Superlinear Convergence of an Infeasible Interior Point Algorithm on the Homogeneous Feasibility Model of a Semi-definite Program

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Abstract

In the literature, superlinear convergence of implementable polynomial-time interior point algorithms to solve semi-definite programs (SDPs) can only be shown by (i) assuming that the given SDP is nondegenerate and modifying these algorithms, or (ii) considering special classes of SDPs, such as the class of linear semi-definite feasibility problems, when a suitable initial iterate is required as well. Otherwise, these algorithms are not easy to implement even though they can be shown to have polynomial iteration complexities and superlinear convergence. These are besides the assumption of strict complementarity imposed on the given SDP. In this paper, we show superlinear convergence of an implementable interior point algorithm that has polynomial iteration complexity when it is used to solve the homogeneous feasibility model of a primal-dual SDP pair that has no special structure imposed. This is achieved by only assuming strict complementarity and the availability of an interior feasible point to the primal SDP. Furthermore, we do not need to modify the algorithm to show this.

Keywords. Semi-definite program; semi-definite linear complementarity problem; homogeneous feasibility model; interior point method; superlinear convergence.

1   Introduction

Many problems in diverse areas, such as optimal control, estimation and signal processing, communications and networks, statistics, and finance, can be modelled well as semi-definite programs (SDPs) [3]. Finding effective and efficient ways to solve this class of problems is hence practically important. Interior point methods (IPMs) have been proven to be successful in solving SDPs - see for example [1, 7, 9, 23]. Research on interior point methods is still ongoing with recent papers, such as [2, 4, 18], proposing contemporary interior point algorithms to solve symmetric cone problems, which include semi-definite programs.

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Among different IPMs, primal-dual path following interior point algorithms are the most successful and most widely studied. In this paper, we focus on an infeasible predictor-corrector primal-dual path following interior point algorithm to solve the homogeneous feasibility model [15] of a primal-dual SDP pair. Global convergence, in particular, polynomial iteration complexity of the algorithm has been shown in [15]. In this paper, we consider the local convergence behavior of the algorithm.

It proves not an easy task to show superlinear convergence of an implementable interior point algorithm that has polynomial iteration complexity on an SDP with minimal assumptions on the problem and no modifications to the algorithm. In the literature, such as [6], in addition to strict complementarity assumption, nondegeneracy assumption at an optimal solution and modifications to the algorithm, such as solving the corrector-step linear system in an iteration repeatedly instead of only once ("narrowing" the central path neighborhood[1]), need to be imposed for superlinear convergence of the interior point algorithm. In [8], without assuming nondegeneracy, the feasible interior point algorithm considered in the paper is shown to have polynomial iteration complexity and superlinear convergence. However, the algorithm is not easy to implement. The idea behind the algorithm considered in [8] to have superlinear convergence when solving an SDP is to force the centrality measure of the kth iterate to converge to zero as k tends to infinity. This can be enforced in practice by for example, repeatedly solving the corrector-step linear system in an iteration, as in [6], so as to "narrow" the neighborhood of the central path in which iterates lie. Therefore, we require existing algorithms to be modified for superlinear convergence. If these algorithms are not modified, then special structure needs to be imposed on the SDP, such as, considering linear semi-definite feasibility problems (LSDFPs), for superlinear convergence [19, 20]. The latter also additionally requires one to choose a suitable initial iterate. In this paper, for the first time, we show superlinear convergence of an implementable interior point algorithm having polynomial iteration complexity on a primal-dual SDP pair by assuming strict complementarity and without special structure imposed on the problem. Furthermore, no modifications to the algorithm, such as repeatedly solving the corrector-step linear system, instead of once, in an iteration, is needed to achieve this, although a suitable initial iterate is required. To show superlinear convergence, we consider the algorithm applied to the homogeneous feasibility model of the primal-dual SDP pair.

In Section 2, we describe the homogeneous feasibility model of a primal-dual SDP pair. Then in Section 3, we express the homogeneous feasibility model as a semi-definite linear complementarity problem. This allows us to apply results in the literature in Section 4 to show superlinear convergence of an implementable interior point algorithm on the homogeneous feasibility model. We conclude the paper with Section 5.

1.1 Notations

The space of symmetric $n \times n$ matrices is denoted by S^n. The cone of positive semi-definite (resp., positive definite) symmetric matrices is denoted by S^n_+ (resp. S^n_{++}). The identity matrix is denoted by $I_{n \times n}$, where n stands for the size of the matrix. We omit the subscript when the size of the identity matrix is clear from the contex. The matrix $E^{ij} \in \Re^{n_1 \times n_2}$ is defined to have 1 in its (i, j) entry, and zero everywhere else.

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[1]We note that this idea is used in [13] to show superlinear convergence of an interior point algorithm on a wide class of conic optimization problems. Also, this is related to the sufficient conditions on behavior of iterates generated by the algorithm as studied in [16, 17] for superlinear convergence.

Given a matrix $G \in \Re^{n_1 \times n_2}$, $\|G\|_F := \sqrt{\mathrm{Tr}(GG^T)}$ is the Frobenius norm of G, where $\mathrm{Tr}(\cdot)$ is the trace of a square matrix. G_{ij} is the entry of G in the ith row and the jth column of G.

Given a vector $x \in \Re^n$, $\|x\|$ refers to its Euclidean norm. Also, $\mathrm{Diag}(x) \in \Re^{n \times n}$ is a square matrix with the entries of $x \in \Re^n$ making up the main diagonal elements of the matrix, with all its other elements equal to zero.

Given $X \in S^n$, svec(X) is defined to be

$$\mathrm{svec}(X) := (X_{11}, \sqrt{2}X_{21}, \ldots, \sqrt{2}X_{n1}, X_{22}, \sqrt{2}X_{32}, \ldots, X_{n-1,n-1}, \sqrt{2}X_{n,n-1}, X_{nn})^T \in \Re^{\tilde{n}},$$

where $\tilde{n} = n(n+1)/2$. svec$(\cdot)$ sets up a one-to-one correspondence between S^n and $\Re^{\tilde{n}}$.

Given functions $f : \Omega \to E$ and $g : \Omega \to \Re_{++}$, where Ω is an arbitrary set and E is a normed vector space with norm $\|\cdot\|$. For a subset $\hat{\Omega} \subseteq \Omega$, we write $f(w) = \mathcal{O}(g(w))$ for all $w \in \hat{\Omega}$ to mean that $\|f(w)\| \leq Mg(w)$ for all $w \in \hat{\Omega}$, where $M > 0$ is a positive constant. Suppose $E = S^n$. Then we write $f(w) = \Theta(g(w))$ if for all $w \in \hat{\Omega}$, $f(w) \in S^n_{++}$, and $f(w) = \mathcal{O}(g(w))$, $f(w)^{-1} = \mathcal{O}(1/g(w))$.

2 A Semi-definite Program and its Homogeneous Feasibility Model

Given $C, A_i \in S^n$, $i = 1, \ldots, m$, and $b = (b_1, \ldots, b_m)^T \in \Re^m$. A (primal) semi-definite program (SDP) is given by

$$\begin{aligned} \min \quad & \mathrm{Tr}(CX) \\ \text{subject to} \quad & \mathrm{Tr}(A_i X) = b_i, \ i = 1, \ldots, m, \\ & X \in S^n_+. \end{aligned} \tag{1}$$

The dual of (1) is given by

$$\begin{aligned} \max \quad & b^T y \\ \text{subject to} \quad & \sum_{i=1}^m y_i A_i + Y = C, \\ & Y \in S^n_+. \end{aligned} \tag{2}$$

Here, $y = (y_1, \ldots, y_m)^T \in \Re^m$.

We impose the following assumptions on primal-dual SDP pair (1)-(2) throughout this paper:

Assumption 2.1 *(a) There exist $X \in S^n_{++}$ and $(y, Y) \in \Re^m \times S^n_{++}$ that is feasible to (1) and (2) respectively.*

(b) $A_1, \ldots, A_m$ are linearly independent.

The above assumptions, in particular, Assumption 2.1(a), ensure that there exists an optimal solution X^* to (1) and an optimal solution (y^*, Y^*) to (2) such that $\mathrm{Tr}(X^*Y^*) = 0$ - see for example, [3].

We now introduce the homogeneous feasibility model, that appears in [15], that gives optimal solutions to (1) and (2). It is given by the following homogeneous system:

$$\mathrm{Tr}(A_i X) = b_i \tau, \ i = 1, \ldots, m, \tag{3}$$

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$$\sum_{i=1}^{m} y_i A_i + Y = \tau C, \tag{4}$$

$$\kappa = b^T y - \text{Tr}(CX), \tag{5}$$

$$X \in S_+^n, Y \in S_+^n, \tau \geq 0, \kappa \geq 0. \tag{6}$$

Observe that (3)-(5) implies that

$$\text{Tr}(XY) + \tau\kappa = 0, \tag{7}$$

from which we obtain, using (6),

$$XY = 0,$$
$$\tau\kappa = 0.$$

Furthermore, observe that a solution to the homogeneous system is readily available and is given by $(X, y, Y, \tau, \kappa) = (0, 0, 0, 0, 0)$. However, we cannot derive optimal solutions to (1) and (2) from this. If there exists a solution $(X^*, y^*, Y^*, \tau^*, \kappa^*)$ to (3)-(6) such that $\kappa^* = 0$ and $\tau^* > 0$, then $(X^*/\tau^*, y^*/\tau^*, Y^*/\tau^*)$ is an optimal solution to primal-dual SDP pair (1)-(2) with zero duality gap. Conversely, if (X^*, y^*, Y^*) is an optimal solution to primal-dual SDP pair (1)-(2) with zero duality gap, then $(X^*, y^*, Y^*, 1, 0)$ solves the system (3)-(6). By Assumption 2.1(a), which ensures an optimal solution to primal-dual SDP pair (1)-(2) with zero duality gap, we see that this optimal solution can be obtained by solving (3)-(6). Interior point algorithms when applied to the homogeneous feasibility model can be used to find optimal solutions to primal-dual SDP pair (1)-(2) by finding a solution $(X^*, y^*, Y^*, \tau^*, \kappa^*)$ to (3)-(6) such that $\kappa^* = 0$ and $\tau^* > 0$. Such an interior point algorithm is discussed in [15], which is also given in Section 4 below. These interior point algorithms necessarily have to be of the infeasible type in that the initial iterate and subsequent iterates generated by the algorithm cannot satisfy (3)-(5). This is so because if an iterate $(X_k, y_k, Y_k, \tau_k, \kappa_k) \in S_{++}^n \times \Re^m \times S_{++}^n \times \Re_{++} \times \Re_{++}$ for some $k \geq 0$ satisfies (3)-(5), then (7) holds, which is impossible since $X_k, Y_k \in S_{++}^n$ and $\tau_k, \kappa_k > 0$.

3 Homogeneous Feasibility Model as a Semi-definite Linear Complementarity Problem

Recall that a semi-definite linear complementarity problem (SDLCP), introduced in [7], is given by:

$$\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = q, \tag{8}$$

$$X^1 Y^1 = 0, \tag{9}$$

$$X^1, Y^1 \in S_+^{n_1}, \tag{10}$$

where $\mathcal{A}_1, \mathcal{B}_1 : S^{n_1} \to \Re^{\tilde{n}_1}$ are linear operators, $q \in \Re^{\tilde{n}_1}$, and $\tilde{n}_1 = n_1(n_1+1)/2$. The following assumptions are imposed on the SDLCP, which we show in Proposition 3.3 to hold for the SDLCP representation of the homogeneous feasibility model of primal-dual SDP (1)-(2). We are going to derive this representation in this section.

Assumption 3.1 *(a) System (8)-(10) is monotone. That is, $\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = 0$ for $X^1, Y^1 \in S^{n_1} \Rightarrow \text{Tr}(X^1 Y^1) \geq 0$.*

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(b) There exists at least one solution to SDLCP (8)-(10).

(c) $\{\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) \; ; \; X^1, Y^1 \in S^{n_1}\} = \Re^{\tilde{n}_1}.$

When SDLCP (8)-(10) is studied in some papers in the literature, instead of Assumption 3.1(b), the following assumption is imposed:

Assumption 3.2 *There exist* $X^1, Y^1 \in S^{n_1}_{++}$ *such that* $\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = q.$

Assumptions 3.1(a), (c) and 3.2 are imposed in [7] where the paper studies feasible interior point algorithms on SDLCP (8)-(10), while [6, 19] assume Assumption 3.1 in the study of infeasible interior point algorithms on SDLCP (8)-(10). In the study of infeasible interior point algorithms on SDLCP (8)-(10), it is not necessary to impose Assumption 3.2. A reason being that we do not need a strictly feasible initial iterate in the algorithm. In this paper, we solve primal-dual SDP pair (1)-(2) by applying an interior point algorithm on homogeneous feasibility model (3)-(6), the interior point algorithm is necessarily infeasible as discussed at the end of the previous section. When we express the homogeneous feasibility model as an SDLCP (to be discussed next), the interior point algorithm that is applied to the homogeneous feasibility model can be equivalently applied to the corresponding SDLCP, and by necessity, the algorithm on the SDLCP is infeasible just like that for the homogeneous feasibility model. Therefore, having Assumption 3.2 imposed on the SDLCP is not suitable and in fact can never hold in our case as there cannot exist an $(X, Y) \in S^n_{++} \times S^n_{++}$ such that (8) holds in the SDLCP obtained from the homogeneous feasibility model. We therefore have Assumption 3.1(b) in its place. In fact, we will see later that when primal-dual SDP pair (1)-(2) satisfies Assumption 2.1, Assumption 3.1 holds for the SDLCP that results from homogeneous feasibility model (3)-(6).

We now proceed to express homogeneous feasibility model (3)-(6) as an SDLCP.

We can write (3) as

$$
\begin{bmatrix}
\operatorname{svec}(A_1)^T & -b_1 \\
\vdots & \vdots \\
\operatorname{svec}(A_m)^T & -b_m
\end{bmatrix}
\begin{bmatrix}
\operatorname{svec}(X) \\
\tau
\end{bmatrix} = 0.
\tag{11}
$$

On the other hand, combining (4) and (5) into one equation, we obtain

$$
\begin{bmatrix}
\operatorname{svec}(A_1) & \dots & \operatorname{svec}(A_m) \\
-b_1 & \dots & -b_m
\end{bmatrix} y +
\begin{bmatrix}
0 & -\operatorname{svec}(C) \\
\operatorname{svec}(C)^T & 0
\end{bmatrix}
\begin{bmatrix}
\operatorname{svec}(X) \\
\tau
\end{bmatrix} +
\begin{bmatrix}
\operatorname{svec}(Y) \\
\kappa
\end{bmatrix} = 0.
\tag{12}
$$

Let us now rewrite (11) and (12) in a more compact form. Let

$$
\mathcal{A} :=
\begin{bmatrix}
\operatorname{svec}(A_1)^T \\
\vdots \\
\operatorname{svec}(A_m)^T
\end{bmatrix},
$$

and recall that $b = (b_1, \dots, b_m)^T$. Then, (11) and (12) can be written as

$$
[\mathcal{A} \; -b]
\begin{bmatrix}
\operatorname{svec}(X) \\
\tau
\end{bmatrix} = 0
\tag{13}
$$

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and

$$\begin{bmatrix} \mathcal{A}^T \\ -b^T \end{bmatrix} y + \begin{bmatrix} 0 & -\mathrm{svec}(C) \\ \mathrm{svec}(C)^T & 0 \end{bmatrix} \begin{bmatrix} \mathrm{svec}(X) \\ \tau \end{bmatrix} + \begin{bmatrix} \mathrm{svec}(Y) \\ \kappa \end{bmatrix} = 0 \qquad (14)$$

respectively.

Now, let the following set of linearly independent vectors in $\Re^{\tilde{n}+1}$, which are orthogonal,

$$\left\{ \begin{bmatrix} \mathrm{svec}(B_1) \\ d_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{svec}(B_{\tilde{n}+1-m}) \\ d_{\tilde{n}+1-m} \end{bmatrix} \right\}$$

spans the orthogonal subspace to the space spanned by

$$\left\{ \begin{bmatrix} \mathrm{svec}(A_1) \\ -b_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{svec}(A_m) \\ -b_m \end{bmatrix} \right\},$$

where $\tilde{n} = n(n+1)/2$.

Let

$$\mathcal{B} := \begin{bmatrix} \mathrm{svec}(B_1)^T \\ \vdots \\ \mathrm{svec}(B_{\tilde{n}+1-m})^T \end{bmatrix}, \quad d := \begin{bmatrix} d_1 \\ \vdots \\ d_{\tilde{n}+1-m} \end{bmatrix}.$$

Then, (14) holds if and only if

$$[d\mathrm{svec}(C)^T \quad -\mathcal{B}\mathrm{svec}(C) \quad \mathcal{B} \quad d] \begin{bmatrix} \mathrm{svec}X \\ \tau \\ \mathrm{svec}(Y) \\ \kappa \end{bmatrix} = 0.$$

The above development implies that (3)-(5) can be rewritten as

$$\begin{bmatrix} \mathcal{A} & -b & 0 & 0 \\ d\mathrm{svec}(C)^T & -\mathcal{B}\mathrm{svec}(C) & \mathcal{B} & d \end{bmatrix} \begin{bmatrix} \mathrm{svec}(X) \\ \tau \\ \mathrm{svec}(Y) \\ \kappa \end{bmatrix} = 0. \qquad (15)$$

We have (15) together with

$$X \in S_+^n, Y \in S_+^n, \tau \geq 0, \kappa \geq 0, \qquad (16)$$

is also the homogeneous feasibility model of the SDP pair (1)-(2).

We now express the homogeneous feasibility model as an SDLCP, by first making the following observation:

Proposition 3.1 *For* $i = 1, \ldots, \tilde{n} + 1 - m,$

$$\begin{bmatrix} d_i \mathrm{svec}(C) \\ -\mathrm{svec}(B_i)^T \mathrm{svec}(C) \end{bmatrix} \in \Re^{\tilde{n}+1}$$

is a linear combination of

$$\begin{bmatrix} \mathrm{svec}(A_1) \\ -b_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{svec}(A_m) \\ -b_m \end{bmatrix}.$$

Proof: First, note that

$$\left\{ \begin{bmatrix} \mathrm{svec}(A_1) \\ -b_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{svec}(A_m) \\ -b_m \end{bmatrix}, \begin{bmatrix} \mathrm{svec}(B_1) \\ d_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{svec}(B_{\tilde{n}+1-m}) \\ d_{\tilde{n}+1-m} \end{bmatrix} \right\}$$

spans $\Re^{\tilde{n}+1}$. Therefore, given

$$\begin{bmatrix} d_i \mathrm{svec}(C) \\ -\mathrm{svec}(B_i)^T \mathrm{svec}(C) \end{bmatrix},$$

there exists u_j^i, $j = 1, \ldots, m$, v_i, v_k^i, for $k = 1, \ldots, \tilde{n} + 1 - m$, $k \neq i$, such that

$$\begin{bmatrix} d_i \mathrm{svec}(C) \\ -\mathrm{svec}(B_i)^T \mathrm{svec}(C) \end{bmatrix} = \sum_{j=1}^m u_j^i \begin{bmatrix} \mathrm{svec}(A_j) \\ -b_j \end{bmatrix} + v_i \begin{bmatrix} \mathrm{svec}(B_i) \\ d_i \end{bmatrix} + \sum_{k=1, k \neq i}^{\tilde{n}+1-m} v_k^i \begin{bmatrix} \mathrm{svec}(B_k) \\ d_k \end{bmatrix}.$$

Multiplying both sides of the above equality by $(\mathrm{svec}(B_l)^T \ d_l)$ and noting that

$$\left\{ \begin{bmatrix} \mathrm{svec}(B_1) \\ d_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{svec}(B_{\tilde{n}+1-m}) \\ d_{\tilde{n}+1-m} \end{bmatrix} \right\}$$

is an orthogonal set which is orthogonal to $\begin{bmatrix} \mathrm{svec}(A_j) \\ -b_j \end{bmatrix}$, $j = 1, \ldots, m$, we first observe that if $l = i$, then $v_i = 0$. If $l \neq i$, then

$$d_i \mathrm{svec}(B_l)^T \mathrm{svec}(C) - d_l \mathrm{svec}(B_i)^T \mathrm{svec}(C) = v_l^i \left\| \begin{bmatrix} \mathrm{svec}(B_l) \\ d_l \end{bmatrix} \right\|^2. \tag{17}$$

On the other hand, we have

$$d_l \mathrm{svec}(B_i)^T \mathrm{svec}(C) - d_i \mathrm{svec}(B_l)^T \mathrm{svec}(C) = v_i^l \left\| \begin{bmatrix} \mathrm{svec}(B_i) \\ d_i \end{bmatrix} \right\|^2. \tag{18}$$

Adding (17) and (18), we get $v_l^i = v_i^l = 0$, and the proposition is proved. ∎

Using Proposition 3.1, by performing row operations on (15), we can write (15) as

$$\begin{bmatrix} \mathcal{A} & -b & 0 & 0 \\ 0 & 0 & \mathcal{B} & d \end{bmatrix} \begin{bmatrix} \mathrm{svec}(X) \\ \tau \\ \mathrm{svec}(Y) \\ \kappa \end{bmatrix} = 0. \tag{19}$$

It is easy to convince ourselves that (19), (16) is also the homogeneous feasibility model of primal-dual SDP pair (1)-(2), just like (3)-(6), and (15), (16).

We are now ready to express the homogeneous feasibility model of primal-dual SDP pair (1)-(2) as SDLCP (8)-(10) by letting $n_1 = n + 1$, $q = 0$, and $\mathcal{A}_1, \mathcal{B}_1$ such that

$$\begin{aligned} (\mathcal{A}_1(X^1))_i &:= \mathrm{Tr}\left(\begin{bmatrix} A_i & 0 \\ 0 & -b_i \end{bmatrix} X^1 \right), \quad i = 1, \ldots, m, \\ (\mathcal{A}_1(X^1))_i &:= \mathrm{Tr}(E^{i-m,n+1} X^1), \quad i = m+1, \ldots, m+n, \\ (\mathcal{A}_1(X^1))_i &:= 0, \quad i = m+n+1, \ldots, \tilde{n}_1, \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
(\mathcal{B}_1(Y^1))_j &:= 0, \quad j = 1, \ldots, m+n, \\
(\mathcal{B}_1(Y^1))_j &:= \mathrm{Tr}\left(\left[\begin{array}{cc} B_{j-(m+n)} & 0 \\ 0 & d_{j-(m+n)} \end{array}\right] Y^1\right), \quad j = m+n+1, \ldots, \tilde{n}_1,
\end{aligned}
\tag{21}
$$

for $X^1, Y^1 \in S^{n_1}$.

Remark 3.1 *Recall that a linear semi-definite feasibility problem (LSDFP) written as SDLCP (8)-(10) is such that in (8), $(\mathcal{A}_1(X^1))_i = 0$ for $i = m_1 + 1, \ldots, \tilde{n}_1$, $(\mathcal{B}_1(Y^1))_j = 0$ for $j = 1, \ldots, m_1$, and $q_i = 0$ for $i = m_1 + 1, \ldots, \tilde{n}_1$ or $q_i = 0$ for $i = 1, \ldots, m_1$. Here, $q = (q_1, \ldots, q_{\tilde{n}_1})^T$. From (20), (21), we therefore see that the SDLCP representation of the homogeneous feasibility model has the structure of an LSDFP with $m_1 = m + n$ and $q = 0$. This observation is important in that we are then able to apply results in the literature, namely [19], to show the main result in the paper in Theorem 4.2.*

The following proposition relates a solution of the homogeneous feasibility model of primal-dual SDP pair (1)-(2) to that of its SDLCP representation:

Proposition 3.2 *We have if (X, Y, τ, κ) satisfies (19), (16), then*

$$
X^1 = \left[\begin{array}{cc} X & 0 \\ 0 & \tau \end{array}\right], \quad Y^1 = \left[\begin{array}{cc} Y & * \\ * & \kappa \end{array}\right],
\tag{22}
$$

where the "$*$" entries in Y^1 are such that $Y^1 \in S_+^{n_1}$, satisfies (8)-(10), where $n_1 = n+1$, $q = 0$, $\mathcal{A}_1, \mathcal{B}_1$ are given by (20), (21) respectively. On the other hand, if (X^1, Y^1) satisfies (8)-(10), where $n_1 = n + 1$, $q = 0$, $\mathcal{A}_1, \mathcal{B}_1$ are given by (20), (21) respectively, then X^1, Y^1 are given by (22), and (X, Y, τ, κ) satisfies (19), (16).

Proof: The proposition is clear based on how $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined in (20) and (21) respectively. ∎

We have another proposition below that shows that if primal-dual SDP pair (1)-(2) satisfies Assumption 2.1, then the SDLCP representation of its homogeneous feasibility model satisfies Assumption 3.1:

Proposition 3.3 *SDLCP (8)-(10) with $n_1 = n + 1$, $q = 0$ and $\mathcal{A}_1, \mathcal{B}_1$ given by (20), (21) respectively, satisfies Assumption 3.1 when primal-dual SDP pair (1)-(2) satisfies Assumption 2.1.*

Proof: We first make two observations. Firstly, if (X, Y, τ, κ) satisfies (19), then $\mathrm{Tr}(XY) + \tau\kappa = 0$. Secondly, the matrix on the left-hand side of (19) has full row rank by Assumption 2.1(b).

Given SDLCP (8)-(10) with $n_1 = n + 1$, $q = 0$ and $\mathcal{A}_1, \mathcal{B}_1$ given by (20), (21) respectively. Suppose

$$
\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = 0
$$

for some $(X^1, Y^1) \in S^{n_1} \times S^{n_1}$. Then X^1 and Y^1 are given by

$$
X^1 = \left[\begin{array}{cc} X & 0 \\ 0 & \tau \end{array}\right], \quad Y^1 = \left[\begin{array}{cc} Y & * \\ * & \kappa \end{array}\right],
$$

where (X, Y, τ, κ) satisfies (19). Hence, by the first observation above, we have $\text{Tr}(X^1 Y^1) = \text{Tr}(XY) + \tau\kappa = 0$. Therefore, Assumption 3.1(a) holds. Furthermore, Assumption 3.1(b) holds since a solution to the given SDLCP is $X^1 = 0, Y^1 = 0$. Finally, the second observation above means that the matrix

$$
\begin{bmatrix}
\mathcal{A} & -b & 0 & 0 \\
0 & 0 & \mathcal{B} & d
\end{bmatrix}
$$

has full row rank. This implies that the matrix $(\mathcal{A}_1 \ \mathcal{B}_1)$, where $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined by (20) and (21) respectively, has full row rank, and hence Assumption 3.1(c) holds as well. ∎

4 An Infeasible Interior Point Algorithm on the Homogeneous Feasibility Model

We describe in this section an infeasible path-following interior point algorithm on homogeneous feasibility model (3)-(6) (or (15), (16) or (19), (16)). It generates iterates following an infeasible central path in a (narrow) neighborhood. This algorithm is a predictor-corrector type algorithm applied to the homogeneous feasibility model, and is first considered in [15].

An infeasible central path to homogeneous feasibility model (3)-(6), $(X^c(\mu), y^c(\mu), Y^c(\mu), \tau^c(\mu), \kappa^c(\mu)) \in S_{++}^n \times \Re^m \times S_{++}^n \times \Re_{++} \times \Re_{++}$, satisfies

$$
\begin{aligned}
X^c(\mu) Y^c(\mu) &= \mu I, \\
\tau^c(\mu)\kappa^c(\mu) &= \mu,
\end{aligned}
$$

besides satisfying

$$
\begin{aligned}
\text{Tr}(A_i X^c(\mu)) - b_i \tau^c(\mu) &= \frac{\mu}{\mu_0}(\text{Tr}(A_i X_0) - b_i \tau_0), \ i = 1, \ldots, m, \\
\sum_{i=1}^{m}(y^c(\mu))_i A_i + Y(\mu) - \tau(\mu)C &= \frac{\mu}{\mu_0}\left(\sum_{i=1}^{m}(y_0)_i A_i + Y_0 - \tau_0 C\right), \\
\kappa(\mu) - b^T y(\mu) + \text{Tr}(CX(\mu)) &= \frac{\mu}{\mu_0}(\kappa_0 - b^T y_0 + \text{Tr}(CX_0))
\end{aligned}
$$

for a given $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in S_{++}^n \times \Re^m \times S_{++}^n \times \Re_{++} \times \Re_{++}$, which is the value taken by $(X^c(\mu), y^c(\mu), Y^c(\mu), \tau^c(\mu), \kappa^c(\mu))$ when $\mu = \mu_0$.

Remark 4.1 *In [19], under Assumption 3.1, the author discusses the existence and uniqueness of infeasible off-central paths, as defined in the paper, to SDLCP (8)-(10). Infeasible central path is an important and special infeasible off-central path. Since the SDLCP representation of the homogeneous feasibility model as detailed in the previous section satisfies Assumption 3.1 (Proposition 3.3), it means that infeasible central path to the SDLCP exists and is unique. This ensures that given an infeasible interior point to the homogeneous feasibility model, the infeasible central path to the model, as defined above, passing through the point, exists and is unique. Furthermore, any of its accumulation points is a solution to the model as is any of the accumulation points of an infeasible off-central path to SDLCP (8)-(10).*

From now onwards, whenever we mention central path, we are referring to the infeasible central path.

Consider the following (narrow) neighborhood of the central path:

$$\mathcal{N}(\beta, \mu) \ := \ \{(X, y, Y, \tau, \kappa) \in S_{++}^n \times \Re^m \times S_{++}^n \times \Re_{++} \times \Re_{++} \ ;$$
$$(\|Y^{1/2}XY^{1/2} - \mu I\|_F^2 + (\tau\kappa - \mu)^2)^{1/2} \leq \beta\mu, \ \mu = (\mathrm{Tr}(XY) + \tau\kappa)/(n+1)\}.$$

In the algorithm described below, iterates generated by the algorithm always stay within a neighborhood of the central path. We consider the dual Helmberg-Kojima-Monteiro (HKM) search direction in the algorithm - see Remark 4.4. Among different search directions used in interior point algorithms on SDLCPs/SDPs, the Alizadeh-Haeberly-Overton (AHO) [1], Helmberg-Kojima-Monteiro (HKM) [5, 7, 9] and Nesterov-Todd (NT) [11, 12] search directions are better known, with the latter two being implemented in SDP solvers, such as SeDuMi [21] and SDPT3 [22].

Solving the following system of equations for $(\Delta X, \Delta y, \Delta Y, \Delta \tau, \Delta \kappa) \in S^n \times \Re^m \times S^n \times \Re \times \Re$ plays an important role in the algorithm:

$$Y^{1/2}(X\Delta Y + \Delta XY)Y^{-1/2} + Y^{-1/2}(\Delta YX + Y\Delta X)Y^{1/2} \ = \ 2(\sigma\mu I - Y^{1/2}XY^{1/2}), \ (23)$$

$$\kappa\Delta\tau + \tau\Delta\kappa \ = \ \sigma\mu - \tau\kappa, \tag{24}$$

$$\mathrm{Tr}(A_i\Delta X) - b_i\Delta\tau \ = \ -\bar{r}_i, \quad i = 1, \ldots, m, \tag{25}$$

$$\sum_{i=1}^m \Delta y_i A_i + \Delta Y - \Delta\tau C \ = \ -\bar{s}, \tag{26}$$

$$\Delta\kappa - b^T\Delta y + \mathrm{Tr}(C\Delta X) \ = \ -\bar{\gamma}, \tag{27}$$

where $\mu = (\mathrm{Tr}(XY) + \tau\kappa)/(n+1)$. Note that it can be shown that $(\Delta X, \Delta y, \Delta Y, \Delta\tau, \Delta\kappa)$ in (23)-(27) is uniquely determined [15].

The infeasible predictor-corrector path-following interior point algorithm on homogeneous feasibility model (3) - (6) is described as follows:

Algorithm 4.1 *(See Algorithm 5.1 of [15]) Given $\beta_1 < \beta_2$ with $\beta_2^2/(2(1-\beta_2)^2) \leq \beta_1 < \beta_2 < \beta_2/(1-\beta_2) < 1$. Choose $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$ with $(n+1)\mu_0 = \mathrm{Tr}(X_0 Y_0) + \tau_0\kappa_0$. For $k = 0, 1, \ldots,$ do (a1) through (a5):*

(a1) Set $X = X_k$, $y = y_k$, $Y = Y_k$, $\tau = \tau_k$, $\kappa = \kappa_k$, and define

$$r_i \ := \ \mathrm{Tr}(A_i X) - b_i\tau, \quad i = 1, \ldots, m,$$

$$s \ := \ \sum_{i=1}^m y_i A_i + Y - \tau C,$$

$$\gamma \ := \ \kappa - b^T y + \mathrm{Tr}(CX).$$

(a2) If $\max\{(\mathrm{Tr}(XY) + \tau\kappa)/\tau^2, |r_1/\tau|, \ldots, |r_m/\tau|, \|s/\tau\|\} \leq \epsilon$, then report X/τ, y/τ and Y/τ as an approximate solution to (1) and (2), respectively, and terminate. If τ is sufficiently small, terminate with no optimal solutions to (1) and (2) with zero duality gap.

(*a*3) [**Predictor Step**] *Find the solution* $(\Delta X_p, \Delta y_p, \Delta Y_p, \Delta \tau_p, \Delta \kappa_p)$ *of the linear system* *(23)-(27), with* $\sigma = 0$, $\bar{r}_i = r_i, i = 1, \ldots, m$, $\bar{s} = s$ *and* $\bar{\gamma} = \gamma$.
Define

$$\overline{X} = X + \overline{\alpha} \Delta X_p, \quad \overline{y} = y + \overline{\alpha} \Delta y_p, \quad \overline{Y} = Y + \overline{\alpha} \Delta Y_p, \quad \overline{\tau} = \tau + \overline{\alpha} \Delta \tau_p, \quad \overline{\kappa} = \kappa + \overline{\alpha} \Delta \kappa_p,$$

where the steplength $\overline{\alpha}$ *satisfies*

$$\alpha_1 \le \overline{\alpha} \le \alpha_2. \tag{28}$$

Here,

$$\alpha_1 = \frac{2}{\sqrt{1 + 4\delta/(\beta_2 - \beta_1)} + 1}, \tag{29}$$

$$\delta = \frac{1}{\mu} \left\| \begin{bmatrix} Y & 0 \\ 0 & \kappa \end{bmatrix}^{1/2} \begin{bmatrix} \Delta X_p & 0 \\ 0 & \Delta \tau_p \end{bmatrix} \begin{bmatrix} \Delta Y_p & 0 \\ 0 & \Delta \kappa_p \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & \kappa \end{bmatrix}^{-1/2} \right\|_F, \tag{30}$$

where

$$\mu = \frac{\text{Tr}(XY) + \tau\kappa}{n + 1},$$

and

$$\alpha_2 = \max\{\tilde{\alpha} \in [0, 1] \; ; \; (X + \alpha\Delta X_p, y + \alpha\Delta y_p, Y + \alpha\Delta Y_p, \tau + \alpha\Delta\tau_p, \kappa + \alpha\Delta\kappa_p)$$
$$\in \mathcal{N}(\beta_2, (1 - \alpha)\mu) \; \forall \; \alpha \in [0, \tilde{\alpha}]\}.$$

(*a*4) [**Corrector Step**] *Find the solution* $(\Delta X_c, \Delta y_c, \Delta Y_c, \Delta \tau_c, \Delta \kappa_c)$ *of the linear system* *(23)-(26), with* $\sigma = 1 - \overline{\alpha}$, $\bar{r}_i = 0, i = 1, \ldots, m$, $\bar{s} = 0$ *and* $\bar{\gamma} = 0$. *Set*

$$X_+ = \overline{X} + \Delta X_c, \quad y_+ = \overline{y} + \Delta y_c, \quad Y_+ = \overline{Y} + \Delta Y_c, \quad \tau_+ = \overline{\tau} + \Delta \tau_c, \quad \kappa_+ = \overline{\kappa} + \Delta \kappa_c,$$
$$\mu_+ = (1 - \overline{\alpha})\mu.$$

(*a*5) *Set*

$$X_{k+1} = X_+, \quad y_{k+1} = y_+, \quad Y_{k+1} = Y_+, \quad \tau_{k+1} = \tau_+, \quad \kappa_{k+1} = \kappa_+,$$
$$\mu_{k+1} = \mu_+.$$

The above algorithm can be easily adapted to solve primal-dual SDP pair (1)-(2) instead of its homogeneous feasibility model. An advantage of applying the algorithm on its homogeneous feasibility model is that we have superlinear convergence of iterates generated by the algorithm, as shown in Theorem 4.2.

For $k = 0, 1, \ldots$, define

$$
\begin{aligned}
(r_k)_i &:= \text{Tr}(A_i X_k) - b_i \tau_k, \quad i = 1, \ldots, m, \\
s_k &:= \sum_{i=1}^{m} (y_k)_i A_i + Y_k - \tau_k C, \\
\gamma_k &:= \kappa_k - b^T y_k + \text{Tr}(C X_k).
\end{aligned}
$$

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Remark 4.2 *For all $k \geq 0$, we have*

$$(X_k, y_k, Y_k, \tau_k, \kappa_k) \in \mathcal{N}(\beta_1, \mu_k).$$

Remark 4.2 holds by [15] - see also [17].

Remark 4.3 *Throughout this paper, we consider Algorithm 4.1 with initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$ such that $X_0 \in S_{++}^n$ is feasible to the primal SDP (1), $(y_0, Y_0) = (0, I)$, $\tau_0 = 1$, and $\kappa_0 = 1$. Therefore, we have $(r_0)_i = 0, i = 1, \ldots, m$, while s_0 and γ_0 are generally nonzero in the algorithm.*

The following theorem provides global convergence and polynomial iteration complexity of Algorithm 4.1. It is taken from [15] and is stated here for the sake of completeness, even though the focus of this paper is to study the local convergence behavior of iterates generated by the algorithm.

Theorem 4.1 *(See Theorem 5.2 of [15]) Assume that in Algorithm 4.1, we choose a starting point of the form $(X_0, y_0, Y_0, \tau_0, \kappa_0) = (I, 0, I, 1, 1)$. Let*

$$\epsilon_0 = \max\{\mathrm{Tr}(X_0 Y_0) + \tau_0 \kappa_0, |(r_0)_1|, \ldots, |(r_0)_m|, \|s_0\|\},$$

and let $\epsilon > 0$ be arbitrary. Then the following statements hold:

(i) *If there exists an optimal solution to (1) and (2) with zero duality gap, then Algorithm 4.1 terminates with an ϵ-approximate solution $(X_k/\tau_k, y_k/\tau_k, Y_k/\tau_k) \in S_+^n \times S_+^n$ with*

$$0 \leq \mathrm{Tr}(X_k Y_k/\tau_k^2) \leq \epsilon, \quad |(r_k)_i/\tau_k| \leq \epsilon, i = 1, \ldots, m, \quad \|s_k/\tau_k\| \leq \epsilon$$

in a finite number of steps $k = K_\epsilon < \infty$.

(ii) *If $\rho^* = \mathrm{Tr}(X^* + Y^*)$, where X^* solves (1) and (y^*, Y^*) solves (2) with zero duality gap, then $K_\epsilon = \mathcal{O}(\sqrt{n} \ln(\rho^* \epsilon_0/\epsilon))$.*

(iii) *For any choice of $\rho > 0$, there is an index $k = \hat{K}_\epsilon = \mathcal{O}(\sqrt{n} \ln(\rho \epsilon_0/\epsilon))$ such that either*

    (iiia) *$(X_k/\tau_k, y_k/\tau_k, Y_k/\tau_k)$ satisfies $0 \leq \mathrm{Tr}(X_k Y_k/\tau_k^2) \leq \epsilon, |(r_k)_i/\tau_k| \leq \epsilon, i = 1, \ldots, m$, $\|s_k/\tau_k\| \leq \epsilon$, or*

    (iiib) *$\tau_k < (1 - \beta_1)/(\rho + 1)$, and in this case there is no solution $X^*, (y^*, Y^*)$ that solves (1) and (2) respectively, with $\mathrm{Tr}(X^* + Y^*) \leq \rho$*

It is easy to convince ourselves that Theorem 4.1 still holds if $X_0, Y_0 \in S_{++}^n, y_0 \in \Re^m, \tau_0 > 0$ and $\kappa_0 > 0$, with $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$.

Algorithm 4.1 can be expressed as an equivalent algorithm, which we describe below, that is used to solve the SDLCP representation of the homogeneous feasibility model, (8)-(10), where $n_1 = n + 1$, $q = 0$, $\mathcal{A}_1, \mathcal{B}_1$ are given by (20), (21) respectively. Before describing the algorithm, we first define an analogous (narrow) neighborhood of the central path of the SDLCP representation:

$$\mathcal{N}_1(\beta, \mu^1) \; := \; \{(X^1, Y^1) \in S_{++}^{n_1} \times S_{++}^{n_1} \; ; \; \|(Y^1)^{1/2} X^1 (Y^1)^{1/2} - \mu^1 I\|_F \leq \beta \mu^1,$$
$$\mu^1 = \mathrm{Tr}(X^1 Y^1)/n_1\}.$$

Similar as before, we have a system of equations for $(\Delta X^1, \Delta Y^1) \in S^{n_1} \times S^{n_1}$ that plays an important role in the algorithm to solve the SDLCP representation, just like the system of equations for Algorithm 4.1:

$$(Y^1)^{1/2}(X^1\Delta Y^1 + \Delta X^1 Y^1)(Y^1)^{-1/2} + (Y^1)^{-1/2}(\Delta Y^1 X^1 + Y^1 \Delta X^1)(Y^1)^{1/2}$$
$$= 2(\sigma\mu^1 I - (Y^1)^{1/2}X^1(Y^1)^{1/2}), \tag{31}$$
$$\mathcal{A}_1(\Delta X^1) + \mathcal{B}_1(\Delta Y^1) = -\overline{r}. \tag{32}$$

Note that it can be shown that $(\Delta X^1, \Delta Y^1)$ obtained from (31), (32) exists and is unique [19, 17].

The corresponding algorithm to Algorithm 4.1 for the SDLCP representation is given by:

Algorithm 4.2 *(See Algorithm 4.1 of [19]; Algorithm 2.1 of [17]) Given $\beta_1 < \beta_2$ with $\beta_2^2/(2(1-\beta_2)^2) \le \beta_1 < \beta_2 < \beta_2/(1 - \beta_2) < 1$. Choose $(X_0^1, Y_0^1) \in S_{++}^{n_1} \times S_{++}^{n_1}$ such that*

$$X_0^1 = \begin{bmatrix} X_0 & 0 \\ 0 & \tau_0 \end{bmatrix}, \quad Y_0^1 = \begin{bmatrix} Y_0 & 0 \\ 0 & \kappa_0 \end{bmatrix}, \tag{33}$$

where $X_0, Y_0, \tau_0, \kappa_0$ are from the initial iterate in Algorithm 4.1. For $k = 0, 1, \ldots,$ do (a1) through (a5):

(a1) *Set $X^1 = X_k^1$, $Y^1 = Y_k^1$, and define*

$$r := \mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1).$$

(a2) *If $\max\{\mathrm{Tr}(X^1 Y^1)/(X^1)_{n_1,n_1}^2, \|r/X_{n_1,n_1}^1\|\} \le \epsilon$, then terminate with the solution (X^1, Y^1). If X_{n_1,n_1}^1 is sufficiently small, terminate with no optimal solutions to (1) and (2) with zero duality gap.*

(a3) **[Predictor Step]** *Find the solution $(\Delta X_p^1, \Delta Y_p^1)$ of the linear system (31), (32), with $\sigma = 0$, $\overline{r} = r$.*
Define

$$\overline{X}^1 = X^1 + \overline{\alpha}\Delta X_p^1, \quad \overline{Y}^1 = Y^1 + \overline{\alpha}\Delta Y_p^1,$$

where the steplength $\overline{\alpha}$ satisfies

$$\alpha_1 \le \overline{\alpha} \le \alpha_2. \tag{34}$$

Here,

$$\alpha_1 = \frac{2}{\sqrt{1 + 4\delta/(\beta_2 - \beta_1)} + 1}, \tag{35}$$

$$\delta = \frac{1}{\mu^1}\|(Y^1)^{1/2}\Delta X_p^1 \Delta Y_p^1 (Y^1)^{-1/2}\|_F, \tag{36}$$

where

$$\mu^1 = \frac{\mathrm{Tr}(X^1 Y^1)}{n_1},$$

and

$$\alpha_2 = \max\{\tilde{\alpha} \in [0, 1] \; ; \; (X^1 + \alpha\Delta X_p^1, Y^1 + \alpha\Delta Y_p^1) \in \mathcal{N}_1(\beta_2, (1 - \alpha)\mu^1) \; \forall \; \alpha \in [0, \tilde{\alpha}]\}.$$

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(a4) [**Corrector Step**] *Find the solution* $(\Delta X_c^1, \Delta Y_c^1)$ *of the linear system (31), (32), with* $\sigma = 1 - \overline{\alpha}$ *and* $\overline{r} = 0$. *Set*

$$X_+^1 = \overline{X}^1 + \Delta X_c^1, \quad Y_+^1 = \overline{Y}^1 + \Delta Y_c^1,$$
$$\mu_+^1 = (1 - \overline{\alpha})\mu^1.$$

(a5) *Set*

$$X_{k+1}^1 = X_+^1, \quad Y_{k+1}^1 = Y_+^1,$$
$$\mu_{k+1}^1 = \mu_+^1.$$

The following proposition relates the iterates in the two algorithms:

Proposition 4.1 *For all* $k \geq 0$,

$$X_k^1 = \begin{bmatrix} X_k & 0 \\ 0 & \tau_k \end{bmatrix}, \quad Y_k^1 = \begin{bmatrix} Y_k & 0 \\ 0 & \kappa_k \end{bmatrix}. \tag{37}$$

Consequently, $\mu_k^1 = \mu_k$ *and* $(X_k^1, Y_k^1) \in \mathcal{N}_1(\beta_1, \mu_k^1)$.

Proof: We show (37) holds by induction on $k \geq 0$. It is clear that (37) holds when $k = 0$, by the choice of X_0^1, Y_0^1. Suppose (37) holds for $k = k_0 \geq 0$. Then, by comparing the system of equations (23)-(27) and (31), (32), it can be seen easily that

$$\Delta X_p^1 = \begin{bmatrix} \Delta X_p & 0 \\ 0 & \Delta \tau_p \end{bmatrix}, \quad \Delta Y_p^1 = \begin{bmatrix} \Delta Y_p & 0 \\ 0 & \Delta \kappa_p \end{bmatrix}$$

satisfy (31), (32) when $\sigma = 0, \overline{r} = r = \mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1)$. Furthermore, the steplength $\overline{\alpha}$ in the $(k_0 + 1)^{\text{th}}$ iteration of both algorithms are the same. These lead to

$$\overline{X}^1 = \begin{bmatrix} \overline{X} & 0 \\ 0 & \overline{\tau} \end{bmatrix}, \quad \overline{Y}^1 = \begin{bmatrix} \overline{Y} & 0 \\ 0 & \overline{\kappa} \end{bmatrix}. \tag{38}$$

With (38), again comparing the system of equations (23)-(27) and (31), (32), it is also easy to see that

$$\Delta X_c^1 = \begin{bmatrix} \Delta X_c & 0 \\ 0 & \Delta \tau_c \end{bmatrix}, \quad \Delta Y_c^1 = \begin{bmatrix} \Delta Y_c & 0 \\ 0 & \Delta \kappa_c \end{bmatrix}$$

satisfy (31), (32) when $\sigma = 1 - \overline{\alpha}$ and $\overline{r} = 0$. Hence, we conclude that (37) holds for $k = k_0 + 1$. Therefore, by induction, (37) holds for all $k \geq 0$. Furthermore, we have

$$\mu_k^1 = \frac{\text{Tr}(X_k^1 Y_k^1)}{n_1} = \frac{\text{Tr}(X_k Y_k) + \tau_k \kappa_k}{n + 1} = \mu_k.$$

Finally, comparing the definition of the neighborhood $\mathcal{N}(\beta, \mu)$ and the neighborhood $\mathcal{N}_1(\beta, \mu^1)$, and that $\mu_k = \mu_k^1$, we see that since $(X_k, y_k, Y_k, \tau_k, \kappa_k) \in \mathcal{N}(\beta_1, \mu_k)$ (Remark 4.2), we have $(X_k^1, Y_k^1) \in \mathcal{N}_1(\beta_1, \mu_k^1)$. ∎

4.1 Superlinear Convergence

We show in this subsection that Algorithm 4.1 applied to the homogeneous feasibility model when the initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0)$ is such that $X_0 \in S_{++}^n$ is feasible to primal SDP (1), $(y_0, Y_0) = (0, I)$, $\tau_0 = 1$ and $\kappa_0 = 1$, leads to superlinear convergence of iterates generated by the algorithm, besides global convergence and polynomial iteration complexity (Theorem 4.1). First, we state an additional assumption, strict complementarity, on primal-dual SDP pair (1)-(2) that is needed for this result to hold. Note that strict complementarity assumption is generally considered the minimal requirement for superlinear convergence of interior point algorithms, as investigated for example in [10].

Assumption 4.1 *There exists an optimal solution X^* to primal SDP (1) and an optimal solution (y^*, Y^*) to dual SDP (2) such that $X^* + Y^* \in S_{++}^n$.*

A consequence of the above assumption on homogeneous feasibility model (3)-(6) is that it has a solution $(X^*, y^*, Y^*, \tau^*, \kappa^*)$ with $X^* + Y^* \in S_{++}^n$ and $\tau^* + \kappa^* > 0$. This then implies that its SDLCP representation has a solution $(X^{1,*}, Y^{1,*})$ such that $X^{1,*} + Y^{1,*} \in S_{++}^{n_1}$, that is, the SDLCP representation has a strictly complementary solution.

We consider local superlinear convergence using Algorithm 4.1 in the sense of

$$\frac{\mu_{k+1}}{\mu_k} \to 0, \quad \text{as } k \to \infty. \tag{39}$$

Consideration of superlinear convergence in the form (39) is typical in the interior point literature, such as [6, 8, 17]. It is closely related to local convergence behavior of iterates, as studied for example in [14].

The following result, which is the main result in this paper, ends this subsection:

Theorem 4.2 *Given an initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0)$ to Algorithm 4.1, with $X_0 \in S_{++}^n$ feasible to primal SDP (1), $(y_0, Y_0) = (0, I)$, $\tau_0 = 1$, and $\kappa_0 = 1$. Iterates generated by Algorithm 4.1 converge superlinearly in the sense of (39).*

Proof: Let us show the result in the theorem by considering iterates generated by Algorithm 4.2 instead. These iterates are related to those generated by Algorithm 4.1 in a close way as shown in Proposition 4.1. We note that Algorithm 4.2 is Algorithm 4.1 in [19], and Assumptions 2.1 and 3.1 in [19] are satisfied for the SDLCP representation of the homogeneous feasibility model (3)-(6) defined in the previous section (Proposition 3.3 and Assumption 4.1). Hence, results in [19] are applicable to our SDLCP representation. The SDLCP representation that Algorithm 4.2 is solving has the structure of an LSDFP (Remark 3.1), and therefore Theorem 5.1 in [19] can be applied on our SDLCP representation provided that Condition (51) in the theorem is satisfied.

Our choice of initial iterate to Algorithm 4.1 leads to an initial iterate, (X_0^1, Y_0^1), to Algorithm 4.2 that satisfies $\mathcal{A}_1(X_0^1) = 0$. Therefore, Condition (51) of Theorem 5.1 in [19] is satisfied, and by the theorem, we have superlinear convergence in the sense that

$$\frac{\mu_{k+1}^1}{\mu_k^1} \to 0, \quad \text{as } k \to \infty.$$

This implies by Proposition 4.1, where we have $\mu_k^1 = \mu_k$, superlinear convergence in the sense of (39) using Algorithm 4.1 to solve the homogeneous feasibility model (3)-(6) for the given initial iterate. ∎

We remark that the conclusion in Theorem 4.2 still holds if X_0 is a strictly feasible solution to primal SDP (1) and $\tau_0 = 1$ without the need for y_0 to be zero, Y_0 to be the identity matrix and $\kappa_0 = 1$. However, we require $Y_0 \in S_{++}^n$ and $\kappa_0 > 0$.

Remark 4.4 *Similar result as Theorem 4.2 also holds when the NT search direction is used in Algorithm 4.1 instead of the dual HKM search direction. The equivalent algorithm on the SDLCP representation in this case is Algorithm 1 in [20]. We can then apply Theorem 4 or Corollary 1 in [20] to conclude superlinear convergence of iterates when the initial iterate to Algorithm 4.1 with the NT search direction comes from a strictly feasible solution to primal SDP (1). The process to show this is analogous to what we have discussed and we will not repeat it again.*

5   Conclusion

In this paper, we show superlinear convergence of an implementable polynomial-time infeasible predictor-corrector primal-dual path following interior point algorithm on the homogeneous feasibility model of a primal-dual SDP pair under the assumption of strict complementarity and a suitable choice of initial iterate to the algorithm. We do not need to modify the algorithm to show this. This result improves on what is known in the IPM literature.

Data Availability Statement

All data in the paper are available from the corresponding author on reasonable request.

There is no conflict of interest in writing the paper.

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