On the Exactness of Dantzig-Wolfe Relaxation for Rank Constrained Optimization Problems

Yongchun Li
H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, USA, ycli@gatech.edu

Weijun Xie
H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, USA, wxie@gatech.edu

This paper studies the rank constrained optimization problem (RCOP) that aims to minimize a linear objective function over intersecting a prespecified closed rank constrained domain set with \( m \) two-sided linear constraints. Replacing the domain set by its closed convex hull offers us a convex Dantzig-Wolfe Relaxation (DWR) of the RCOP. Our goal is to characterize necessary and sufficient conditions under which the DWR and RCOP are equivalent in the sense of extreme point, convex hull, and objective value. More precisely, we develop the first-known necessary and sufficient conditions about when the DWR feasible set matches that of RCOP for any \( m \) linear constraints from two perspectives: (i) extreme point exactness–all extreme points in the DWR feasible set belong to that of the RCOP; and (ii) convex hull exactness–the DWR feasible set is identical to the closed convex hull of RCOP feasible set. From the optimization view, we also investigate: (iii) objective exactness–the optimal values of the DWR and RCOP coincide for any \( m \) linear constraints and a family of linear objective functions. We derive the first-known necessary and sufficient conditions of objective exactness when the DWR admits four favorable classes of linear objective functions, respectively. From the primal perspective, this paper presents how our proposed conditions refine and extend the existing exactness results in the quadratically constrained quadratic program (QCQP) and fair unsupervised learning.

Key words: Rank Constraint; Dantzig-Wolfe Relaxation; Extreme point Exactness; Convex Hull Exactness; Objective Exactness; QCQP; Fair Unsupervised Learning.
1. Introduction

This paper studies the Rank Constrained Optimization Problem (RCOP) of the form:

\[
\text{(RCOP)} \quad V_{\text{opt}} := \min_{X \in \mathcal{X}} \left\{ \langle A_0, X \rangle : b_l^i \leq \langle A_i, X \rangle \leq b_u^i, \forall i \in [m] \right\},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product of two matrices, \( \mathcal{X} \) is the domain set including the rank constraint, and for each \( i \in [m] \), the lower and upper bounds of the \( i \)th two-sided linear constraint can be negative infinite or positive infinite (i.e., \( -\infty \leq b_l^i \leq b_u^i \leq +\infty \)). Throughout this paper, we let \( \tilde{m} \) denote the dimension of technology matrices \( \{A_i\}_{i \in [m]} \) in RCOP (1), i.e., the maximum number of linearly independent matrices in the set \( \{A_i\}_{i \in [m]} \), and we must have \( \tilde{m} \leq m \). The domain set \( \mathcal{X} \) in RCOP (1) is quite generic and can be adapted to different application settings, e.g., we let \( \mathcal{X} := \{X \in S_n^{+1} : \text{rank}(X) \leq 1\} \) for the quadratically constrained quadratic program (QCQP). Formally, we define the domain set \( \mathcal{X} \) as

\[
\mathcal{X} := \{X \in Q : \text{rank}(X) \leq k, F_j(X) \leq 0, \forall j \in [t]\},
\]

where the matrix space \( Q \) can denote positive semidefinite matrix space \( S_n^+ \), symmetric matrix space \( S_n \), or non-symmetric matrix space \( \mathbb{R}^{n \times p} \) with \( n \leq p, k \leq n \) being positive integers, and for each \( j \in [t] \), function \( F_j(\cdot) : Q \to \mathbb{R} \) can be nonconvex. The flexibility of domain set \( \mathcal{X} \) allows the proposed RCOP framework (1) to deliver significant modeling flexibility. Section 1.2 summarizes several interesting examples in machine learning and optimization that fall into RCOP (1).

Albeit versatile, the underlying rank-\( k \) constraint dramatically complicates RCOP (1) which often turns out to be a nonconvex bilinear program. Consequently, many researchers have compromised to use easy-to-solve relaxation problems, for example, dropping the rank constraint or replacing it with the nuclear norm. Alternatively, we leverage the closed convex hull of domain set \( \mathcal{X} \) (i.e., \( \text{conv}(\mathcal{X}) \)) to obtain a stronger convex relaxation of the RCOP (1), which refers to the Dantzig-Wolfe Relaxation (DWR) in literature (see, e.g., Conforti et al. 2014), i.e., we consider the following relaxation problem

\[
\text{(DWR)} \quad V_{\text{rel}} := \min_{X \in \text{conv}(\mathcal{X})} \left\{ \langle A_0, X \rangle : b_l^i \leq \langle A_i, X \rangle \leq b_u^i, \forall i \in [m] \right\}.
\]

Consequently, we have \( V_{\text{rel}} \leq V_{\text{opt}} \). The DWR (3) can be solved the off-the-shelf solvers such as Gurobi and Mosek or the Dantzig-Wolfe decomposition algorithm as long as the separation problem over the domain set \( \mathcal{X} \) can be done effectively via a solvable convex program or even a mixed-integer convex program.

This work focuses on studying the exactness of DWR (3). Albeit the existing efforts on establishing the DWR exactness conditions for QCQP (see, e.g., Burer and Ye 2020, Kilınç-Karzan and
Wang 2021, Wang and Kılınç-Karzan 2022, Sojoudi and Lavaei 2014, Azuma et al. 2022 and references therein), to our best knowledge, the existing results for QCQP can neither directly apply nor be simply extended to our RCOP (1). Analyzing the QCQP exactness from dual perspective comes into the focus in literature, which often relies on the Slater condition or more strict assumptions, e.g., Wang and Kılınç-Karzan (2022) assumed the dual set of DWR to be polyhedral. More importantly, existing works on QCQP exactness often require that the \( m \) linear constraints in RCOP (1) are prespecified. From a primal and geometric angle, this paper first develops the necessary and sufficient conditions concerning when DWR (3) is equivalent to the general RCOP (1) for any \( m \) linear constraints. To be specific, we study the DWR exactness when intersecting the domain set \( \mathcal{X} \) with any \( m \) linear constraints in RCOP (1), i.e., data \( \{(b_i, A_i, b^u_i)\}_{i \in [m]} \) are arbitrary instead of being presumed. Besides advancing the state-of-the-art DWR exactness for QCQP, this paper throws new light on understanding the DWR exactness for some applications examples of our RCOP (1) beyond QCQP, including fair unsupervised learning, as detailed in Table 1.

It is important to note that when the rank constrained domain set \( \mathcal{X} \) in (2) enjoys some desired properties, one can obtain an explicit description of its closed convex hull. For example, we have \( \text{conv}(\mathcal{X}) = S_{n+1}^+ \) in the QCQP. The set \( \text{conv}(\mathcal{X}) \) will be specified later for more RCOP application examples. In recent years, given a rank constrained set \( \mathcal{X} \), research into deriving \( \text{conv}(\mathcal{X}) \) has mainly investigated the perspective technique (Bertsimas et al. 2021, De Rosa and Khajavirad 2022, Wei et al. 2022) and majorization technique (Kim et al. 2021) based on different conditions. Particularly, the majorization that requires a permutation-invariant domain set \( \mathcal{X} \) can be very useful for describing \( \text{conv}(\mathcal{X}) \), as the rank-\( k \) constraint that is independent of the permutation of eigenvalues or singular values is naturally symmetric. Hence, building on these exciting results, we suppose that the closed convex hull of the domain set \( \mathcal{X} \) is known or the separation over set \( \text{conv}(\mathcal{X}) \) can be done effectively via off-the-shelf solvers.

Observe that the feasible sets of RCOP (1) and DWR (3) are considered in a separate manner, the intersection of \( m \) linear constraints with the domain set \( \mathcal{X} \) and its closed convex hull \( \text{conv}(\mathcal{X}) \), respectively. For the sake of notational convenience, let us define these two feasible sets as

\[
\mathcal{C} := \{ \mathbf{X} \in \mathcal{X} : b_i^l \leq \langle A_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \}, \quad \mathcal{C}_{\text{rel}} := \{ \mathbf{X} \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle A_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \},
\]

where \( \text{conv}(\mathcal{C}) \subseteq \mathcal{C}_{\text{rel}} \) always holds. In this way, RCOP (1) and DWR (3) can be equivalently recast as \( \min_{\mathbf{X} \in \mathcal{C}} \langle A_0, \mathbf{X} \rangle \) and \( \min_{\mathbf{X} \in \mathcal{C}_{\text{rel}}} \langle A_0, \mathbf{X} \rangle \), respectively.

1.1. Three Notions of DWR Exactness: Extreme Point, Convex Hull, and Objective Value

Our notions defining the exactness of DWR (3) encompass both geometric and optimization perspectives, including extreme point exactness, convex hull exactness, and objective exactness, where
the first two concepts center on the feasible set and the last one highlights the optimal value. We propose necessary and sufficient conditions under which DWR (3) achieves the three notions of exactness for any \( m \) linear constraints of dimension \( \tilde{m} \) in RCOP (1), respectively. To be specific, given the rank constrained domain set \( \mathcal{X} \), \textit{extreme point exactness} represents that all the extreme points of the feasible set of DWR (3) are contained in the feasible set of RCOP (1), i.e., \( \text{ext}(C_{\text{rel}}) \subseteq C \) for any \( \tilde{m} \)-dimensional linear constraints; \textit{convex hull exactness} seeks that the DWR set coincides with the closed convex hull of RCOP feasible set, i.e., \( C_{\text{rel}} = \overline{\text{conv}}(C) \) for any \( \tilde{m} \)-dimensional linear constraints; and \textit{objective exactness} is another commonly-used concept in literature, meaning that DWR (3) yields the same objective value as the original RCOP (1), i.e., \( V_{\text{opt}} = V_{\text{rel}} \) for any \( \tilde{m} \)-dimensional linear constraints and a given family of linear objective functions. For example, we consider the objective exactness for any linear objective function such that the DWR optimal value is bounded from below, i.e., \( V_{\text{rel}} > -\infty \). The objective exactness implying the zero relaxation gap highly relies on the linear objective function, so we make a further analysis of two special yet intriguing families of functions as detailed in Section 3.

These exactness notions are strongly related as illustrated in Figure 1. The convex hull exactness is the strongest notion and implies the other two. The convex hull exactness reduces to the extreme point exactness for a bounded set \( C_{\text{rel}} \). We show that if DWR (3) yields a finite objective value, i.e., \( V_{\text{rel}} > -\infty \) and its feasible set \( C_{\text{rel}} \) contains no line, then the extreme point exactness is equivalent to the objective exactness for any linear objective function satisfying \( V_{\text{rel}} > -\infty \). If the objective exactness holds for any linear objective function, then the convex hull exactness naturally follows.

**Figure 1** The relation among three DWR exactness notions for any \( \tilde{m} \)-dimensional linear constraints.

**1.2. Scope and Flexibility of our RCOP Framework (1)**

This subsection presents some generic RCOPs from optimization, statistics, and machine learning fields. Therefore, our RCOP is applicable to different problems in various areas.
Quadratically Constrained Quadratic Program (QCQP) When $k = 1$. The QCQP has been widely-used in many application areas including optimal power flow (Josz et al. 2016, Low 2013), sensor network problems (Bertrand and Moonen 2011, Khobahi et al. 2019), signal processing (Huang and Palomar 2014, Gharanjik et al. 2016), and others. A QCQP has the following generic form:

\[(\text{QCQP}) \quad \min_{x \in \mathbb{R}^n} \left\{ x^\top Q_0 x + q_0^\top x : b_i^l \leq x^\top Q_i x + q_i^\top x \leq b_i^u, \forall i \in [m] \right\},\]

where matrices $Q_0, Q_1, \ldots, Q_m$ are symmetric but may not be positive semidefinite. Introducing the matrix variable $X := \begin{pmatrix} 1 & x^\top \end{pmatrix}$, the equivalent formulation below of the QCQP in matrix form can be viewed a special case of our RCOP (1) with $m + 1$ linear constraints

\[(\text{QCQP}) \quad \min_{X \in \mathcal{X}} \left\{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m], X_{11} = 1 \right\}, \mathcal{X} := \{ X \in S_+^{n+1} : \text{rank}(X) \leq 1 \},\]

where $A_0 = \begin{pmatrix} 0 & q_0^\top / 2 \\ q_0 / 2 & Q_0 \end{pmatrix}$ and for each $i \in [m]$, $A_i = \begin{pmatrix} 0 & q_i^\top / 2 \\ q_i / 2 & Q_i \end{pmatrix}$. We notice that since the closed convex hull is $\text{conv}(\mathcal{X}) = S_+^{n+1}$, the corresponding DWR of the QCQP (6) reduces to the well-known SDP relaxation in literature, i.e.,

\[(\text{DWR-QCQP}) \quad \min_{X \in \text{conv}(\mathcal{X})} \left\{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m], X_{11} = 1 \right\}, \text{conv}(\mathcal{X}) := S_+^{n+1}.

It is worthy of mentioning that we may strengthen the SDP relaxation by incorporate more induced side constraints to the domain set $\mathcal{X}$.

Fair Unsupervised Learning When $k \geq 1$. It has been recently recognized that the conventional unsupervised learning produces biased learning results when dealing with sensitive attributes, such as gender, race, and education level. The fairness has recently attracted significant attention. For example, Fair PCA (FPCA) was studied in Samadi et al. (2018), Tantipongpipat et al. (2019) and Fair SVD (FSVD) was proposed in Buet-Golfouse and Utyagulov (2022). Formally, FPCA in Tantipongpipat et al. (2019) admits the following form

\[(\text{FPCA}) \quad \max_{(z,X) \in \mathbb{R} \times \mathcal{X}} \left\{ z : z \leq \langle A_i, X \rangle, \forall i \in [m] \right\}, \mathcal{X} := \{ X \in S_+^n : \text{rank}(X) \leq k, \|X\|_2 \leq 1 \},\]

where $\| \cdot \|_2$ denotes the spectral norm (i.e., the largest singular value) of a matrix and matrices $A_1, \cdots, A_m \in S_+^n$ denote the sample covariance matrices from $m$ different groups. Note that the FSVD has a similar framework as FPCA except that matrices $A_1, \cdots, A_m \in \mathbb{R}^{n \times p}$ are non-symmetric and the corresponding domain set is $\mathcal{X} \subseteq \mathbb{R}^{n \times p}$. Simple calculations show that the closed convex hull of domain set $\mathcal{X}$ admits a closed form and thus its DWR can be written as

\[(\text{DWR-FPCA}) \quad \max_{(z,X) \in \mathbb{R} \times \text{conv}(\mathcal{X})} \left\{ z : z \leq \langle A_i, X \rangle, \forall i \in [m] \right\}, \text{conv}(\mathcal{X}) = \{ X \in S_+^n : \text{tr}(X) \leq k, \|X\|_2 \leq 1 \}.

In a similar vein, the DWR of FSVD can be obtained. For FPCA/FSVD, the detailed derivation of their DWRs and the significance of our results are delegated to Section 3.
1.3. Review of Relevant Work

As far as we are concerned, existing works on the DWR exactness in literature mainly focus on the QCQP with a rank-$k:= 1$ constraint and Fair PCA with a rank-$k \geq 1$ constraint.

**QCQP.** For the QCQP (5), the existing literature on its DWR exactness is extensive and focuses particularly on the sufficient conditions of the objective exactness and convex hull exactness given $m$ specific linear constraints, i.e., $\{(b_i, A_i, b^*_i)\}_{i \in [m]}$ are specified. Please see the excellent survey by Kılınç-Karzan and Wang (2021) and references therein. We first rigorously define the extreme point exactness which is of great interest as the extreme point may attain the optimum of DWR (3). In addition, our proposed conditions are necessary and sufficient to guarantee the three notions of DWR exactness for any data $\{(b_i, A_i, b^*_i)\}_{i \in [m]}$.

**QCQP with $m \leq 2$ constraints.** Early works have used the S-lemma to explore specific problems of QCQP (5) that admit the DWR exactness, which can date back to Yakubovich (1971). It is known that the QCQP (5) with one or two quadratic constraints can yield the DWR exactness under some mild assumptions. For example, the DWR achieves the objective exactness for the Trusted Region Subproblem (TRS), Generalized TRS (GTRS), and two-sided GTRS under Slater condition (Yakubovich 1971, Pólik and Terlaky 2007, Wang and Xia 2015), a class of QCQP with $m = 1$ quadratic constraint. Beyond the objective exactness, it is proven by Ho-Nguyen and Kilinc-Karzan (2017), Kılınç-Karzan and Wang (2021) that set $C_{rel}$ provides an explicit convex hull of the original feasible set $C$ for TRS and GTRS under Slater condition. When the quadratic coefficient matrix $Q_1$ in QCQP (5) is nonzero, the convex hull exactness also holds for the two-sided GTRS (Joyce and Yang 2021) which does not require Slater condition. Our proposed conditions can advance the DWR exactness results discussed above by getting rid of extra assumptions such that the Slater condition is no longer required.

The general QCQP (5) with $m = 2$ quadratic constraints has its own interesting applications (see, e.g., Powell and Yuan 1991, Ai and Zhang 2009). However, it may not always have DWR exactness. In this case, existing works have attempted to examine the sufficient conditions under which the DWR objective exactness holds. For example, Ye and Zhang (2003) showed that the DWR objective exactness holds if one of the two constraints is non-binding under both primal and dual Slater conditions. In a similar vein, Ben-Tal and Den Hertog (2014) extended this sufficient condition to the case that one of nonzero dual variables equaled zero at optimality for the simultaneously diagonalizable QCQP. Together, these two studies recognize the importance of binding constraints and nonzero dual variables in determining objective exactness. We propose necessary and sufficient conditions for the DWR objective exactness regarding the number of binding constraints and nonzero dual variables, respectively. Importantly, we can extend the results in Ye and Zhang (2003), Ben-Tal and Den Hertog (2014) to the general RCOP (1).
QCQP with \( m \) constraints. Another thread of work on QCQP (5) aims to develop sufficient conditions for its DWR exactness given \( m \) constraints in contrast to the previously discussed ones, which address one or two constraints. Burer and Yang (2015) proved the zero optimality gap of the DWR provided that the feasible set of QCQP (5) builds on a unit ball and the other non-intersecting constraints. When the QCQP (5) admits bipartite graph structures, several studies have proven the DWR objective exactness (see, e.g., Azuma et al. 2022, Sojoudi and Lavaei 2014, Kim and Kojima 2003). Besides, for the diagonal QCQP in which matrices \( Q_0 \) and \( \{Q_i\}_{i \in [m]} \) in (5) are diagonal, Burer and Ye (2020), Locatelli (2020) proposed sufficient conditions of DWR objective exactness. Particularly, Burer and Ye (2020) extended the results to the general QCQP (5), providing the first-known sufficient condition of DWR objective exactness. Recently, a seminal study in Wang and Kılınç-Karzan (2022) proposed the sufficient conditions for the objective exactness and convex hull exactness under the assumption that the Lagrangian dual feasible set is strictly feasible and polyhedral. Their follow-up work relaxed the polyhedral assumption (Wang and Kılınç-Karzan 2020) and proved that the sufficient conditions are necessary for the convex hull exactness whenever the polar cone to the Lagrange dual set is facially exposed. While we were finishing our paper, we have noted that very recent studies by Dey et al. (2022), Blekherman et al. (2022) gave an explicit description of the convex hull of \( m = 3 \) quadratic constraints via aggregations under mild conditions on matrices \( \{Q_i\}_{i \in [m]} \). Most studies reviewed here have investigated the DWR exactness for QCQP from its dual perspective and relied on KKT conditions and hence on strong duality (Azuma et al. 2022, Burer and Ye 2020, Locatelli 2020, Wang and Kılınç-Karzan 2022, Wang and Kılinc-Karzan 2020).

Fair Unsupervised Learning. The DWR exactness of the Fair PCA (FPCA) has been studied in an interesting work by Tantipongpipat et al. (2019), where the authors proved the extreme point exactness for any \( m = 2 \) different groups of covariance matrices in FPCA. Our proposed conditions successfully extends this result to the convex hull exactness for any \( \tilde{m} = 2 \) linearly different groups of covariance matrices. We further go beyond the positive semi-definite set to cover the Fair SVD (FSVD) problem, where we first derive the convex hull exactness for any \( \tilde{m} = 3 \) linearly different groups of data matrices in FSVD.

1.4. Summary of Main Contributions and Organization

As far as we are concerned, this paper is the first one to study three notions of DWR exactness on the general RCOP (1) for any \( m \) linear constraints of dimension \( \tilde{m} \), derive simultaneously necessary and sufficient conditions, where our analyses are mainly from the primal perspective, geometrically interpretable, and do not depend on the Lagrangian dual of DWR (3) except for a special case of objective exactness. The main contributions and an outline of the reminders of the paper are summarized below:
(i) Section 2 derives the necessary and sufficient conditions about when DWR (3) achieves the extreme point exactness and convex hull exactness, respectively.

- In Section 2.1, we first prove that any extreme point in set $C_{\text{rel}}$ is on the face of $\text{conv}(\mathcal{X})$ of dimension at most $\tilde{m}$. Using this result, we derive a necessary and sufficient condition for the extreme point exactness in Section 2.2.

- We show in Section 2.3 that each extreme direction of set $C_{\text{rel}}$ is contained in an at most $(\tilde{m}+1)$-dimensional face of the recession cone of $\text{conv}(\mathcal{X})$. If domain set $\mathcal{X}$ contains all crucial faces in $\text{conv}(\mathcal{X})$ and in its recession cone on which extreme points and directions of set $C_{\text{rel}}$ reside, then set $\text{conv}(C)$ must also contain these extreme points and directions; therefore, it is natural to develop a sufficient condition for convex hull exactness. Besides, we show that this condition becomes necessary and sufficient when domain set $\mathcal{X}$ is conic.

- Section 2.4 presents how our proposed conditions refine and extend the existing exactness results for several simple QCQP problems, which are specifically the first six problems in Table 1. This contributes to the literature in two aspects: our exactness conditions generate fresh insight into understanding the DWR exactness for these problems without Slater condition; and we first prove the extreme point exactness for a novel IQP-2 problem that refers to a QCQP (5) with an inhomogeneous objective function and two homogeneous constraints.

(ii) Section 3 investigates the necessary and sufficient conditions of the objective exactness for DWR (3) which admits four favorable classes of linear objective functions.

- As illustrated in Figure 1, since the objective exactness for any linear objective function and for any linear objective function satisfying $V_{\text{rel}} > -\infty$ reduce to the extreme point exactness and convex hull exactness, respectively, we directly extend the proposed conditions in Section 2 to show objective exactness under these two settings in Section 3.1.

- Section 3.2 develops a relaxed necessary and sufficient condition for objective exactness for any linear objective function given a finite DWR optimal value and bounded optimal set.

- Section 3.3 further investigates the necessary and sufficient condition for objective exactness for any linear objective function given a finite DWR optimal value and bounded optimal set, and a fixed number of nonzero optimal Lagrangian multipliers.

- The objective exactness conditions can be applied to the remaining problems of Table 1 present in literature, where we recover the objective exactness for the QCQP with multiple constraints, prove the convex hull exactness of Fair SVD, and generalize the other interesting results by either providing stronger exactness results or using less strict assumptions.

(iii) Section 4 concludes this paper.

Notation: The following notation is used throughout the paper. For a closed convex set $D$ and a positive integer $d$, we let $\mathcal{F}^d(D)$ denote the collection of all no larger than $d$–dimensional face of
### Table 1: Example Applications of Our Proposed Conditions

<table>
<thead>
<tr>
<th>Application</th>
<th>Problem</th>
<th>Setting</th>
<th>DWR Exactness</th>
<th>Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>QCQP (5) with ( m = 1 ) quadratic constraint ((k = 1))</td>
<td>QCQP-1</td>
<td>single constraint</td>
<td>extreme point (Corollary 3)</td>
<td>–</td>
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<td></td>
<td>TRS</td>
<td>single ball constraint</td>
<td>convex hull (Corollary 4)</td>
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<td></td>
<td>GTRS</td>
<td>single inequality constraint</td>
<td>convex hull (Corollary 5)</td>
<td>–</td>
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<tr>
<td>Two-sided GTRS</td>
<td>single two-sided quadratic constraint</td>
<td>convex hull (Corollary 6)</td>
<td>( Q_1 \neq 0 )</td>
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</tr>
<tr>
<td>QCQP (5) with ( m = 2 ) quadratic constraints ((k = 1))</td>
<td>HQP-2</td>
<td>homogeneous QCQP</td>
<td>extreme point (Corollary 7)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>IQP-2</td>
<td>inhomogeneous objective with homogeneous constraints</td>
<td>extreme point (Corollary 8)</td>
<td>–</td>
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<tr>
<td></td>
<td>–</td>
<td>inequality constraints and one constraint is not binding</td>
<td>objective (Corollary 10)</td>
<td>( V_{\text{rel}} &gt; -\infty ); bounded optimal set(^{ii})</td>
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<tr>
<td></td>
<td>–</td>
<td>inequality constraints and an optimal dual variable is zero</td>
<td>objective (Corollary 16)</td>
<td>( V_{\text{rel}} &gt; -\infty ); bounded optimal set; relaxed Slater condition</td>
</tr>
<tr>
<td>QCQP (5) with ( m ) inequality quadratic constraints ((k = 1))</td>
<td>–</td>
<td>all off-diagonal elements are sign-definite</td>
<td>objective (Part (i) of Corollary 15)</td>
<td>the underlying circle structures</td>
</tr>
<tr>
<td></td>
<td>–</td>
<td>diagonal QCQP with sign-definite linear terms</td>
<td>objective (Part (ii) of Corollary 15)</td>
<td>–</td>
</tr>
<tr>
<td>Fair Unsupervised Learning ((k \geq 1))</td>
<td>Fair PCA</td>
<td>( m = 2 ) groups</td>
<td>convex hull (Corollary 11)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Fair SVD</td>
<td>( m = 3 ) groups</td>
<td>convex hull (Corollary 13)</td>
<td>–</td>
</tr>
</tbody>
</table>

\(^i\) “–” denotes either empty assumption or no specified name of the problem
\(^{ii}\) The set of all optimal solutions to DWR is bounded

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\( D, \) let \( \text{aff}(D) \) denote the affine hull of set \( D, \) let \( \text{dim}(D) \) denote the dimension of set \( D, \) let \( \text{rec}(D) \) denote the recession cone of set \( D \) when it is unbounded, and let \( \text{ri}(D) \) denote the relative interior of set \( D. \) Given \( m \) matrices \( \{ A_i \}_{i \in [m]} \), their linear span is defined by \( \text{span}(\{ A_i \}_{i \in [m]}) := \{ \sum_{i \in [m]} \alpha_i A_i : \alpha \in \mathbb{R}^m \}. \) For a matrix \( X, \) let \( \| X \|_2 \) denote its spectral norm (i.e., the largest singular value), let \( \| X \|_1 \) denote its nuclear norm (i.e., the sum of singular values) and \( \| X \|_1 = \text{tr}(X) \) when \( X \in \mathcal{S}_+^n. \) or. Additional notation will be introduced later as needed.
2. A Geometric View of DWR Exactness: Necessary and Sufficient Conditions for Extreme Point Exactness and Convex Hull Exactness

This section investigates necessary and sufficient conditions on the DWR exactness from a geometric view of its feasible set: extreme point and convex hull. Specifically, extremal point exactness guarantees all extreme points in the set $C_{rel}$ to belong to the original feasible set $C$ of RCOP (1), i.e., $\text{ext}(C_{rel}) \subseteq C$ and convex hull exactness seeks $C_{rel}$ to equal the closed convex hull of set $C$, i.e., $C_{rel} = \overline{\text{conv}}(C)$. Note that when the convex hull of set $C$ is closed, we have $\overline{\text{conv}}(C) = \text{conv}(C)$.

2.1. Where Extreme Points of Set $C_{rel}$ are Located at Set $\overline{\text{conv}}(\mathcal{X})$?

In this subsection, we study some properties of extreme points in set $C_{rel}$, which offers important insights into understanding the extreme point exactness. Observe that given a domain set $\mathcal{X}$, set $C_{rel}$ in (4) is constructed by intersecting its closed convex hull (i.e., $\overline{\text{conv}}(\mathcal{X})$) with any $m$ linear constraints, so we are interested in what faces of the set $\overline{\text{conv}}(\mathcal{X})$ contain all extreme points in the intersection set $C_{rel}$. We expect the result to be independent of the linear constraints as data $\{(b_i^l, A_i, b_i^u)\}_{i \in [m]}$ can be arbitrary. To begin with, let us define the face and its dimension:

**Definition 1 (Face, Proper Face, Exposed Face, & Dimension)** A convex subset $F$ of a closed convex set $D$ is called a face of $D$ if for any line segment $[a, b] \subseteq D$ such that $F \cap (a, b) \neq \emptyset$, we have $[a, b] \subseteq F$. A nonempty face $F$ of $D$ is called a proper face if $F \subsetneq D$. If a face $F$ can be represented as the intersection of set $\overline{\text{conv}}(\mathcal{X})$ with its supporting hyperplane, then it is called exposed face. The dimension of a face is equal to the dimension of its affine hull.

Throughout the paper, for any closed convex set $D$, we let $\mathcal{F}^m(D)$ denote a collection of faces in this set with dimension no larger than $m$. Some faces with different dimensions are illustrated in Figure 2. Furthermore, these faces are proper and exposed.

**Figure 2** Examples of faces with different dimensions highlighted by red color

The example below illustrates the basic idea of identifying where extreme points of set $C_{rel}$ are.
**Example 1** Consider domain set \( \mathcal{X} := \{ \mathbf{X} \in S_+^2 : \text{rank}(\mathbf{X}) \leq 1, X_{12} = 0, \| \mathbf{X} \|_2 \leq 1 \} \) and \( m = 1 \) linear constraint \( X_{11} \leq 0.5 \). Then we have \( \overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) = \{ \mathbf{X} \in S_+^2 : X_{12} = 0, \| \mathbf{X} \|_2 \leq 1 \} \) and the two corresponding feasible sets defined in (4) are

\[
\mathcal{C} := \{ \mathbf{X} \in \mathcal{X} : X_{11} \leq 0.5 \}, \quad \mathcal{C}_{\text{rel}} := \{ \mathbf{X} \in \overline{\text{conv}}(\mathcal{X}) : X_{11} \leq 0.5 \}.
\]

In this example, we see that the matrix in domain set \( \mathcal{X} \) is of size two by two, positive semi-definite, and diagonal. Thus, we can equivalently recast \( \mathcal{X} \) as a two–dimensional set, i.e., \( \mathcal{X} := \{ \mathbf{x} \in \mathbb{R}_+^2 : \| \mathbf{x} \|_0 \leq 1, \| \mathbf{x} \|_\infty \leq 1 \} \), as shown in Figure 3(a). Then the convex hull of domain set \( \mathcal{X} \) can be readily derived and is closed, i.e., \( \overline{\text{conv}}(\mathcal{X}) := \{ \mathbf{x} \in \mathbb{R}_+^2 : \| \mathbf{x} \|_1 \leq 1 \} \), which is presented in Figure 3(b). Furthermore, the red solid line and the red shadow area in Figure 3(c) and Figure 3(d) represent sets \( \mathcal{C} \) and \( \mathcal{C}_{\text{rel}} \) that are obtained by intersecting sets \( \mathcal{X} \), \( \overline{\text{conv}}(\mathcal{X}) \) with a halfspace \( \{ \mathbf{x} \in \mathbb{R}^2 : x_1 \leq 0.5 \} \), respectively. We observe in Figure 3 that given \( m = 1 \) linear constraint, each extreme point of \( \mathcal{C}_{\text{rel}} \) is contained by a point or an edge in set \( \overline{\text{conv}}(\mathcal{X}) \) which is exactly a zero– or one–dimensional face by Definition 1. Specifically, for the set \( \mathcal{C}_{\text{rel}} \) in Figure 3(d), extreme points \( a_1 \) and \( a_2 \) belong to zero–dimensional faces of \( \overline{\text{conv}}(\mathcal{X}) \); and extreme points \( a_3 \) and \( a_4 \) belong to one–dimensional faces of \( \overline{\text{conv}}(\mathcal{X}) \). Motivated by this observation in Example 1, we will show the identity between the dimension of \( m \) linear constraints (i.e., \( \tilde{m} \leq m \)) and the largest dimension of faces in set \( \overline{\text{conv}}(\mathcal{X}) \) among which contain all extreme points in set \( \mathcal{C}_{\text{rel}} \) in the lemma below.

\[\text{Lemma 1} \quad \text{For a closed convex set } D \subseteq \mathbb{R}^n, \text{ each extreme point of the nonempty intersection set } \mathcal{H} := D \cap \{ \mathbf{x} \in \mathbb{R}^n : b_i^l \leq \langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i^u, \forall i \in [m] \} \text{ is contained in a face of set } D \text{ with dimension no larger than } \tilde{m}, \text{ where } \tilde{m} \leq m \text{ denotes the number of linearly independent vectors } \{ \mathbf{a}_i \}_{i \in [m]} \text{.} \]

\[\text{Proof.} \quad \text{We use the induction to prove this result. Suppose that set } D \text{ lies in a dimension-}d \text{ affine space. When } d \leq \tilde{m}, \text{ it is trivial to verify the statement since set } D \text{ itself is a } d\text{–dimensional face and } \mathcal{H} \subseteq D. \text{ Suppose the result holds for any } d \in [\bar{d} - 1] \text{ with } \bar{d} \geq \tilde{m} + 1. \text{ We will show that the result can be extended to the case } d := \bar{d} \text{ by contradiction. Let } \mathbf{\hat{x}} \text{ be an extreme point in set } \mathcal{H}. \]
Suppose the face with the smallest dimension in set $D$ containing $\hat{x}$ has a dimension greater than $\hat{m}$, denoted by $F^{d'} \subseteq D$ and $d' \geq \hat{m} + 1$. Then there are two cases to be discussed depending on whether $d' = \hat{d}$ or not.

(i) Suppose $d' < \hat{d}$. Since $F^{d'}$ is a $d' \leq \hat{d} - 1$ dimensional closed convex set and $\hat{x}$ is also an extreme point of the intersection set $F^{d'} \cap \{x \in \mathbb{R}^n : b_i^l \leq \langle a_i, x \rangle \leq b_i^u, \forall i \in [m]\}$, then by induction, $\hat{x}$ belongs to a face with its dimension up to $\hat{m}$ in set $F^{d'}$. It is known that a face of a face of a closed convex set is also a face of this set (see section 18 in Rockafellar 1972). Thus, the face in set $F^{d'}$ including the extreme point $\hat{x}$ that is at most $\hat{m}$-dimensional is also a face of set $D$, which contradicts with the fact that $d' \geq \hat{m} + 1$ is the smallest dimension of faces in set $D$ including $\hat{x}$.

(ii) Suppose $d' = \hat{d}$. Since set $D$ itself is the one and only one $\hat{d}$-dimensional face, i.e., $F^{d'} = F^{\hat{d}} = D$, then $\hat{x}$ does not belong to any proper face of $D$. According to the proposition 3.1.5 in Hiriart-Urruty and Lemaréchal (2004), the relative boundary of a closed convex set is equal to the union of all the exposed proper faces of this set. Therefore, $\hat{x}$ must be in the relative interior of set $D$ and there exists a scalar $\alpha > 0$ such that

$$B(\hat{x}, \alpha) \cap \text{aff}(D) \subseteq D,$$

where $B(\hat{x}, \alpha) := \{x \in \mathbb{R}^n : ||\hat{x} - x||_2 \leq \alpha\}$.

Given $\hat{d} \geq \hat{m} + 1$, for any $m$ vectors $\{a_i\}_{i \in [m]}$ of dimension $\hat{m}$, the intersection set $H^1 := \text{aff}(D) \cap \{x \in \mathbb{R}^n : \langle a_i, x \rangle = \langle a_i, \hat{x} \rangle, \forall i \in [m]\}$ has dimension at least one and thus cannot be a singleton. Clearly, $H \subseteq H^1$. Suppose $x^* \in H^1$ is a distinct point from $\hat{x}$. Then let us construct the vector $y := \hat{x} - x^*$ and thus we have $\langle a_i, y \rangle = 0$ for any $i \in [m]$. In addition, there exists a small scalar $0 < \epsilon \leq \alpha$ such that two vectors $\hat{x} \pm \epsilon y / ||y||_2$ belong to set $B(\hat{x}, \alpha) \cap \text{aff}(D) \subseteq D$. Hence, it is clear that both points $\hat{x} \pm \epsilon y / ||y||_2$ belong to the set $H$ and we have $\hat{x} = \frac{1}{2}(\hat{x} + \epsilon y / ||y||_2) + \frac{1}{2}(\hat{x} - \epsilon y / ||y||_2)$, which contradicts with the fact that $\hat{x}$ is an extreme point of set $H$.

This completes the proof. \qed

For the ease of analysis, Lemma 1 considers the intersection set in a vector space, which in fact can cover any matrix-based set by reshaping a matrix into a long vector, since the result in Lemma 1 is independent of the vector length $n$ and holds for any vectors $\{a_i\}_{i \in [m]}$. Therefore, a natural generalization follows.

**Corollary 1** For a closed convex set $D$ in matrix space $Q$, each extreme point of the nonempty intersection set $H := D \cap \{X \in Q : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m]\}$ is contained in a face of set $D$ with dimension no larger than $\hat{m}$, where $\hat{m}$ denotes the number of linearly independent vectors transformed by matrices $\{A_i\}_{i \in [m]}$. 
The result in Corollary 1 can be applied to the extreme point characterization of set $C_{rel}$ by letting $D := \text{conv}(\mathcal{X})$ and $\mathcal{H} := C_{rel}$. Given a domain set $\mathcal{X}$, when intersecting its closed convex hull with any $\tilde{m}$-dimensional linear constraints, Corollary 1 implies that only $F_{\tilde{m}}(\text{conv}(\mathcal{X}))$ plays a critical role in generating the extreme points in the DWR set $C_{rel}$, which motivates us to explore necessary and sufficient conditions for the DWR exactness by studying those no larger than $\tilde{m}$-dimensional faces of set $\text{conv}(\mathcal{X})$. It should be emphasized that the results in Lemma 1 and Corollary 1 only require a closed convex set $D$, and thus can be applied to any closed domain set $\mathcal{X}$ defined in (2).

2.2. The Necessary and Sufficient Condition for Extreme Point Exactness

By exploring Corollary 1, this subsection presents a necessary and sufficient condition under which the DWR problem (3) achieves the extreme point exactness, i.e., $\text{ext}(C_{rel}) \subseteq C$.

We will use Example 1 to showcase our main idea. We observe in Figure 3 that the extreme points $a_1, a_2, a_4$ of set $C_{rel}$ belong to set $C$ as the points and edges (i.e., zero and one-dimensional faces) where they locate in $\text{conv}(\mathcal{X})$ belong to domain set $\mathcal{X}$. By contrast, the one-dimensional face in set $\text{conv}(\mathcal{X})$ including the extreme point $a_3$ which does not belong to set $C$ is not contained in domain set $\mathcal{X}$. Motivated by this example, whether an extreme point of set $C_{rel}$ belongs to set $C$ highly depends on whether its related face in set $\text{conv}(\mathcal{X})$ is a subset of domain set $\mathcal{X}$, which offers a necessary and sufficient condition for the extreme point exactness as below.

**Theorem 1** Given a nonempty closed domain set $\mathcal{X}$, the followings are equivalent.

(a) **Inclusive Face:** Any no larger than $\tilde{m}$-dimensional face of set $\text{conv}(\mathcal{X})$ is contained in set $\mathcal{X}$, i.e., $F_{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$;

(b) **Extreme Point Exactness:** All the extreme points of set $C_{rel}$ belong to set $C$ for any $m$ linear constraints of dimension $\tilde{m}$ in RCOP (1), i.e., $\text{ext}(C_{rel}) \subseteq C$.

**Proof.** Let us prove the two directions of the equivalence, respectively.

(i) $F_{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \implies \text{ext}(C_{rel}) \subseteq C$.

Let $\hat{\mathcal{X}}$ be an extreme point in set $C_{rel}$. According to Corollary 1, we have $\hat{\mathcal{X}} \in F_{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$. It follows that $\hat{\mathcal{X}} \in \mathcal{X} \cap C_{rel} \subseteq C$. Therefore, $\text{ext}(C_{rel}) \subseteq C$ holds.

(ii) $\text{ext}(C_{rel}) \subseteq C \implies F_{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$.

Suppose that $F \in F_{\tilde{m}}(\text{conv}(\mathcal{X}))$ is a face not contained in set $\mathcal{X}$. Let $\hat{\mathcal{X}}$ be a point satisfying $\hat{\mathcal{X}} \in F \setminus \mathcal{X}$. Since face $F$ has dimension of no larger than $\tilde{m}$, we can then construct the $m$ matrices $\{A_i\}_{i \in [m]}$ of dimension $\dim(\text{aff}(F))$, i.e., we let $\mathcal{H} := \{\mathcal{X} \in \text{aff}(\mathcal{Q}) : \langle A_i, \mathcal{X} \rangle = \langle A_i, \hat{\mathcal{X}} \rangle, \forall i \in [m]\}$ such that the intersection $\text{aff}(F) \cap \mathcal{H} = F \cap \mathcal{H} := \{\hat{\mathcal{X}}\}$ is a singleton. Then
we show that is an extreme point of the set \( C_{\text{rel}} \). If not, then \( \tilde{X} \) can be written as the convex combination below

\[
\tilde{X} = \alpha X_1 + (1 - \alpha)X_2, \quad 0 < \alpha < 1,
\]

where \( X_1, X_2 \in C_{\text{rel}} \). Since \( X_1, X_2 \in C_{\text{rel}} \subseteq \mathcal{F}(\mathcal{X}) \) and \( \tilde{X} \in F \), according to Definition 1 of a face, \( X_1 \) and \( X_2 \) also belong to \( F \). In addition, we have \( X_1, X_2 \in C_{\text{rel}} \subseteq H \) and thus \( X_1, X_2 \in F \cap H \), which contradicts with the fact that the intersection set \( F \cap H \) is a singleton.

However, the extreme point \( \tilde{X} \) of set \( C_{\text{rel}} \) does not belong to \( C \) as \( \tilde{X} \notin \mathcal{X} \), which contradicts the fact that \( \text{ext}(C_{\text{rel}}) \subseteq C \). Therefore, \( \mathcal{F}^m(\mathcal{F}(\mathcal{X})) \) is contained in set \( \mathcal{X} \).

We remark that (i) set \( \mathcal{F}^m(\mathcal{F}(\mathcal{X})) \) refers to the set including all faces in set \( \mathcal{F}(\mathcal{X}) \) up to dimension \( m \) and equals \( \mathcal{F}(\mathcal{X}) \) itself when set \( \mathcal{F}(\mathcal{X}) \) is of dimension less than \( m \); and (ii) Part (a) in Theorem 1 serves as a necessary and sufficient condition of the extreme point exactness from the perspective of faces.

As aforementioned in Figure 1, the extreme point exactness sheds light on the convex hull exactness, where a compact set \( C_{\text{rel}} \) is exactly the convex combination of its extreme points. The relation among them motivates us to further investigate the convex hull exactness by leveraging the faces in set \( \mathcal{F}(\mathcal{X}) \).

### 2.3. Necessary and Sufficient Conditions for Convex Hull Exactness

In this subsection, we study the necessary and sufficient conditions on convex hull exactness of DWR (3), the strongest concept of exactness, which may fail even when the extreme point exactness holds. An example below illustrates this phenomenon.

**Example 2** Consider domain set \( \mathcal{X} := \{ X \in \mathcal{S}^2_{\mathbb{R}} : \text{rank}(X) \leq 1 \} \). Then \( \mathcal{F}(\mathcal{X}) = \mathcal{F}(\mathcal{S}^2_{\mathbb{R}}) := \mathcal{S}^2_{\mathbb{R}} \).

Let us construct the following intersection sets with \( m = 2 \) linear equality constraints.

\[
\mathcal{C} := \{ X \in \mathcal{S}^2_{\mathbb{R}} : X_{12} = 0, X_{11} = 1, \text{rank}(X) \leq 1 \}, \quad \mathcal{C}_{\text{rel}} := \{ X \in \mathcal{S}^n_{\mathbb{R}} : X_{12} = 0, X_{11} = 1 \}.
\]

In this example, the domain set is equivalent to \( \mathcal{X} = \{ X \in \mathbb{R}^{2 \times 2} : X_{12} = 0, X_{11} = 1, X_{22} = 0, X_{12} \geq 0, X_{22} \geq 0 \} \). We see that both sets \( \mathcal{X} \) and \( \mathcal{F}(\mathcal{X}) \) are unbounded, as shown in Figure 4(a) and Figure 4(b), respectively. We also see that

(i) Set \( \mathcal{X} \) is a three–dimensional surface, i.e., the boundary of its convex hull;

(ii) Any zero or one–dimensional face in \( \mathcal{F}(\mathcal{X}) \) is contained in set \( \mathcal{X} \) and \( \mathcal{F}(\mathcal{X}) \) contains no two–dimensional face, which are thus trivially contained in set \( \mathcal{X} \);

(iii) The only three–dimensional face in \( \mathcal{F}(\mathcal{X}) \) is itself and does not belong to set \( \mathcal{X} \).
Hence, set $\mathcal{X}$ contains any face of $\overline{\text{conv}}(\mathcal{X})$ with dimension no larger than two, i.e., $\mathcal{F}^2(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$. When intersecting set $\mathcal{X}$ with two linear equality constraint $X_{12} = 0, X_{11} = 1$, we have that $m = \tilde{m} = 2$. The resulting set $\mathcal{C}$ is a singleton, as marked a red solid point in Example 2, while set $\mathcal{C}_{\text{rel}}$ illustrated in Figure 4(d) is a ray.

In this example, we have that (i) the extreme point exactness holds, i.e., $\text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C}$, which is consistent with the result in Theorem 1; and (ii) the closed convex hull of set $\mathcal{C}$ is itself which is not identical to set $\mathcal{C}_{\text{rel}}$, i.e., the convex hull exactness fails, $\overline{\text{conv}}(\mathcal{C}) \neq \mathcal{C}_{\text{rel}}$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example2.png}
\caption{Illustration of Sets in Example 2 with $\mathcal{X} = \{X \in S^2_+ : \text{rank}(X) \leq 1\}$ and $\text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C}$, $\overline{\text{conv}}(\mathcal{C}) \neq \mathcal{C}_{\text{rel}}$.}
\end{figure}

As seen from Example 2, the convex hull exactness requires stronger conditions than that of extreme point exactness in Theorem 1. One exemplary condition is that set $\mathcal{C}_{\text{rel}}$ is bounded, which enables us to derive a necessary and sufficient condition as below. Note that $\overline{\text{conv}}(\mathcal{C}) = \text{conv}(\mathcal{C})$ when set $\mathcal{C}$ is compact.

**Theorem 2** Given a nonempty closed domain set $\mathcal{X}$, suppose that set $\mathcal{C}_{\text{rel}}$ is bounded. Then the followings are equivalent.

(a) **Inclusive Face**: Any no larger than $\tilde{m}$–dimensional face of set $\overline{\text{conv}}(\mathcal{X})$ is contained in the domain set $\mathcal{X}$, i.e., $\mathcal{F}^\tilde{m}(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$;

(b) **Extreme Point Exactness**: All the extreme points of set $\mathcal{C}_{\text{rel}}$ belong to set $\mathcal{C}$ for any $m$ linear constraints of dimension $\tilde{m}$ in RCOP (1), i.e., $\text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C}$;

(c) **Convex Hull Exactness**: The feasible set $\mathcal{C}_{\text{rel}}$ is equal to the convex hull of set $\mathcal{C}$ for any $m$ linear constraints of dimension $\tilde{m}$ in RCOP (1), i.e., $\mathcal{C}_{\text{rel}} = \text{conv}(\mathcal{C})$.

**Proof.** According to Theorem 1, we have $(a) \iff (b)$. It remains to prove $(b) \iff (c)$.

First, we observe that set $\mathcal{C} \subseteq \mathcal{C}_{\text{rel}}$ is also compact and thus $\text{conv}(\mathcal{C})$ is compact. Besides, since the compact convex set $\mathcal{C}_{\text{rel}}$ can be written as the convex hull of its extreme points, i.e., $\mathcal{C}_{\text{rel}} = \text{conv}(\text{ext}(\mathcal{C}_{\text{rel}}))$, it is clear that $\mathcal{C}_{\text{rel}} = \text{conv}(\mathcal{C}) \iff \text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C}$. This completes the proof. \qed
Interestingly, Theorem 2 provides a unified necessary and sufficient condition for three notions of DWR exactness when set $C_{rel}$ is bounded, since here the extreme point exactness is equivalent to the convex hull exactness and objective exactness for any linear objective function as shown in Figure 1. This result can be explained by the fact that a compact convex set can be described by the convex combination of its extreme points and there is always an extreme point attaining the finite optimal value when the feasible set is compact.

A considerable amount of literature has investigated the sufficient conditions for DWR exactness (see, e.g., Azuma et al. 2022, Kilınç-Karzan and Wang 2021, Pólik and Terlaky 2007); however, such studies remain narrow in focus dealing only with QCQP, a special case of our RCOP (1). Compared to them, the results of Theorem 2 have the following interesting aspects: (i) Beyond the scope of QCQP, we propose the first-known unified and general necessary and sufficient condition of DWR exactness in the sense that it simultaneously holds for all the three notions for exactness and can be applied to our RCOP (1), which is proven useful for covering and extending many existing results; (ii) Most exactness conditions proposed in the literature rely on some restricted assumptions, e.g., the structural dual set of DWR (3) is required to be polyhedral in Kilınç-Karzan and Wang (2021), which prevents the results from being applied to general QCQP. In contrast, our proposed condition in Theorem 2 is quite mild, which is the compactness of the feasible set of DWR (3), respectively. In particular, for QCQP, the feasible set of its corresponding DWR (3) with $k = 1$ is closed and contains no line, which naturally allows the equivalence of extreme point exactness and objective exactness for any objective function with finite $V_{rel}$; (iii) As mentioned before, the results in Theorem 1 and Theorem 2 are very general and are applicable to any closed nonconvex domain set $X$; (iv) From a novel yet geometric perspective, we show that the DWR exactness only depends on some crucial faces in set $X$, i.e., those containing extreme points of set $C_{rel}$.

When dealing with the convex hull exactness in Theorem 2, we assume that set $C_{rel}$ is bounded. The next theorem provides a sufficient condition for the convex hull exactness under the unbounded setting. The key ingredient that distinguishes an unbounded closed convex set from a compact one is the extreme direction as defined below; and Example 2 shows that the necessary and sufficient condition in Theorem 2 may be insufficient to guarantee the convex hull exactness of an unbounded set $C_{rel}$. Particularly, in Example 2, the unbounded set $C_{rel}$ is a ray while $C$ is a singleton, as the domain set $X$ does not contain this ray. Similar to exploring the extreme points of set $C_{rel}$ in Lemma 1, this observation intuitively motivates us to study the relationship between the extreme directions of unbounded set $C_{rel}$ and the set $\mathcal{Coav}(X)$. The example below illustrates their relationship.
Definition 2 (Extreme Ray, Extreme Direction) A half-line or line face of a convex set is called an extreme ray. The direction of an extreme ray is called an extreme direction.

Example 3 Consider domain set \( \mathcal{X} := \{ X \in \mathbb{S}^2_+ : \text{rank}(X) \leq 1, X_{12} = 0 \} \) and \( m = 1 \) linear constraint \( X_{11} \leq X_{22} \). Then we have \( \overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) = \{ X \in \mathbb{S}^2_+ : X_{12} = 0 \} \) and \( \mathcal{C} := \{ X \in \mathcal{X} : X_{11} \leq X_{22} \}, \ C_{\text{rel}} := \{ X \in \overline{\text{conv}}(\mathcal{X}) : X_{11} \leq X_{22} \} \).

Analogous to Example 1, the domain set \( \mathcal{X} \) here can be recast into the two-dimensional vector space, i.e., \( \mathcal{X} := \{ (X_{11}, X_{22}) \in \mathbb{R}^2_+ : X_{11}X_{22} = 0 \} \) and the linear constraint is \( X_{11} \leq X_{22} \). Therefore, we have \( \overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) := \mathbb{R}^2_+ \). The corresponding sets in vector space are displayed in Figure 5, where set \( \mathcal{C} \) is the red vertical line in Figure 5(c) and the red shadow area in Figure 5(d) represents set \( \mathcal{C}_{\text{rel}} \). We also observe that (i) any zero– or one–dimensional face of \( \text{conv}(\mathcal{X}) \) belongs to \( \mathcal{X} \), i.e., \( \mathcal{F}^1(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \); (ii) when intersecting \( \overline{\text{conv}}(\mathcal{X}) \) with a linear constraint \( X_{11} \leq X_{22} \), the resulting set \( \mathcal{C}_{\text{rel}} \) satisfies the extreme point exactness, i.e., \( \text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C} \); (iii) the recession cones of sets \( \overline{\text{conv}}(\mathcal{C}) \) and \( \mathcal{C}_{\text{rel}} \) are themselves; (iv) however, the convex hull exactness does not hold since the extreme direction \((1,1)\) of set \( \mathcal{C}_{\text{rel}} \) is not contained by the recession cone of set \( \overline{\text{conv}}(\mathcal{C}) \); and (v) for the recession cone of set \( \mathcal{C}_{\text{rel}} \), its one–dimensional faces are extreme directions.

The lemma below shows the equivalence between the unbounded one–dimensional faces of the recession cone of a closed convex set and its extreme directions. Recall that the recession cone is a convex combination of extreme directions.

**Lemma 2** Suppose that a convex set \( D \) is closed and unbounded. Then the extreme directions of its recession cone (i.e., \( \text{rec}(D) \)) are equivalent to the one–dimensional faces in \( \text{rec}(D) \).

**Proof.** For an unbounded, closed, and convex set \( D \), its recession cone \( \text{rec}(D) \) is a closed convex cone (see Rockafellar 1972, theorem 8.2), then the conclusion directly follows from the definition of extreme directions. \( \square \)
To ensure that sets $C_{rel}$ and $\overline{\text{conv}}(S)$ admit the same recession cone, we first study the properties of faces in the recession cone of set $\overline{\text{conv}}(X)$ that contain the extreme directions of all intersection set $C_{rel}$ for any $\tilde{m}$-dimensional constraints. According to Lemma 2, it suffices to focus on the one-dimensional faces of set $\text{rec}(C_{rel})$.

**Lemma 3** For a closed convex set $D \subseteq \mathbb{R}^n$ in vector space, suppose that the nonempty intersection set $\mathcal{H} := D \cap \{ x \in \mathbb{R}^n : b^i \leq \langle a_i, x \rangle \leq b^i_\text{rel}, \forall i \in [m] \}$. Then any extreme direction of the recession cone of set $\mathcal{H}$ is contained in a face of the recession cone of set $D$ with dimension no larger than $\tilde{m} + 1$, where $\tilde{m} \leq m$ denotes the number of linearly independent vectors $\{ a_i \}_{i \in [m]}$.

**Proof.** According to the equivalence between extreme directions and one-dimensional faces of the recession cone of a closed convex set in Lemma 2, it suffices to show that any one-dimensional face $F$ in the recession cone $\text{rec}(\mathcal{H})$ is contained in $\mathcal{F}_{\tilde{m} + 1}(\text{rec}(D))$. Similar to Lemma 1, we use the induction to prove this result. If the recession cone of set $D$ is $d$-dimensional with $d \leq \tilde{m} + 1$, then the result trivially holds.

Suppose $\mathcal{F}^1(\text{rec}(\mathcal{H})) \subseteq \mathcal{F}_{\tilde{m} + 1}(\text{rec}(D))$ for any $d \in [\tilde{d} - 1]$ with $\tilde{d} \geq \tilde{m} + 2$. Then let us prove the case $d = \tilde{d}$ by contradiction. If there is a one-dimensional face $F$ in $\text{rec}(\mathcal{H})$ that is not contained in $\mathcal{F}_{\tilde{m} + 1}(\text{rec}(D))$, then we let $d'$ denote the smallest dimension of the faces in $\text{rec}(D)$ that contain face $F$ and we must have $d' > \tilde{m} + 1$. Next, we split the proof into two parts depending on whether $d' < \tilde{d}$ or $d' = \tilde{d}$.

(i) If $d' < \tilde{d}$, then following the induction and the similar analysis in Lemma 1, the result holds.

(ii) If $d' = \tilde{d}$, then we must have that face $F$ belongs to the relative interior of $\text{rec}(D)$. Suppose there are two distinct points $x_1, x_2 \in F$ and without loss of generality, suppose that the first $\tilde{m}$ vectors $\{ a_i \}_{i \in [\tilde{m}]}$ are linearly independent. Given $n \geq \tilde{d} = d' > \tilde{m} + 1$, there is a nonzero vector $\Delta \neq 0 \in \mathbb{R}^n$ such that

$$\langle a_i, \Delta \rangle = 0, \forall i \in [\tilde{m}], \langle x_1 - x_2, \Delta \rangle = 0,$$

where the last equation implies that $\Delta$ is orthogonal to the face $F$, i.e., $\Delta \perp F$.

Besides, it is known (see, e.g., Luc 1990) that the recession cone of the intersection set $\mathcal{H}$ is equal to the intersection of the recession cones of set $D$ and the linear system $\{ x \in \mathbb{R}^n : b^i \leq \langle a_i, x \rangle \leq b^i_\text{rel}, \forall i \in [m] \}$. Since $x_1 \in F \subseteq \text{ri}(\text{rec}(D))$, there is a small scalar $\epsilon > 0$ such that $x_1 \pm \epsilon \Delta \in \text{rec}(D)$. In addition, we have $\langle a_i, x_1 \pm \epsilon \Delta \rangle = \langle a_i, x_1 \rangle$. It follows that $x_1 \pm \epsilon \Delta \in \text{rec}(\mathcal{H})$ and $x_1 = \frac{1}{2}(x_1 + \epsilon \Delta) + \frac{1}{2}(x_1 - \epsilon \Delta)$. This implies $x_1 \pm \epsilon \Delta \in F$ according to Definition 1 of the one-dimensional face $F$. This contradicts the fact $\Delta \perp F$.

Therefore, this result $\mathcal{F}^1(\text{rec}(\mathcal{H})) \subseteq \mathcal{F}_{\tilde{m} + 1}(\text{rec}(D))$ holds for the case $d = \tilde{d}$.  \(\square\)
Similar to Lemma 1, the result in Lemma 3 can be generalized to any matrix-based set.

**Corollary 2** For a closed convex set $D$ in matrix space $Q$, each extreme direction of the nonempty intersection set $\mathcal{H} := D \cap \{X \in Q : b_i^j \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m]\}$ is contained in a face of set $\text{rec}(D)$ with dimension no larger than $\bar{m} + 1$, where $\bar{m} \leq m$ denotes the dimension of matrices $\{A_i\}_{i \in [m]}$.

When intersecting the set $\overline{\text{conv}}(\mathcal{X})$ with multiple linear constraints, Lemma 1 together with Lemma 2 show that the faces of set $\overline{\text{conv}}(\mathcal{X})$ and its recession cone play an important role in determining the extreme points and extreme directions of the intersection set $\mathcal{C}_{\text{rel}}$. Next we show a sufficient condition about when the unbounded set $\mathcal{C}_{\text{rel}}$ achieves convex hull exactness.

**Theorem 3** Given a nonempty closed domain set $\mathcal{X}$, the following statement (a) implies (b).

(a) **Inclusive Face**: The (Minkowski) sum of any no larger than $\bar{m}$-dimensional face of set $\overline{\text{conv}}(\mathcal{X})$ and any no larger than $\bar{m} + 1$-dimensional face of the recession cone $\text{rec}(\overline{\text{conv}}(\mathcal{X}))$ is contained in the domain set $\mathcal{X}$, i.e., $\mathcal{F}^{\bar{m}}(\overline{\text{conv}}(\mathcal{X})) + \mathcal{F}^{\bar{m} + 1}(\text{rec}(\overline{\text{conv}}(\mathcal{X}))) \subseteq \mathcal{X}$;

(b) **Convex Hull Exactness**: The feasible set $\mathcal{C}_{\text{rel}}$ is equal to the closed convex hull of set $\mathcal{C}$ for any $m$ linear constraints of dimension $\bar{m}$ in RCOP (1), i.e., $\mathcal{C}_{\text{rel}} = \overline{\text{conv}}(\mathcal{C})$.

**Proof.** As $\mathcal{F}^{\bar{m}}(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$, according to Lemma 1 and Theorem 1, any extreme point $\mathcal{X}$ of set $\mathcal{C}_{\text{rel}}$ belongs to $\mathcal{F}^{\bar{m}}(\overline{\text{conv}}(\mathcal{X}))$ and also to set $\mathcal{C}$. For any extreme direction $D$ in set $\mathcal{C}_{\text{rel}}$, according to Lemma 3 and its corollary, belongs to $\mathcal{F}^{\bar{m} + 1}(\text{rec}(\overline{\text{conv}}(\mathcal{X})))$.

Since $\mathcal{F}^{\bar{m}}(\overline{\text{conv}}(\mathcal{X})) + \mathcal{F}^{\bar{m} + 1}(\text{rec}(\overline{\text{conv}}(\mathcal{X}))) \subseteq \mathcal{X}$, we have $\mathcal{X} + \alpha D \in \mathcal{X}$ for any $\alpha \geq 0$. As $D \in \text{rec}(\mathcal{C}_{\text{rel}})$ is also a recession direction of the $m$ linear constraints, we have that $D$ is a also recession direction in set $\overline{\text{conv}}(\mathcal{C})$ according to the definition of recession directions. Therefore, we have $\text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C}$ and $\text{rec}(\mathcal{C}_{\text{rel}}) \subseteq \text{rec}(\overline{\text{conv}}(\mathcal{C}))$, which implies that $\mathcal{C}_{\text{rel}} = \overline{\text{conv}}(\mathcal{C})$. \hfill \qed

The following example shows that the sufficient condition in Theorem 3, unfortunately, may be not necessary for the convex hull exactness.

**Example 4** Suppose that the domain set $\mathcal{X} := \{x \in \mathbb{R}_+^2 : \text{rank}(x) \leq 1, (x_1 - 1)^2 + (x_2 - 1)^2 \geq 0.5^2\}$.

Then the domain set is equivalent to $\mathcal{X} := \mathbb{R}_+^2 \setminus \{x \in \mathbb{R}_+^2 : (x_1 - 1)^2 + (x_2 - 1)^2 < 0.5^2\}$. We see that $\overline{\text{conv}}(\mathcal{X}) := \mathbb{R}_+^2$ since set $\mathcal{X}$ is defined as removing an open ball away from the interior of two-dimensional nonnegative orthant. Since $\overline{\text{conv}}(\mathcal{X}) \neq \mathcal{X}$ and both of them are two-dimensional, set $\mathcal{X}$ does not contain the two-dimensional face in $\overline{\text{conv}}(\mathcal{X})$.

Since set $\overline{\text{conv}}(\mathcal{X})$ is conic, our sufficient condition in Theorem 3 reduces to $\mathcal{F}^{\bar{m} + 1}(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$. Therefore, the condition $\mathcal{F}^{\bar{m} + 1}(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$ does not hold even when $\bar{m} = 1$. However, when intersecting sets $\mathcal{X}$ and $\overline{\text{conv}}(\mathcal{X})$ with any $m = \bar{m} = 1$ linear constraint $\mathcal{H}$, respectively, it is evident that the convex hull exactness always holds, due to the fact that $\overline{\text{conv}}(\mathcal{X} \cap \mathcal{H}) = \overline{\text{conv}}(\mathcal{X}) \cap \mathcal{H}$. \hfill \diamond
Interestingly, we show that the sufficient condition in Theorem 3 becomes necessary when the domain set $\mathcal{X}$ is closed and conic. For example, the domain set $\mathcal{X}$ of QCQP only has a rank-1 constraint and is thus conic. In this case, the following properties of the closed convex hull of set $\mathcal{X}$ hold, i.e.,

$$\text{conv}(\mathcal{X}) = \text{conv}(\mathcal{X}) = S_n^+, \quad \text{rec}(\text{conv}(\mathcal{X})) = \text{conv}(\mathcal{X}),$$

which simplifies the sufficient condition in Theorem 3 to be $F_{\tilde{m}+1}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$.

**Theorem 4** Suppose that the closed domain set $\mathcal{X}$ is conic, i.e., for any $\alpha \geq 0$ and $X \in \mathcal{X}$, we have $\alpha X \in \mathcal{X}$. Then the followings are equivalent:

(a) **Inclusive Face:** Any no larger than $(\tilde{m}+1)$-dimensional face of set $\text{conv}(\mathcal{X})$ is contained in the domain set $\mathcal{X}$, i.e., $F_{\tilde{m}+1}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$;

(b) **Convex Hull Exactness:** The feasible set $\mathcal{C}_{\text{rel}}$ is equal to the closed convex hull of set $\mathcal{C}$ for any $m$ linear constraints of dimension $\tilde{m}$ in RCOP (1), i.e., $\mathcal{C}_{\text{rel}} = \text{conv}(\mathcal{C})$.

**Proof.** Since $\text{conv}(\mathcal{X})$ is a closed convex cone, we have that $F_{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq F_{\tilde{m}+1}(\text{conv}(\mathcal{X}))$ and $\text{rec}(\text{conv}(\mathcal{X})) = \text{conv}(\mathcal{X})$. Thus, $F_{\tilde{m}}(\text{conv}(\mathcal{X})) + F_{\tilde{m}+1}(\text{conv}(\mathcal{X})) = F_{\tilde{m}+1}(\text{conv}(\mathcal{X}))$. According to Theorem 3, it remains to show the necessity of the condition $F_{\tilde{m}+1}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$. That is, suppose that set $\mathcal{C}_{\text{rel}}$ achieves the convex hull exactness for any $m$ linear constraints of dimension $\tilde{m}$ in RCOP (1). According to Theorem 1, we must have $F_{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$. Next we show that set $\mathcal{X}$ contains all $(\tilde{m}+1)$-dimensional faces in its convex hull by contradiction.

Suppose there is an $(\tilde{m}+1)$-dimensional face $F$ of $\text{conv}(\mathcal{X})$ that is not contained in $\mathcal{X}$. Since the $\text{conv}(\mathcal{X})$ is a closed convex cone, given $F \not\subseteq \mathcal{X}$, there must exist a nonzero direction $D \in F$ that is not included by $\mathcal{X}$, i.e., $\alpha D \not\in \mathcal{X}$ for all $\alpha > 0$.

Since $\dim(F) = \tilde{m}+1$ and $\text{conv}(\mathcal{X}) \subseteq Q$, we have $\dim(Q) \geq \tilde{m}+1$ and there exist $\tilde{m}$ independent matrices $\{A_i\}_{i \in [m]}$ to construct $\mathcal{H} := \{X \in Q : \langle A_i, X \rangle = 0, \forall i \in [m] \}$ such that the intersection $F \cap \mathcal{H} := \{\alpha D : \forall \alpha \geq 0\}$ is a ray. Now we intersect these $m$ linear equalities of dimension $\tilde{m}$ with
sets $\mathcal{X}$ and $\overline{\mathcal{C}}$ to obtain $\mathcal{C}$ and $\mathcal{C}_{\text{rel}}$, respectively, i.e., $\mathcal{C} = \mathcal{H} \cap \mathcal{X}$ and $\mathcal{C}_{\text{rel}} = \mathcal{H} \cap \overline{\mathcal{C}}$. Since $\mathbf{D}$ is an extreme direction of set $\mathcal{F} \cap \mathcal{H}$, $\mathbf{D}$ is also an extreme direction of set $\mathcal{C}_{\text{rel}}$.

However, this extreme direction does not belong to set $\overline{\mathcal{C}}$ since it does not belong to set $\mathcal{X}$. This contradicts that $\mathcal{C}_{\text{rel}} = \overline{\mathcal{C}}$ and completes the proof. $\square$

We remark that the above necessary and sufficient condition can be used to show the convex hull exactness for the QCQP, in which the domain set is defined as $\mathcal{X} := \{ \mathbf{X} \in \mathbb{S}^n_+ : \text{rank}(\mathbf{X}) \leq 1 \}$ and thus conic. For Example 2, as a special case of the set $\mathcal{X}$ of the QCQP with $n = 2$ in which the domain set $\mathcal{X}$ contains any no larger than two-dimensional face of $\overline{\mathcal{C}}$, the convex hull exactness fails $\mathcal{C}_{\text{rel}} \neq \overline{\mathcal{C}}$ in Example 2 since there are $\tilde{m} = 2$-dimensional linear constraints. Below we show that if there is $m = \tilde{m} = 1$ linear constraint, then according to Theorem 4, DWR (3) must achieve the convex hull exactness.

**Example 5** Let us consider an unbounded set $\mathcal{C}_{\text{rel}}$ built on the same domain set $\mathcal{X} := \{ \mathbf{X} \in \mathbb{S}^2_+ : \text{rank}(\mathbf{X}) \leq 1 \}$ as Example 2 and $m = \tilde{m} = 1$ linear constraint, which provide

$$
\mathcal{C} := \{ \mathbf{X} \in \mathbb{S}^2_+ : X_{12} \leq 0, \text{rank}(\mathbf{X}) \leq 1 \}, \quad \mathcal{C}_{\text{rel}} := \{ \mathbf{X} \in \mathbb{S}^2_+ : X_{12} \leq 0 \}.
$$

We see that set $\mathcal{X}$ satisfies $\mathcal{F}^2(\overline{\mathcal{C}}) \subseteq \mathcal{X}$ as indicated in Example 2. When intersecting set $\mathcal{X}$ with a linear constraint $X_{12} \leq 0$, we have that $m = \tilde{m} = 1$. According to Theorem 4, we must have convex hull exactness, i.e., $\overline{\mathcal{C}} = \mathcal{C}_{\text{rel}}$. The resulting set $\mathcal{C}$ is exactly the bottom surface of domain set $\mathcal{X}$, as marked in red in Figure 7(a). Furthermore, the red area in Figure 7(b) illustrates the corresponding set $\mathcal{C}_{\text{rel}}$. It is seen that (i) both $\mathcal{C}$ and $\mathcal{C}_{\text{rel}}$ are unbounded; (ii) the convex hull of set $\mathcal{C}$ is half closed and half open, as $X_{12}$ cannot attain zero; and (iii) the closed convex hull of set $\mathcal{C}$ is equal to $\mathcal{C}_{\text{rel}}$. $\diamond$

![Figure 7](image)

**Figure 7** Illustration of Sets in Example 5 with $\mathcal{X} := \{ \mathbf{X} \in \mathbb{S}^2_+ : \text{rank}(\mathbf{X}) \leq 1 \}$ and we have $\overline{\mathcal{C}} = \mathcal{C}_{\text{rel}}$.

The proposed necessary and sufficient conditions in this subsection have revealed a significant connection between the DWR exactness and the faces of set $\overline{\mathcal{C}}$. The next subsection demonstrates the DWR exactness using these conditions in some important applications.
2.4. Application of Our Proposed Exactness Conditions to QCQP (5)

As an important special case of our RCOP (1), this subsection investigates extreme point exactness and convex hull exactness of DWR for several QCQP problems. We first study what faces of the closed convex hull (i.e., $\overline{\text{conv}}(X)$) are included in the domain set $\mathcal{X}$ of QCQP (5). It is important to note that for a particular application example, we specify their related sets $\mathcal{X}, \overline{\text{conv}}(\mathcal{X}), \mathcal{C}, \mathcal{C}_{rel}$.

As mentioned in Section 2.4, the domain set of QCQP only involves a rank-one constraint, i.e., $\mathcal{X} := \{X \in S_{++}^{n+1} : \text{rank}(X) \leq 1\}$, whose convex hull is closed and line-free and exactly equal to the positive semidefinite cone, i.e., $\overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) := S_{++}^{n+1}$. The domain set $\mathcal{X}$ in Example 2 is in fact a special case of QCQP with $n+1 = 2$. Note that the set $\mathcal{X}$ in Example 2 contains all one- or two–dimensional faces of set $\overline{\text{conv}}(\mathcal{X})$ as shown in Figure 4(a) and Figure 4(b). Interestingly, this result can be extended to any solely rank-$k$ constrained positive semidefinite matrix set as below.

It is worth of mentioning that our proof idea is different from that of Pataki (1998), since we focus on the faces of set $\overline{\text{conv}}(\mathcal{X})$ and we deal with the linear constraints.

**Lemma 4** Suppose that $\mathcal{X} = \{X \in S_{++}^{n+1} : \text{rank}(X) \leq k\}$, i.e., $\mathcal{Q} = S_{++}^{n+1}$ and $t = 0$ in (2), then we have that $\overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) := S_{++}^{n+1}$ and any no larger than $\frac{k(k+3)}{2}$-dimensional face of $\overline{\text{conv}}(\mathcal{X})$ is contained in $\mathcal{X}$, i.e., $\mathcal{F}_{\frac{k(k+3)}{2}}(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$.

**Proof.** The equivalence of $\overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) := S_{++}^{n+1}$ is because any positive semi-definite matrix can be viewed as a convex combination of multiple rank-one matrices. Now we are ready to prove the facial inclusion result.

First, it is known that any extreme point of set $\overline{\text{conv}}(\mathcal{X})$ belongs to set $\mathcal{X}$ which is closed. Let us denote by $F$ a face of $\overline{\text{conv}}(\mathcal{X})$ with dimension $d \leq \frac{k}{2}(k + 3)$. Suppose $F$ is not contained in set $\mathcal{X}$, then there exists a matrix $\hat{X} \in F$ with rank $r$ such that $r > k$, i.e., $\hat{X} \notin \mathcal{X}$. Let $\hat{X} = Q\Lambda Q^T$ denote the eigen-decomposition of matrix $\hat{X}$ where $\Lambda \in S_{++}^r$ is a positive-definite diagonal matrix and $Q \in \mathbb{R}^{(n+1) \times r}$ is rank-$r$.

Since face $F$ is of dimension $d$, there are $d+1$ distinct and linearly independent points in $F$, denoted by $\{X_i\}_{i \in [d+1]}$ such that

$$F \subseteq X_1 + \text{span}\{X_1 - X_2, X_1 - X_3, \cdots, X_1 - X_{d+1}\} = \text{aff}(X_1, X_2, \cdots, X_{d+1}).$$

In addition, given $r \geq k+1$, we have $\frac{r}{2}(r + 1) \geq \frac{k+1}{2}(k + 2) > \frac{k}{2}(k + 3) \geq d$; hence, there is a nonzero symmetric matrix $\Delta \in S^r$ satisfying

$$\langle \Delta, Q^T(X_1 - X_i)Q \rangle = 0, \forall i \in [2, d+1],$$

which means that $Q\Delta Q^T \neq 0$ is a matrix orthogonal to the face $F$, i.e., $Q\Delta Q^T \perp F$. 
Then let us construct two matrices $X^+$ and $X^-$ as below
\[
X^+ := \hat{X} + \delta Q \Delta Q^\top, \quad X^- := \hat{X} - \delta Q \Delta Q^\top,
\]
where $\delta > 0$. It is clear that $X^+$ and $X^-$ have nonzero eigenvalues identical to $\Lambda + \delta \Delta$ and $\Lambda - \delta \Delta$, respectively. Since $\Lambda \in S_n^{++}$, we can always find $\delta$ small enough such that $X^+, X^- \in \text{conv}(\mathcal{X})$.

According to Definition 1 of the face $F$, we conclude that $X^+, X^- \in F$ as $cX^+ = 1/2X^++1/2X^-$, which contradicts with the fact that $Q \Delta Q^\top \perp F$. Therefore, any face of dimension no larger than $k_2(k+3)$ in $\text{conv}(X)$ belongs to $\mathcal{X}$. □

The facial inclusion result of Lemma 4 enables us to develop sufficient conditions for the extreme point exactness and convex hull exactness of DWR (3) for QCQP with a rank-$k = 1$ constraint; more importantly, if the extreme point does not hold, we can derive a upper bound for the rank among all the extreme points in the feasible set $C_{\text{rel}}$. These results are summarized below.

\textbf{Theorem 5} Suppose that $\mathcal{X} := \{X \in S_n^{n+1}: \text{rank}(X) \leq 1\}$, then we have

(i) If $\tilde{m} \leq 2$, DWR (3) admits extreme point exactness;

(ii) If $\tilde{m} \leq 1$, DWR (3) admits the convex hull exactness;

(iii) If $\tilde{m} \geq 3$, then each extreme point in set $C_{\text{rel}}$ has a rank $r^*$ satisfying $r^*(r^*+1) \leq 2\tilde{m}$.

\textbf{Proof.} The proof includes two parts.

(i) According to Part (i) in Lemma 4 with $k = 1$ and Theorem 1, we have the extreme point exactness.

(ii) Following the sufficient condition in Theorem 3, the convex hull exactness holds when $\tilde{m} \leq 1$.

(iii) We prove the rank bound by contradiction. Suppose there is an extreme point $\hat{X} \in \text{ext}(C_{\text{rel}})$ such that $r^*(r^*+1) > 2\tilde{m} \iff \tilde{m} \leq \frac{r^*-1}{2}(r^*+2)$. According to Lemma 1, the extreme point $\hat{X}$ is contained in a face of set $\text{conv}(\mathcal{X})$ with dimension at most $\tilde{m} \leq \frac{r^*-1}{2}(r^*+2)$. Using Part (i) in Lemma 4 with $k = r^*-1$, the extreme point $\hat{X}$ should have rank $r^*-1$, a contradiction. □

Surprisingly, the exactness results in Part (i) and Part (ii) of Theorem 5 do not require any additional assumption, and they hold for general QCQP. Besides, the rank bound in Part (iii) of Theorem 5 recovers the classical result in QCQP, which has been independently proved by Barvinok (1995), Pataki (1998). Since set $C_{\text{rel}}$ for QCQP contains no line, the extreme point exactness for QCQP implies the objective exactness for any linear objective function with $V_{\text{rel}} > -\infty$.

Many results have been developed for the DWR exactness of QCQP (5) with one or two quadratic constraints. Our Theorem 5 with minor extra efforts can immediately recover or generalize those results and our proof does not rely on strong duality or Slater condition. For the sake of page limit, we show the DWR exactness for QCQP with one quadratic constraint (QCQP-1) and homogeneous
QCQP with two quadratic constraints (HQP-2) for the demonstrative purpose. Beyond these known results, our Theorem 5 can also prove a new exactness result for inhomogeneous QCQP with two independent homogeneous constraints (IQP-2).

**QCQP with One Constraint (QCQP-1).** Let us first focus on the QCQP-1

$$(\text{QCQP-1}) \min_{x \in \mathbb{R}^n} \{ x^\top Q_0 x + q_0^\top x + b : b_1^l \leq x^\top Q_1 x + q_1^\top x \leq b_1^u \},$$

which can be formulated as a special case of our RCOP (1) with two linearly dependent constraints (i.e., $\tilde{m} = 2$) as below

$$\min_{X \in \mathcal{X}} \{ \langle A_0, X \rangle : b_1^l \leq \langle A_1, X \rangle \leq b_1^u, X_{11} = 1 \}, \quad \mathcal{X} := \{ X \in S_n^{++} : \text{rank}(X) \leq 1 \},$$

where $A_0 = \begin{pmatrix} b & q_0^\top/2 \\ q_0/2 & Q_0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & q_1^\top/2 \\ q_1/2 & Q_1 \end{pmatrix}$. Hence, according to Part (i) of our Theorem 5, we have the following conclusion for QCQP-1.

**Corollary 3** For QCQP-1, its corresponding DWR admits the extreme point exactness.

QCQP-1 covers many important and challenging quadratic optimization problems such as trust region subproblem and its variants, which have attracted much attention in various applications from different domains such as robust optimization (Ben-Tal et al. 2009), regularization problem (e.g., ridge regression) (Hoerl and Kennard 1970, Tikhonov and Arsenin 1977, Xie and Deng 2020), and forming subproblems in signal preprocessing (Huang and Sidiropoulos 2016). Beyond the objective exactness that has been mainly studied in literature, we show that these special cases of QCQP-1 may possess the extreme point exactness and convex hull exactness, which deserves further discussion in the following.

**Trust Region Subproblem (TRS).** The classical TRS, a special case of QCQP-1, is to minimize a quadratic objective over a ball ($x^\top x \leq 1$), where TRS arises naturally in trust region methods for nonlinear programming (Conn et al. 2000). Albeit being nonconvex, the TRS problem is known to achieve DWR objective exactness and strong duality (see a survey Pólik and Terlaky 2007). Recently, Burer (2015) also explicitly described the convex hull of feasible set of the TRS based on second-order cone. Our result of QCQP-1 in Corollary 3 implies that the TRS problem has convex hull exactness given the bounded feasible set.

**Corollary 4 (TRS)** The convex hull exactness holds for the TRS problem.

**Proof.** Since the DWR of TRS problem has a bounded feasible set, thus the convex hull exactness follows from the equivalence between Part (b) and Part (c) in Theorem 2. \qed
Generalized TRS (GTRS). Replacing the ball constraint in TRS by an arbitrary one-sided quadratic inequality constraint (i.e., $x^TQ_1x + q_1^Tx \leq b_1$) leads to the classic GTRS problem:

\[
\text{(GTRS)} \quad \min_{\mathbf{x} \in \mathcal{A}} \{ \langle \mathbf{A}, \mathbf{X} \rangle : \langle \mathbf{A}_1, \mathbf{X} \rangle \leq b_1, X_{11} = 1 \}, \quad \mathcal{X} := \{ \mathbf{X} \in \mathcal{S}^{n+1} : \text{rank}(\mathbf{X}) \leq 1 \}, \tag{7}
\]

which satisfies $\tilde{m} = 2$ and thus admits the extreme point exactness as a special of QCQP-1. Note that the set $\mathcal{C}_{\text{rel}}$ corresponding to GTRS (7) can be unbounded, which often results in the failure of convex hull exactness; see, e.g., Example 2 and Example 5. On the other hand, the special linear constraint $X_{11} = 1$ inspires us to prove that its corresponding DWR of GTRS admits the convex hull exactness in the corollary below. This is a rather unexpected result. It is worth of mentioning that our result strengthens the one in Kılınç-Karzan and Wang (2021) which relying on the assumption that the dual of DWR of the GTRS is strictly feasible.

**Corollary 5 (GTRS)** *The convex hull exactness holds for the GTRS problem.*

**Proof.** For the GTRS problem (7), its corresponding sets $\mathcal{C}$ and $\mathcal{C}_{\text{rel}}$ are equal to

\[
\mathcal{C} := \{ \mathbf{X} \in \mathcal{S}^{n+1} : \text{rank}(\mathbf{X}) \leq 1, X_{11} = 1, \langle \mathbf{A}_1, \mathbf{X} \rangle \leq b_1 \}, \quad \mathcal{C}_{\text{rel}} := \{ \mathbf{X} \in \mathcal{S}^{n+1} : X_{11} = 1, \langle \mathbf{A}_1, \mathbf{X} \rangle \leq b_1 \},
\]

where $\mathbf{A}_1 = \begin{pmatrix} 0 & Q_1^T/2 \\ q_1/2 & Q_1 \end{pmatrix}$.

According to representation theorem, it suffices to prove that $\mathcal{C}_{\text{rel}}$ and $\overline{\text{conv}}(\mathcal{C})$ have the same extreme points and recession cones in order to show their equivalence. They always have the same extreme points based on Theorem 5. This also implies that if $\overline{\text{conv}}(\mathcal{C})$ is empty, then $\mathcal{C}_{\text{rel}}$ must be also empty. Thus, without loss of generality, we can assume both sets are nonempty and unbounded. Hence, it remains to show that these two sets share the same recession cone. To prove this statement, we first observe that the recession cone of $\mathcal{C}_{\text{rel}}$ is equal to

\[
\text{rec}(\mathcal{C}_{\text{rel}}) = \{ \mathbf{X} \in \mathcal{S}^{n+1}_+ : X_{11} = 0, \langle \mathbf{A}, \mathbf{X} \rangle \leq 0 \} = \left\{ \mathbf{X} \in \mathcal{S}^{n+1}_+ : \exists \mathbf{Y} \in \mathcal{S}^n_+, \mathbf{X} = \begin{pmatrix} 0 & 0^T \\ 0 & \mathbf{Y} \end{pmatrix}, \langle Q_1, \mathbf{Y} \rangle \leq 0 \right\},
\]

where the second equation is due to the fact that $X_{1j} = X_{j1} = 0$ for all $j \in [n+1]$. According to Part (ii) in Theorem 5, the set $\text{rec}(\mathcal{C}_{\text{rel}})$ satisfies $\tilde{m} = 1$ and thus we have

\[
\text{rec}(\mathcal{C}_{\text{rel}}) = \overline{\text{conv}} \left( \left\{ \mathbf{X} \in \mathcal{S}^{n+1}_+ : \exists \mathbf{Y} \in \mathcal{S}^n_+, \mathbf{X} = \begin{pmatrix} 0 & 0^T \\ 0 & \mathbf{Y} \end{pmatrix}, \text{rank}(\mathbf{Y}) \leq 1, \langle Q_1, \mathbf{Y} \rangle \leq 0 \right\} \right) = \overline{\text{conv}} \left( \left\{ \mathbf{X} \in \mathcal{S}^{n+1}_+ : \text{rank}(\mathbf{X}) \leq 1, \langle \mathbf{A}_1, \mathbf{X} \rangle \leq 0, X_{11} = 0 \right\} \right).
\]

Next we show that any rank-one direction in recession cone $\text{rec}(\mathcal{C}_{\text{rel}})$ is also a direction in set $\overline{\text{conv}}(\mathcal{C})$ which implies the recession cone equivalence. Let us denote by $\mathbf{D} := \begin{pmatrix} 0 & 0^T \\ 0 & \mathbf{y} \mathbf{y}^T \end{pmatrix}$ a rank-one direction in $\text{rec}(\mathcal{C}_{\text{rel}})$ where $\mathbf{y} \neq 0 \in \mathbb{R}^n$. Then we must have $\langle Q_1, \mathbf{y} \mathbf{y}^T \rangle \leq 0$. There are two cases to be discussed depending on whether $\langle Q_1, \mathbf{y} \mathbf{y}^T \rangle - |q_1^T \mathbf{y}|$ equals zero or not.
(i) Suppose that \( \langle Q_1, yy^\top \rangle - |q_1^\top y| \neq 0 \). Since matrix \( D \) is sign-invariant with respect to \( y \), we can let \( y := -y \) if \( q_1^\top y > 0 \). Thus, WLOG, we always have \( \langle Q_1, yy^\top \rangle + q_1^\top y < 0 \). Then there is a scalar \( \bar{\gamma} > 0 \) such that for any \( \gamma \geq \bar{\gamma} \geq 1 \), we have

\[
\left\langle A_1, \begin{pmatrix} 1 \gamma y \gamma^2 yy^\top \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \ q_1^\top / 2 \ q_1 \end{pmatrix}, \begin{pmatrix} 1 \gamma y \gamma^2 yy^\top \end{pmatrix} \right\rangle \leq \gamma \left( \langle Q_1, yy^\top \rangle + q_1^\top y \right) \leq b_1, \tag{8}
\]

where the first inequality is due to \( \langle Q_1, yy^\top \rangle \leq 0 \).

Let us define a matrix \( \hat{X} := \begin{pmatrix} 1 \ 2\bar{\gamma} y / \gamma^2 yy^\top \end{pmatrix} \). According to the result in (8), we have \( \hat{X} \in C \).

For any \( \alpha \geq 0 \), matrix \( \hat{X} + \alpha D \) can be written as the following convex combination

\[
\hat{X} + \alpha D = \frac{\alpha}{\alpha + \bar{\gamma}^2} \begin{pmatrix} 1 \gamma y \gamma^2 yy^\top \end{pmatrix} + \frac{\bar{\gamma}^2}{\alpha + \bar{\gamma}^2} \begin{pmatrix} (\alpha + 2\bar{\gamma}^2) / \bar{\gamma} \gamma y \gamma^2 yy^\top \end{pmatrix},
\]

where both rank-one matrices \( X_1, X_2 \) belong to set \( C \) because \( \bar{\gamma}, (\alpha + 2\bar{\gamma}^2) / \bar{\gamma} \geq \bar{\gamma} \), and the inequalities (8) hold. It follows that \( \hat{X} + \alpha D \in \overline{\text{conv}}(C) \) for any \( \alpha \geq 0 \), implies that \( D \) is also a recession direction of \( \overline{\text{conv}}(C) \).

(ii) Suppose that \( \langle Q_1, yy^\top \rangle - |q_1^\top y| = 0 \), then we have \( q_1^\top y = \langle Q_1, yy^\top \rangle = 0 \). Then let us consider two following subcases depending on whether \( Q_1 y = 0 \) holds or not.

a) Suppose that \( Q_1 y = 0 \). Let \( \bar{X} = \begin{pmatrix} \bar{y}^\top \end{pmatrix} \) denote a feasible solution in set \( C \), i.e.,

\[
\left\langle A_1, \bar{X} \right\rangle = \langle Q_1, \bar{y} \bar{y}^\top \rangle + q_1^\top \bar{y} \leq b_1, \text{ then for any } \alpha \geq 0, \text{ we have }
\]

\[
\hat{X} + \alpha D = \frac{1}{2} \begin{pmatrix} 1 \ 1 \end{pmatrix} \begin{pmatrix} \bar{y} + \sqrt{\alpha} y \ (\bar{y} + \sqrt{\alpha} y)(\bar{y} + \sqrt{\alpha} y)^\top \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \ 1 \end{pmatrix} \begin{pmatrix} \bar{y} - \sqrt{\alpha} y \ (\bar{y} - \sqrt{\alpha} y)(\bar{y} - \sqrt{\alpha} y)^\top \end{pmatrix},
\]

where both rank-one matrices \( X_1, X_2 \) above lie in set \( C \) since \( \langle A_1, X_1 \rangle = \langle A_1, X_2 \rangle = \langle A_1, \bar{X} \rangle \). This implies that \( D \) is a recession direction of \( \overline{\text{conv}}(C) \).

b) Suppose that \( Q_1 y \neq 0 \). Then we can decompose \( y = y_1 + y_2 \) such that \( y_1^\top Q_1 y_1 > 0 \), \( y_2^\top Q_1 y_2 < 0 \), and \( y_1^\top Q_1 y_2 = 0 \). Then let us construct a new matrix \( D_\epsilon \in \text{rec}(C_{\text{rel}}) \) for any \( \epsilon > 0 \) as below, which satisfies \( \langle Q_1, (y + \epsilon y_2)(y + \epsilon y_2)^\top \rangle - |q_1^\top (y + \epsilon y_2)| < 0 \) and \( \langle Q_1, (y + \epsilon y_2)(y + \epsilon y_2)^\top \rangle \leq 0 \)

\[
D_\epsilon := \begin{pmatrix} 0 \ y^\top \end{pmatrix} \begin{pmatrix} 0 \ y + \epsilon y_2 \end{pmatrix} \begin{pmatrix} 0 \ y + \epsilon y_2 \end{pmatrix}^\top.
\]

Following the similar proof of Part (i), we can show that for any \( \epsilon > 0 \), matrix \( D_\epsilon \) is a recession direction of \( \overline{\text{conv}}(C) \). Since the recession cone of set \( \overline{\text{conv}}(C) \) is closed, which indicates that the limit \( \lim_{\epsilon \to 0} D_\epsilon = D \) is also a recession direction in set \( \overline{\text{conv}}(C) \).

This completes the proof. \( \square \)
Two-sided GTRS. As an extension of GTRS, the two-sided GTRS problem has a two-sided quadratic constraint (i.e., $-\infty < b_1^l \leq x^\top Q_1 x + q_1^\top x \leq b_1^u < +\infty$), which has been successfully applied to signal processing (see Huang and Palomar 2014 and references therein). Using S-lemma, the objective exactness for the DWR of the two-sided GTRS has been established under Slater assumption (see survey by Wang and Xia 2015 and references therein), which is equivalent to the extreme point exactness. According to Part (i) in our Theorem 5, we can readily derive the first-known DWR extreme point exactness of the two-sided GTRS without assuming Slater condition. Recent work by Joyce and Yang (2021) showed that the two-sided GTRS has the convex hull exactness give that that matrix $Q_1$ in QCQP (5) is nonzero, which can be also recovered by our framework. It should be noted that the convex hull exactness may not hold for the general two-sided GTRS (see Example 2).

**Corollary 6 (Two-Sided GTRS)** For the two-sided GTRS problem, we have

(i) The extreme point exactness holds;

(ii) The convex hull exactness holds when $Q_1 \neq 0$.

**Proof.** Part (i) can be obtained by simply following the proof of QCQP-1 in Corollary 3.

Next let us prove Part (ii). First, the corresponding sets $C$ and $C_{rel}$ for two-sided GTRS are

\[
C := \{ X \in S_{++}^{n+1} : \text{rank}(X) \leq 1, X_{11} = 1, b_1^l \leq \langle A_1, X \rangle \leq b_1^u \},
\]

\[
C_{rel} := \{ X \in S_{++}^{n+1} : X_{11} = 1, b_1^l \leq \langle A_1, X \rangle \leq b_1^u, X_{11} = 0 \},
\]

where $A_1 = \begin{pmatrix} 0 & q_1^\top / 2 \\ q_1 / 2 & Q_1 \end{pmatrix}$. Set $C_{rel}$ has the extreme point exactness by Part (i) in Theorem 5. Following the analysis in Corollary 5, the recession cone of set $C_{rel}$ is equivalent to

\[
\text{rec}(C_{rel}) = \text{conv} \left( \{ X \in S_{++}^{n+1} : \text{rank}(X) \leq 1, \langle A_1, X \rangle = 0, X_{11} = 0 \} \right).
\]

For any rank-one direction $D := \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$ in $\text{rec}(C_{rel})$, according to proposition 3 in Joyce and Yang (2021), $D$ is also a direction in $\text{conv}(C)$ when $Q_1 \neq 0$. Thus, $\text{rec}(C_{rel}) = \text{rec}(\text{conv}(C))$. $\square$

**Homogeneous QCQP with Two Independent Constraints (HQP-2).** Another important special case is the homogeneous QCQP with two independent constraints without linear terms, denoted as HQP-2, which recently has been applied to robust receive beamforming (Khabbazibamenj et al. 2010, Huang and Palomar 2010) and signal processing (Huang and Palomar 2014). The HQP-2 admit the following form

\[
\text{(HQP-2)} \quad \min_{x \in \mathbb{R}^n} \{ x^\top Q_1 x : b_1^l \leq x^\top Q_1 x \leq b_1^u, b_2^l \leq x^\top Q_2 x \leq b_2^u \}. \quad (9)
\]
The HQP-2 is clearly different from the general QCQP-1 since its equivalent rank-one constrained formulation does not have the auxiliary constraint \( X_{11} = 1 \)

\[
\min_{X \in \mathcal{X}} \left\{ \langle A, X \rangle : b_1^l \leq \langle A_1, X \rangle \leq b_1^u, b_2^l \leq \langle A_2, X \rangle \leq b_2^u \right\}, \quad \mathcal{X} := \{ X \in S^n_+ : \text{rank}(X) \leq 1 \},
\]

where \( A = Q_0, A_1 = Q_1, \) and \( A_2 = Q_2. \)

Part (i) in Theorem 5 implies the extreme point exactness of HQP-2. This result has been first proven by Polyak (1998) and years later, Ye and Zhang (2003) used the matrix rank-one decomposition procedure to reprove it. It is worthy of mentioning that both proof relies on the strong duality assumption. In contrast, we provide a quite different proof that manages to relax the strong duality assumption. Note that we may not obtain convex hull exactness of HQP-2 since set \( \mathcal{C}_{\text{rel}} \) can be unbounded and its recession cone can be different from those in \( \overline{\text{conv}}(\mathcal{C}) \) (see, e.g., the sets \( \mathcal{C}_{\text{rel}} \) in Figure 4 for an illustration).

**Corollary 7 (HQP-2)** For HQP-2, its corresponding DWR admits the extreme point exactness.

**Inhomogeneous QCQP with Two Independent Homogeneous Constraints (IQP-2).**

Our results can be also applied to a QCQP problem that minimize an inhomogeneous quadratic function over the intersection of two homogeneous constraints (IQP-2), i.e.,

\[
(IQP-2) \quad \min_{x \in \mathbb{R}^n} \left\{ x^\top Q_0 x + q_0^\top x : b_1^l \leq x^\top Q_1 x \leq b_1^u, b_2^l \leq x^\top Q_2 x \leq b_2^u \right\},
\]

Different from HQP-2, the DWR of IQP-2 cannot get rid of the additional constraint \( X_{11} = 1 \) due to the extra linear term in the objective function of IQP-2 (10), resulting in \( \tilde{m} = 3 \) independent constraints. For a general QCQP with \( \tilde{m} \geq 3 \), the corresponding DWR may not have extreme point exactness as shown in Part (iii) of Theorem 5. However, we show that the extreme point exactness still holds for IQP-2. This surprising result may be due to the \( X_{11} = 1 \) constraint.

**Corollary 8 (IQP-2)** For IQP-2, its corresponding DWR admits the extreme point exactness.

Proof. The feasible set \( \mathcal{C}_{\text{rel}} \) of the DWR of IQP-2 problem is equal to

\[
\mathcal{C}_{\text{rel}} = \left\{ X \in S_+^{n+1} : X_{11} = 1, b_1^l \leq \langle A_1, X \rangle \leq b_1^u, b_2^l \leq \langle A_2, X \rangle \leq b_2^u \right\}
= \left\{ \begin{pmatrix} 1 & y^\top \\ y & Y \end{pmatrix} : Y \in S_+^{n}, Y \succeq yy^\top, b_1^l \leq \langle Q_1, Y \rangle \leq b_1^u, b_2^l \leq \langle Q_2, Y \rangle \leq b_2^u \right\},
\]

where \( A_1 = \begin{pmatrix} 0 & 0^\top \\ 0 & Q_1 \end{pmatrix} \) and \( A_2 = \begin{pmatrix} 0 & 0^\top \\ 0 & Q_2 \end{pmatrix} \). Next we show by contradiction that any extreme point in set \( \mathcal{C}_{\text{rel}} \) is rank-one and belongs to set \( \mathcal{C} \), i.e., \( Y = yy^\top \) holds. The proof includes two steps below.
(i) Suppose that the extreme point \( \left( \frac{1}{\sqrt{r}} y^\top Y \right) \in \text{ext}(C_{\text{rel}}) \) has a rank-\( r \) submatrix \( Y \) and \( r \geq 2 \). Then we have \( \frac{r}{2} (r + 1) \geq 3 \) and \( Y \succeq yy^\top \) and \( Y \neq yy^\top \). Following the proof of Lemma 4, there is a symmetric matrix \( Y' \in S^n \) such that \( \left( \frac{1}{\sqrt{r}} y^\top Y \pm \delta Y' \right) \in C_{\text{rel}} \) for some small \( \delta > 0 \), which indicates that any extreme point \( \left( \frac{1}{\sqrt{r}} y^\top Y \right) \in \text{ext}(C_{\text{rel}}) \) has a rank-one matrix \( Y \).

(ii) Suppose \( \left( \frac{1}{\sqrt{r}} y^\top Y \right) \in \text{ext}(C_{\text{rel}}) \) has a rank-one submatrix \( Y \) but \( Y \succeq yy^\top \) and \( Y \neq yy^\top \). Then \( Y \) can be written as \( \delta yy^\top \) where \( \delta > 1 \). Since the linear constraints are independent of the variable \( y \), then two matrices that belong to set \( C_{\text{rel}} \) can be readily constructed as below

\[
\left( \begin{array}{c}
1 \\
\sqrt{\delta} y & Y
\end{array} \right), \left( \begin{array}{c}
1 \\
0 & Y
\end{array} \right),
\]

whose convex combination includes this extreme point. A contradiction.

Therefore, any extreme point \( \left( \frac{1}{\sqrt{r}} y^\top Y \right) \) in set \( C_{\text{rel}} \) satisfies \( Y = yy^\top \) and belongs to \( C \). This completes the proof. \( \square \)

3. An Optimality View of DWR Exactness: Necessary and Sufficient Conditions for Objective Exactness

The objective exactness (i.e., \( V_{\text{opt}} = V_{\text{rel}} \)) is another common way to show whether DWR (3) matches the original RCOP (1) (see, e.g., Azuma et al. 2022, Burer and Ye 2020, Kılınç-Karzan and Wang 2021). The main difference of objective exactness from the previously ones is that it depends on a given linear objective function in RCOP (1). To illustrate this difference, let us review Example 1, where two optimal values \( V_{\text{opt}} = V_{\text{rel}} = -0.5 \) coincide with the objective function being \(-X_{11}\), while the extreme point exactness and convex hull exactness do not hold. Therefore, in this section, we study the objective exactness of DWR (3) for any \( m \) linear constraints of dimension \( \tilde{m} \) concerning four favorable families of linear objective functions in RCOP (1) specified as follows and derive their corresponding necessary and sufficient conditions.

**Setting I.** Any linear objective function;

**Setting II.** Any linear objective function such that \( V_{\text{opt}} > -\infty \);

**Setting III.** Any linear objective function such that \( V_{\text{opt}} > -\infty \), the set of optimal solutions is bounded, and the binding constraints are of dimension \( m \);

**Setting IV.** Any linear objective function such that the relaxed Slater condition holds, \( V_{\text{opt}} > -\infty \), the set of optimal solutions is bounded, and there are \( m^* \) nonzero optimal Lagrangian multipliers corresponding to the optimal DWR.

Note that \( m \) and \( m^* \) are used throughout this section to indicate the number of linearly independent matrices from binding constraints of DWR (3) and the smallest number of nonzero Lagrangian...
multipliers among all the optimal DWR dual solutions. For comparison purpose, the four key notations of the linear constraints in DWR (3) are listed as Table 2. Please note that we will show $m^* \leq m \leq \tilde{m} \leq m$.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>the number of linear constraints</td>
</tr>
<tr>
<td>$\tilde{m}$</td>
<td>dimension of technology matrices in all $m$ linear constraints</td>
</tr>
<tr>
<td>$m$</td>
<td>dimension of technology matrices in binding constraints</td>
</tr>
<tr>
<td>$m^*$</td>
<td>the number of nonzero optimal Lagrangian multipliers</td>
</tr>
</tbody>
</table>

Table 2 Notations about linear constraints in DWR (3).

As illustrated in Figure 1, the objective exactness under setting (I) is equivalent to the convex hull exactness; and given a line-free set $C_{\text{rel}}$, the objective exactness under setting (II) is equivalent to the extreme point exactness, which, based on the results in previous section, can directly give rise to the necessary and sufficient conditions for objective exactness. The remaining two settings (III) and (IV) focus on two special yet intriguing families of linear objective functions from analyzing primal and dual perspectives, respectively. It is important to note that although the necessary and sufficient conditions for objective exactness under settings (III) and (IV) build on some assumptions of the linear constraints of DWR (3), our results can at once cover and extend the relevant existing ones in the celebrated papers (Ye and Zhang 2003, Ben-Tal and Den Hertog 2014).

3.1. Objective Exactness Under Settings (I) and (II): Necessary and Sufficient Conditions

This subsection presents necessary and sufficient conditions for objective exactness of DWR (3) under settings (I) and (II) using their equivalence to convex hull exactness and extreme point exactness, respectively.

A natural extension of the convex hull exactness result in Theorem 3 is providing a sufficient condition of the objective exactness for any linear objective function as shown below.

**Theorem 6** Given a nonempty closed domain set $\mathcal{X}$, the following statement (a) implies (b).
(a) **Inclusive Face:** The (Minkowski) sum of any no larger than $\tilde{m}$-dimensional face of set $\text{conv}(\mathcal{X})$ and any no larger than $\tilde{m} + 1$-dimensional face of the recession cone $\text{rec}(\text{conv}(\mathcal{X}))$ is contained in the domain set $\mathcal{X}$, i.e., $\mathcal{F}^{\tilde{m}}(\text{conv}(\mathcal{X})) + \mathcal{F}^{\tilde{m}+1}(\text{rec}(\text{conv}(\mathcal{X}))) \subseteq \mathcal{X}$;  
(b) **Objective Exactness:** The DWR (3) has the same optimal value as RCOP (1) (i.e., $V_{\text{opt}} = V_{\text{rel}}$) for any linear objective function and any $m$ linear constraints of dimension $\tilde{m}$.

**Proof.** The objective exactness for any linear objective function is equivalent to the convex hull exactness and thus completes the proof. □
Besides, following the analysis in Theorem 4, the sufficient condition in Theorem 6 becomes necessary when set $\mathcal{X}$ being closed and conic.

**Theorem 7** Suppose that the domain set $\mathcal{X}$ is closed and conic, i.e., for any $\alpha \geq 0$ and $X \in \mathcal{X}$, we have $\alpha X \in \mathcal{X}$. Then the followings are equivalent.

(a) **Inclusive Face**: Any no larger than $(\tilde{m} + 1)$–dimensional face of set $\text{conv}(\mathcal{X})$ is contained in the domain set $\mathcal{X}$, i.e., $\mathcal{F}^{\tilde{m} + 1}(\text{conv}(\mathcal{X})) \subset \mathcal{X}$;

(b) **Objective Exactness**: The DWR (3) has the same optimal value as problem (1) (i.e., $V_{\text{opt}} = \mathcal{V}_{\text{rel}}$) for any linear objective function and any $m$ linear constraints of dimension $\tilde{m}$.

We remark that since the domain set $\mathcal{X} = \{X \in S_n^+ : \text{rank}(X) \leq 1\}$ of the QCQP (6) is closed and conic, according to Theorem 7, its DWR can yield the same optimal value for any linear objective function and $\tilde{m}$-dimensional linear constraints if and only if $\mathcal{F}^{\tilde{m} + 1}(\text{conv}(\mathcal{X})) \subset \mathcal{X}$ holds.

Let us now consider the objective exactness of the DWR with finite optimal value, i.e., $\mathcal{V}_{\text{rel}} > -\infty$. We show that when set $\mathcal{C}_{\text{rel}}$ is line-free, the objective exactness under this setting is equivalent to the extreme point exactness, i.e., the following necessary and sufficient condition holds.

**Theorem 8** Given a nonempty closed domain set $\mathcal{X}$, the followings are equivalent.

(a) **Inclusive Face**: Any no larger than $\tilde{m}$–dimensional face of set $\text{conv}(\mathcal{X})$ is contained in the domain set $\mathcal{X}$, i.e., $\mathcal{F}^{\tilde{m}}(\text{conv}(\mathcal{X})) \subset \mathcal{X}$;

(b) **Objective Exactness**: The DWR (3) has the same optimal value as problem (1) (i.e., $V_{\text{opt}} = \mathcal{V}_{\text{rel}}$) for any linear objective function such that $\mathcal{V}_{\text{rel}} > -\infty$ and any $m$ linear constraints of dimension $\tilde{m}$ such that set $\mathcal{C}_{\text{rel}}$ contains no line.

**Proof.** According to Theorem 1, we have that statement $(a) \iff$ extreme point exactness, so it suffices to show that extreme point exactness $\iff$ objective exactness for any linear objective function such that $\mathcal{V}_{\text{rel}} > -\infty$ given a line-free set $\mathcal{C}_{\text{rel}}$. According to the theorem 18.5 in Rockafellar (1972), a closed, convex, line-free set can be represented as sum of a convex combination of extreme points and a conic combination of extreme directions. Next, we divide the proof into two parts.

(i) $\text{ext}(\mathcal{C}_{\text{rel}}) \subset \mathcal{C} \implies V_{\text{opt}} = \mathcal{V}_{\text{rel}} > -\infty$. Given an objective function $\langle A_0, X \rangle$, since $V_{\text{opt}} \geq V_{\text{rel}} > -\infty$, we have $\langle A_0, D \rangle \geq 0$ for any $D \in \text{rec}(\mathcal{C}_{\text{rel}})$ and $\langle A_0, D \rangle \geq 0$ for any $D \in \text{rec}(\text{conv}(\mathcal{C}))$. In addition, both sets $\mathcal{C}_{\text{rel}}$ and $\text{conv}(\mathcal{C})$ are line-free. Thus, it suffices to restrict RCOP (1) and its DWR (3) as $V_{\text{opt}} = \min_X \{\langle A_0, X \rangle : X \in \text{ext}(\text{conv}(\mathcal{C}))\}$ and $\mathcal{V}_{\text{rel}} = \min_X \{\langle A_0, X \rangle : X \in \text{ext}(\mathcal{C}_{\text{rel}})\}$. Since $\text{ext}(\mathcal{C}_{\text{rel}}) \subset \mathcal{C}$, we must have $V_{\text{opt}} = \mathcal{V}_{\text{rel}} > -\infty$.

(ii) $V_{\text{opt}} = \mathcal{V}_{\text{rel}} > -\infty \implies \text{ext}(\mathcal{C}_{\text{rel}}) \subset \mathcal{C}$. For any exposed point $\hat{X}$ in set $\mathcal{C}_{\text{rel}}$, there exists a supporting hyperplane $\{X \in \text{aff}(Q) : \langle \hat{A}_0, X \rangle = \mathcal{V}_{\text{rel}} > -\infty\}$ of set $\mathcal{C}_{\text{rel}}$ which only intersects set $\mathcal{C}_{\text{rel}}$ at $\hat{X}$ and satisfies $\langle \hat{A}_0, X \rangle > \mathcal{V}_{\text{rel}}$ for any $X \in \mathcal{C}_{\text{rel}}$ and $X \neq \hat{X}$. Therefore, by setting
the linear objective function \( \langle \hat{A}_0, X \rangle \) in RCOP (1), \( \hat{X} \) is the unique optimal solution to DWR (3). Since \( V_{\text{opt}} = V_{\text{rel}} > -\infty \) and \( C \subseteq C_{\text{rel}} \), we conclude that \( \hat{X} \in C \).

For a non-exposed point \( \hat{X} \) in the closed convex set \( C_{\text{rel}} \), according to Straszewicz’s theorem in Rockafellar (1972), there exists a sequence of exposed points \( \{X_\ell\}_{\ell=1,2,\ldots,\infty} \) in set \( C_{\text{rel}} \) such that \( \lim_{\ell \to \infty} X_\ell = \hat{X} \). Using the result above, we can show \( \{X_\ell\}_{\ell=1,2,\ldots,\infty} \subseteq C \). Since set \( C \) is closed, we must have \( \hat{X} \in C \) as it is the limit point of a sequence in set \( C \). This proves that \( \text{ext}(C_{\text{rel}}) \subseteq C \). □

We remark that (i) the line-free assumption on set \( C_{\text{rel}} \) in Theorem 8 ensures the non-emptiness of extreme points, which is quite mild. For example, if the domain set \( \mathcal{X} \) in (2) builds on the positive semidefinite matrices (i.e., \( Q = S_+^n \)), the corresponding set \( C_{\text{rel}} \) naturally contains no line.

Example 6 below confirms that if set \( C_{\text{rel}} \) contains a line, the condition in Theorem 8 is insufficient to guarantee the objective exactness; and (ii) the conditions for objective exactness in Theorem 6 - Theorem 8 can be applicable to any closed domain set \( \mathcal{X} \).

**Example 6** Consider the domain set \( \mathcal{X} := \{x \in \mathbb{R}^2 : x_1 \in \{-1, 1\}\} \). Then set \( \text{conv}(\mathcal{X}) = \text{conv}(\mathcal{X}) := \{x \in \mathbb{R}^2 : x_1 \in [-1, 1]\} \). When intersecting with \( m = 1 \) linear constraint \( x_1 \geq 0 \), we obtain sets

\[
C = \{x \in \mathbb{R}^2 : x_1 = 1\}, \quad C_{\text{rel}} = \{x \in \mathbb{R}^2 : x_1 \in [0, 1]\},
\]

as shown the first horizontal red line in Figure 8(c) and \( C_{\text{rel}} \) marked by red hatching area in Figure 8(d), respectively. For this example, we see from Figure 8 that: (i) Any zero- or one-dimensional face of \( \text{conv}(\mathcal{X}) \) is contained in \( \mathcal{X} \), i.e., \( \mathcal{F}^1(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \); (ii) Since \( \text{ext}(C_{\text{rel}}) = \emptyset \), the extreme point exactness trivially holds, i.e., \( \text{ext}(C_{\text{rel}}) \subseteq C \); (iii) Set \( C_{\text{rel}} \) contains lines; and (iv) If we set the objective function in (1) to be \( x_1 \), then \( V_{\text{rel}} = 0 \) is finite; however, the objective exactness fails to hold, \( V_{\text{opt}} = 1 > 0 = V_{\text{rel}} \). Hence, when set \( C_{\text{rel}} \) contains a line, the extreme point exactness is not equivalent to the objective exactness for any linear objective function such that \( V_{\text{rel}} > -\infty \).

Consequently, the condition in Theorem 8 requires set \( C_{\text{rel}} \) to be line-free. □

![Figure 8](image_url)  
**Figure 8** Illustration of Sets in Example 6 and \( C_{\text{rel}} \nsubseteq \text{conv}(C) \).
Next we apply Theorem 8 to show the objective exactness for a special QCQP problem, known as Simultaneously Diagonalizable QCQP (SD-QCQP), i.e., the matrices \( \{Q_0, Q_1, \cdots, Q_m\} \) in QCQP (5) are simultaneously diagonalizable. Ben-Tal and Den Hertog (2014) first proved that the DWR of SD-QCQP with one-sided inequality constraint admitted the objective exactness under the assumption that \( V_{\text{rel}} > -\infty \), which is a special case of QCQP-1 in Corollary 3. In fact, we can further generalize their result by showing that the objective exactness for SD-QCQP with \( V_{\text{rel}} > -\infty \) and a two-sided constraint always holds.

**Corollary 9 (SD-QCQP)** For the SD-QCQP with a two-sided constraint, its DWR admits the objective exactness for any linear objective function such that \( V_{\text{rel}} > -\infty \).

**Proof.** As SD-QCQP is a special case of QCQP-1 and the corresponding set \( C_{\text{rel}} \) is always line-free, the conclusion follows by Corollary 3 and Theorem 8. \( \square \)

The objective exactness for the convex relaxations (e.g., SDP, SOCP) of the QCQP has been widely studied in literature, and most of research focuses on \( V_{\text{opt}} = V_{\text{rel}} \) for a specific QCQP problem (6) with the prespecified linear constraints and objective function, i.e., data \( \{A_i\}_{i \in \{0\} \cup \{1, \cdots, n\}} \) are fixed. Quite differently, our proposed objective exactness refers to \( V_{\text{opt}} = V_{\text{rel}} \) for any \( \tilde{m} \)-dimensional linear constraints and for any linear objective function satisfying some desired properties (e.g., \( V_{\text{rel}} > -\infty \)). Note that the QCQP (6) has a domain set \( \mathcal{X} = \{X \in S_{++}^n : \text{rank}(X) \leq 1\} \) to be conic and closed (i.e., satisfies the condition in Theorem 7) and its corresponding set \( C_{\text{rel}} \) is always line-free; hence, our Theorem 7 and Theorem 8 can be directly applied to the general QCQP problems, which provide, for the first time, the necessary and sufficient conditions for the objective exactness under Setting I and Setting II, respectively. Specifically, the corresponding conditions that \( F^{\tilde{m}+1}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \) and \( F^{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \) are intuitively obtained from a geometric perspective. When applying to the QCQP (6), our proposed conditions only involve the domain set \( \mathcal{X} \) and are regardless of the linear objective function and linear constraints. In contrast, the literature on objective exactness usually studies a specific QCQP problem with conditions on the linear objective function and/or the linear constraints of the QCQP.

### 3.2. Objective Exactness Under Setting (III): Relaxed Necessary and Sufficient Condition based on Binding Constraints

Our proposed necessary and sufficient condition in Theorem 8 guarantees the objective exactness when DWR (3) is equipped with any linear objective function such that \( V_{\text{rel}} > -\infty \). Beyond that, when this proposed condition fails, the objective exactness may still hold for the DWR with favorable objective functions (see, e.g., Example 7 below). This motivates us to study a relaxed
necessary and sufficient condition for DWR objective exactness given \( m \)-dimensional binding constraints at optimality, which covers and extends the objective exactness of two major applications in fair unsupervised learning: fair PCA and fair SVD.

Throughout this subsection, for ease of the analysis, we consider only one-sided linear constraints for RCOP (1) in which, without loss of generality, we let \( b_i^l = -\infty \) for each \( i \in [m] \). In fact, any \( i \)-th linear constraint of RCOP (1) can be recast as two linearly one-sided inequality constraints of dimension one as shown below since matrices \( \{ A_i, -A_i \} \) have dimension of one.

\[
 b_i^l \leq \langle A_i, X_i \rangle \leq b_i^u \iff \langle -A_i, X_i \rangle \leq -b_i^l, \langle A_i, X_i \rangle \leq b_i^u.
\]

We begin with an example illustrating why the dimension of binding constraints of the DWR plays an important role in improving the condition for objective exactness in Theorem 8.

**Example 7** Following the same domain set \( \mathcal{X} = \{ X \in S^2_+ : \text{rank}(X) \leq 1, X_{12} = 0 \} \) and its convex hull \( \text{conv}(\mathcal{X}) = \{ X \in S^2_+ : X_{12} = 0 \} \) as Example 3, we consider \( m = \tilde{m} = 2 \) linear constraints \( X_{11} \leq X_{22} \) and \( X_{22} \leq 1 \) in RCOP (1). Then we have

\[
 C := \{ X \in \mathcal{X} : X_{11} \leq X_{22}, X_{22} \leq 1 \}, \quad C_{\text{rel}} := \{ X \in \text{conv}(\mathcal{X}) : X_{11} \leq X_{22}, X_{22} \leq 1 \}.
\]

Note that sets \( \mathcal{X} \) and \( \text{conv}(\mathcal{X}) \) are illustrated in Figure 5(a) and Figure 5(b), respectively, by projecting them onto a two-dimension vector space with respect to \( (X_{11}, X_{22}) \). Similarly, the corresponding sets \( \text{conv}(C) \) and \( C_{\text{rel}} \) are presented in Figure 9 below.

Since any zero– or one–dimensional face of \( \text{conv}(\mathcal{X}) \) belongs to set \( \mathcal{X} \) as presented in Figure 5, according to Theorem 8, when there is \( \tilde{m} \leq 1 \)-dimensional linear constraint, the objective exactness holds for any objective function such that \( V_{\text{rel}} > -\infty \). However, the result in Theorem 8 fails to hold in this example since there are \( \tilde{m} \geq 2 \)-dimensional linear constraints.

Albeit powerful, the sufficient condition in Theorem 8 may not rule out the possibility of attaining the objective exactness in this example. For instance, if we set the objective function to be \( X_{22} \), then the objective exactness holds and \( V_{\text{opt}} = V_{\text{rel}} = 0 \) with the optimal solution at point \( a_1 \), where there is only one binding constraint at optimality. Thus, the binding constraints at optimality may play a key factor in determining the objective exactness, instead of using all the constraints. □

Motivated by Example 7, we derive a relaxed necessary and sufficient condition for objective exactness under the setting that \( V_{\text{opt}} > -\infty \) and there is an optimal solution to DWR falling on the intersection of at most \( m \)-dimensional linearly equality constraints. We observe that \( m \leq \tilde{m} \leq m \).

The relaxed condition is based on the vector projection, which is defined below.
Yongchun Li and Weijun Xie: On the Exactness of Dantzig-Wolfe Relaxation for Rank Constrained Optimization Problems

(a) $\text{conv}(C)$

(b) $C_{\text{rel}}$

Figure 9 Illustration of Sets in Example 7 and $C_{\text{rel}} \neq \text{conv}(C)$.

Definition 3 (Vector Projection) For any nonzero vector $a_1 \in \mathbb{R}^n$ and a set $C \in \mathbb{R}^n$, we let $\text{proj}_C(a_1)$ denote the orthogonal projection of $a_1$ onto $C$, called “vector projection”. In particular, $\text{proj}_C(a_1)$ is parallel to set $C$.

By vectorizing a matrix, the vector projection can be straightforwardly applied to our matrix set.

Theorem 9 Given a nonempty closed domain set $\mathcal{X}$, suppose $b_i = -\infty$ for any $i \in [m]$ in RCOP (1). Then the followings are equivalent.

(a) Inclusive Face: Any no larger than $m$-dimensional face of set $\text{conv}(\mathcal{X})$ is contained in the domain set $\mathcal{X}$, i.e., $\mathcal{F}_{\text{in}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$;

(b) Objective Exactness: The DWR (3) has the same optimal value as RCOP (1) (i.e., $V_{\text{opt}} = V_{\text{rel}}$) for any linear objective function and any $m$ linear constraints such that (i) $V_{\text{rel}} > -\infty$, (ii) the optimal set of the DWR (3) is bounded, (iii) there are $m$-dimensional linearly binding constraints indexed by $T \subseteq [m]$, and (iv) matrices $\{A_j - \text{proj}_H(A_j)\}_{j \in [m] \setminus T}$ are parallel with the same direction and $H := \text{span}(\{A_i\}_{i \in T})$.

Proof. We split the proof into two parts.

Part I. Let us first show that statement (a) implies statement (b). As there are only $m$-dimensional linear binding constraints indexed by $T$, the DWR (3) can equivalently reduce to the one with these binding constraints. Since the optimal set is nonempty and bounded, there exists an optimal solution $X^*$ of DWR (3). We let $\widehat{C}_{\text{rel}}$ denote the intersection set of these $m$-dimensional binding constraints with set $\text{conv}(\mathcal{X})$, i.e., $\widehat{C}_{\text{rel}} = \{X \in \text{conv}(\mathcal{X}) : \langle A_j, X \rangle = b_j, \forall j \in T\}$ and $X^* \in \widehat{C}_{\text{rel}}$.

We first show that

Claim 1 For any nonzero direction $D$ in the orthogonal complement of space $H$ (i.e., $H^\perp$), if there exists some $\ell \in [m] \setminus T$ such that $A_\ell - \text{proj}_H(A_\ell) \neq 0$ and $\langle A_\ell, D \rangle \leq 0$, then $\langle A_j, D \rangle \leq 0$ for all $j \in [m] \setminus T$.

Proof. As $D \in H^\perp$, for each $j \in [m] \setminus T$, we have 

$$\langle A_j, D \rangle = \langle \text{proj}_H(A_j) + A_j - \text{proj}_H(A_j), D \rangle = \langle A_j - \text{proj}_H(A_j), D \rangle$$
\[ c_{jl}(A_\ell - \text{proj}_H(A_\ell), D) = c_{jl}(A_\ell, D) \leq 0, \quad \exists c_{jl} \geq 0, \]

where the second and last equation is from the fact that \( \langle \text{proj}_H(A_j), D \rangle = 0 \) for all \( j \in [m] \setminus T \), the third one is because the matrices \( \{A_j - \text{proj}_H(A_j)\}_{j \in [m] \setminus T} \) are parallel with the same direction, and the last one stems from the fact that \( \text{proj}_H(A_\ell) \) is parallel with \( H \) following Definition 3, \( D \) is parallel with \( H^\perp \), and thus \( \langle \text{proj}_H(A_\ell, D) \rangle = 0 \). Therefore, given a direction \( D \) in the \( \ell \)-th non-binding constraint, it must be a direction for all non-binding constraints.

Next, we prove that

**Claim 2** Set \( \hat{C}_{\text{rel}} \) contains no line.

**Proof.** We prove it by contradiction. Suppose that set \( \hat{C}_{\text{rel}} \) contains a line. There exists a nonzero direction \( D \) such that \( \langle A_0, D \rangle = 0 \) and \( D \) is also a direction for some non-binding constraint (otherwise, we can use the direction \( -D \)). According to Claim 1, \( D \) is a nonzero direction for all non-binding constraints. Thus, \( \{X^* + \alpha D : \alpha \geq 0\} \) is a subset of optimal solutions, contradicting that the optimal set is bounded.

Using the results in Claim 1 and Claim 2, there are two cases to be discussed.

Case 1. Suppose that \( X^* \) is an extreme point of set \( \hat{C}_{\text{rel}} \). According to Claim 2 and the representation theorem 18.5 in Rockafellar (1972), set \( \hat{C}_{\text{rel}} \) has at least one extreme point. Since there are \( m \)-dimensional binding constraints, Corollary 1 shows that any extreme point of set \( \hat{C}_{\text{rel}} \) belongs to \( \mathcal{F}_{\text{ext}}(\text{conv}(X)) \). Since \( \mathcal{F}_{\text{ext}}(\text{conv}(X)) \subseteq X \), the extreme point \( X^* \) also belongs to set \( C \), which implies the objective exactness.

Case 2. Suppose that \( X^* \notin \text{ext} (\hat{C}_{\text{rel}}) \). Then, using the representation theorem 18.5 in Rockafellar (1972), the optimal solution \( X^* \in \hat{C}_{\text{rel}} \) can be represented by the convex combination below.

\[
X^* = \sum_{t \in [\tau]} \alpha_t X_t + \sum_{t \in [\tau]} \beta_t Y_t,
\]

where \( \{X_1, \ldots, X_\tau\} \) denote extreme points of set \( \hat{C}_{\text{rel}} \) which belong to \( \mathcal{F}_{\text{ext}}(\text{conv}(X)) \) by Corollary 1, \( \{Y_1, \ldots, Y_\tau\} \) denote extreme directions of set \( \hat{C}_{\text{rel}} \), \( \alpha \in \mathbb{R}^{\tau}_{++}, \sum_{t \in [\tau]} \alpha_t = 1 \), and \( \beta \in \mathbb{R}^{\tau} \). Since \( X^* \in \text{arg min}_{X \in \hat{C}_{\text{rel}}} \langle A_0, X \rangle \), we must have \( \langle A_0, X_t \rangle = \langle A_0, X^* \rangle \) for all \( t \in [\tau] \) and \( \langle A_0, Y_t \rangle = 0 \) for all \( t \in [\tau] \).

According to Claim 1 and the presumption that the set of optimal solutions is bounded and optimal value is finite, for any extreme direction \( Y_t \), it cannot be a feasible direction for any of the non-binding constraints. Thus, \( \langle A_j, Y_t \rangle \geq 0 \) for all \( j \in [m] \setminus T \) and at least one of inequality is strict. Hence, for any \( j \in [m] \setminus T \), we must have \( \langle A_j, X^* \rangle \geq \sum_{t \in [\tau]} \alpha_t \langle A_j, X_t \rangle \).

It follows that there exists an extreme point \( X_t \) such that \( X_t - X^* \) is a direction for all non-binding constraints according to Claim 1. Therefore, set \( X_t \in \mathcal{C}_{\text{rel}} \) and \( \langle A_0, X_t \rangle = \langle A_0, X_t \rangle \).
Thus, $X_l$ is an optimal extreme point solution in set $\tilde{C}_{rel}$. According to Case 1, we must have $X_l \in \mathcal{X}$ along with the fact that $X_l$ is feasible to all linear constraints, which proves the objective exactness.

**Part II.** When Part (b) holds, we can simply set all the non-binding constraints to be trivial with all constraint coefficients and right-hand sides being zeros. Since the optimal set is supposed to be bounded, there is no line in set $C_{rel}$. Then following the similar argument as Theorem 8, we can complete the proof.

The objective exactness of Example 7 with objective function $X_{22}$ can be confirmed by the result above. In fact, since $F^1(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$ and there is only $m = 1$ binding constraint at the unique optimal solution $a_2$ and only one non-binding constraint in Figure 9, according to Theorem 9, the objective exactness must hold. Besides, we remark that

(i) Part (a) in Theorem 9 serves as a necessary and sufficient condition of the objective exactness of DWR (3) with a finite optimal value, bounded optimal set, and at most $\tilde{m}$-dimensional binding constraints at optimality, provided that the non-binding constraints are parallel in the projected space. This condition can be somehow more general than that in Theorem 8 since $m \leq \tilde{m}$ implies that $F^m(\text{conv}(\mathcal{X})) \subseteq F^{\tilde{m}}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$;

(ii) Theorem 9 provides a fresh geometric angle for the significance of the binding constraints for determining objective exactness;

(iii) It is worthy of mentioning that the necessary and sufficient condition in Theorem 9 requires a bounded DWR optimal set. The unbounded optimal set may not always preserve the objective exactness as shown in Example 8 below. Besides, the boundedness of set $C_{rel}$ implies that the optimal set is also bounded;

(iv) The assumption on the non-binding constraints adopted in Theorem 9 is critical to ensure the objective exactness. Otherwise, violating this assumption may not always have the objective exactness. An interesting observation is that if there is only one non-binding constraint, then the assumption naturally holds;

(v) When $\mathcal{X} := \{X \in S_n^+ : \text{rank}(X) \leq 1\}$, our Theorem 9 generalizes the classical QCQP result in Ye and Zhang (2003) with two quadratic inequality constraints, where the authors showed that the DWR objective exactness holds if one of them is not binding under the assumption that the DWR and its dual both satisfy the Slater condition. Their assumption implies the boundedness of both the DWR optimal value and optimal set. Hence, our Theorem 9 is more general even under their setting, summarized in Corollary 10. For example, the result in Corollary 10 can cover the case that DWR has the singleton feasible set, while Ye and Zhang (2003) cannot; and

(vi) The result of Theorem 9 holds for any closed set $\mathcal{X}$ regardless of rank constraint.
Example 8 Following the same domain set $\mathcal{X} = \{ X \in S^2_+ : \text{rank}(X) \leq 1, X_{12} = 0 \}$ and its closed convex hull $\overline{\text{conv}}(\mathcal{X}) = \{ X \in S^2_+ : X_{12} = 0 \}$ as Example 3, we consider two linear constraints $X_{11} \leq X_{22}$ and $X_{22} \geq 1$. The corresponding feasible sets $\mathcal{C}$ and $\mathcal{C}_{\text{rel}}$ of problem (1) and its DWR (3) are

$$
\mathcal{C} := \{ X \in \mathcal{X} : X_{11} \leq X_{22}, X_{22} \geq 1 \}, \quad \mathcal{C}_{\text{rel}} := \{ X \in \overline{\text{conv}}(\mathcal{X}) : X_{11} \leq X_{22}, X_{22} \geq 1 \}.
$$

which are marked by red line and shadow area in Figure 10(a) and Figure 10(b), respectively.

By setting the objective function to be $X_{22} - X_{11}$, the DWR achieves the finite optimal value $V_{\text{rel}} = 0$ and there is an optimal solution with only one binding constraint, i.e., $m = 1$. Since only one constraint is not binding, the parallel assumption required by Theorem 9 holds naturally as indicated in Part (iv) above. Although $F^1(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$, we have $V_{\text{opt}} = 1 > V_{\text{rel}} = 0$, i.e., the objective exactness fails. This is possibly because the DWR optimal set $\{ (X_{11}, X_{22}) : X_{11} = X_{22}, X_{22} \geq 1 \}$ is unbounded, violating the boundedness assumption in Part (b) of Theorem 9.

\[ \phi \]

Corollary 10 For the QCQP (5) with two quadratic inequality constraints, suppose that its DWR yields finite optimal value and bounded optimal set. Then the objective exactness holds if one of the quadratic inequality constraints is not binding.

Proof. This result holds according to Theorem 9 and the fact $F^2(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X}$ from Lemma 4. \qed

Let us now turn to two application problems in fair unsupervised learning whose DWRs can achieve objective exactness by leveraging the binding constraint-based condition in Theorem 9.

Fair PCA (FPCA). Fair PCA (FPCA) extends the conventional PCA to the dataset concerning different groups to achieve the fairness. The seminal work (Samadi et al. 2018) presented a real-world example to show that the conventional PCA can cause gender bias and introduced the notion of FPCA to improve the fairness of the conventional PCA. Their follow-up work (Tantipongpipat et al. 2019) proposed the following rank-$k$ constrained formulation for FPCA that seeks to optimize the dimensionality reduction over all groups in a fair way:

$$
(FPCA) \quad V_{\text{opt}} = \max_{(z, X) \in \mathbb{R} \times \mathcal{X}} \{ z : z \leq \langle A_i, X \rangle, \forall i \in [m] \}, \quad (11)
$$
where there are \( m \) different groups, \( A_i \in S^n_i \) denotes the covariance matrix of the \( i \)-th group for each \( i \in [m] \), and domain set \( \mathcal{X} := \{ X \in S^n_+ : \text{rank}(X) \leq k, ||X||_2 \leq 1 \} \). It is important to note that FPCA fits in our RCOP framework (1) by observing that at optimality, \( z \) equals to at least one of \( \{ \langle A_i, X \rangle \}_{i \in [m]} \), i.e., FPCA (11) is equivalent to

\[
\text{FPCA} \quad V_{opt} = \max_{j \in [m]} \max_{X \in \mathcal{X}} \{ \langle A_j, X \rangle : \langle A_j, X \rangle \leq \langle A_i, X \rangle, \forall i \in [m] \}.
\]

For ease of analysis, we focus on FPCA (11) with variable \( z \). Before proceeding to the DWR of FPCA (11), it is important to show the closed convex hull of \( \mathcal{X} \) and its facial inclusion result.

**Lemma 5** Suppose the domain set \( \mathcal{X} := \{ X \in S^n_+ : \text{rank}(X) \leq k, ||X||_2 \leq 1 \} \), then we have

(i) \( \overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) := \{ X \in S^n_+ : \text{tr}(X) \leq k, ||X||_2 \leq 1 \} \);

(ii) Any no larger than one–dimensional face of \( \overline{\text{conv}}(\mathcal{X}) \) is contained in \( \mathcal{X} \), i.e., \( \mathcal{F}^1(\overline{\text{conv}}(\mathcal{X})) \subseteq \mathcal{X} \).

**Proof.** **Part** (i). The equality \( \overline{\text{conv}}(\mathcal{X}) = \text{conv}(\mathcal{X}) \) is because set \( \mathcal{X} \) is compact. Suppose \( X = Q \text{Diag}(\lambda)Q^\top \) denotes the eigen-decomposition, then the set \( \mathcal{X} \) is equivalent to

\[
\mathcal{X} := \{ X \in S^n_+ : \lambda \in \mathbb{R}^n_+, ||\lambda||_0 \leq k, ||\lambda||_\infty \leq 1, QQ^\top = I_n, X = Q \text{Diag}(\lambda)Q^\top \}.
\]

It is known that \( \text{conv} \left( \{ \lambda \in \mathbb{R}^n_+ : ||\lambda||_0 \leq k, ||\lambda||_\infty \leq 1 \} \right) = \{ \lambda \in \mathbb{R}^n_+ : ||\lambda||_1 \leq k, ||\lambda||_\infty \leq 1 \} \) (see, e.g., Argyriou et al. 2012). Hence, we have

\[
\text{conv}(\mathcal{X}) = \text{conv} \left( \{ X \in S^n_+ : \lambda \in \text{conv} \{ \lambda \in \mathbb{R}^n_+ : ||\lambda||_0 \leq k, ||\lambda||_\infty \leq 1 \}, QQ^\top = I_n, X = Q \text{Diag}(\lambda)Q^\top \} \right) = \text{conv} \left( \{ X \in S^n_+ : \lambda \in \mathbb{R}^n_+ : ||\lambda||_1 \leq k, ||\lambda||_\infty \leq 1, QQ^\top = I_n, X = Q \text{Diag}(\lambda)Q^\top \} \right) = \{ X \in S^n_+ : \text{tr}(X) \leq k, ||X||_2 \leq 1 \},
\]

where the last equation stems from projecting out variables \( \lambda, Q \) and the identities: \( ||\lambda||_1 = \text{tr}(X) \) and \( ||\lambda||_\infty = ||X||_2 \) for any \( X \in S^n_+ \).

**Part** (ii). If there exists a one–dimensional face \( F \subseteq \overline{\text{conv}}(\mathcal{X}) \) not contained in \( \mathcal{X} \), then there is a matrix \( \hat{X} \in F \setminus \mathcal{X} \) that must have a rank \( r \) greater than \( k \). Let \( \hat{X} = Q_1 \Lambda_1 Q_1^\top + Q_2 \Lambda_2 Q_2^\top \) denote the resulting eigen-decomposition, where the eigenvalues attaining one compose the diagonal matrix \( \Lambda_1 \in S^n_+ \) and the other fractional eigenvalues are in \( \Lambda_2 \in S^n_+ \). It is clear that \( r_1 + r_2 = r \) and \( r_2 \geq 2 \). Following the proof in Lemma 4, as \( \frac{r_2}{r_1} (r_2 + 1) \geq 3 \), we can construct a symmetric matrix \( \Delta \in S^{r_2} \) which satisfies \( \text{tr}(\Delta) = 0 \) and \( Q_2 \Delta Q_2^\top \perp F \) such that \( \hat{X} \pm Q_2 \Delta Q_2^\top \in F \) for significantly small \( \delta > 0 \). This contradicts with the fact that \( Q_2 \Delta Q_2^\top \perp F \). \( \square \)
According to the Part (i) in Lemma 5, the DWR of FPCA (11) is defined by

$$V_{\text{rel}} = \max_{(z, X) \in \mathbb{R} \times \text{conv}(\mathcal{X})} \left\{ z : z \leq \langle A_i, X \rangle, \forall i \in [m] \right\}, \quad (12)$$

where \(\text{conv}(\mathcal{X}) = \{ X \in S^n_* : \text{tr}(X) \leq k, \|X\|_2 \leq 1 \} \). Part (ii) in Lemma 5 indicates that the domain set \(\mathcal{X}\) of FPCA (11) satisfies \(\mathcal{F}^1(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}\). In the following result, we show by using Theorem 8 and Theorem 9 that the convex hull exactness holds for FPCA when \(\tilde{m} = 2\), i.e., where there are two independent covariance matrices from \(m\) groups. Note that the feasible set in DWR (12) contains no line and DWR (12) has finite optimal value.

**Corollary 11** For FPCA (11) with \(\tilde{m} = 2\), its DWR (12) admits the convex hull exactness.

**Proof.** WLOG, the feasible sets of the FPCA (11) and DWR (12) with \(\tilde{m} = 2\) can be written as

$$C := \left\{ (z, X) \in \mathbb{R} \times S^n_* : z \leq \langle A_1, X \rangle, z \leq \langle A_2, X \rangle, \text{rank}(X) \leq k, \|X\|_2 \leq 1 \right\},$$

$$C_{\text{rel}} := \left\{ (z, X) \in \mathbb{R} \times S^n_* : z \leq \langle A_1, X \rangle, z \leq \langle A_2, X \rangle, \text{tr}(X) \leq k, \|X\|_2 \leq 1 \right\}.$$

We see that the recession cone of set \(C_{\text{rel}}\) in DWR (12) is \(\text{rec}(C_{\text{rel}}) := \{(z, 0) : z \leq 0\}\). We observe that \(\text{rec}(C_{\text{rel}}) = \text{rec}(\text{conv}(C))\). Thus, to prove \(C_{\text{rel}} = \text{conv}(C)\), we only need to show the extreme point exactness of the set \(C_{\text{rel}}\). For any extreme point \((\bar{z}, \bar{X}) \in \text{ext}(C_{\text{rel}})\), two cases are discussed below.

(i) If only one constraint is binding at \((\bar{z}, \bar{X})\), given \(\mathcal{F}^1(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}\) in Lemma 5, using the extreme point exactness result in Theorem 1, we have \(\bar{X}\) is rank-\(k\) and \((\bar{z}, \bar{X}) \in C\).

(ii) If both constraints are binding at \((\bar{z}, \bar{X})\), then \(\bar{X}\) is also an extreme point of set \(\{ X \in \text{conv}(\mathcal{X}) : \langle A_1, X \rangle = \langle A_2, X \rangle \}\). Similarly, this implies that \(\bar{X} \in \mathcal{X}\). Thus, \((\bar{z}, \bar{X}) \in C\). \(\square\)

We remark that (i) although set \(C_{\text{rel}}\) in DWR (12) is unbounded (as variable \(z\) can be \(-\infty\)), its convex hull exactness still holds with \(\tilde{m} = 2\) groups since the recession cones of \(C_{\text{rel}}\) and \(\text{conv}(C)\) coincide; (ii) our result extends the objective exactness with \(\tilde{m} = 2\) for FPCA in Tantipongpipat et al. (2019), where the authors proved that the upper bound of the rank of all extreme points in the set \(C_{\text{rel}}\) is linear in \(\sqrt{\tilde{m}}\) and the rank bound becomes \(k\) when \(\tilde{m} = 2\); and (iii) this result can be further generalized to objective exactness when there are \(m = 2\)-dimensional binding constraints of DWR (12) at optimality according to our Theorem 9, which is summarized in Corollary 12. Note that DWR (12) always produces finite optimal value and bounded optimal set.

**Corollary 12** For FPCA (11), suppose that (i) its DWR (12) has an optimal solution with \(m \leq 2\)-dimensional linear binding constraints and (ii) matrices \(\{A_j - \text{proj}_H(A_j)\}\) for all the non-binding constraints are parallel with the same direction, where set \(H\) denotes the space spanned by the binding covariance matrices. Then its DWR (12) admits the objective exactness.
Fair SVD. Another significant application of our relaxed condition in Theorem 9 is the fair SVD (FSVD), which can be formulated as

$$\text{FSVD} \quad \max_{(z,X) \in \mathbb{R} \times \mathcal{X}} \{ z : z \leq \langle A_i, X \rangle, \forall i \in [m] \}, \quad (13)$$

where $A_i \in \mathbb{R}^{n \times p}$ denotes the data matrix of the $i$-th group for each $i \in [m]$ and domain set $\mathcal{X} := \{ X \in \mathbb{R}^{n \times p} : \text{rank}(X) \leq k, ||X||_2 \leq 1 \}$. Different from FPCA (11), the FSVD (13) aims to seek a fair representation learning of $m$ different data matrices that are non-symmetric. Similar to Lemma 5 for FPCA (11), the next lemma characterizes the convex hull and facial inclusion property of the domain set $\mathcal{X}$ in FSVD (13).

**Lemma 6** Suppose that domain set $\mathcal{X} := \{ X \in \mathbb{R}^{n \times p} : \text{rank}(X) \leq k, ||X||_2 \leq 1 \}$, then we have

(i) $\text{conv}(\mathcal{X}) = \text{conv}(\mathcal{X}^*) := \{ X \in \mathbb{R}^{n \times p} : ||X||_* \leq k, ||X||_2 \leq 1 \}$, where $|| \cdot ||_*$ is the nuclear norm;

(ii) Any no larger than two–dimensional face of $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$, i.e., $F^2(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$.

**Proof.** The proof is similar to Lemma 5 and is thus omitted. □

Note that the domain set $\mathcal{X}$ in FPCA (11) contains any at most one–dimensional face in its convex hull, and we show that the convex hull exactness holds when there are two independent groups in Corollary 11. According to Part (ii) of Lemma 6, it is natural to extend the convex hull exactness for FSVD up to three independent groups (i.e., $\tilde{m} \leq 3$) as below.

**Corollary 13** For the FSVD (13) with $\tilde{m} \leq 3$, its DWR admits the convex hull exactness.

**Proof.** The proof directly follows from Lemma 6 and Corollary 11. □

We remark that this is the first-known convex hull exactness result for FSVD when there are three groups and can be further extended to objective exactness when its DWR has at most $m = 3$-dimensional binding constraints.

**Corollary 14** For FPCA (13), suppose that (i) its DWR has an optimal solution with $m \leq 3$-dimensional binding constraints, and (ii) matrices $\{ A_j - \proj_{\mathcal{H}}(A_j) \}$ for all the non-binding constraints are parallel with the same direction with $\mathcal{H}$ denoting the space spanned by binding covariance matrices. Then its DWR admits the objective exactness.

### 3.3. Objective Exactness Under Setting (IV): Relaxed Necessary and Sufficient Condition based on the Nonzero Optimal Lagrangian Multipliers

The previous subsection shows whether the DWR problem (3) achieves objective exactness depends on its binding constraints. Motivated by the fact that binding constraints may also have zero-value
Lagrangian multipliers, this subsection aims to further relax the necessary and sufficient condition for objective exactness by leveraging the Lagrangian multipliers of DWR (3). This result provides us the flexibility to cover and generalize the objective exactness results for more applications present in the literature. Similar to the previous subsection, we still focus on the one-sided linear constrained RCOP (1).

We first show an example of the DWR in which the objective exactness holds and the dimension of binding constraints is strictly larger than the number of nonzero optimal Lagrangian multipliers (i.e., $m > m^*$). More importantly, in this example, both conditions in Theorem 8 and Theorem 9 fail to cover the objective exactness of the DWR. This motivates us to further relax the condition in Theorem 9 from the perspective of Lagrangian multipliers.

**Example 9** Suppose domain set $\mathcal{X} := \{X \in S^2_+ : \text{rank}(X) \leq 1, X_{12} = 0\}$ same as Example 3 and $m = m = 2$ linear constraints $X_{11} \leq X_{22}, 2X_{11} \leq X_{22}$. Hence, we have sets $\mathcal{C}$ and $\mathcal{C}_{\text{rel}}$ defined as

$$
\mathcal{C} = \{X \in \mathcal{X} : X_{11} \leq X_{22}, 2X_{11} \leq X_{22}\}, \quad \mathcal{C}_{\text{rel}} = \{X \in \text{conv}(\mathcal{X}) : X_{11} \leq X_{22}, 2X_{11} \leq X_{22}\}.
$$

Set $\mathcal{C}$ in this example is the vertical red solid line as shown in Figure 11(a) and set $\mathcal{C}_{\text{rel}}$ is presented in red shadow area in Figure 11(b). Note that both sets are unbounded.

If we set the objective function of the DWR to be $X_{22}$, then the objective exactness holds, i.e., $V_{\text{rel}} = V_{\text{opt}} = 0$ with the same optimal point $a_1$. It can be seen that point $a_1$ falls on two linearly independent constraints. However, as $\mathcal{F}^2(\text{conv}(\mathcal{X})) \not\subseteq \mathcal{X}$ and there are $m = 2$-dimensional linear binding constraints, Theorem 8 and Theorem 9 cannot be used to show the objective exactness in this example. On the other hand, there is an optimal dual solution (i.e., Lagrangian multipliers $\mu^* = (0, 1)^\top$) of the DWR that has only $m^* = 1$ nonzero Lagrangian multiplier achieving the same objective value. In fact, we observe that the smallest number of nonzero optimal Lagrangian multipliers is upper bounded by the dimension of binding constraints. Therefore, to leverage the number of nonzero optimal Lagrangian multipliers, we are motivated to derive another necessary and sufficient condition for objective exactness, further relaxing the one based on binding constraints in Theorem 9.

Next we obtain a geometric interpretation of the Lagrangian multipliers of DWR (3) based on the normal cone of set $\mathcal{C}_{\text{rel}}$.

**Definition 4 (Normal Cone)** Let $D \subseteq \mathbb{R}^n$ be a closed convex set. The normal cone of set $D$ at $x$ is defined by $N_D(x) = \{g \in \mathbb{R}^n : g^\top(y - x) \leq 0, \forall y \in D\}$.

**Proposition 1** Suppose that $V_{\text{rel}} > -\infty$ in DWR (3), then the followings must hold:
Yongchun Li and Weijun Xie: On the Exactness of Dantzig-Wolfe Relaxation for Rank Constrained Optimization Problems

Figure 11 Illustration of Sets in Example 9 and $C_{rel} \neq \overline{conv}(C)$.

(i) A feasible solution $X^* \in C_{rel}$ is optimal to DWR (3) if and only if $-A_0 \in N_{C_{rel}}(X^*)$;

(ii) Suppose that $b^i = -\infty$ for all $i \in [m]$ in RCOP (1), and DWR (3) satisfies the relaxed Slater condition, i.e., $C_{rel} \cap ri(\overline{conv}(X)) \neq \emptyset$. Then for any optimal solution $X^*$ to DWR, there exists $\mu^* \in \mathbb{R}^m_+$ such that $supp(\mu^*) \subseteq T^*$ and

$-A_0 - \sum_{i \in T^*} \mu^*_i A_i \in N_{\overline{conv}(X)}(X^*)$, 

where $T^* \subseteq [m]$ denotes the index set of binding constraints of dimension $m$, and $N_{\overline{conv}(X)}(X^*)$ denotes the normal cone of set $\overline{conv}(X)$ at $X^*$.

Proof. Part (i). Since the feasible set $C_{rel}$ of the DWR is closed and convex, then the optimality condition is $(A_0, X^*) \leq (A_0, X)$ for any $X \in C_{rel}$, which is equivalent to $-A_0 \in N_{C_{rel}}(X^*)$ according to Definition 4 of the normal cone.

Part (ii). Since DWR (3) satisfies the relaxed Slater condition, the normal cone of the intersection set $C_{rel}$ is equal to the intersection of normal cones (Burachik and Jeyakumar 2005). Therefore, $-A_0 \in N_{C_{rel}}(X^*)$ if and only if there exists $\mu^* \in \mathbb{R}^m_+$ such that

$-A_0 \in N_{C_{rel}}(X^*) = N_{\overline{conv}(X)} \cap \{X \in \mathbb{Q} : (A_i, X) \leq b_i^*, \forall i \in [m]\}(X^*) = \sum_{i \in T^*} \mu^*_i A_i + N_{\overline{conv}(X)}(X^*)$.

Note that for each $i \in [m] \setminus T^*$, we have $\mu^*_i = 0$. This completes the proof.

We remark that the relaxed Slater condition in Proposition 1 is meant to provide an explicit description of the normal cone of the intersection set $C_{rel}$ and this condition can be further relaxed (see, e.g., Burachik and Jeyakumar 2005), which is omitted in this paper due to page limit. Part (ii) of Proposition 1 implies that $m^* \leq m$ holds, which enables us to improve Theorem 9.

One-sided QCQP. To illustrate Proposition 1, we study the one-sided QCQP (6), where we let $b^i = -\infty$ for all $i \in [m]$. Recent studies reveal the DWR of one-sided QCQP can achieve objective exactness when the matrix coefficients $\{A_0\} \cup \{A_i\}_{i \in [m]}$ exhibit favorable properties. For instance, Kim and Kojima (2003) proved that $V_{rel} = V_{opt} > -\infty$ when all the off-diagonal elements of matrices $A_0$ and $A_1, \cdots, A_m$ are nonpositive. Mapping the matrix coefficients of the one-sided
QCQP into an undirected graph $G$ that there is an edge in $G$ if any of $(A_0)_{ij}$ and $(A_1)_{ij}, \ldots, (A_m)_{ij}$ is nonzero, the work (Sojoudi and Lavaei 2014) generalized Kim and Kojima (2003), and proved that $V_{rel} = V_{opt} > -\infty$ when all the off-diagonal elements of matrices $A_0$ and $A_1, \ldots, A_m$ are sign-definite, i.e., for any pair $(i,j)$ with $i \neq j$, $(A_0)_{ij}$ and $(A_1)_{ij}, \ldots, (A_m)_{ij}$ are either all nonpositive or all nonnegative and the signs of matrices further satisfy

$$\prod_{i \in [S]} \sigma_{S(i), S(i+1)} = (-1)^{|S|}, \quad \forall S \text{ is a cycle of graph } G,$$

where we let $S(i)$ denote the $i$th node in the cycle $S$, let $S(|S| + 1) = S(1)$, and let $\sigma_{ij} = -1$ if $(A_0)_{ij}, (A_1)_{ij}, \ldots, (A_m)_{ij}$ are all nonpositive and $\sigma_{ij} = 1$ if $(A_0)_{ij}, (A_1)_{ij}, \ldots, (A_m)_{ij}$ are all positive.

In recent follow-up works (Bur er and Ye 2020, Kılınç-Karzan and Wang 2021), the authors reproved the result in Sojoudi and Lavaei (2014) from different angles when applying to the diagonal one-sided QCQP. Our following theorem provides a unified analysis of these cases that achieve objective exactness based on the normal cone-based optimality condition in Proposition 1.

**Corollary 15** Suppose that $b_i' = -\infty$ for all $i \in [m]$ in QCQP (6). Then its DWR admits the objective exactness for any linear objective function such that $V_{rel} > -\infty$ when any of the following conditions holds:

(i) All the off-diagonal elements of matrices $A_0, A_1, \ldots, A_m$ are sign-definite and for each cycle $S$ in graph $G$, we have $\prod_{i \in [S]} \sigma_{S(i), S(i+1)} = (-1)^{|S|}$ with $S(|S| + 1) = S(1)$;

(ii) Matrices $Q_0, Q_1, \ldots, Q_m$ are diagonal and vectors $q_0, q_1, \ldots, q_m$ are sign-definite. Note that $\{Q_0\} \cup \{Q_i\}_{i \in [m]}$ are symmetric, $A_i = \begin{pmatrix} 0 & q_i/2 \\ q_i/2 & Q_i \end{pmatrix}$ and for each $i \in \{0\} \cup [m]$.

**Proof.** We first prove Part (i) by contradiction and the other one follows a similar analysis.

**Part (i).** Suppose that there is no rank-1 optimal solution of the corresponding DWR. Consider a DWR optimal solution $X^* \in S_+^{n+1}$ with rank($X^*$) > 1. If the optimal solution does not exist, the proof still follows by passing to the limit since both sets $X$ and $C_{rel}$ are closed. Thus, without loss of generality, we assume that the set of optimal solutions is nonempty. Since rank($X^*$) > 1, there exists a $2 \times 2$ principal submatrix $X_{T,T}^* \in S_+^2$ of $X^*$ satisfying $(X_{T(1),T(2)}^*)^2 < X_{T(1),T(1)}^* X_{T(2),T(2)}^*$. WLOG, suppose $T = \{1,2\}$. Let us construct a new solution $\tilde{X}$ such that $\tilde{X}_{1,2} = -\sigma_{ij} \sqrt{X_{1,1}^* X_{2,2}^*}$, $\tilde{X}_{1,1} = X_{1,1}^*$, $\tilde{X}_{1,1} = X_{1,2}^*$ and $\tilde{X}_{ij} = X_{ij}^*$ for any $(i,j) \in ([n+1] \times [n+1]) \setminus (T \times T)$. Then we consider an equivalent truncated DWR problem with a focus on the variables indexed by $T \times T$ as

$$(\text{Truncated DWR}) \quad \langle \hat{A}_0, X^* \rangle := \min_{X \in \text{conv}(\hat{X})} \left\{ \langle \hat{A}_0, X \rangle + c_0 : \langle \hat{A}_i, X \rangle + c_i \leq b_i, \forall i \in [m] \right\}, \quad (14)$$

where $\hat{X} := \{ X \in S_+^2 : X_{12} = X_{11} X_{22} \}$ and $\text{conv}(\hat{X}) = S_+^2$, $\hat{A}_0 \in S^2 = (A_0)_{T,T}$ and $\hat{A}_i \in S^2 = (A_i)_{T,T}$ for each $i \in [m]$, and $c_0 = \langle \hat{A}_0, X^* \rangle - \langle \hat{A}_0, X_{T,T}^* \rangle$ and $c_i = \langle \hat{A}_i, X^* \rangle - \langle \hat{A}_i, X_{T,T}^* \rangle$ for each $i \in [m]$. 

For the truncated DWR problem (14), without loss of generality, the submatrix \( X_{T,T}^\ast \) is optimal and leads to the objective value \( \langle A_0, X^\ast \rangle \). We also see that \( X_{T,T}^\ast \) is in the interior of \( \text{conv}(\hat{X}) \) and satisfied the relaxed Slater condition. According to Proposition 1, there must exist an \( \mu^\ast \in \mathbb{R}_+^m \) such that

\[
-\hat{A}_0 \in \sum_{i \in [m]} \mu_i^\ast \hat{A}_i + \mathcal{N}_{\text{conv}(\hat{X})}(X_{T,T}^\ast) = \sum_{i \in [m]} \mu_i^\ast \hat{A}_i,
\]

where the equation is because \( X_{T,T}^\ast \) is in the interior of \( \text{conv}(\hat{X}) \). Next we discuss two cases depending on whether \( (\hat{A}_0)_{12} \) attains zero or not to show that \( \hat{X}_{T,T} \) is also optimal to the truncated DWR (14).

(a) Suppose that \( (\hat{A}_0)_{12} \neq 0 \). Since \( (\hat{A}_0)_{12} \) has the same sign with \( (\hat{A}_1)_{12}, \ldots, (\hat{A}_m)_{12} \), given \( \mu^\ast \in \mathbb{R}_+^m \), the optimality condition \(-\hat{A}_0 = \sum_{i \in [m]} \mu_i^\ast \hat{A}_i \) cannot hold, a contradiction.

(b) Suppose that \( (\hat{A}_0)_{12} = 0 \). In this case, changing \( X_{12}^\ast \) does not affect the objective value. As \( (X_{12}^\ast)^2 < X_{1,1}^\ast X_{2,2}^\ast \), submatrix \( \hat{X}_{T,T} \) is feasible and attains the same optimal value for the truncated DWR (14) and thus is optimal.

Following this scheme to adjust the solution \( \hat{X} \), we can either find an alternative DWR feasible rank-one matrix \( \tilde{X} \) such that for any \( i \neq j \), \( \tilde{X}_{ij} = -\sigma_{ij} \sqrt{\hat{X}_{ii} \hat{X}_{jj}} \) or arrive at a contradiction as part (a). In the first case, given the condition that for each cycle \( S \) of graph \( \mathcal{G} \), we have \( \prod_{i \in [|S|]} \sigma_{S(i),S(i+1)} = (-1)^{|S|} \), following the similar argument in Sojoudi and Lavaei (2014), there exists a vector \( \mathbf{x}^\ast \) such that \( \hat{X} = \begin{bmatrix} 1 & \mathbf{x}^\ast & \mathbf{x}^\ast (\mathbf{x}^\ast)^\top \end{bmatrix} \), implying the superiority of a rank-one matrix, a contradiction.

**Part (ii).** For any feasible solution \( X^\ast := \begin{bmatrix} 1 & \mathbf{x}^\ast & \mathbf{x}^\ast Y^\ast \end{bmatrix} \) to the DWR of the one-sided QCQP, suppose that for some \( i \in [n] \), we have \( Y_{ii}^\ast > (x_i^\ast)^2 \). Following the similar proof as **Part (i)**, we can construct a truncated problem of the DWR that only involves with variables \( (x_i,Y_{ii}) \) with the feasible set \( \hat{X} := \{(x,y) \in \mathbb{R}^2 : x^2 = y\} \). Since \( q_0, q_1, \ldots, q_n \) are sign-definite, we can either obtain a contradiction or adjust \( x_i^\ast \) and construct a pair of new optimal solution \( (\sigma_i \sqrt{Y_{ii}^\ast}, Y_{ii}^\ast) \), where \( \sigma_i = -1 \) if \( (q_0)_i, (q_1)_i, \ldots, (q_m)_i \geq 0 \) and \( \sigma_i = 1 \), otherwise. It follows that there exists a rank-one optimal solution, i.e., \( Y_{ii}^\ast = (x_i^\ast)^2 \) for all \( i \in [n] \), a contradiction. \( \square \)

Finally, we conclude this subsection by showing a necessary and sufficient condition for the DWR objective exactness by analyzing the number of its nonzero optimal Lagrangian multipliers.

**Theorem 10** Given a nonempty closed domain set \( \mathcal{X} \), suppose \( b_l^i = -\infty \) for any \( i \in [m] \) in RCOP (1). Then the followings are equivalent.

(a) **Inclusive Face:** Any no larger than \( m^\ast \)-dimensional face of set \( \text{conv}(\mathcal{X}) \) is contained in the domain set \( \mathcal{X} \), i.e., \( \mathcal{F}^{m^\ast}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \);
(b) **Objective Exactness:** The DWR (3) has the same optimal value as problem (1) (i.e., \( \mathbf{V}_{\text{opt}} = \mathbf{V}_{\text{rel}} \)) for any linear objective function and \( m \) linear constraints such that (i) the relaxed Slater condition holds, (ii) \( \mathbf{V}_{\text{rel}} > -\infty \), (iii) the DWR optimal set is bounded, (iv) there are \( m^* \) nonzero optimal Lagrangian multipliers corresponding to the DWR indexed by set \( T \), and (v) matrices \( \{ \mathbf{A}_j - \text{proj}_H(\mathbf{A}_j) \}_{j \in [m] \setminus T} \) are parallel with the same direction with \( H := \text{span}(\{ \mathbf{A}_i \}_{i \in T}) \).

**Proof.** We split the proof into two parts.

(i) We first prove (a) \( \implies \) (b). Let \( \mathbf{X}^* \) denote a primal optimal solution of DWR (3), corresponding to the optimal dual solution \( \mathbf{\mu}^* \in \mathbb{R}^m_+ \) with only \( m^* \) nonzero Lagrangian multipliers. We let \( F^d \subseteq \text{conv}(\mathcal{X}) \) denote the smallest-dimension face of set \( \text{conv}(\mathcal{X}) \) containing \( \mathbf{X}^* \) which has dimension \( d \). Then there are two cases to be discussed depending on the dimension \( d \).

1. Suppose that \( d \leq m^* \). Then according to Part (a), we have that \( \mathbf{X}^* \in F^d \subseteq \mathcal{X} \) and the objective exactness of the DWR directly follows, i.e., \( \mathbf{V}_{\text{opt}} = \mathbf{V}_{\text{rel}} \).

2. Suppose that \( d > m^* \). Given the relaxed Slater condition, using Part (ii) of Proposition 1, the optimal solution \( \mathbf{X}^* \) satisfies

\[
-A_0 - \sum_{i \in T} \mu_i^* \mathbf{A}_i \in \mathcal{N}_{\text{conv}(\mathcal{X})}(\mathbf{X}^*),
\]

where \( \mathbf{\mu}^* \) denotes the dual optimal solution with set \( T \) being its support and \( |T| = m^* \).

Since \( F^d \) is the smallest face containing \( \mathbf{X}^* \), we must have \( \mathbf{X}^* \in \text{ri}(F^d) \). According to Definition 4 of normal cone, it follows that \( \langle \mathbf{V}, \mathbf{X}^* - \mathbf{\hat{X}} \rangle = 0 \) for any \( \mathbf{\hat{X}} \in F^d, \mathbf{\hat{V}} \in \mathcal{N}_{\text{conv}(\mathcal{X})}(\mathbf{X}^*) \). Then let us define a set \( S = F^d \cap \{ \mathbf{X} \in \mathcal{Q} : \langle \mathbf{A}_i, \mathbf{X} \rangle = b_i, \forall i \in T \} \) and for any \( \mathbf{\hat{X}} \in S \) and some \( \mathbf{\hat{V}} \in \mathcal{N}_{F^d}(\mathbf{X}^*) \), we have

\[
-A_0 = \sum_{i \in [m^*]} \mu_i^* \mathbf{A}_i + \mathbf{\hat{V}}, \quad \langle \mathbf{A}_i, \mathbf{X}^* \rangle = \langle \mathbf{A}_i, \mathbf{\hat{X}} \rangle = b_i, \forall i \in T, \quad \langle \mathbf{\hat{V}}, \mathbf{X}^* - \mathbf{\hat{X}} \rangle = 0,
\]

where the first equation is from the optimality condition (15). Therefore, we can conclude that \( \langle A_0, \mathbf{\hat{X}} \rangle = \langle A_0, \mathbf{X}^* \rangle = \mathbf{V}_{\text{rel}} \) for any \( \mathbf{\hat{X}} \in S \).

Following the analysis of Theorem 9, there must exist an extreme point \( \mathbf{X}_t \) in set \( S \) which is feasible and optimal to the DWR. In addition, using Corollary 1, \( \mathbf{X}_t \) must belong to some \( m^* \)-dimensional face of face \( F^d \), which is thus contained in \( \mathcal{F}_{m^*}(\text{conv}(\mathcal{X})) \subseteq \mathcal{X} \).

It follows that \( \mathbf{X}_t \in \mathcal{C} \) and \( \mathbf{V}_{\text{opt}} = \mathbf{V}_{\text{rel}} \).

(ii) We next prove (b) \( \implies \) (a). Since set \( \mathcal{C}_{\text{rel}} \) has a nonempty relative interior, set \( \text{conv}(\mathcal{X}) \) must have the nonempty relative interior \( \mathbf{\hat{X}} \). That is, there exists a ball \( B(\mathbf{\hat{X}}, \epsilon) \) with radius \( \epsilon > 0 \) such that \( B(\mathbf{\hat{X}}, \epsilon) \cap \text{aff}(\text{conv}(\mathcal{X})) \subseteq \text{conv}(\mathcal{X}) \).

Given a \( d \)-dimension face \( F^d \) of set \( \text{conv}(\mathcal{X}) \) with \( d \leq \min\{ \text{dim}(\mathcal{X}) - 1, m^* \} \), we must have \( \mathbf{\hat{X}} \not\in F^d \), otherwise \( F^d = \text{conv}(\mathcal{X}) \). Then for any point \( \mathbf{X} \in F^d \), let us can construct \( d \)-dimensional linear inequalities in the \( \text{aff}(\mathcal{X}) \) whose corresponding hyperplanes pass the point.
$\mathbf{X}$ and are tangent to the ball $B(\widehat{\mathbf{X}}, \| \mathbf{X} - \widehat{\mathbf{X}} \| / 2)$ and whose intersection includes the ball $B(\widehat{\mathbf{X}}, \| \mathbf{X} - \widehat{\mathbf{X}} \| / 2)$ and let the rest $m - d$ number of inequalities have zero coefficient and zero right-hand sides. Thus, point $\mathbf{X}$ is an extreme point of set $\mathcal{C}_{rel}$ and $\mathcal{C}_{rel}$ has nonempty relative interior. Following the proof in Part (II) of Theorem 8, we have that $\mathbf{X} \in \mathcal{C} \subseteq \mathcal{X}$. Hence, $F^d \subseteq \mathcal{X}$.

Now suppose that $d = \dim(\mathcal{X})$, i.e., $F^d = \overline{\text{conv}}(\mathcal{X})$. Then for any $\mathbf{X} \in F^d$ but $\mathbf{X} \neq \widehat{\mathbf{X}}$, we can do the same procedure as the previous part of the proof. Since $\overline{\text{conv}}(\mathcal{X}) \setminus \{ \widehat{\mathbf{X}} \} \subseteq \mathcal{X}$ and set $\mathcal{X}$ is closed, we must have $\overline{\text{conv}}(\mathcal{X}) = \mathcal{X}$. This completes the proof. $\square$

For the necessary and sufficient condition in Theorem 10 based on Lagrangian multipliers, we remark that

(i) This condition, together with that provided by Theorem 9, provide a unified primal and dual analysis for the DWR objective exactness.

(ii) From Part (ii) in Proposition 1, we see that the smallest number of nonzero optimal Lagrangian multipliers is upper bounded by the dimension of linear binding constraints, i.e., $m^* \leq m$. Hence, the condition in Theorem 10 can be more general than that in Theorem 9 based on binding constraints since $F^m(\text{conv}(\mathcal{X})) \subseteq F^m(\text{conv}(\mathcal{X})) \subseteq \mathcal{X}$, which aligns with the findings in Example 9. On the other hand, in Theorem 10 we need the relaxed Slater condition, but Theorem 9 does not need; and

(iii) Analogous to Corollary 10, it is interesting to apply Theorem 10 to the QCQP (5) with two quadratic inequality constraints and generalize the result in Ben-Tal and Den Hertog (2014). In fact, Corollary 16 below proves the objective exactness for the QCQP with two quadratic inequality constraints when there is only one nonzero Lagrangian multiplier, while Ben-Tal and Den Hertog (2014) proved the objective exactness for the SD-QCQP under the same conditions. It is worthy of mentioning that the work in Ben-Tal and Den Hertog (2014) assumed the Slater condition, finite optimal value, and bounded optimal set, which clearly satisfies the presumption in Part (b) of Theorem 10.

**Corollary 16** For the QCQP (5) with two quadratic inequality constraints, suppose that its DWR satisfies the relaxed Slater condition and yields finite optimal value and bounded optimal set. Then the objective exactness holds if one of the optimal Lagrangian multipliers is zero.

**Proof.** For the QCQP with two quadratic inequality constraints in matrix form, as indicated in Corollary 10, its DWR has the additional constraint $X_{11} = 1$ but its domain set $\mathcal{X} = \{ \mathbf{X} \in S^{n+1}_+ : \text{rank}(\mathbf{X}) \leq 1 \}$ contains any face of $\overline{\text{conv}}(\mathcal{X})$ with dimension two or less. Therefore, the objective exactness directly follows from Theorem 10 if one of the optimal Lagrangian multipliers corresponding to two quadratic constraints is equal to zero. $\square$
4. Conclusion

In this paper, we study the rank constrained optimization problem (RCOP). Replacing its domain set including a rank constraint by the closed convex hull offers us a convex Dantzig-Wolfe Relaxation (DWR) of the RCOP. We have derived the first-known necessary and sufficient conditions for three notions of DWR exactness. Our proposed conditions have identified, for the first time, the effect of the domain set on determining the DWR exactness. Specifically, the DWR exactness relies on how many faces in the closed convex hull of the domain set are contained in the original domain set. There are many potential applications of our results to provide an exact DWR relaxation. For example, we can analyze the sparse constrained machine learning problems where the zero-norm constraint can be viewed as the rank constraint of a diagonal matrix. We are working on studying the rank bound when the domain set is defined by spectral functions and developing an efficient column generation solution approach.

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