lemma 6).

Therefore, we have $|C| \leq 5$. From the maximality property of C we find $|C| \geq 4$. In case of $|C| = 4$, again due to the maximality property of C, G must be the graph G_1, and if we have $|C| = 5$, then G must be one of the graphs G_2 and G_3 (see Fig. 7).

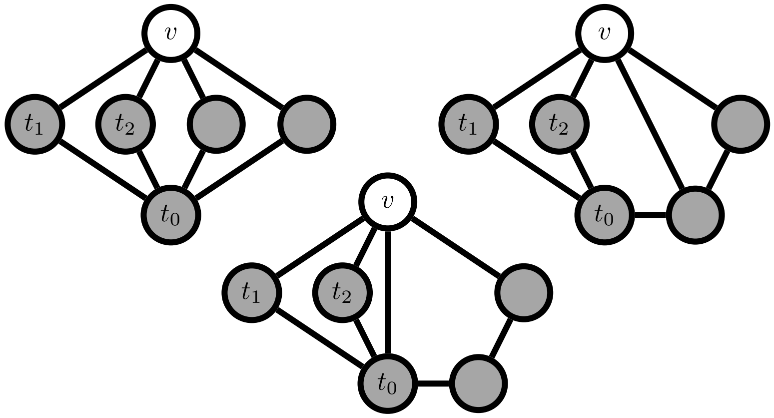


FIGURE 7. The graphs G_1 (top left), G_2 (top right, and G_3 (bottom). Terminals are gray.

Using (16), we find that for each of the three graphs

$$c(\delta(v)) \geq c(\delta(t_1)) + c(\delta(t_2)) - c(\delta(t_0)) = \gamma$$

holds. Thus, in each of the three cases the inequality $cx \geq \gamma$ is even valid for $\mathrm{CUT}_+(G)$ (as $\delta(v)$ is the only non-trivial cut in G that is not a T-Steiner cut). Since none of the graphs G_1, G_2, and G_3 has a prism or a pyramid as a minor (due to $m = 8$), Corollary 1 implies $\gamma = 2$. However, each of the three graphs has a terminal t of degree larger than two, which then contradicts (16).

Case (7): $\tau = 5$, $n = 7$, $m = 9$

Due to $m = n+2$, and as both non-terminal nodes $v_1, v_2 \in V(G) \setminus T$ have degree equal to four, all terminal nodes have degrees equal to two. Therefore, an arbitrary ear decomposition will have v_1 and v_2 in its initial cycle with both ears having v_1 and v_2 as their end nodes. Thus, G consists of four disjoint paths each having v_1 and v_2 as its end node, hence G is one of the graphs shown in Fig. 8.

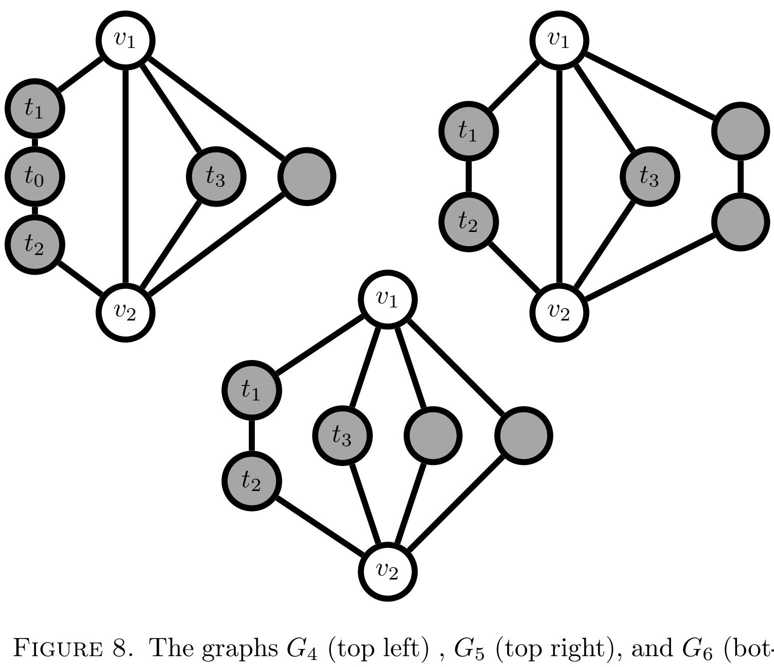


FIGURE 8. The graphs G_4 (top left) , G_5 (top right), and G_6 (bottom). Terminals are dark.

From the equations in (16) we deduce $c(t_1v_1) = c(t_2v_2)$ for each of the graphs G_4, G_5, and G_6 (where for G_4 we additionally exploit $c(\delta\{t_0, t_1\}) \geq \gamma$ and $c(\delta\{t_0, t_2\}) \geq \gamma$).

For G_4 we thus have $c(\delta(v_1)) = c(\delta(\{v_1, t_1, t_0, t_2\})) \geq \gamma$ as well as $c(\delta(v_2)) = c(\delta(\{v_2, t_2, t_0, t_1\})) \geq \gamma$, and for G_5 and G_6 we find $c(\delta(v_1)) = c(\delta(\{v_1, t_1, t_2\})) \geq \gamma$ as well as $c(\delta(v_2)) = c(\delta(\{v_2, t_2, t_1\})) \geq \gamma$.

As additionally $c(\delta\{v_1, v_2\}) \geq c(\delta(t_3)) = \gamma$ holds for G_4, G_5, and G_6, the inequality $cx \geq \gamma$ is in fact valid for $\mathrm{CUT}_+(G, V(G))$. Since none of the graphs G_4, G_5, and G_6 has a prism or a pyramid as a minor (as we have $m = 9$, and each of the three graphs has only two nodes of degree larger than two), Corollary 1 implies $\gamma = 2$ and c is the all-one vector (since each edge is contained in at least one root).

As for both G_4 and G_5 the edge v_1v_2 is not contained in any cut with two edges, we conclude $G = G_6$. However, the only cuts with exactly two edges in G_6 are the five cuts $\delta(t)$ for $t \in T$ and the cut $\delta(\{t_1, t_2\})$ which contradicts the existence of a root basis (due to $m = 9$). □

We conclude this section by the following characterization of the facet defining inequalities of Steiner degree at most five for cut dominants that follows immediately from Theorem 6.

Definition 2. *If H is a cactus and C is a cycle in H, then we denote by* $\deg(C)$ *the number of nodes in C that are connected to nodes outside of C (i.e., they are cut nodes of G), and we define the* defect *of C to be*

$$\max\{0, 3 - \deg(C)\}.$$

Corollary 2. *For each connected graph G the facet defining inequalities of Steiner degree at most five for* $\mathrm{CUT}_+(G)$ *are the inequalities*

$$x(E(H)) \quad \geq \quad 1$$

for each spanning tree H in G with at most five leaves, and the inequalities

$$\sum_{\substack{e \in E(H) \\ e \text{ is in some cycle of } H}} x_e + \sum_{\substack{e \in E(H) \\ e \text{ is in no cycle of } H}} 2 \cdot x_e \quad \geq \quad 2$$

for each spanning cactus H of G for which the number of leaves plus the sum of the defects of the cycles is at most five (which in particular implies that H has no more than three cycles).

One consequence of Corollary 2 is that the only vertices of the subtour relaxation polytope that have Steiner degree at most five are the incidence vectors of the Hamiltonian cycles (i.e., the vertices of the traveling salesman polytope itself), which in fact have Steiner degree three.

9. Conclusion

More than five terminals. For six terminals or more the irreducible facet inducing Steiner graphs are considerably more involved than they are for up to five terminals (where we only have single edges and cycles of length at most five according to Theorem 6). In fact, at this point we do not know the complete list of irreducible facet inducing Steiner graphs with six terminals. Figure 9 shows a complete list of those ones with six or seven nodes. As one sees from the examples (c), (d), and (e) in Figure 9, in contrast to the situation with at most five terminals, irreducible facet inducing Steiner graphs in general do have non-terminal nodes as well. Note that (b) and (d) induce vertices of the subtour relaxation polytope that have Steiner degree six.

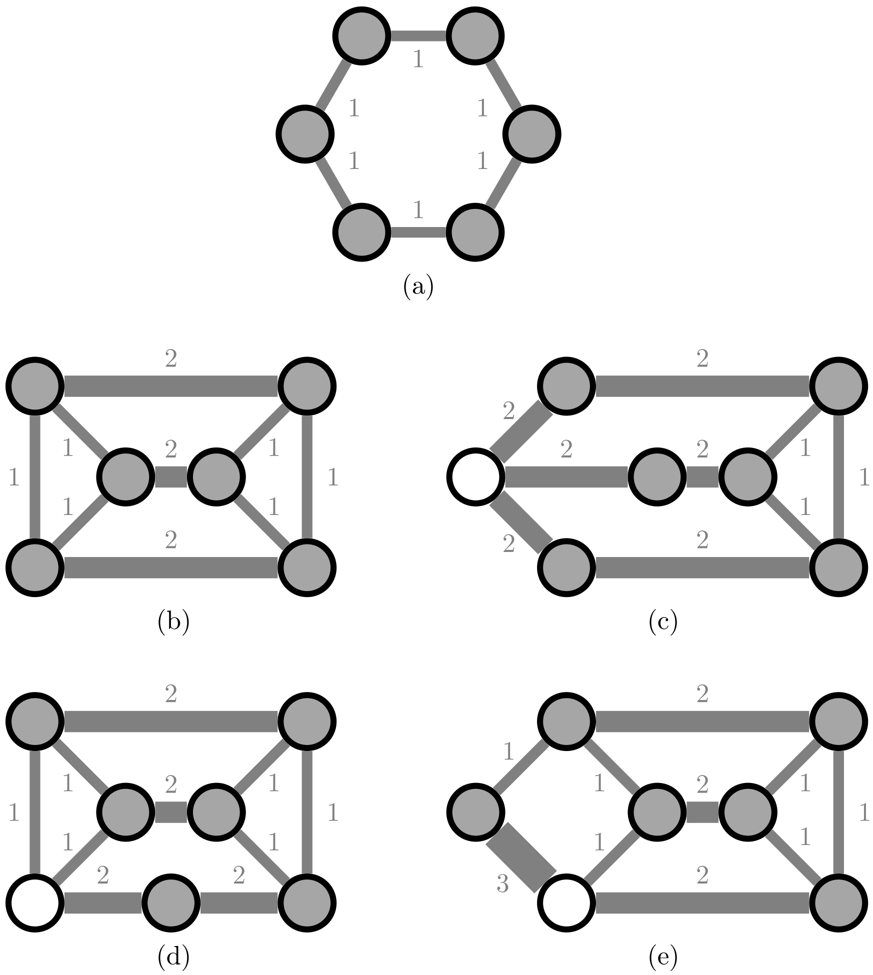


Figure 9. The irreducible facet inducing Steiner graphs with six terminals and at most seven nodes (again, terminals are dark); the cycle (a) has right-hand-side two, the other ones (b)–(e) have right-hand-side four (note that (b) is a prism).

Unique facet weights. As a consequence of Theorem 6 (and the remarks made in connection with the definitions of Steiner trees and Steiner cacti inequalities) for up to five terminals the facet weights of each facet inducing Steiner graph are uniquely determined. We do not expect a similar result to hold for general numbers of terminals.

Upwards validity. Turning a non-terminal node into a terminal can turn a facet-defining Steiner cut inequality into an invalid one, as the inequality arising from (b) in Figure 9 by subdividing one of the triangle-edges by a non-terminal node shows. However, we are only aware of exmples exhibiting that effect where the non-terminal node has degree two. Therefore, one might ask the question whether every *irreducible* facet defining Steiner cut inequality for $\mathrm{CUT}_+(G,T)$ in fact is valid (and thus facet defining) for $\mathrm{CUT}_+(G)$. Again, Theorem 6 at least shows that this holds for $|T| \leq 5$.

Computing the facets in polynomial time. It appears conceivable that one can, for every given Steiner graph (G,T) with $|T| \leq 5$, compute all Steiner subtrees and Steiner subcactii of (G,T) in time that is bounded polynomially in the total size of in- and output (for a survey on enumerating s-t-paths and other structures see [8]). This would imply via Theorem 6 that for Steiner graphs (G, T) with $|T| \leq 5$ one can compute a list if all facet defining inequalities for $\mathrm{CUT}_+(G, T)$ in output-polynomial time. A general question would be whether Theorem 4 opens up possibilites for a corresponding output-polynomial time algorithm for every fixed number of terminals.

More powerful operations. Theorem 6 shows that for each Steiner graph (G, T) with $|T| \leq 5$ the facet defining inequalities for $\mathrm{CUT}_+(G, T)$ can be constructed from the facet defining inequalities for $\mathrm{CUT}_+(K_\tau)$ with $\tau \leq |T|$ via iterated applications of the two operations *gluing* and *subdividing*. For more than five terminals, the corresponding result is not true, in general. Thus, the question for a more powerful set of operations arises that would allow for a similar result for arbitrary numbers of terminals. In fact, it may appear tempting to believe that the polyhedral structure of $\mathrm{CUT}_+(G, T)$ in some sense is a refinement of the polyhedral structure of $\mathrm{CUT}_+(K_{|T|})$.

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