Approximation Hierarchies for Copositive Cone over Symmetric Cone and Their Comparison

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Abstract

We first provide an inner-approximation hierarchy described by a sum-of-squares (SOS) constraint for the copositive (COP) cone over a general symmetric cone. The hierarchy is a generalization of that proposed by Parrilo (2000) for the usual COP cone (over a nonnegative orthant). We also discuss its dual. Second, we characterize the COP cone over a symmetric cone using the usual COP cone. By replacing the usual COP cone appearing in this characterization with the inner- or outer-approximation hierarchy provided by de Klerk and Pasechnik (2002) or Yıldırım (2012), we obtain an inner- or outer-approximation hierarchy described by semidefinite but not by SOS constraints for the COP matrix cone over the direct product of a nonnegative orthant and a second-order cone. We then compare them with the existing hierarchies provided by Zuluaga et al. (2006) and Lasserre (2014). Theoretical and numerical examinations imply that we can numerically increase a depth parameter, which determines an approximation accuracy, in the approximation hierarchies derived from de Klerk and Pasechnik (2002) and Yıldırım (2012), particularly when the nonnegative orthant is small. In such a case, the approximation hierarchy derived from Yıldırım (2012) can yield nearly optimal values numerically. Combining the proposed approximation hierarchies with existing ones, we can evaluate the optimal value of COP programming problems more accurately and efficiently.

Key words. Approximation hierarchy, Copositive cone, Completely positive cone, Symmetric cone, Copositive programming

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1 Introduction

In this study, we focus on the copositivity and its dual, the complete positivity of tensors, which include matrices and, more generally, linear transformations, over a symmetric cone. Typically, the nonnegative orthant, second-order cone, and semidefinite cone and their direct product are symmetric cones. A symmetric cone plays a significant role in optimization [16] and often appears in the modeling of realistic problems [35, 37]. The copositivity and complete positivity over a nonnegative orthant, i.e., those in the typical sense, are of particular importance. They have been deeply studied [45, 49] and exploited in the convex conic reformulation of many NP-hard problems [5, 8, 41, 42]. In addition, the complete positivity over other symmetric cones has been used in the convex conic reformulation of rank-constrained semidefinite programming (SDP) [3] and polynomial optimization problems over a symmetric cone [30], whose applications include polynomial SDP [31], which often appears in system and control theory [25, 26, 27], and polynomial second-order cone programming [29, 33]. Moreover, Gowda [20] discussed (weighted) linear complementarity problems over a symmetric cone, in which the copositivity over a symmetric cone of linear transformations was exploited. For convenience, the cones of copositive (COP) and completely positive (CP) tensors over a closed cone $\mathbb{K}$ are hereafter called the COP and CP cones (over $\mathbb{K}$), respectively.

As the copositivity and complete positivity appear in the reformulation of such formidable problems, the COP and CP cones are difficult to handle [12]. Thus, to guarantee copositivity or complete positivity, we must consider sufficient and necessary conditions that can be handled efficiently.

To achieve this objective, many types of approximation hierarchies have been proposed [1, 7, 11, 13, 14, 19, 23, 24, 28, 32, 39, 40, 50, 52, 53, 54]. An approximation hierarchy, e.g., $\{\mathcal{K}_r\}_r$, gradually approaches the COP or CP cone from the inside or outside as the depth parameter r, which determines the approximation accuracy, increases and, in a sense, agrees with the cone in the limit. By defining each $\mathcal{K}_r$ as a set represented by nonnegative, second-order cone, or semidefinite constraints, we can tentatively handle the copositivity or complete positivity on a computer using methods such as primal-dual interior point methods [2, Chap. 11].

However, most of these works provided approximation hierarchies for the usual COP and CP cones, and few studies [13, 23, 32, 54] have considered those for the COP or CP cones over a closed cone $\mathbb{K}$ other than a nonnegative orthant. Zuluaga et al. [54] provided an inner-approximation hierarchy described by the sum-of-squares (SOS) constraints, which reduce to semidefinite constraints, for the COP cone over a pointed semialgebraic cone. The term "semialgebraic" means that the set is defined by finitely many nonnegative constraints of homogeneous polynomials. The class of pointed semialgebraic cones includes the nonnegative orthant, second-order cone, and semidefinite cone or their direct product, which are also symmetric cones, in theory. However, the semidefiniteness of a matrix of size n is characterized by the nonnegativity of all the $2^n - 1$ principal minors. That is, the semidefinite cone is a pointed semialgebraic cone; however, the semialgebraic representation requires an exponential number of nonnegative constraints in its size. In such a case, their approximation hierarchy is no longer tractable even if semidefinite constraints can describe it. Lasserre [32] provided an outer-approximation hierarchy described by a semidefinite constraint for the COP cone over a general closed convex cone $\mathbb{K}$; however, the hierarchy is implementable only for the case in which the moment of a finite Borel measure dependent on $\mathbb{K}$ is obtainable. (The following case of $\mathbb{K}$ being the direct product of a nonnegative orthant and a second-order cone is an example in which the moment can be theoretically obtained.) The study by Dickinson and Povh [13] is a variant of that of Lasserre [32], which considered the special case in which $\mathbb{K}$ is included in a nonnegative orthant to provide a tighter approximation than Lasserre [32].

This study aims to provide approximation hierarchies for the COP cone over a symmetric cone and compare them with existing ones. First, we provide an inner-approximation hierarchy described by an SOS constraint. It is a generalization of the approximation hierarchy proposed by Parrilo [39] for the usual COP cone. We call the proposed approximation hierarchy the NN-type inner-approximation hierarchy. Moreover, we discuss its dual to provide an outer-approximation hierarchy for the CP cone over a symmetric cone and provide its more explicit expression for the case in which the symmetric cone is a nonnegative orthant.

Second, we characterize the COP cone over a symmetric cone using the usual COP cone. The basic idea for providing an approximation hierarchy is to replace the usual COP cone appearing in this characterization with its approximation hierarchy. In general, the induced sequence is defined by the intersection of *infinitely* many sets and is not even guaranteed to converge to the COP cone over a symmetric cone. However, by exploiting the inner-approximation hierarchy given by de Klerk and Pasechnik [11] or outer-approximation hierarchy given by Yıldırım [52], we obtain an inner- or outer-approximation hierarchy described by *finitely* many semidefinite but not by SOS constraints for the cone of COP matrices (COP matrix cone) over the direct product of a nonnegative orthant and one second-order cone. Hereafter, we call the proposed inner- and outer-approximation hierarchies the dP- and Yıldırım-type approximation hierarchies, respectively.

As mentioned, Zuluaga et al.'s (ZVP-type) inner-approximation hierarchy [54] and Lasserre's (Lasserre-type) outer-approximation hierarchy [32] are applicable to the COP matrix cone over the direct product of a nonnegative orthant and second-order cone. Then, we theoretically and numerically compare the proposed approximation hierarchies with existing ones. We determined that we can numerically increase a depth parameter in the dP- and Yıldırım-type approximation hierarchies, particularly when the nonnegative orthant is small. In particular, the Yıldırım-type outer-approximation hierarchy has a higher numerical stability than the Lasserre-type one and can approach nearly optimal values of COP programming (COPP) problems numerically.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the notation and concepts used in this study. In Sect. 3, we provide an SOS-based NN-type inner-approximation hierarchy for the COP cone over a general symmetric cone and discuss its dual. In Sect. 4, as generalizations of the approximation hierarchies given by de Klerk and Pasechnik [11] and Yıldırım [52], we provide dP- and Yıldırım-type approximation hierarchies described by finitely many semidefinite constraints for the COP matrix cone over the direct product of a nonnegative orthant and second-order cone. We also discuss their concise expressions. In Sect. 5, we introduce the existing ZVP- and Lasserre-type approximation hierarchies that are applicable to the COP matrix cone over the direct product of a nonnegative orthant and second-order cone and compare them with the proposed approximation hierarchies theoretically. In Sect. 6, we compare the approximation hierarchies numerically by solving optimization problems obtained by approximating the COP cone and investigate the effect of the concise expressions mentioned in Sect. 4. Finally, Sect. 7 provides concluding remarks.

2 Preliminaries

2.1 Notation

We use $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^{n\times m}$, $\mathbb{S}^n$, and $\mathbb{S}^n_+$ to denote the set of nonnegative integers, set of real numbers, set of real $n \times m$ matrices, space of $n \times n$ symmetric matrices, and set of positive semidefinite matrices in $\mathbb{S}^n$, respectively. For a finite set I, we use $\mathbb{R}^I$ and $\mathbb{S}^I$ to denote the $|I|$-dimensional Euclidean space with elements indexed by I and space of $|I|\times|I|$ symmetric matrices with columns and rows indexed by I, respectively. Similarly, let $\mathbb{S}^I_+$ denote the set of positive semidefinite matrices in $\mathbb{S}^I$. We use $\boldsymbol{e}_i$ to denote the vector with an ith element of 1 and the remaining elements of 0, whose size is determined from the context. In addition, we use $\mathbf{0}$, $\mathbf{1}$, $\boldsymbol{O}$, $\boldsymbol{I}$, and $\boldsymbol{E}$ to denote the zero vector, vector with all elements 1, zero matrix, identity matrix, and matrix with all elements 1, respectively. We sometimes use a subscript, such as $\mathbf{1}_n$ and $\boldsymbol{I}_n$, to specify the size. Although all vectors that appear in this paper are column vectors, for notational convenience, the difference between a column and row may not be stated if it is clear from the context. The Euclidean space $\mathbb{R}^n$ is endowed with the usual transpose inner product and $\|\cdot\|_2$ denotes the induced norm. We use S^n and $\Delta^n_=$ to denote the n-dimensional unit sphere and standard simplex in $\mathbb{R}^{n+1}$, i.e.,

$$
\begin{aligned}
S^n &= \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\|_2 = 1\}, \\
\Delta^n_= &= \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \boldsymbol{x}^\top \mathbf{1} = 1 \text{ and } x_i \geq 0 \text{ for all } i = 1, \dots, n+1\},
\end{aligned}
$$

respectively. For a set $\mathcal{X}$, we use $|\mathcal{X}|$, $\mathrm{conv}(\mathcal{X})$, $\mathrm{cone}(\mathcal{X})$, $\mathrm{cl}(\mathcal{X})$, $\mathrm{int}(\mathcal{X})$, and $\partial(\mathcal{X})$ to denote the cardinality, convex hull, conical hull, closure, interior, and boundary of $\mathcal{X}$, respectively. For two finite-dimensional real vector spaces $\mathbb{V}$ and $\mathbb{W}$, we use $\mathrm{Hom}(\mathbb{V},\mathbb{W})$ to denote the set of linear mappings from $\mathbb{V}$ to $\mathbb{W}$. We use $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ to denote the floor and ceiling functions, respectively.

We call a nonempty set $\mathcal{K}$ in a finite-dimensional real vector space a cone if $\alpha x \in \mathcal{K}$ for all $\alpha > 0$ and $x \in \mathcal{K}$. For a cone $\mathcal{K}$ in a finite-dimensional real inner product space, $\mathcal{K}^*$ denotes its dual cone, i.e., the set of x such that the inner product between x and y is greater than or equal to 0 for all $y \in \mathcal{K}$. A cone $\mathcal{K}$ is said to be pointed if it contains no lines. The following properties of a cone and its dual are well known:

Theorem 2.1 ([6, Sect. 2.6.1])**.** *Let $\mathcal{K}$ be a cone.*

(i) If $\mathcal{K}$ is pointed, closed, and convex, $\mathcal{K}^*$ has a nonempty interior.

(ii) If $\mathcal{K}$ is convex, $(\mathcal{K}^*)^* = \mathrm{cl}(\mathcal{K})$ holds. If $\mathcal{K}$ is also closed, $(\mathcal{K}^*)^* = \mathcal{K}$ holds.

For a polynomial f, we use $\deg(f)$ to denote the degree of f. Let $H^{n,m}$ be the set of homogeneous polynomials in n variables of degree m with real coefficients. We then define $\Sigma^{n,2m} := \mathrm{conv}\{\theta^2 \mid \theta \in H^{n,m}\}$. $\Sigma^{n,2m}$ is known to be a closed convex cone [47, Proposition 3.6]. For $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$, we define $\boldsymbol{\alpha}! := \prod_{i=1}^n \alpha_i$, $|\boldsymbol{\alpha}| := \sum_{i=1}^n \alpha_i$, and $\boldsymbol{x}^{\boldsymbol{\alpha}} := \prod_{i=1}^n x_i^{\alpha_i}$. In addition, we define

$$\begin{aligned}
\mathbb{I}^n_{=m} &:= \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid |\boldsymbol{\alpha}| = m\},\\
\mathbb{I}^n_{\leq m} &:= \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid |\boldsymbol{\alpha}| \leq m\},\\
\mathbb{N}^m_n &:= \{(i_1,\dots,i_m) \in \mathbb{N}^m \mid 1 \leq i_k \leq n \text{ for all } k = 1,\dots,m\}.
\end{aligned}$$

Under this notation, $\mathbb{R}^{\mathbb{I}^n_{=m}}$ is linear isomorphic to $H^{n,m}$ by the mapping $(\theta_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mapsto \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \theta_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$. Let $\mathfrak{S}_m$ be the symmetric group of order m. Then, the group $\mathfrak{S}_m$ acts on the set $\mathbb{N}^m_n$ by $\sigma \cdot (i_1,\dots,i_m) = (i_{\sigma(1)},\dots,i_{\sigma(m)})$. As mentioned in [14, Sect. 4], a bijection exists between the set $\mathbb{I}^n_{=m}$ and a complete set of the representatives of $\mathfrak{S}_m$-orbits in $\mathbb{N}^m_n$. As the set

$$\{(i_1,\dots,i_m) \mid 1 \leq i_1 \leq \cdots \leq i_m \leq n\} \tag{1}$$

is a complete set of the representatives of $\mathfrak{S}_m$-orbits in $\mathbb{N}^m_n$, we define $[\boldsymbol{\alpha}] \in \mathbb{N}^m_n$ as the element of (1) corresponding to $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$, i.e.,

$$[\boldsymbol{\alpha}] = (\underbrace{1,\dots,1}_{\alpha_1 \text{ factors}},\dots,\underbrace{n,\dots,n}_{\alpha_n \text{ factors}}).$$

2.2 Euclidean Jordan algebra and symmetric cone

A finite-dimensional real vector space $\mathbb{E}$ equipped with a bilinear mapping (product) $\circ : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is said to be a Jordan algebra if the following two conditions hold for all $x, y \in \mathbb{E}$:

(J1) $x \circ y = y \circ x$

(J2) $x \circ ((x \circ x) \circ y) = (x \circ x) \circ (x \circ y)$

In this study, we assume that a Jordan algebra has an identity element e for the product. A Jordan algebra $(\mathbb{E}, \circ)$ is said to be Euclidean if there exists an associative inner product $\bullet$ on $\mathbb{E}$ such that

(J3) $(x \circ y) \bullet z = x \bullet (y \circ z)$

for all $x, y, z \in \mathbb{E}$. Throughout this study, we fix an associative inner product $\bullet$ on a Euclidean Jordan algebra $(\mathbb{E}, \circ)$ and regard $(\mathbb{E}, \circ, \bullet)$ as a finite-dimensional real inner product space.

Let $(\mathbb{E}, \circ, \bullet)$ be a Euclidean Jordan algebra. An element $c \in \mathbb{E}$ is called an idempotent if $c \circ c = c$. In addition, an idempotent c is called primitive if it is nonzero and cannot be written as the sum of two nonzero idempotents. Two elements $c, d \in \mathbb{E}$ are called orthogonal if $c \circ d = 0$. The system $c_1, \dots, c_k$ is called a complete system of orthogonal idempotents if each c_i is an idempotent, $c_i \circ c_j = 0$ if $i \neq j$, and $\sum_{i=1}^{k} c_i = e$. In addition, if each c_i is also primitive, the system is called a Jordan frame. Each Jordan frame is known to consist of exactly rk elements, where rk is the rank of the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ and the rank depends only on the algebra [15, Sect. III.1]. Here, for the convenience of the proofs, we consider an ordered Jordan frame and let $\mathfrak{F}(\mathbb{E})$ be the set of ordered Jordan frames, i.e.,

$$\mathfrak{F}(\mathbb{E}) = \{(c_1, \dots, c_{\mathrm{rk}}) \mid \text{The system } c_1, \dots, c_{\mathrm{rk}} \text{ is a Jordan frame}\}.$$

Note that $\mathfrak{F}(\mathbb{E})$ is a compact subset in $\mathbb{E}^{\mathrm{rk}}$ [15, Exercise IV.5]. Each element of $\mathbb{E}$ can be decomposed into a linear combination of a Jordan frame [15, Theorem III.1.2]. In particular, for each $x \in \mathbb{E}$, there exist $x_1, \dots, x_{\mathrm{rk}} \in \mathbb{R}$ and $(c_1, \dots, c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$ such that

$$x = \sum_{i=1}^{\mathrm{rk}} x_i c_i. \tag{2}$$

The symmetric cone $\mathbb{E}_+$ associated with the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ is defined as $\mathbb{E}_+ \coloneqq \{x \circ x \mid x \in \mathbb{E}\}$. Note that when each $x \in \mathbb{E}_+$ is decomposed into the form (2), all coefficients x_i are nonnegative. Conversely, for any nonnegative scalers $x_1, \dots, x_{\mathrm{rk}}$ and $(c_1, \dots, c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$, it follows that $\sum_{i=1}^{\mathrm{rk}} x_i c_i \in \mathbb{E}_+$.

We now show some examples of the symmetric cones frequently used in this paper.

Example 2.2 (nonnegative orthant)**.** *Let $\mathbb{E}$ be an n-dimensional Euclidean space $\mathbb{R}^n$. If we set $\boldsymbol{x} \circ \boldsymbol{y} \coloneqq (x_1 y_1, \dots, x_n y_n)$ and $\boldsymbol{x} \bullet \boldsymbol{y} \coloneqq \boldsymbol{x}^\top \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}$, then $(\mathbb{E}, \circ, \bullet)$ is a Euclidean Jordan algebra, and the induced symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i = 1, \dots, n\}$. The set $\mathfrak{F}(\mathbb{E})$ of ordered Jordan frames is $\{(\boldsymbol{e}_{\sigma(1)}, \dots, \boldsymbol{e}_{\sigma(n)}) \mid \sigma \in \mathfrak{S}_n\}$.*

Example 2.3 (second-order cone)**.** *Let $\mathbb{E}$ be an n-dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$. If we set $\boldsymbol{x} \circ \boldsymbol{y} := (\boldsymbol{x}^\top \boldsymbol{y}, x_1 \boldsymbol{y}_{2:n} + y_1 \boldsymbol{x}_{2:n})$ and $\boldsymbol{x} \bullet \boldsymbol{y} := \boldsymbol{x}^\top \boldsymbol{y}$ for $\boldsymbol{x} = (x_1, \boldsymbol{x}_{2:n})$, $\boldsymbol{y} = (y_1, \boldsymbol{y}_{2:n}) \in \mathbb{E}$, then $(\mathbb{E}, \circ, \bullet)$ is a Euclidean Jordan algebra, and the induced symmetric cone $\mathbb{E}_+$ is the second-order cone $\mathbb{L}^n = \{(x_1, \boldsymbol{x}_{2:n}) \in \mathbb{R}^n \mid x_1 \geq \|\boldsymbol{x}_{2:n}\|_2\}$. The set $\mathfrak{F}(\mathbb{E})$ of ordered Jordan frames is*

$$\left\{ \left(\frac{1}{2} \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -\boldsymbol{v} \end{pmatrix} \right) \;\middle|\; \boldsymbol{v} \in S^{n-2} \right\}.$$

Consider the case in which a Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ can be written as the direct product (sum) of two Euclidean Jordan algebras $(\mathbb{E}_i, \circ_i, \bullet_i)$ with rank rk_i and identity element e_i $(i = 1, 2)$. Note that the following discussion can be directly extended to the case of finitely many Euclidean Jordan algebras. The product $\circ$ and associative inner product $\bullet$ are defined as

$$(x_1, x_2) \circ (y_1, y_2) := (x_1 \circ_1 y_1, x_2 \circ_2 y_2),$$
$$(x_1, x_2) \bullet (y_1, y_2) := x_1 \bullet_1 y_1 + x_2 \bullet_2 y_2$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2$, and the identity element e of $\mathbb{E}$ is (e_1, e_2). In the following, we derive the set of ordered Jordan frames of $\mathbb{E}$.

Lemma 2.4. *For a primitive idempotent $f = (f_1, f_2) \in \mathbb{E}$, exactly one of the following two statements holds:*

(a) f_1 is a primitive idempotent of $\mathbb{E}_1$ and $f_2 = 0$.

(b) $f_1 = 0$ and f_2 is a primitive idempotent of $\mathbb{E}_2$.

Proof. We assume that $f_1 \neq 0$. If $f_2 \neq 0$, then f can be decomposed into the sum of two nonzero elements $(f_1, 0)$ and $(0, f_2)$. The two elements are actually idempotents of $\mathbb{E}$, which contradicts f being primitive. Thus, we have $f_2 = 0$. Because $f = (f_1, 0)$ is a primitive idempotent, f_1 is also a primitive idempotent. This case falls under the case (a).

Next, we assume that $f_1 = 0$. In this case, as in the above discussion, we note that f_2 is a primitive idempotent. This case falls under the case (b). □

Proposition 2.5. *For $(f_1, \ldots, f_{\mathrm{rk}_1 + \mathrm{rk}_2}) \in \mathbb{E}^{\mathrm{rk}_1 + \mathrm{rk}_2}$, $(f_1, \ldots, f_{\mathrm{rk}_1 + \mathrm{rk}_2}) \in \mathfrak{F}(\mathbb{E})$ if and only if there exists a partition (I_1, I_2) of $\{1, \ldots, \mathrm{rk}_1 + \mathrm{rk}_2\}$ such that the following two conditions hold:*

(i) $f_i = (f_{1i}, 0) \in \mathbb{E}$ for all $i \in I_1$ and $(f_{1i})_{i \in I_1} \in \mathfrak{F}(\mathbb{E}_1)$.

(ii) $f_i = (0, f_{2i}) \in \mathbb{E}$ for all $i \in I_2$ and $(f_{2i})_{i \in I_2} \in \mathfrak{F}(\mathbb{E}_2)$.

Proof. We first prove the "if" part. Because $(f_1, \dots, f_{\mathrm{rk}_1+\mathrm{rk}_2})$ satisfying the two conditions is evidently a complete system of orthogonal idempotents, we prove only that each f_i is primitive. To prove this, showing that $(f, 0) \in \mathbb{E}$ is primitive if $f \in \mathbb{E}_1$ is primitive is sufficient. Assume that $(f, 0)$ can be written as the sum of two idempotents (f_1, f_2) and (g_1, g_2), i.e.,

$$(f, 0) = (f_1, f_2) + (g_1, g_2). \tag{3}$$

First, (3) implies that $f_2 = -g_2$. In addition, we note that f_1, g_1, f_2, and g_2 are idempotents because (f_1, f_2) and (g_1, g_2) are idempotents. Therefore, $f_2 = g_2 = 0$. Second, (3) implies again that $f = f_1 + g_1$. As f_1 and g_1 are idempotents and f is primitive, either f_1 or g_1 must be 0. We can assume that $f_1 = 0$ without loss of generality. We then obtain $(f_1, f_2) = 0$, which implies that $(f, 0)$ is primitive.

We now prove the "only if" part. Let $(f_1, \dots, f_{\mathrm{rk}_1+\mathrm{rk}_2}) \in \mathfrak{F}(\mathbb{E})$ and set $f_i = (f_{1i}, f_{2i})$ for each i. Then, each $f_i = (f_{1i}, f_{2i})$ falls into exactly one of the two cases in Lemma 2.4; thus, we define

$$\begin{aligned} I_1 &\coloneqq \{i \in \{1, \dots, \mathrm{rk}_1 + \mathrm{rk}_2\} \mid f_{1i} \text{ is a primitive idempotent and } f_{2i} = 0\}, \\ I_2 &\coloneqq \{i \in \{1, \dots, \mathrm{rk}_1 + \mathrm{rk}_2\} \mid f_{1i} = 0 \text{ and } f_{2i} \text{ is a primitive idempotent}\}. \end{aligned}$$

Evidently, (I_1, I_2) is a partition of $\{1, \dots, \mathrm{rk}_1 + \mathrm{rk}_2\}$. In the following, we show that $(f_{1i})_{i \in I_1} \in \mathfrak{F}(\mathbb{E}_1)$ and $(f_{2i})_{i \in I_2} \in \mathfrak{F}(\mathbb{E}_2)$. From the assumption on $(f_1, \dots, f_{\mathrm{rk}_1+\mathrm{rk}_2})$, it follows that

$$e = (e_1, e_2) = \sum_{i=1}^{\mathrm{rk}_1+\mathrm{rk}_2} f_i = \left(\sum_{i \in I_1} f_{1i}, \sum_{i \in I_2} f_{2i} \right),$$

from which we obtain $\sum_{i \in I_1} f_{1i} = e_1$ and $\sum_{i \in I_2} f_{2i} = e_2$. In addition, it follows that $0 = f_i \circ f_j = (f_{1i} \circ_1 f_{1j}, f_{2i} \circ_2 f_{2j})$ for any $i \neq j$. In particular, we have $f_{1i} \circ_1 f_{1j} = 0$ for all $i \neq j \in I_1$ and $f_{2i} \circ_2 f_{2j} = 0$ for all $i \neq j \in I_2$. □

2.3 Symmetric tensor space

Let $(\mathbb{V}, (\cdot, \cdot))$ be an n-dimensional real inner product space. Note that $\mathbb{V}$ can be identified with the dual space $\mathrm{Hom}(\mathbb{V}, \mathbb{R})$ by the natural isomorphism $x \mapsto (x, \cdot)$. We use

$$\mathbb{V}^{\otimes m} \coloneqq \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{m \text{ factors}}$$

to denote the tensor space of order m over $\mathbb{V}$.

Let $v_1, \dots, v_n$ be a basis for $\mathbb{V}$. Then, $\tilde{v}_{i_1 \cdots i_m} \coloneqq v_{i_1} \otimes \cdots \otimes v_{i_m}$ $((i_1, \dots, i_m) \in \mathbb{N}_n^m)$ form a basis for $\mathbb{V}^{\otimes m}$. That is, each $\mathcal{A} \in \mathbb{V}^{\otimes m}$ can be written in the following form:

$$\mathcal{A} = \sum_{i_1, \dots, i_m = 1}^{n} \mathcal{A}_{i_1 \cdots i_m} \tilde{v}_{i_1 \cdots i_m} \tag{4}$$

with coefficients $\mathcal{A}_{i_1\cdots i_m} \in \mathbb{R}$. For $\sigma \in \mathfrak{S}_m$, the linear transformation π_σ on $\mathbb{V}^{\otimes m}$ is defined by $\pi_\sigma(\tilde{v}_{i_1\cdots i_m}) \coloneqq \tilde{v}_{i_{\sigma(1)}\cdots i_{\sigma(m)}}$. The definition of π_σ does not depend on the choice of the basis for $\mathbb{V}$. Then,

$$\mathcal{S}^{n,m}(\mathbb{V}) \coloneqq \{\mathcal{A} \in \mathbb{V}^{\otimes m} \mid \pi_\sigma \mathcal{A} = \mathcal{A} \text{ for all } \sigma \in \mathfrak{S}_m\}$$

denotes the symmetric tensor space of order m over $\mathbb{V}$, which is a subspace of $\mathbb{V}^{\otimes m}$. Note that the symmetric tensor $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$ with the form (4) depends only on the coefficients in the form of $\mathcal{A}_{[\boldsymbol{\alpha}]}$ ($\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$). Let $\mathscr{S}$ be the linear transformation on $\mathbb{V}^{\otimes m}$ defined as

$$\mathscr{S} \coloneqq \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \pi_\sigma.$$

Then, $\mathscr{S}\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$ for each $\mathcal{A} \in \mathbb{V}^{\otimes m}$ and $\mathscr{S}\tilde{v}_{[\boldsymbol{\alpha}]}$ ($\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$) form a basis for $\mathcal{S}^{n,m}(\mathbb{V})$. For $x \in \mathbb{V}$, let

$$x^{\otimes m} \coloneqq \underbrace{x \otimes \cdots \otimes x}_{m \text{ factors}} \in \mathcal{S}^{n,m}(\mathbb{V}).$$

In particular, we consider the case of $\mathbb{V} = \mathbb{R}^n$ with the canonical basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ and write $\mathcal{S}^{n,m}(\mathbb{R}^n)$ as $\mathcal{S}^{n,m}$. Then, each element $\mathcal{A} \in \mathcal{S}^{n,m}$ can be considered a multi-dimensional array; thus, we write $\mathcal{A}_{i_1\cdots i_m}$ for the $(i_1, \dots, i_m)$th element of $\mathcal{A}$. Note that the symmetric tensor space $\mathcal{S}^{n,2}$ of order two equals the space $\mathbb{S}^n$ of the symmetric matrices.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{V}^{\otimes m}$ induced by that on $\mathbb{V}$. That is, it satisfies $\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m \rangle = \prod_{i=1}^m (x_i, y_i)$ for $x_1 \otimes \cdots \otimes x_m$, $y_1 \otimes \cdots \otimes y_m \in \mathbb{V}^{\otimes m}$. We write $\|\cdot\|_{\mathrm{F}}$ for the norm on $\mathbb{V}^{\otimes m}$ induced by the inner product $\langle \cdot, \cdot \rangle$.

Using the inner product, we note that $\mathcal{S}^{n,m}(\mathbb{V})$ and $H^{n,m}$ are linear isomorphic. Indeed, let $\phi \in \mathrm{Hom}(\mathbb{V}, \mathbb{R}^n)$ be the linear isomorphism induced by the basis $v_1, \dots, v_n$. Then, the mapping $\psi : \mathcal{S}^{n,m}(\mathbb{V}) \to H^{n,m}$ defined by

$$\psi(\mathcal{A}) \coloneqq \langle \mathcal{A}, \phi^{-1}(\boldsymbol{x})^{\otimes m} \rangle \tag{5}$$

is a linear isomorphism.

In the following proofs, for convenience, we fix an orthonormal basis $v_1, \dots, v_n$ for $\mathbb{V}$ arbitrarily. The following lemma describes a property of the inner product on the (symmetric) tensor space.

Lemma 2.6. *For $\mathcal{A} \in \mathbb{V}^{\otimes m}$ and $\mathcal{B} \in \mathcal{S}^{n,m}(\mathbb{V})$, it follows that $\langle \mathscr{S}\mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{A}, \mathcal{B} \rangle$.*

Proof. Using the orthonormal basis $v_1, \dots, v_n$ for $\mathbb{V}$, we write $\mathcal{A}$ and $\mathcal{B}$ in the form (4).

It then follows from the symmetry of $\mathcal{B}$ that

$$\begin{aligned}
\langle \mathscr{S}\mathcal{A}, \mathcal{B}\rangle &= \left\langle \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \pi_\sigma \mathcal{A}, \mathcal{B}\right\rangle \\
&= \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{B}_{i_{\sigma(1)}\cdots i_{\sigma(m)}} \\
&= \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{B}_{i_1\cdots i_m} \\
&= \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \langle \mathcal{A}, \mathcal{B}\rangle \\
&= \langle \mathcal{A}, \mathcal{B}\rangle.
\end{aligned}$$

□

Consider the case of $\mathbb{V} = \mathbb{R}^n$ with the canonical basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. The following lemma provides an orthonormal basis for the symmetric tensor space $\mathcal{S}^{n,m}$ and the representation of the elements of $\mathcal{S}^{n,m}$ with the orthonormal basis.

Lemma 2.7. *We now define* $\mathcal{F}_{\boldsymbol{\alpha}} := \sqrt{m!\,/\boldsymbol{\alpha}!}\mathscr{S}\tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}$ *for each* $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$. *Then,* $(\mathcal{F}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$ *is an orthonormal basis for* $\mathcal{S}^{n,m}$. *In addition, using the orthonormal basis, we can represent each* $\mathcal{A} \in \mathcal{S}^{n,m}$ *as* $\mathcal{A} = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \sqrt{m!\,/\boldsymbol{\alpha}!}\mathcal{A}_{[\boldsymbol{\alpha}]}\mathcal{F}_{\boldsymbol{\alpha}}$.

Proof. Note that $\dim \mathcal{S}^{n,m} = |\mathbb{I}^n_{=m}|$. Because the linear independence of $(\mathcal{F}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$ is clear, showing that it is orthonormal is sufficient. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m}$. We first consider the case of $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. Let $(i_1, \ldots, i_m) := [\boldsymbol{\alpha}]$ and $(j_1, \ldots, j_m) := [\boldsymbol{\beta}]$. Then, as $(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)$, there exists $k_0 \in \{1, \ldots, m\}$ such that $i_{k_0} \neq j_{k_0}$, which implies that the number of values i_{k_0} in the vector $(i_1, \ldots, i_m)$ is not equal to that in the vector $(j_1, \ldots, j_m)$. Using Lemma 2.6, we then have

$$\begin{aligned}
\langle \mathcal{F}_{\boldsymbol{\alpha}}, \mathcal{F}_{\boldsymbol{\beta}}\rangle &= \left\langle \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathscr{S}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}, \sqrt{\frac{m!}{\boldsymbol{\beta}!}}\mathscr{S}\tilde{\boldsymbol{e}}_{j_1\cdots j_m}\right\rangle \\
&= \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\sqrt{\frac{m!}{\boldsymbol{\beta}!}}\langle \mathscr{S}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}, \tilde{\boldsymbol{e}}_{j_1\cdots j_m}\rangle \\
&= \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\sqrt{\frac{m!}{\boldsymbol{\beta}!}}\left\langle \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \tilde{\boldsymbol{e}}_{i_{\sigma(1)}\cdots i_{\sigma(m)}}, \tilde{\boldsymbol{e}}_{j_1\cdots j_m}\right\rangle \\
&= \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\sqrt{\frac{m!}{\boldsymbol{\beta}!}}\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\prod_{k=1}^{m} \boldsymbol{e}_{i_{\sigma(k)}}^\top \boldsymbol{e}_{j_k} \\
&= 0.
\end{aligned}$$

Second, we consider the case of $\boldsymbol{\alpha} = \boldsymbol{\beta}$. Let $(i_1, \ldots, i_m) \coloneqq [\boldsymbol{\alpha}]$. Then,

$$\begin{aligned}\|\mathcal{F}_{\boldsymbol{\alpha}}\|_{\mathrm{F}} &= \frac{m!}{\boldsymbol{\alpha}!}\langle \mathscr{S}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}, \tilde{\boldsymbol{e}}_{i_1\cdots i_m}\rangle \\ &= \frac{m!}{\boldsymbol{\alpha}!}\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\prod_{k=1}^{m}\boldsymbol{e}_{i_k}^{\top}\boldsymbol{e}_{i_{\sigma(k)}} \\ &= 1.\end{aligned}$$

Therefore, $(\mathcal{F}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$ is an orthonormal basis for $\mathcal{S}^{n,m}$.

In addition, for $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$, let

$$\mathbb{N}_n^m[\boldsymbol{\alpha}] \coloneqq \{(i_1, \ldots i_m) \in \mathbb{N}_n^m \mid (i_{\sigma(1)}, \ldots, i_{\sigma(m)}) = [\boldsymbol{\alpha}] \text{ for some } \sigma \in \mathfrak{S}_m\}.$$

From the symmetry of $\mathcal{A}$, it then follows that

$$\begin{aligned}\mathcal{A} &= \sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m}\tilde{\boldsymbol{e}}_{i_1\cdots i_m} \\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}\ \sum_{(i_1,\ldots,i_m)\in\mathbb{N}_n^m[\boldsymbol{\alpha}]} \mathcal{A}_{i_1\cdots i_m}\tilde{\boldsymbol{e}}_{i_1\cdots i_m} \\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}_{[\boldsymbol{\alpha}]}\left(\sum_{(i_1,\ldots,i_m)\in\mathbb{N}_n^m[\boldsymbol{\alpha}]} \tilde{\boldsymbol{e}}_{i_1\cdots i_m}\right) \\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}_{[\boldsymbol{\alpha}]}\left(\frac{1}{\boldsymbol{\alpha}!}\sum_{\sigma\in\mathfrak{S}_m}\pi_\sigma\tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}\right) \\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathcal{A}_{[\boldsymbol{\alpha}]}\left(\sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathscr{S}\tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}\right) \\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathcal{A}_{[\boldsymbol{\alpha}]}\mathcal{F}_{\boldsymbol{\alpha}}.\end{aligned}$$

□

For convenience, we write the coefficients of $\mathcal{A} \in \mathcal{S}^{n,m}$ with respect to the orthonormal basis $(\mathcal{F}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$ taken in Lemma 2.7 as $(\mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$. That is, each $\mathcal{A} \in \mathcal{S}^{n,m}$ is written as

$$\mathcal{A} = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}}\mathcal{F}_{\boldsymbol{\alpha}}. \tag{6}$$

Because $\mathcal{S}^{n,m}$ and $H^{n,m}$ are linear isomorphic by the mapping (5), for each $\mathcal{A} \in \mathcal{S}^{n,m}$, there exists $\theta \in H^{n,m}$ such that $\langle \mathcal{A}, \boldsymbol{x}^{\otimes m}\rangle = \theta(\boldsymbol{x})$. The following lemma links the coefficients $(\mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$ with the coefficients of θ.

Lemma 2.8. *Suppose that* $\mathcal{A} \in \mathcal{S}^{n,m}$ *and* $(\theta_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ *satisfy*

$$\langle \mathcal{A}, \boldsymbol{x}^{\otimes m}\rangle = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \theta_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}. \tag{7}$$

Then, $\mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} = \sqrt{\boldsymbol{\alpha}!\,/m!}\theta_{\boldsymbol{\alpha}}$ *for all* $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$.

Proof. Let $\mathcal{A}$ be in the form (6). It then follows from Lemma 2.6 that

$$\begin{aligned}
\langle \mathcal{A}, \boldsymbol{x}^{\otimes m}\rangle &= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \left\langle \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}} \mathscr{S}\widetilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}, \boldsymbol{x}^{\otimes m} \right\rangle \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}} \langle \underbrace{\boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_1}_{\alpha_1 \text{ factors}} \otimes \cdots \otimes \underbrace{\boldsymbol{e}_n \otimes \cdots \otimes \boldsymbol{e}_n}_{\alpha_n \text{ factors}}, \boldsymbol{x}^{\otimes m}\rangle \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}} \boldsymbol{x}^{\boldsymbol{\alpha}}.
\end{aligned}$$

As this equals $\sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \theta_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ and by comparing the coefficients, we obtain the desired result. ☐

2.4 Copositive and completely positive cones

Let $\mathbb{K}$ be a closed cone in an n-dimensional real inner product space $(\mathbb{V}, (\cdot,\cdot))$. Then, we define

$$\begin{aligned}
\mathcal{COP}^{n,m}(\mathbb{K}) &:= \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V}) \mid \langle \mathcal{A}, x^{\otimes m}\rangle \ge 0 \text{ for all } x \in \mathbb{K}\}, \\
\mathcal{CP}^{n,m}(\mathbb{K}) &:= \operatorname{conv}\{x^{\otimes m} \mid x \in \mathbb{K}\}
\end{aligned}$$

and call them the COP and CP cones (over $\mathbb{K}$), respectively. In the case in which $\mathbb{V} = \mathbb{R}^n$ and $\mathbb{K} = \mathbb{R}^n_+$, the COP and CP cones reduce to the COP tensor cone, written as $\mathcal{COP}^{n,m}$ and the CP tensor cone given by, for example, [44, 46]. In the case of $m = 2$ and $\mathbb{V} = \mathbb{R}^n$, under the identification between $\mathcal{S}^{n,2}$ and $\mathbb{S}^n$, we have

$$\begin{aligned}
\mathcal{COP}(\mathbb{K}) &:= \mathcal{COP}^{n,2}(\mathbb{K}) = \{\boldsymbol{A} \in \mathbb{S}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \ge 0 \text{ for all } \boldsymbol{x} \in \mathbb{K}\}, \\
\mathcal{CP}(\mathbb{K}) &:= \mathcal{CP}^{n,2}(\mathbb{K}) = \operatorname{conv}\{\boldsymbol{x}\boldsymbol{x}^\top \mid \boldsymbol{x} \in \mathbb{K}\},
\end{aligned}$$

which are the COP and CP matrix cones [21], respectively. Because we considered both the tensor and matrix cases, we generally omit the terms "tensor" and "matrix."

We now discuss the duality between $\mathcal{COP}^{n,m}(\mathbb{K})$ and $\mathcal{CP}^{n,m}(\mathbb{K})$.

Proposition 2.9. *Let* $\mathbb{K}$ *be a closed cone in* $\mathbb{V}$.

(i) $\mathcal{COP}^{n,m}(\mathbb{K}) = \mathcal{CP}^{n,m}(\mathbb{K})^*$.

(ii) If $\mathbb{K}$ *is also pointed and convex, then* $\mathcal{CP}^{n,m}(\mathbb{K})$ *is a closed convex cone.*

Proof. We first prove (i). Let $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{K})$. Then, for any $x_i \in \mathbb{K}$ and $\lambda_i \geq 0$ such that $\sum_i \lambda_i = 1$, we have

$$\left\langle \mathcal{A}, \sum_i \lambda_i x_i^{\otimes m} \right\rangle = \sum_i \lambda_i \langle \mathcal{A}, x_i^{\otimes m} \rangle. \tag{8}$$

As $\langle \mathcal{A}, x_i^{\otimes m} \rangle$ and λ_i are nonnegative for all i, (8) is also nonnegative, which implies that $\mathcal{A} \in \mathcal{CP}^{n,m}(\mathbb{K})^*$. Conversely, let $\mathcal{A} \in \mathcal{CP}^{n,m}(\mathbb{K})^*$. For any $x \in \mathbb{K}$, because $x^{\otimes m} \in \mathcal{CP}^{n,m}(\mathbb{K})$, $\langle \mathcal{A}, x^{\otimes m} \rangle \geq 0$ follows from the definition of dual cones. Therefore, we obtain $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{K})$.

We now prove (ii). The convexity of $\mathcal{CP}^{n,m}(\mathbb{K})$ follows from its definition. In addition, $\mathcal{CP}^{n,m}(\mathbb{K})$ is a cone because $\mathbb{K}$ is a cone. To prove the closedness, let $\{\mathcal{A}_k\}_k \subseteq \mathcal{CP}^{n,m}(\mathbb{K})$ and suppose it converges to some $\mathcal{A}_\infty \in \mathcal{S}^{n,m}(\mathbb{V})$. Note that $\mathcal{CP}^{n,m}(\mathbb{K})$ can be represented as $\mathcal{CP}^{n,m}(\mathbb{K}) = \operatorname{cone}\{x^{\otimes m} \mid x \in \mathbb{K}\}$ as it is a convex cone containing zero (the origin). Therefore, by Carathéodory's theorem for cones [4, Exercise B.1.7], every element of $\mathcal{CP}^{n,m}(\mathbb{K})$ can be written as the sum of at most $d := \dim \mathcal{S}^{n,m}(\mathbb{V})$ elements in the form of $x^{\otimes m}$ with $x \in \mathbb{K}$; for every k, there exist x_{ki} with $x_{ki} \in \mathbb{K}$ $(i = 1, \ldots, d)$ such that $\mathcal{A}_k = \sum_{i=1}^d x_{ki}^{\otimes m}$. As $\operatorname{int}(\mathbb{K}^*)$ is nonempty under the assumption on $\mathbb{K}$, we take $a \in \operatorname{int}(\mathbb{K}^*)$ arbitrarily. Then, we obtain

$$\langle \mathcal{A}_k, a^{\otimes m} \rangle = \sum_{i=1}^d \langle x_{ki}^{\otimes m}, a^{\otimes m} \rangle = \sum_{i=1}^d (x_{ki}, a)^m \to \langle \mathcal{A}_\infty, a^{\otimes m} \rangle \ (k \to \infty).$$

Given that $x_{ki} \in \mathbb{K}$, $(x_{ki}, a) \geq 0$ for any k and i. Therefore, $\{(x_{ki}, a)\}_k$ is bounded for each i.

Then, $\{x_{ki}\}_k$ is bounded for each i. To observe this, let $\{k(l) \mid l \in \mathbb{N}\}$ denote the set of indices k such that $x_{ki} \neq 0$. Showing the boundedness of $\{x_{k(l)i}\}_l$ is sufficient. Let $B := \mathbb{K} \cap \{y \in \mathbb{V} \mid (y, a) = 1\}$. Note that B is compact. Then, for each l, there exist $\alpha_{li} > 0$ and $y_{li} \in B$ such that $x_{k(l)i} = \alpha_{li} y_{li}$. Given that $(x_{k(l)i}, a) = \alpha_{li}$, the sequence $\{\alpha_{li}\}_l$ is bounded. Combining it with the boundedness of $\{y_{li}\}_l \subseteq B$ leads to the boundedness of $\{x_{k(l)i}\}_l = \{\alpha_{li} y_{li}\}_l$.

Thus, by taking a subsequence, if necessary, we assume that $\{x_{ki}\}_k$ converges to some $x_{\infty i}$ for each i. The closedness of $\mathbb{K}$ implies that $x_{\infty i} \in \mathbb{K}$. Therefore,

$$\mathcal{A}_\infty = \lim_{k \to \infty} \mathcal{A}_k = \lim_{k \to \infty} \sum_{i=1}^d x_{ki}^{\otimes m} = \sum_{i=1}^d x_{\infty i}^{\otimes m} \in \mathcal{CP}^{n,m}(\mathbb{K}),$$

which means that $\mathcal{CP}^{n,m}(\mathbb{K})$ is closed. □

Corollary 2.10. *Let $\mathbb{K}$ be a pointed closed convex cone. Then, $\mathcal{COP}^{n,m}(\mathbb{K})$ and $\mathcal{CP}^{n,m}(\mathbb{K})$ are dual to each other.*

Proof. It follows from (ii) in Proposition 2.9 that $\mathcal{CP}^{n,m}(\mathbb{K})$ is a closed convex cone. Taking the dual of (i) in Proposition 2.9, we obtain $\mathcal{COP}^{n,m}(\mathbb{K})^* = \mathcal{CP}^{n,m}(\mathbb{K})$. $\square$

In this study, we focused only on the case in which $\mathbb{K}$ is a symmetric cone, which is a pointed closed convex cone. In this case, the consequence of Corollary 2.10 is applicable to $\mathcal{COP}^{n,m}(\mathbb{K})$ and $\mathcal{CP}^{n,m}(\mathbb{K})$.

2.5 Homogeneous polynomial function on inner product space

Let $(\mathbb{V}, (\cdot,\cdot))$ be an n-dimensional real inner product space. A homogeneous polynomial function of degree m on $\mathbb{V}$ is the mapping $\mathbb{V} \ni x \mapsto \langle \mathcal{A}, x^{\otimes m}\rangle$ for some $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$. $H^{n,m}(\mathbb{V})$ denotes the set of homogeneous polynomial functions of degree m on $\mathbb{V}$, i.e., $H^{n,m}(\mathbb{V}) = \{\langle \mathcal{A}, x^{\otimes m}\rangle \mid \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})\}$. As $H^{n,m} = \{\langle \mathcal{A}, \boldsymbol{x}^{\otimes m}\rangle \mid \mathcal{A} \in \mathcal{S}^{n,m}\}$, $H^{n,m}(\mathbb{R}^n)$ agrees with $H^{n,m}$.

As the definition of $\Sigma^{n,2m}$, $\Sigma^{n,2m}(\mathbb{V})$ denotes the set of sums of squares of homogeneous polynomial functions of degree m on $\mathbb{V}$. To represent the set $\Sigma^{n,2m}(\mathbb{V})$ more explicitly, we prove the following lemma.

Lemma 2.11. *$\langle \mathcal{A}, x^{\otimes m}\rangle^2 = \langle \mathscr{S}(\mathcal{A} \otimes \mathcal{A}), x^{\otimes 2m}\rangle$ for any $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$.*

Proof. Fix an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{V}$ arbitrarily. In addition, using the basis, we write $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$ in the form (4). Given that

$$\langle \mathcal{A}, x^{\otimes m}\rangle = \sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m} \prod_{k=1}^{m} (x, v_{i_k}),$$

we have

$$\langle \mathcal{A}, x^{\otimes m}\rangle^2 = \sum_{i_1,\ldots,i_{2m}=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{A}_{i_{m+1}\cdots i_{2m}} \prod_{k=1}^{2m} (x, v_{i_k}). \tag{9}$$

Moreover, as $\mathcal{A} \otimes \mathcal{A} = \sum_{i_1,\ldots,i_{2m}=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{A}_{i_{m+1}\cdots i_{2m}} \tilde{v}_{i_1\cdots i_{2m}}$, $\langle \mathcal{A} \otimes \mathcal{A}, x^{\otimes 2m}\rangle$ agrees with (9). Therefore, by applying Lemma 2.6, we obtain the desired result. $\square$

Using Lemma 2.11, we can express $\Sigma^{n,2m}(\mathbb{V})$ as $\mathrm{conv}\{\langle \mathscr{S}(\mathcal{A} \otimes \mathcal{A}), x^{\otimes 2m}\rangle \mid \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})\}$. Through the isomorphism $H^{n,2m}(\mathbb{V}) \ni \langle \mathcal{A}, x^{\otimes 2m}\rangle \mapsto \mathcal{A} \in \mathcal{S}^{n,2m}(\mathbb{V})$, the set $\Sigma^{n,2m}(\mathbb{V})$ is mapped onto the set $\mathrm{conv}\{\mathscr{S}(\mathcal{A} \otimes \mathcal{A}) \mid \mathcal{A} \in \mathcal{S}^{n,2m}(\mathbb{V})\}$ denoted by $\mathrm{SOS}^{n,2m}(\mathbb{V})$. We define $\mathrm{MOM}^{n,2m}(\mathbb{V}) \coloneqq \mathrm{SOS}^{n,2m}(\mathbb{V})^*$ and call it the moment cone. Because $\mathrm{SOS}^{n,2m}(\mathbb{V})$ is a closed convex cone [9, Lemma 2.2], $\mathrm{SOS}^{n,2m}(\mathbb{V})$ and $\mathrm{MOM}^{n,2m}(\mathbb{V})$ are dual to each other.

3 Sum-of-squares-based inner-approximation hierarchy

Let $(\mathbb{E}, \circ, \bullet)$ be a Euclidean Jordan algebra of dimension n. In this section, we aim to provide an inner-approximation hierarchy described by an SOS constraint for the COP cone $\mathcal{COP}^{n,m}(\mathbb{E}_+)$. In Sect. 3.1, we first provide an inner-approximation hierarchy for the cone of homogeneous polynomials that are nonnegative over a symmetric cone in $\mathbb{R}^n$. Using the results in Sect. 3.1, we provide the desired approximation hierarchy in Sect. 3.2. In Sect. 3.3, we discuss its dual. In the following, we fix an orthonormal basis $v_1, \dots, v_n$ for the given Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ and let $\phi : \mathbb{E} \to \mathbb{R}^n$ be the associated isometry.

3.1 Case: homogeneous polynomial

If we define $\boldsymbol{x} \lozenge \boldsymbol{y} := \phi(\phi^{-1}(\boldsymbol{x}) \circ \phi^{-1}(\boldsymbol{y}))$ and $\boldsymbol{x} \blacklozenge \boldsymbol{y} := \boldsymbol{x}^\top \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then $(\mathbb{R}^n, \lozenge, \blacklozenge)$ is also a Euclidean Jordan algebra. Hereafter, to emphasize that $\mathbb{R}^n$ is a Euclidean Jordan algebra and $(\mathbb{R}^n, \lozenge, \blacklozenge)$ depends on the choice of ϕ, we write the Euclidean Jordan algebra $\mathbb{R}^n$ as $\phi(\mathbb{E})$. We note that the symmetric cone $\phi(\mathbb{E})_+$ associated with the Euclidean Jordan algebra $(\phi(\mathbb{E}), \lozenge, \blacklozenge)$ satisfies

$$\phi(\mathbb{E})_+ = \{\boldsymbol{x} \lozenge \boldsymbol{x} \mid \boldsymbol{x} \in \phi(\mathbb{E})\} = \{\phi(x) \lozenge \phi(x) \mid x \in \mathbb{E}\} = \phi(\mathbb{E}_+). \tag{10}$$

Here, we derive an inner-approximation hierarchy for the cone of homogeneous polynomials that are nonnegative over the symmetric cone $\phi(\mathbb{E}_+)$.

Proposition 3.1. *Let*

$$\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+)) := \{\theta \in H^{n,m} \mid \theta(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \in \phi(\mathbb{E}_+)\}$$

be the cone of homogeneous polynomials of degree m that are nonnegative over the symmetric cone $\phi(\mathbb{E}_+)$. For each $r \in \mathbb{N}$, we define

$$\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+)) := \{\theta \in H^{n,m} \mid (\boldsymbol{x}^\top \boldsymbol{x})^r \theta(\boldsymbol{x} \lozenge \boldsymbol{x}) \in \Sigma^{n,2(r+m)}\}.$$

Then, each $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$ is a closed convex cone, and the sequence $\{\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))\}_r$ satisfies the following two conditions:

(i) $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+)) \subseteq \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r+1}(\phi(\mathbb{E}_+)) \subseteq \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$ *for all* $r \in \mathbb{N}$.

(ii) $\operatorname{int} \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+)) \subseteq \bigcup_{r=0}^{\infty} \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$.

In the following, the notation "$\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+)) \uparrow \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$" is used to represent the two conditions mentioned in Proposition 3.1; however, the notation is not limited to the sequence $\{\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))\}_r$. To prove Proposition 3.1, we exploit Reznick's Positivstellensatz, which is described as follows:

Theorem 3.2 ([48, Theorem 3.12])**.** *Let* $\theta \in H^{n,2m}$*. If* $\theta(\boldsymbol{x}) > 0$ *for all* $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$*, then there exists* $r_0 \in \mathbb{N}$ *such that* $(\boldsymbol{x}^\top \boldsymbol{x})^{r_0}\theta(\boldsymbol{x}) \in \Sigma^{n,2(r_0+m)}$*.*

Proof of Proposition 3.1. Note that $\theta(\boldsymbol{x}\Diamond\boldsymbol{x}) \in H^{n,2m}$ for each $\theta \in H^{n,m}$ because the product $\Diamond$ is bilinear; thus, the set $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$ is well-defined for each $r \in \mathbb{N}$. $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$ a closed convex cone follows from the counterpart properties of $\Sigma^{n,2(r+m)}$. In the following, we prove $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+)) \uparrow \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$.

To prove (i), let $\theta \in \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$. Then, there exist $p_1, \dots, p_N \in H^{n,r+m}$ such that

$$(\boldsymbol{x}^\top \boldsymbol{x})^r \theta(\boldsymbol{x}\Diamond\boldsymbol{x}) = \sum_{i=1}^{N} p_i^2(\boldsymbol{x}). \tag{11}$$

Using this, we obtain

$$(\boldsymbol{x}^\top \boldsymbol{x})^{r+1} \theta(\boldsymbol{x}\Diamond\boldsymbol{x}) = \sum_{i=1}^{N}\sum_{j=1}^{n} (x_j p_i(\boldsymbol{x}))^2 \in \Sigma^{n,2(r+m+1)},$$

which means that $\theta \in \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r+1}(\phi(\mathbb{E}_+))$. Now, we assume that $\theta \notin \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. Then, from (10), it follows that there exists $\widetilde{\boldsymbol{x}} \in \phi(\mathbb{E})$ such that $\theta(\widetilde{\boldsymbol{x}}\Diamond\widetilde{\boldsymbol{x}}) < 0$. As $\widetilde{\boldsymbol{x}} \neq \mathbf{0}$ and $\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{x}} > 0$, we have $(\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{x}})^r\theta(\widetilde{\boldsymbol{x}}\Diamond\widetilde{\boldsymbol{x}}) < 0$. However, (11) implies that $(\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{x}})^r\theta(\widetilde{\boldsymbol{x}}\Diamond\widetilde{\boldsymbol{x}})$ must take a nonnegative value, which is a contradiction. Therefore, we obtain $\theta \in \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$.

To prove (ii), let $\theta \in \operatorname{int}\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. Then, as $\phi(\mathbb{E}_+)$ is a closed cone, it follows that $\theta(\boldsymbol{y}) > 0$ for all $\boldsymbol{y} \in \phi(\mathbb{E}_+) \setminus \{\mathbf{0}\}$, i.e., $\theta(\boldsymbol{x}\Diamond\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \phi(\mathbb{E}) \setminus \{\mathbf{0}\} = \mathbb{R}^n \setminus \{\mathbf{0}\}$ (see [54, Observation 1], for example). Thus, by Theorem 3.2, there exists $r_0 \in \mathbb{N}$ such that $(\boldsymbol{x}^\top\boldsymbol{x})^{r_0}\theta(\boldsymbol{x}\Diamond\boldsymbol{x}) \in \Sigma^{n,2(r_0+m)}$, which means that $\theta \in \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r_0}(\phi(\mathbb{E}_+)) \subseteq \bigcup_{r=0}^{\infty}\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$. □

Note that the set $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$ is defined by the SOS constraint. This constraint can be written as a semidefinite constraint of size $|\mathbb{I}^n_{=r+m}| = \binom{n+r+m+1}{n-1}$, which is polynomial in n and m for each fixed $r \in \mathbb{N}$.

3.2 Case: symmetric tensor

In this subsection, we translate the result of Proposition 3.1 to the case of symmetric tensors. Given that $(x \bullet x)^r\langle \mathcal{A}, (x \circ x)^{\otimes m}\rangle \in H^{n,2(r+m)}(\mathbb{E})$ for each $r \in \mathbb{N}$ and $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$, there exists a unique $\mathcal{A}^{(r)} \in \mathcal{S}^{n,2(r+m)}(\mathbb{E})$ such that

$$(x \bullet x)^r\langle \mathcal{A}, (x \circ x)^{\otimes m}\rangle = \langle \mathcal{A}^{(r)}, x^{\otimes 2(r+m)}\rangle. \tag{12}$$

Using the symmetric tensor $\mathcal{A}^{(r)}$, we define

$$\mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+) := \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}^{(r)} \in \mathrm{SOS}^{n,2(r+m)}(\mathbb{E})\}.$$

Theorem 3.3. *Using the orthonormal basis $v_1, \dots, v_n$ for $\mathbb{E}$, we define $\psi \in \mathrm{Hom}(\mathcal{S}^{n,m}(\mathbb{E}), H^{n,m})$ in the same manner as for* (5)*. Then,*

(i) $\psi(\mathcal{COP}^{n,m}(\mathbb{E}_+)) = \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$.

(ii) $\psi(\mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+)) = \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$.

(iii) Each $\mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+)$ is a closed convex cone, and the sequence $\{\mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+)\}_r$ satisfies $\mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+) \uparrow \mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Proof. We first prove (i). Let $\theta = \psi(\mathcal{A}) \in \psi(\mathcal{COP}^{n,m}(\mathbb{E}_+))$ and $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. Then, for any $x \in \mathbb{E}$, we have

$$\begin{aligned}\theta(\phi(x)\Diamond\phi(x)) &= \langle \mathcal{A}, \phi^{-1}(\phi(x)\Diamond\phi(x))^{\otimes m}\rangle \\ &= \langle \mathcal{A}, (x \circ x)^{\otimes m}\rangle \\ &\geq 0,\end{aligned}$$

using $\phi^{-1}(\phi(x)\Diamond\phi(x)) = x \circ x$ and $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. Therefore, we have $\theta \in \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. Conversely, let $\theta \in \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$ and $\mathcal{A} := \psi^{-1}(\theta) \in \mathcal{S}^{n,m}(\mathbb{E})$. Then, for any $x \in \mathbb{E}$, in the same manner as the discussion above, we obtain $\langle \mathcal{A}, (x \circ x)^{\otimes m}\rangle = \theta(\phi(x)\Diamond\phi(x)) \geq 0$, using $\phi(x)\Diamond\phi(x) \in \phi(\mathbb{E}_+)$. Therefore, $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$, and thus, $\theta = \psi(\mathcal{A}) \in \psi(\mathcal{COP}^{n,m}(\mathbb{E}_+))$.

Second, we prove (ii). Let $\theta = \psi(\mathcal{A}) \in \psi(\mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+))$ and $\mathcal{A} \in \mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+)$. As $\mathcal{A} \in \mathcal{K}^{n,m}_{\mathrm{NN},r}(\mathbb{E}_+)$, there exist $\mathcal{A}_1, \dots, \mathcal{A}_N \in \mathcal{S}^{n,r+m}(\mathbb{E})$ such that $(x \bullet x)^r\langle \mathcal{A}, (x \circ x)^{\otimes m}\rangle = \sum_{i=1}^N \langle \mathcal{A}_i, x^{\otimes(r+m)}\rangle^2$. As $\langle \mathcal{A}_i, \phi^{-1}(\boldsymbol{x})^{\otimes(r+m)}\rangle \in H^{n,r+m}$ for each i, it follows that

$$\begin{aligned}(\boldsymbol{x}^\top\boldsymbol{x})^r\theta(\boldsymbol{x}\Diamond\boldsymbol{x}) &= (\phi^{-1}(\boldsymbol{x}) \bullet \phi^{-1}(\boldsymbol{x}))^r\langle \mathcal{A}, \phi^{-1}(\boldsymbol{x}\Diamond\boldsymbol{x})^{\otimes m}\rangle \\ &= (\phi^{-1}(\boldsymbol{x}) \bullet \phi^{-1}(\boldsymbol{x}))^r\langle \mathcal{A}, (\phi^{-1}(\boldsymbol{x}) \circ \phi^{-1}(\boldsymbol{x}))^{\otimes m}\rangle \\ &= \sum_{i=1}^N \langle \mathcal{A}_i, \phi^{-1}(\boldsymbol{x})^{\otimes(r+m)}\rangle^2 \in \Sigma^{n,2(r+m)}.\end{aligned}$$

Therefore, we have $\theta \in \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$. Conversely, let $\theta \in \widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+))$. Then, there exist $\mathcal{A}_1, \dots, \mathcal{A}_N \in \mathcal{S}^{n,r+m}(\mathbb{E})$ such that $(\boldsymbol{x}^\top\boldsymbol{x})^r\theta(\boldsymbol{x}\Diamond\boldsymbol{x}) = \sum_{i=1}^N \langle \mathcal{A}_i, \phi^{-1}(\boldsymbol{x})^{\otimes(r+m)}\rangle^2$. Let $\mathcal{A} := \psi^{-1}(\theta) \in \mathcal{S}^{n,m}(\mathbb{E})$. Then, in the same manner as in (i), we have

$$\begin{aligned}(x \bullet x)^r\langle \mathcal{A}, (x \circ x)^{\otimes m}\rangle &= (\phi(x)^\top\phi(x))^r\theta(\phi(x)\Diamond\phi(x)) \\ &= \sum_{i=1}^N \langle \mathcal{A}_i, x^{\otimes(r+m)}\rangle^2 \in \Sigma^{n,2(r+m)}(\mathbb{E}).\end{aligned}$$

Finally, (iii) can be proven by the linear isomorphism of ψ and $\widetilde{\mathcal{K}}^{n,m}_{\mathrm{NN},r}(\phi(\mathbb{E}_+)) \uparrow \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. □

Note that $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$ does not depend on the choice of the orthonormal basis $v_1, \dots, v_n$ or isometry ϕ. We call the sequence $\{\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)\}_r$ the NN-type inner-approximation hierarchy.

Remark 3.4. *The NN-type inner-approximation hierarchy can be extended to the SOS cones proposed by Papp and Alizadeh [38]. Let $\mathbb{A}$ and $\mathbb{B}$ be real inner product spaces of dimensions l and n, respectively, and let $\diamond : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ be a bilinear mapping. The SOS cone is then defined as $\Sigma_\diamond := \mathrm{conv}\{x_i \diamond x_i \mid x_i \in \mathbb{A}\}$. Note that each element of $\Sigma_\diamond$ can be written as the sum of at most n elements $x_1 \diamond x_1, \dots, x_n \diamond x_n$ such that $x_1, \dots, x_n \in \mathbb{A}$, by Carathéodory's theorem for cones. If $(\mathbb{A}, \mathbb{B}, \diamond)$ is formally real, or equivalently, if $\Sigma_\diamond$ is proper [38, Theorem 3.3], then*

$$\mathcal{K}_r^{n,m}(\Sigma_\diamond) := \left\{ \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{B}) \;\middle|\; \begin{array}{l} \left(\sum_{i=1}^n x_i \bullet_{\mathbb{A}} x_i\right)^r \left\langle \mathcal{A}, \left(\sum_{i=1}^n x_i \diamond x_i\right)^{\otimes m} \right\rangle \\ \qquad \in \Sigma^{ln,2(r+m)}(\mathbb{A}^n) \end{array} \right\}$$

is a closed convex cone for each $r \in \mathbb{N}$ and satisfies $\mathcal{K}_r^{n,m}(\Sigma_\diamond) \uparrow \mathcal{COP}^{n,m}(\Sigma_\diamond)$, where $\bullet_{\mathbb{A}}$ denotes the inner product on $\mathbb{A}$.

3.3 Dual of $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$

Next, we discuss the dual cone of $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$. By considering its dual, we can provide an outer-approximation hierarchy for the CP cone $\mathcal{CP}^{n,m}(\mathbb{E}_+)$. Although the closure hull operator is generally required to describe the dual cone, we succeeded in removing it for the case in which the symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}_+^n$. Throughout this subsection, using the orthonormal basis $v_1, \dots, v_n$ fixed at the beginning of Sect. 3, we write each element of $\mathcal{S}^{n,m}(\mathbb{E})$ in the form (4).

Lemma 3.5. *There exists a unique linear mapping $\mathcal{C}^{(r)} : \mathcal{S}^{n,2(r+m)}(\mathbb{E}) \to \mathcal{S}^{n,m}(\mathbb{E})$ satisfying the following equation:*

$$\langle \mathcal{A}, \mathcal{C}^{(r)}(\mathcal{X}) \rangle = \langle \mathcal{A}^{(r)}, \mathcal{X} \rangle \text{ for all } \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \text{ and } \mathcal{X} \in \mathcal{S}^{n,2(r+m)}(\mathbb{E}), \tag{13}$$

where $\mathcal{A}^{(r)}$ is defined as (12).

Proof. We first prove existence. If we write $x \in \mathbb{E}$ as $\sum_{i=1}^n x_i v_i$, then

$$
\begin{aligned}
&\langle \mathcal{A}^{(r)}, x^{\otimes 2(r+m)} \rangle \\
&\quad = (x \bullet x)^r \langle \mathcal{A}, (x \circ x)^{\otimes m} \rangle \\
&\quad = \left(\sum_{i=1}^n x_i^2 \right)^r \sum_{i_1,\dots,i_m=1}^n \mathcal{A}_{i_1\cdots i_m} \prod_{l=1}^m \left(\sum_{j,k=1}^n x_j x_k v_j \circ v_k \right) \bullet v_{i_l} \\
&\quad = \sum_{i_1,\dots,i_m=1}^n \mathcal{A}_{i_1\cdots i_m} \left\{ \sum_{\boldsymbol{\alpha}\in\mathbb{I}_{=r}^n} \sum_{\substack{j_1,\dots,j_m=1\\k_1,\dots,k_m=1}}^n \left(\frac{r!}{\boldsymbol{\alpha}!} \prod_{l=1}^m (v_{j_l} \circ v_{k_l}) \bullet v_{i_l} \right) \boldsymbol{x}^{2\boldsymbol{\alpha}+\sum_{l=1}^m (\boldsymbol{e}_{j_l}+\boldsymbol{e}_{k_l})} \right\}. \qquad (14)
\end{aligned}
$$

Given that $\langle \mathcal{A}^{(r)}, \mathcal{X} \rangle$ is obtained by replacing each $\boldsymbol{x}^{\boldsymbol{\gamma}}$ in (14) with $\mathcal{X}_{[\boldsymbol{\gamma}]}$, $\langle \mathcal{A}^{(r)}, \mathcal{X} \rangle$ equals

$$
\sum_{i_1,\dots,i_m=1}^n \mathcal{A}_{i_1\cdots i_m} \left\{ \sum_{\boldsymbol{\alpha}\in\mathbb{I}_{=r}^n} \sum_{\substack{j_1,\dots,j_m=1\\k_1,\dots,k_m=1}}^n \left(\frac{r!}{\boldsymbol{\alpha}!} \prod_{l=1}^m (v_{j_l} \circ v_{k_l}) \bullet v_{i_l} \right) \mathcal{X}_{[2\boldsymbol{\alpha}+\sum_{l=1}^m (\boldsymbol{e}_{j_l}+\boldsymbol{e}_{k_l})]} \right\}.
$$

Using this equation, we define

$$
\mathcal{C}^{(r)}(\mathcal{X})_{i_1\cdots i_m} := \sum_{\boldsymbol{\alpha}\in\mathbb{I}_{=r}^n} \sum_{\substack{j_1,\dots,j_m=1\\k_1,\dots,k_m=1}}^n \left(\frac{r!}{\boldsymbol{\alpha}!} \prod_{l=1}^m (v_{j_l} \circ v_{k_l}) \bullet v_{i_l} \right) \mathcal{X}_{[2\boldsymbol{\alpha}+\sum_{l=1}^m (\boldsymbol{e}_{j_l}+\boldsymbol{e}_{k_l})]}
$$

for each $(i_1,\dots,i_m) \in \mathbb{N}_n^m$ and define $\mathcal{C}^{(r)}(\mathcal{X}) := \sum_{i_1,\dots,i_m=1}^n \mathcal{C}^{(r)}(\mathcal{X})_{i_1\cdots i_m} \tilde{v}_{i_1\cdots i_m}$. We note that $\mathcal{C}^{(r)}(\mathcal{X})$ is linear with respect to $\mathcal{X}$ and $\mathcal{C}^{(r)}(\mathcal{X}) \in \mathcal{S}^{n,m}(\mathbb{V})$ for each $\mathcal{X} \in \mathcal{S}^{n,2(r+m)}(\mathbb{E})$. Moreover, it directly follows from the definition of $\mathcal{C}^{(r)}(\mathcal{X})$ that $\langle \mathcal{A}, \mathcal{C}^{(r)}(\mathcal{X}) \rangle = \langle \mathcal{A}^{(r)}, \mathcal{X} \rangle$.

To prove uniqueness, we take two linear mappings $\mathcal{C}_1^{(r)}$ and $\mathcal{C}_2^{(r)}$ satisfying (13) arbitrarily. Then, for any $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$ and $\mathcal{X} \in \mathcal{S}^{n,2(r+m)}(\mathbb{E})$, as $\langle \mathcal{A}, \mathcal{C}_1^{(r)}(\mathcal{X}) \rangle = \langle \mathcal{A}, \mathcal{C}_2^{(r)}(\mathcal{X}) \rangle = \langle \mathcal{A}^{(r)}, \mathcal{X} \rangle$, we have $\langle \mathcal{A}, \mathcal{C}_1^{(r)}(\mathcal{X}) - \mathcal{C}_2^{(r)}(\mathcal{X}) \rangle = 0$. Because $\mathcal{A}$ is arbitrary, it follows that $\mathcal{C}_1^{(r)}(\mathcal{X}) = \mathcal{C}_2^{(r)}(\mathcal{X})$. In addition, because $\mathcal{X}$ is also arbitrary, we obtain $\mathcal{C}_1^{(r)} = \mathcal{C}_2^{(r)}$. □

Note that, although $\mathcal{C}^{(r)}$ is constructed with the orthonormal basis $v_1,\dots,v_n$ for $\mathbb{E}$ in the proof of Lemma 3.5, it does not depend on the choice of basis.

Proposition 3.6. *Using the linear mapping $\mathcal{C}^{(r)}$ defined in Lemma 3.5, we define $\mathcal{C}_r^{n,m}(\mathbb{E}_+) := \{\mathcal{C}^{(r)}(\mathcal{X}) \mid \mathcal{X} \in \mathrm{MOM}^{n,2(r+m)}(\mathbb{E})\}$. Then, it follows that $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+) = \mathcal{C}_r^{n,m}(\mathbb{E}_+)^*$, and thus, $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)^* = \mathrm{cl}\, \mathcal{C}_r^{n,m}(\mathbb{E}_+)$.*

Proof. Let $\mathcal{A} \in \mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$. Then, $\mathcal{A}^{(r)} \in \mathrm{SOS}^{n,2(r+m)}(\mathbb{E})$ by definition. For any $\mathcal{X} \in \mathrm{MOM}^{n,2(r+m)}(\mathbb{E})$, it follows from Lemma 3.5 and the duality between $\mathrm{MOM}^{n,2(r+m)}(\mathbb{E})$ and $\mathrm{SOS}^{n,2(r+m)}(\mathbb{E})$ that $\langle \mathcal{A}, \mathcal{C}^{(r)}(\mathcal{X})\rangle = \langle \mathcal{A}^{(r)}, \mathcal{X}\rangle \geq 0$, which means that $\mathcal{A} \in \mathcal{C}_r^{n,m}(\mathbb{E}_+)^*$.

Conversely, let $\mathcal{A} \in \mathcal{C}_r^{n,m}(\mathbb{E}_+)^*$. Then, for any $\mathcal{X} \in \mathrm{MOM}^{n,2(r+m)}(\mathbb{E})$, because $\mathcal{C}^{(r)}(\mathcal{X}) \in \mathcal{C}_r^{n,m}(\mathbb{E}_+)$, it follows that $\langle \mathcal{A}^{(r)}, \mathcal{X}\rangle = \langle \mathcal{A}, \mathcal{C}^{(r)}(\mathcal{X})\rangle \geq 0$. Therefore, $\mathcal{A}^{(r)} \in \mathrm{MOM}^{n,2(r+m)}(\mathbb{E})^* = \mathrm{SOS}^{n,2(r+m)}(\mathbb{E})$, which means that $\mathcal{A} \in \mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$.

Given that $\mathcal{C}_r^{n,m}(\mathbb{E}_+)$ is a convex cone, by taking the dual, we have $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)^* = \mathrm{cl}\, \mathcal{C}_r^{n,m}(\mathbb{E}_+)$. □

However, we could not prove that $\mathcal{C}_r^{n,m}(\mathbb{E}_+)$ itself is closed, and the closure hull operator is required to describe the dual of $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$. Therefore, whether $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{E}_+)$ and $\mathcal{C}_r^{n,m}(\mathbb{E}_+)$ itself are dual to each other is an open problem for a general symmetric cone $\mathbb{E}_+$. The difficulty originates from a lack of understanding of the moment cone $\mathrm{MOM}^{n,2m}(\mathbb{E})$.

As a special case, we consider the symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}_+^n$. In such a case, we can understand the structure of the moment cone. We show that the moment cone $\mathrm{MOM}^{n,2m}(\mathbb{R}^n)$ can be characterized by the semidefiniteness of a "moment matrix," which has been an open problem posed by [9, 10]. By exploiting this result, the closedness of $\mathcal{C}_r^{n,m}(\mathbb{R}_+^n)$ (written as $\mathcal{C}_r^{n,m}$ hereafter) is shown.

Definition 3.7. *For $\mathcal{X} \in \mathcal{S}^{n,2m}$, let $\boldsymbol{M}^{n,m}(\mathcal{X})$ be a matrix in $\mathbb{S}^{\mathbb{I}_{=m}^n}$ with the $(\boldsymbol{\alpha},\boldsymbol{\beta})$th element $\mathcal{X}_{[\boldsymbol{\alpha}+\boldsymbol{\beta}]}$ for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{I}_{=m}^n$. Then, we define* $\mathrm{MOM}^{n,2m} := \{\mathcal{X} \in \mathcal{S}^{n,2m} \mid \boldsymbol{M}^{n,m}(\mathcal{X}) \in \mathbb{S}_+^{\mathbb{I}_{=m}^n}\}$.

We aim to show that $\mathrm{MOM}^{n,2m}$ is dual to $\mathrm{SOS}^{n,2m}(\mathbb{R}^n)$, i.e., agrees with $\mathrm{MOM}^{n,2m}(\mathbb{R}^n)$.

Lemma 3.8. *For $\mathcal{A} \in \mathcal{S}^{n,m}$, let $\boldsymbol{\theta}$ be the element of $\mathbb{R}^{\mathbb{I}_{=m}^n}$ satisfying (7). Then, $\langle \mathscr{S}(\mathcal{A}\otimes\mathcal{A}), \mathcal{X}\rangle = \boldsymbol{\theta}^\top \boldsymbol{M}^{n,m}(\mathcal{X})\boldsymbol{\theta}$ for all $\mathcal{X} \in \mathcal{S}^{n,2m}$.*

Proof. Let $(\mathcal{F}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}_{=m}^n}$ be the orthonormal basis for $\mathcal{S}^{n,m}$ defined in Lemma 2.7. From Lemma 2.11 and (7), we have

$$\langle \mathscr{S}(\mathcal{A}\otimes\mathcal{A}), \boldsymbol{x}^{\otimes 2m}\rangle = \langle \mathcal{A}, \boldsymbol{x}^{\otimes m}\rangle^2 = \sum_{\boldsymbol{\gamma}\in\mathbb{I}_{=2m}^n} \left(\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{I}_{=m}^n \\ \boldsymbol{\gamma}=\boldsymbol{\alpha}+\boldsymbol{\beta}}} \theta_{\boldsymbol{\alpha}}\theta_{\boldsymbol{\beta}} \right) \boldsymbol{x}^{\boldsymbol{\gamma}}.$$

Therefore, from Lemma 2.8, when we represent $\mathscr{S}(\mathcal{A}\otimes\mathcal{A})$ with the orthonormal basis $(\mathcal{F}_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathbb{I}_{=2m}^n}$ for $\mathcal{S}^{n,2m}$, the coefficient $\mathscr{S}(\mathcal{A}\otimes\mathcal{A})_{\boldsymbol{\gamma}}$ can be written as

$$\mathscr{S}(\mathcal{A}\otimes\mathcal{A})_{\boldsymbol{\gamma}}^{\mathcal{F}} = \sqrt{\frac{\boldsymbol{\gamma}!}{(2m)!}} \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{I}_{=m}^n \\ \boldsymbol{\gamma}=\boldsymbol{\alpha}+\boldsymbol{\beta}}} \theta_{\boldsymbol{\alpha}}\theta_{\boldsymbol{\beta}}$$

for each $\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}$. In addition, from Lemma 2.7, each $\mathcal{X} \in \mathcal{S}^{n,2m}$ can be written as

$$\mathcal{X} = \sum_{\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}} \sqrt{\frac{(2m)!}{\boldsymbol{\gamma}!}} \mathcal{X}_{[\boldsymbol{\gamma}]} \mathcal{F}_{\boldsymbol{\gamma}}.$$

Then, we have

$$\begin{aligned}
\langle \mathscr{S}(\mathcal{A} \otimes \mathcal{A}), \mathcal{X} \rangle &= \sum_{\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}} \left(\sqrt{\frac{\boldsymbol{\gamma}!}{(2m)!}} \sum_{\substack{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m} \\ \boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}}} \theta_{\boldsymbol{\alpha}} \theta_{\boldsymbol{\beta}} \right) \left(\sqrt{\frac{(2m)!}{\boldsymbol{\gamma}!}} \mathcal{X}_{[\boldsymbol{\gamma}]} \right) \\
&= \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m}} \mathcal{X}_{[\boldsymbol{\alpha} + \boldsymbol{\beta}]} \theta_{\boldsymbol{\alpha}} \theta_{\boldsymbol{\beta}} \\
&= \boldsymbol{\theta}^\top \boldsymbol{M}^{n,m}(\mathcal{X}) \boldsymbol{\theta}.
\end{aligned}$$

□

Proposition 3.9. $\mathrm{MOM}^{n,2m}$ *and* $\mathrm{SOS}^{n,2m}(\mathbb{R}^n)$ *are dual to each other. In particular, it follows that* $\mathrm{MOM}^{n,2m} = \mathrm{MOM}^{n,2m}(\mathbb{R}^n)$.

Proof. Because $\mathrm{SOS}^{n,2m}(\mathbb{R}^n)$ is a closed convex cone, showing that $\mathrm{MOM}^{n,2m} = \mathrm{SOS}^{n,2m}(\mathbb{R}^n)^*$ is sufficient. Let $\mathcal{X} \in \mathrm{MOM}^{n,2m}$. For each $\mathcal{A} \in \mathrm{SOS}^{n,2m}(\mathbb{R}^n)$, there exist $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(k)} \in \mathcal{S}^{n,m}$ such that $\mathcal{A} = \sum_{i=1}^k \mathscr{S}(\mathcal{A}^{(i)} \otimes \mathcal{A}^{(i)})$. For each $\mathcal{A}^{(i)}$, let $\boldsymbol{\theta}^{(i)} = (\theta^{(i)}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ be such that it satisfies (7). Then, it follows from Lemma 3.8 and $\boldsymbol{M}^{n,m}(\mathcal{X}) \in \mathbb{S}^{\mathbb{I}^n_{=m}}_+$ that

$$\langle \mathcal{A}, \mathcal{X} \rangle = \sum_{i=1}^k \langle \mathscr{S}(\mathcal{A}^{(i)} \otimes \mathcal{A}^{(i)}), \mathcal{X} \rangle = \sum_{i=1}^k (\boldsymbol{\theta}^{(i)})^\top \boldsymbol{M}^{n,m}(\mathcal{X}) \boldsymbol{\theta}^{(i)} \geq 0,$$

which means that $\mathcal{X} \in \mathrm{SOS}^{n,2m}(\mathbb{R}^n)^*$.

Conversely, suppose that $\mathcal{X} \in \mathrm{SOS}^{n,2m}(\mathbb{R}^n)^*$. We take $\boldsymbol{\theta} = (\theta_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ arbitrarily and let $\mathcal{A}$ be the element of $\mathcal{S}^{n,m}$ satisfying (7). Then, it follows from Lemma 3.8 and $\mathscr{S}(\mathcal{A} \otimes \mathcal{A}) \in \mathrm{SOS}^{n,2m}(\mathbb{R}^n)$ that $\boldsymbol{\theta}^\top \boldsymbol{M}^{n,m}(\mathcal{X}) \boldsymbol{\theta} = \langle \mathscr{S}(\mathcal{A} \otimes \mathcal{A}), \mathcal{X} \rangle \geq 0$. Therefore, $\mathcal{X} \in \mathrm{MOM}^{n,2m}$. □

Using Proposition 3.9, we show the closedness of $\mathcal{C}^{n,m}_r$. When the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ is that given by Example 2.2, the $(i_1, \ldots, i_m)$th element of $\mathcal{C}^{(r)}(\mathcal{X}) \in \mathcal{S}^{n,m}$ defined in Lemma 3.5 is

$$\mathcal{C}^{(r)}(\mathcal{X})_{i_1 \cdots i_m} = \sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=r}} \frac{r!}{\boldsymbol{\alpha}!} \mathcal{X}_{[2\boldsymbol{\alpha} + 2\sum_{l=1}^m \boldsymbol{e}_{i_l}]}. \tag{15}$$

Lemma 3.10. *Suppose that* $\mathcal{X} \in \mathrm{MOM}^{n,2m}$*, i.e.,* $\boldsymbol{M}^{n,m}(\mathcal{X}) \in \mathbb{S}_+^{\mathbb{I}_{=m}^n}$*. Using* $\mathcal{X}$*, we define* $\mathcal{X}' \in \mathcal{S}^{n,2m}$ *as*

$$\mathcal{X}'_{[\boldsymbol{\gamma}]} \coloneqq \begin{cases} \mathcal{X}_{[\boldsymbol{\gamma}]} & (\textit{if all of the elements of } \boldsymbol{\gamma} \textit{ are even}), \\ 0 & (\textit{if some of the elements of } \boldsymbol{\gamma} \textit{ are odd}) \end{cases}$$

for each $\boldsymbol{\gamma} \in \mathbb{I}_{=2m}^n$*. It then follows that* $\boldsymbol{M}^{n,m}(\mathcal{X}') \in \mathbb{S}_+^{\mathbb{I}_{=m}^n}$*.*

Proof. The set $\mathbb{I}_{=m}^n$ is partitioned as two disjoint sets $\mathbb{I}_{=m,\mathrm{even}}^n$ and $\mathbb{I}_{=m,\mathrm{odd}}^n$, where

$$\begin{aligned} \mathbb{I}_{=m,\mathrm{even}}^n &\coloneqq \{\boldsymbol{\alpha} \in \mathbb{I}_{=m}^n \mid \text{All of the elements of } \boldsymbol{\alpha} \text{ are even}\}, \\ \mathbb{I}_{=m,\mathrm{odd}}^n &\coloneqq \{\boldsymbol{\alpha} \in \mathbb{I}_{=m}^n \mid \text{Some of the elements of } \boldsymbol{\alpha} \text{ are odd}\}. \end{aligned}$$

In addition, the elements of $\{0,1\}^n \setminus \{\boldsymbol{0}\}$ are ordered as $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{2^n-1}$ (e.g., $\boldsymbol{\delta}_1 = (1,0,\dots,0)$). Then, let

$$\mathbb{I}_{=m,\mathrm{odd},i}^n \coloneqq \{\boldsymbol{\alpha} \in \mathbb{I}_{=m,\mathrm{odd}}^n \mid \text{All of the elements of } \boldsymbol{\alpha} - \boldsymbol{\delta}_i \text{ are even}\}$$

for $i = 1, \dots, 2^n - 1$. Note that the parity between $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}_i$ agrees for every $\boldsymbol{\alpha} \in \mathbb{I}_{=m,\mathrm{odd},i}^n$ and that the sets $\mathbb{I}_{=m,\mathrm{odd},i}^n$ ($i = 1, \dots, 2^n - 1$) are disjoint. Then, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}_{=m}^n$, all elements of $\boldsymbol{\alpha} + \boldsymbol{\beta}$ are even if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to the same set of $\mathbb{I}_{=m,\mathrm{even}}^n, \mathbb{I}_{=m,\mathrm{odd},1}^n, \dots, \mathbb{I}_{=m,\mathrm{odd},2^n-1}^n$. Therefore, from the definition of $\mathcal{X}'$, we note that

$$\boldsymbol{M}^{n,m}(\mathcal{X}')_{IJ} = \begin{cases} \boldsymbol{M}^{n,m}(\mathcal{X})_{IJ} & (\text{if } I = J), \\ \boldsymbol{O} & (\text{otherwise}) \end{cases}$$

for I, $J = \mathbb{I}_{=m,\mathrm{even}}^n, \mathbb{I}_{=m,\mathrm{odd},1}^n, \dots, \mathbb{I}_{=m,\mathrm{odd},2^n-1}^n$, where $\boldsymbol{M}^{n,m}(\mathcal{X}')_{IJ}$ is the submatrix obtained by extracting the rows of $\boldsymbol{M}^{n,m}(\mathcal{X}')$ indexed by I and columns indexed by J. Because $\boldsymbol{M}^{n,m}(\mathcal{X})$ is semidefinite, $\boldsymbol{M}^{n,m}(\mathcal{X}')$ is as well. □

Theorem 3.11. $\mathcal{C}_r^{n,m}$ *is a closed convex cone, and thus,* $\mathcal{K}_{\mathrm{NN},r}^{n,m}(\mathbb{R}_+^n)$ *and* $\mathcal{C}_r^{n,m}$ *are dual to each other.*

Proof. Showing the closedness of $\mathcal{C}_r^{n,m}$ is sufficient. Let $\{\mathcal{A}_k\}_k \subseteq \mathcal{C}_r^{n,m}$ and suppose that the sequence converges to some $\mathcal{A}_\infty \in \mathcal{S}^{n,m}$. For each k, there exists $\mathcal{X}_k \in \mathrm{MOM}^{n,2(r+m)}$ such that $\mathcal{A}_k = \mathcal{C}^{(r)}(\mathcal{X}_k)$. We note from (15) that $\mathcal{C}^{(r)}(\mathcal{X}_k)$ is independent of $(\mathcal{X}_k)_{[\boldsymbol{\gamma}]}$ such that some elements of $\boldsymbol{\gamma} \in \mathbb{I}_{=2(r+m)}^n$ are odd. Thus, by Lemma 3.10, we can assume that such $(\mathcal{X}_k)_{[\boldsymbol{\gamma}]}$ is 0 without loss of generality. As $\mathcal{C}^{(r)}(\mathcal{X}_k) \to \mathcal{A}_\infty$ ($k \to \infty$), we observe that

$$\|\mathcal{C}^{(r)}(\mathcal{X}_k)\|_{\mathrm{F}} = \sqrt{\sum_{i_1,\dots,i_m=1}^n \left(\sum_{\boldsymbol{\alpha} \in \mathbb{I}_{=r}^n} \frac{r!}{\boldsymbol{\alpha}!} (\mathcal{X}_k)_{[2\boldsymbol{\alpha} + 2\sum_{l=1}^m \boldsymbol{e}_{i_l}]} \right)^2} \to \|\mathcal{A}_\infty\|_{\mathrm{F}}.$$

Note that each $(\mathcal{X}_k)_{[2\boldsymbol{\alpha}+2\sum_{l=1}^m \boldsymbol{e}_{i_l}]}$ is nonnegative because it is a diagonal element of the semidefinite matrix $\boldsymbol{M}^{n,r+m}(\mathcal{X}_k)$. Therefore, $\{(\mathcal{X}_k)_{[2\boldsymbol{\alpha}+2\sum_{l=1}^m \boldsymbol{e}_{i_l}]}\}_k$ is bounded for each $(i_1,\dots,i_m) \in \mathbb{N}_n^m$ and $\boldsymbol{\alpha} \in \mathbb{I}_{=r}^n$, and $\{\mathcal{X}_k\}_k$ is as well. Thus, by taking a subsequence if necessary, we assume that the sequence $\{\mathcal{X}_k\}_k$ converges to some $\mathcal{X}_\infty$. Given that $\{\mathcal{X}_k\}_k \subseteq \mathrm{MOM}^{n,2(r+m)}$ and $\mathrm{MOM}^{n,2(r+m)}$ is closed, we have $\mathcal{X}_\infty \in \mathrm{MOM}^{n,2(r+m)}$. Therefore,

$$\mathcal{A}_\infty = \lim_{k\to\infty} \mathcal{C}^{(r)}(\mathcal{X}_k) = \mathcal{C}^{(r)}(\mathcal{X}_\infty) \in \mathcal{C}_r^{n,m},$$

which implies that $\mathcal{C}_r^{n,m}$ is closed. □

4 Approximation hierarchies exploiting those for the usual copositive cone

In this section, we provide other approximation hierarchies for the COP cone over a symmetric cone by exploiting those for the usual COP cone. Let $(\mathbb{E},\circ,\bullet)$ be a Euclidean Jordan algebra of dimension n. In addition, for $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$ and an ordered Jordan frame $(c_1,\dots,c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$, let $\mathcal{A}(c_1,\dots,c_{\mathrm{rk}}) \in \mathcal{S}^{\mathrm{rk},m}$ be the tensor with the $(i_1,\dots,i_m)$th element $\langle \mathcal{A}, c_{i_1}\otimes\cdots\otimes c_{i_m}\rangle$. (The tensor is guaranteed to be symmetric by the symmetry of $\mathcal{A}$.)

The following lemma is key to providing the desired approximation hierarchies.

Lemma 4.1.

$$\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{(c_1,\dots,c_{\mathrm{rk}})\in\mathfrak{F}(\mathbb{E})} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_{\mathrm{rk}}) \in \mathcal{COP}^{\mathrm{rk},m}\}. \tag{16}$$

Proof. Let $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. For any $(c_1,\dots,c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$ and $\boldsymbol{x} \in \mathbb{R}_+^{\mathrm{rk}}$, we have

$$\begin{aligned}\langle \mathcal{A}(c_1,\dots,c_{\mathrm{rk}}), \boldsymbol{x}^{\otimes m}\rangle &= \sum_{i_1,\dots,i_m=1}^{\mathrm{rk}} \langle \mathcal{A}, c_{i_1}\otimes\cdots\otimes c_{i_m}\rangle x_{i_1}\cdots x_{i_m}\\ &= \left\langle \mathcal{A}, \left(\sum_{i=1}^{\mathrm{rk}} x_i c_i\right)^{\otimes m}\right\rangle,\end{aligned}$$

which is nonnegative given that $\sum_{i=1}^{\mathrm{rk}} x_i c_i \in \mathbb{E}_+$.

Conversely, suppose that $\mathcal{A}$ belongs to the right-hand side set of (16). For any $x \in \mathbb{E}_+$, there exist $(x_1,\dots,x_{\mathrm{rk}}) \in \mathbb{R}_+^{\mathrm{rk}}$ and $(c_1,\dots,c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$ such that $x = \sum_{i=1}^{\mathrm{rk}} x_i c_i$. As $\mathcal{A}(c_1,\dots,c_{\mathrm{rk}}) \in \mathcal{COP}^{\mathrm{rk},m}$, we can show $\langle \mathcal{A}, x^{\otimes m}\rangle \geq 0$, i.e., $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$ in the same manner as the above discussion. □

However, the characterization of the COP cone over a symmetric cone is somewhat redundant because the Jordan frames we considered are ordered, and thus, the same set appears multiple times on the right-hand side of (16). To solve this problem, we provide a more concise characterization of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Definition 4.2. *For each $\sigma \in \mathfrak{S}_n$, let $\mathscr{P}_\sigma$ be the linear transformation on $\mathcal{S}^{n,m}$ defined by $(\mathscr{P}_\sigma \mathcal{A})_{i_1\cdots i_m} := \mathcal{A}_{\sigma(i_1)\cdots\sigma(i_m)}$ for each $\mathcal{A} \in \mathcal{S}^{n,m}$ and $(i_1, \ldots, i_m) \in \mathbb{N}_n^m$. A set $\mathcal{K} \subseteq \mathcal{S}^{n,m}$ is said to be permutation-invariant if $\mathscr{P}_\sigma \mathcal{K} = \mathcal{K}$ for all $\sigma \in \mathfrak{S}_n$.*

Note that because $\mathscr{P}_\sigma$ is invertible and $\mathscr{P}_\sigma^{-1} = \mathscr{P}_{\sigma^{-1}}$ for all $\sigma \in \mathfrak{S}_n$, Definition 4.2 can be equivalently stated such that $\mathscr{P}_\sigma \mathcal{A} \in \mathcal{K}$ holds for all $\mathcal{A} \in \mathcal{K}$ and $\sigma \in \mathfrak{S}_n$.

Lemma 4.3. *$\mathcal{COP}^{n,m}$ is permutation-invariant.*

Proof. Let $\mathcal{A} \in \mathcal{COP}^{n,m}$. Then, for any $\sigma \in \mathfrak{S}_n$ and $\boldsymbol{x} \in \mathbb{R}_+^n$, we have

$$\begin{aligned}\langle \mathscr{P}_\sigma \mathcal{A}, \boldsymbol{x}^{\otimes m} \rangle &= \sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{\sigma(i_1)\cdots\sigma(i_m)} x_{i_1} \cdots x_{i_m} \\ &= \sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m} x_{\sigma^{-1}(i_1)} \cdots x_{\sigma^{-1}(i_m)} \\ &= \langle \mathcal{A}, (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})^{\otimes m} \rangle \\ &\geq 0,\end{aligned}$$

for which the last inequality follows from the fact that $(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \in \mathbb{R}_+^n$ if $\boldsymbol{x} \in \mathbb{R}_+^n$. □

Lemma 4.4. *Let $\mathcal{K} \subseteq \mathcal{S}^{\mathrm{rk},m}$ be a permutation-invariant set and let $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$. If $\mathcal{A}(c_1, \ldots, c_{\mathrm{rk}}) \in \mathcal{K}$ for a given $(c_1, \ldots, c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$, then $\mathcal{A}(c_{\sigma(1)}, \ldots, c_{\sigma(\mathrm{rk})}) \in \mathcal{K}$ for all $\sigma \in \mathfrak{S}_{\mathrm{rk}}$.*

Proof. Because $\mathcal{K}$ is permutation-invariant, we have $\mathscr{P}_\sigma \mathcal{A}(c_1, \ldots, c_{\mathrm{rk}}) \in \mathcal{K}$. Then, the $(i_1, \ldots, i_m)$th element of $\mathscr{P}_\sigma \mathcal{A}(c_1, \ldots, c_{\mathrm{rk}})$ is

$$\mathcal{A}(c_1, \ldots, c_{\mathrm{rk}})_{\sigma(i_1)\cdots\sigma(i_m)} = \langle \mathcal{A}, c_{\sigma(i_1)} \otimes \cdots \otimes c_{\sigma(i_m)} \rangle,$$

which equals the $(i_1, \ldots, i_m)$th element of $\mathcal{A}(c_{\sigma(1)}, \ldots, c_{\sigma(\mathrm{rk})})$. Therefore, we have

$$\mathcal{A}(c_{\sigma(1)}, \ldots, c_{\sigma(\mathrm{rk})}) = \mathscr{P}_\sigma \mathcal{A}(c_1, \ldots, c_m) \in \mathcal{K}.$$

□

The symmetric group $\mathfrak{S}_{\mathrm{rk}}$ acts on the set $\mathfrak{F}(\mathbb{E})$ by $\sigma \cdot (c_1, \ldots, c_{\mathrm{rk}}) = (c_{\sigma(1)}, \ldots, c_{\sigma(\mathrm{rk})})$. Let $\mathfrak{F}_{\mathrm{c}}(\mathbb{E})$ be a complete set of representatives of $\mathfrak{S}_{\mathrm{rk}}$-orbits in $\mathfrak{F}(\mathbb{E})$. Then, using Lemmas 4.3 and 4.4, we obtain the following, more concise, characterization of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$, compared with that using Lemma 4.1.

Proposition 4.5. *Let $\mathcal{K} \subseteq \mathcal{S}^{\mathrm{rk},m}$ be a permutation-invariant set. Then, the set*

$$\bigcap_{(c_1,\dots,c_{\mathrm{rk}})\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_{\mathrm{rk}}) \in \mathcal{K}\}$$

is the same regardless of the choice of $\mathfrak{F}$ with $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$. In particular, the claim holds when $\mathcal{K} = \mathcal{COP}^{\mathrm{rk},m}$, and it follows that

$$\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{(c_1,\dots,c_{\mathrm{rk}})\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_{\mathrm{rk}}) \in \mathcal{COP}^{\mathrm{rk},m}\}. \tag{17}$$

The idea for providing inner- and outer-approximation hierarchies for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$ is to approximate $\mathcal{COP}^{\mathrm{rk},m}$ on the right-hand side set in (17) from the inside and outside.

4.1 Inner-approximation hierarchy

Throughout this subsection, we only consider the case of $m = 2$. Generally, even if $\{\mathcal{I}_r^{\mathrm{rk}}\}_r$ is an inner-approximation hierarchy for $\mathcal{COP}^{\mathrm{rk},2}$, i.e., $\mathcal{I}_r^{\mathrm{rk}} \uparrow \mathcal{COP}^{\mathrm{rk},2}$, the sequence obtained by replacing $\mathcal{COP}^{\mathrm{rk},2}$ on the right-hand side set in (17) with $\mathcal{I}_r^{\mathrm{rk}}$ is not guaranteed to be an inner-approximation hierarchy for $\mathcal{COP}^{n,2}(\mathbb{E}_+)$. However, if we choose the polyhedral inner-approximation hierarchy provided by de Klerk and Pasechnik [11] as $\{\mathcal{I}_r^{\mathrm{rk}}\}_r$, the induced sequence is indeed an inner-approximation hierarchy for $\mathcal{COP}^{n,2}(\mathbb{E}_+)$. The hierarchy $\{\mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}}\}_r$ provided by de Klerk and Pasechnik [11] is defined as

$$\mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}} \coloneqq \{\boldsymbol{A} \in \mathbb{S}^{\mathrm{rk}} \mid (\boldsymbol{x}^\top \boldsymbol{1})^r \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \text{ has only nonnegative coefficients}\}$$

and satisfies $\mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}} \uparrow \mathcal{COP}^{\mathrm{rk},2}$.

The following theorem plays an important role in proving that the sequence induced by $\{\mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}}\}_r$ converges to $\mathcal{COP}^{n,2}(\mathbb{E}_+)$.

Theorem 4.6 ([11, Corollary 3.5]; also see [43, Theorem 1])**.** *Let $\boldsymbol{A} \in \mathrm{int}(\mathcal{COP}^{\mathrm{rk},2})$, and set $L \coloneqq \max_{1\le i,j\le \mathrm{rk}}|A_{ij}|$ and $\lambda \coloneqq \min_{\boldsymbol{x}\in\Delta_{=}^{\mathrm{rk}-1}} \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0$. If $r \in \mathbb{N}$ satisfies $r > L/\lambda - 2$, then $\boldsymbol{A} \in \mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}}$.*

Proposition 4.7. *We fix a set $\mathfrak{F}$ with $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$. Let*

$$\mathcal{I}_{\mathrm{dP},r}^{n}(\mathbb{E}_+) \coloneqq \bigcap_{(c_1,\dots,c_{\mathrm{rk}})\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,2}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_{\mathrm{rk}}) \in \mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}}\}. \tag{18}$$

Then, each $\mathcal{I}_{\mathrm{dP},r}^{n}(\mathbb{E}_+)$ is a closed convex cone and $\mathcal{I}_{\mathrm{dP},r}^{n}(\mathbb{E}_+) \uparrow \mathcal{COP}^{n,2}(\mathbb{E}_+)$.

Proof. Given that each $\mathcal{I}_{\mathrm{dP},r}^{\mathrm{rk}}$ is a closed convex cone and that the mapping $\mathcal{A} \mapsto \langle \mathcal{A}, c_i \otimes c_j\rangle$ is linear for each $(c_1,\dots,c_{\mathrm{rk}}) \in \mathfrak{F}$ and $i,j = 1,\dots,\mathrm{rk}$, $\mathcal{I}_{\mathrm{dP},r}^{n}(\mathbb{E}_+)$ is a closed convex cone. In the following, we prove $\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+) \uparrow \mathcal{COP}^{n,2}(\mathbb{E}_+)$. As the sequence $\{\mathcal{I}^{\mathrm{rk}}_{\mathrm{dP},r}\}_r$ satisfies $\mathcal{I}^{\mathrm{rk}}_{\mathrm{dP},r} \subseteq \mathcal{I}^{\mathrm{rk}}_{\mathrm{dP},r+1} \subseteq \mathcal{COP}^{\mathrm{rk},2}$ for all $r \in \mathbb{N}$, it follows from Proposition 4.5 that the sequence $\{\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)\}_r$ satisfies $\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+) \subseteq \mathcal{I}^n_{\mathrm{dP},r+1}(\mathbb{E}_+) \subseteq \mathcal{COP}^{n,2}(\mathbb{E}_+)$ for all $r \in \mathbb{N}$. Next, let $\mathcal{A} \in \operatorname{int}\mathcal{COP}^{n,2}(\mathbb{E}_+)$. Using $\mathcal{A}$, we define

$$L(\mathcal{A}; c_1, \dots, c_{\mathrm{rk}}) \coloneqq \max_{i \le i,j \le \mathrm{rk}} |\langle \mathcal{A}, c_i \otimes c_j \rangle|,$$
$$\lambda(\mathcal{A}; c_1, \dots, c_{\mathrm{rk}}) \coloneqq \min_{\boldsymbol{x} \in \Delta^{\mathrm{rk}-1}_{=}} \boldsymbol{x}^\top \mathcal{A}(c_1, \dots, c_{\mathrm{rk}}) \boldsymbol{x}$$

for each $(c_1, \dots, c_{\mathrm{rk}}) \in \mathfrak{F}(\mathbb{E})$, and also define

$$L(\mathcal{A}; \widetilde{\mathfrak{F}}) \coloneqq \sup_{(c_1,\dots,c_{\mathrm{rk}}) \in \widetilde{\mathfrak{F}}} L(\mathcal{A}; c_1, \dots, c_{\mathrm{rk}}),$$
$$\lambda(\mathcal{A}; \widetilde{\mathfrak{F}}) \coloneqq \inf_{(c_1,\dots,c_{\mathrm{rk}}) \in \widetilde{\mathfrak{F}}} \lambda(\mathcal{A}; c_1, \dots, c_{\mathrm{rk}})$$

for $\widetilde{\mathfrak{F}} \in \{\mathfrak{F}, \mathfrak{F}(\mathbb{E})\}$. Given that $\mathcal{A} \in \operatorname{int}\mathcal{COP}^{n,2}(\mathbb{E}_+)$ and that the sets $\Delta^{\mathrm{rk}-1}_{=}$ and $\mathfrak{F}(\mathbb{E})$ are compact, we have $L(\mathcal{A}; \mathfrak{F}(\mathbb{E})) < +\infty$ and $\lambda(\mathcal{A}; \mathfrak{F}(\mathbb{E})) > 0$. Because $L(\mathcal{A}; \mathfrak{F}) \le L(\mathcal{A}; \mathfrak{F}(\mathbb{E}))$ and $\lambda(\mathcal{A}; \mathfrak{F}) \ge \lambda(\mathcal{A}; \mathfrak{F}(\mathbb{E}))$, we obtain $L(\mathcal{A}; \mathfrak{F}) < +\infty$ and $\lambda(\mathcal{A}; \mathfrak{F}) > 0$. Now, let $r_0 \coloneqq \lceil L(\mathcal{A}; \mathfrak{F}) / \lambda(\mathcal{A}; \mathfrak{F}) \rceil \in \mathbb{N}$ and fix $(c_1, \dots, c_{\mathrm{rk}}) \in \mathfrak{F}$ arbitrarily. Then, because

$$r_0 > \frac{L(\mathcal{A}; \mathfrak{F})}{\lambda(\mathcal{A}; \mathfrak{F})} - 2 \ge \frac{L(\mathcal{A}; c_1, \dots, c_{\mathrm{rk}})}{\lambda(\mathcal{A}; c_1, \dots, c_{\mathrm{rk}})} - 2,$$

Theorem 4.6 implies that $\mathcal{A}(c_1, \dots, c_{\mathrm{rk}}) \in \mathcal{I}^{\mathrm{rk}}_{\mathrm{dP},r_0}$. Because r_0 is independent of the choice of $(c_1, \dots, c_{\mathrm{rk}}) \in \mathfrak{F}$, we obtain $\mathcal{A} \in \mathcal{I}^n_{\mathrm{dP},r_0}(\mathbb{E}_+) \subseteq \bigcup_{r=0}^\infty \mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)$. □

Remark 4.8. *As can be seen from the proof of Proposition 4.7, if a non-decreasing sequence* $\{\mathcal{I}^{\mathrm{rk}}_r\}_r$ *satisfies* $\mathcal{I}^{\mathrm{rk}}_{\mathrm{dP},r} \subseteq \mathcal{I}^{\mathrm{rk}}_r \subseteq \mathcal{COP}^{\mathrm{rk},2}$ *for all* $r \in \mathbb{N}$*, the sequence obtained by replacing* $\mathcal{COP}^{\mathrm{rk},2}$ *on the right-hand side set in* (17) *with* $\mathcal{I}^{\mathrm{rk}}_r$ *is also an inner-approximation hierarchy for* $\mathcal{COP}^{n,2}(\mathbb{E}_+)$*. In addition, Proposition 4.7 can be extended to the case of general* m *by using the polyhedral inner-approximation hierarchy provided by Iqbal and Ahmed [28], which is a generalization of that provided by de Klerk and Pasechnik [11], for* $\mathcal{COP}^{\mathrm{rk},m}$.

However, note that $\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)$ is defined as the intersection of the *infinitely* many sets in general even if m, the order of tensors, is limited to 2. This means that the inner-approximation hierarchy induces a semi-infinite conic constraint. Because the COP cone $\mathcal{COP}^{n,2}(\mathbb{E}_+)$ can also be described by a semi-infinite constraint, initially, the approximation hierarchy seems to be useless. However, when the symmetric cone $\mathbb{E}_+$ is the direct product of a nonnegative orthant and *one* second-order cone, each $\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)$ can be represented by *finitely* many semidefinite constraints.

Let $(\mathbb{E}_1, \circ_1, \bullet_1)$ and $(\mathbb{E}_2, \circ_2, \bullet_2)$ be the Euclidean Jordan algebras associated with the nonnegative orthant $\mathbb{R}^{n_1}_+$ and second-order cone $\mathbb{L}^{n_2}$ shown in Examples 2.2 and 2.3, respectively. Then, $\mathbb{E} \coloneqq \mathbb{E}_1 \times \mathbb{E}_2 = \mathbb{R}^{n_1+n_2}$ is the Euclidean Jordan algebra with the induced symmetric cone $\mathbb{R}^{n_1}_+ \times \mathbb{L}^{n_2}$ and rank $\mathrm{rk} = n_1 + 2$. We set $n \coloneqq n_1 + n_2$ and reindex $(1, \dots, \mathrm{rk})$ as $(11, \dots, 1n_1, 21, 22)$, i.e., $1i \coloneqq i$ for $i = 1, \dots, n_1$ and $2i \coloneqq n_1 + i$ for $i = 1, 2$. Section 4.2 uses this notation. In addition,

$$\mathfrak{F} = \left\{ \left(\begin{pmatrix} \boldsymbol{e}_{11} \\ \boldsymbol{0}_{n_2} \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{e}_{1n_1} \\ \boldsymbol{0}_{n_2} \end{pmatrix}, \begin{pmatrix} \boldsymbol{0}_{n_1} \\ 1/2 \\ \boldsymbol{v}/2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{0}_{n_1} \\ 1/2 \\ -\boldsymbol{v}/2 \end{pmatrix} \right) \middle| \, \boldsymbol{v} \in S^{n_2-2} \right\} \tag{19}$$

is a set satisfying $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$ (see Example 2.2, Example 2.3, and Proposition 2.5).

Under the identification between $\mathcal{S}^{n,2}$ and $\mathbb{S}^n$, the inner-approximation hierarchy (18) with $\mathfrak{F}$ the set (19) is

$$\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+) = \bigcap_{\boldsymbol{v} \in S^{n_2-2}} \{\boldsymbol{A} \in \mathbb{S}^n \mid f_r(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}) \text{ has only nonnegative coefficients}\},$$

where

$$\begin{aligned} f(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}) &\coloneqq \begin{pmatrix} 2\sum_{i=1}^{n_1} x_{1i}\boldsymbol{e}_{1i} \\ x_{21} + x_{22} \\ (x_{21} - x_{22})\boldsymbol{v} \end{pmatrix}^\top \boldsymbol{A} \begin{pmatrix} 2\sum_{i=1}^{n_1} x_{1i}\boldsymbol{e}_{1i} \\ x_{21} + x_{22} \\ (x_{21} - x_{22})\boldsymbol{v} \end{pmatrix}, \\ f_r(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}) &\coloneqq (\boldsymbol{x}^\top \boldsymbol{1})^r f(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}). \end{aligned} \tag{20}$$

Note that $f(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v})$ is doubled for the convenience of the following calculation; however, the set $\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)$ is unchanged. Let $\boldsymbol{A} \in \mathbb{S}^n$ be partitioned as follows:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}^{(11)} & \boldsymbol{A}^{(121)} & (\boldsymbol{A}^{(122)})^\top \\ (\boldsymbol{A}^{(121)})^\top & A^{(2121)} & (\boldsymbol{A}^{(2122)})^\top \\ \boldsymbol{A}^{(122)} & \boldsymbol{A}^{(2122)} & \boldsymbol{A}^{(2222)} \end{pmatrix}$$

with $\boldsymbol{A}^{(11)} \in \mathbb{S}^{n_1}$, $\boldsymbol{A}^{(121)} \in \mathbb{R}^{n_1}$, $\boldsymbol{A}^{(122)} \in \mathbb{R}^{(n_2-1)\times n_1}$, $A^{(2121)} \in \mathbb{R}$, $\boldsymbol{A}^{(2122)} \in \mathbb{R}^{n_2-1}$, and $\boldsymbol{A}^{(2222)} \in \mathbb{S}^{n_2-1}$. In addition, for such $\boldsymbol{A} \in \mathbb{S}^n$ and

$$\boldsymbol{\alpha} = (\underbrace{\alpha_{11}, \dots, \alpha_{1n_1}}_{\coloneqq \boldsymbol{\alpha}_1}, \alpha_{21}, \alpha_{22}) \in \mathbb{N}^{\mathrm{rk}},$$

we let

$$\begin{aligned} M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) &\coloneqq 4\{\boldsymbol{\alpha}_1^\top \boldsymbol{A}^{(11)} \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_1^\top \operatorname{diag}(\boldsymbol{A}^{(11)})\} + 4(\alpha_{21} + \alpha_{22})\boldsymbol{\alpha}_1^\top \boldsymbol{A}^{(121)} \\ &\quad + (\alpha_{21} + \alpha_{22})(\alpha_{21} + \alpha_{22} - 1)A^{(2121)}, \\ \boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha}) &\coloneqq 2(\alpha_{21} - \alpha_{22})\boldsymbol{A}^{(122)}\boldsymbol{\alpha}_1 + (\alpha_{21} - \alpha_{22})(\alpha_{21} + \alpha_{22} - 1)\boldsymbol{A}^{(2122)}, \\ \boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) &\coloneqq \{(\alpha_{21} - \alpha_{22})^2 - (\alpha_{21} + \alpha_{22})\}\boldsymbol{A}^{(2222)} \end{aligned}$$

and define

$$\boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) := \begin{pmatrix} M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})^\top \\ \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha}) \end{pmatrix} \in \mathbb{S}^{n_2},$$

where $\operatorname{diag}(\boldsymbol{A}^{(11)}) \in \mathbb{R}^{n_1}$ is the vector of the diagonal elements of $\boldsymbol{A}^{(11)}$. Then, after some calculations, we have

$$f_r(\boldsymbol{x};\boldsymbol{A},\boldsymbol{v}) = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^{\mathrm{rk}}_{=r+2}} \frac{r!}{\boldsymbol{\alpha}!} \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Therefore,

$$\begin{aligned} \mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+) &= \bigcap_{\boldsymbol{\alpha}\in\mathbb{I}^{\mathrm{rk}}_{=r+2}} \bigcap_{\boldsymbol{v}\in S^{n_2-2}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \right\} \\ &= \bigcap_{\boldsymbol{\alpha}\in\mathbb{I}^{\mathrm{rk}}_{=r+2}} \{\boldsymbol{A}\in\mathbb{S}^n \mid \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \in \mathcal{COP}(\partial\mathbb{L}^{n_2})\} \\ &= \bigcap_{\boldsymbol{\alpha}\in\mathbb{I}^{\mathrm{rk}}_{=r+2}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{array}{l} \text{There exists } t_{\boldsymbol{\alpha}}\in\mathbb{R} \text{ such that} \\ \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) - t_{\boldsymbol{\alpha}} \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I}_{n_2-1} \end{pmatrix} \in \mathbb{S}^{n_2}_+ \end{array} \right\}, \end{aligned} \tag{21}$$

where the last equation follows from [51, Corollary 6]. In summary, $\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)$ can be described by $|\mathbb{I}^{\mathrm{rk}}_{=r+2}|$ semidefinite constraints, which is bounded by rk^{r+2} and whose size is n_2. We call the sequence $\{\mathcal{I}^n_{\mathrm{dP},r}(\mathbb{E}_+)\}_r$ the dP-type inner-approximation hierarchy.

Remark 4.9. *We can make the expression* (21) *more concise. First, we show that the number of constraints in* (21) *can be halved. For* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \alpha_{21}, \alpha_{22}) \in \mathbb{I}^{\mathrm{rk}}_{=r+2}$*, let* $\widetilde{\boldsymbol{\alpha}} := (\boldsymbol{\alpha}_1, \alpha_{22}, \alpha_{21}) \in \mathbb{I}^{\mathrm{rk}}_{=r+2}$*. As* $M^{(11)}(\boldsymbol{A},\widetilde{\boldsymbol{\alpha}}) = M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha})$*,* $\boldsymbol{M}^{(21)}(\boldsymbol{A},\widetilde{\boldsymbol{\alpha}}) = -\boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})$*, and* $\boldsymbol{M}^{(22)}(\boldsymbol{A},\widetilde{\boldsymbol{\alpha}}) = \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha})$ *for each* $\boldsymbol{A}\in\mathbb{S}^n$*, we have*

$$\begin{aligned} &\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\widetilde{\boldsymbol{\alpha}}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2} \\ &\iff \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \begin{pmatrix} M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha}) & -\boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})^\top \\ -\boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha}) \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2} \\ &\iff \begin{pmatrix} 1 \\ -\boldsymbol{v} \end{pmatrix}^\top \begin{pmatrix} M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})^\top \\ \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha}) \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2}. \end{aligned} \tag{22}$$

When $\boldsymbol{v}$ *takes every element of* S^{n_2-2}*,* $-\boldsymbol{v}$ *also takes every element of* S^{n_2-2}*. Thus,* (22) *is equivalent to*

$$\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2}.$$

That is, taking the intersection with respect to $\boldsymbol{\alpha} \in \mathbb{I}^{\mathrm{rk}}_{=r+2}$ *with* $\alpha_{21} \le \alpha_{22}$ *in* (21) *is sufficient. This can be explained by the fact that the ordering of a Jordan frame can be ignored because* $\mathcal{I}^{\mathrm{rk}}_{\mathrm{dP},r}$ *is permutation-invariant; thus, we can apply Proposition 4.5.*

Next, we show that some semidefinite constraints in (21) *can be written as second-order cone or nonnegative constraints. If* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \alpha_{21}, \alpha_{22}) \in \mathbb{I}^{\mathrm{rk}}_{=r+2}$ *satisfies*

$$\alpha_{21} = \frac{1}{2}k(k-1),\ \alpha_{22} = \frac{1}{2}k(k+1)\ (k = 0, \dots, \lfloor\sqrt{r+2}\rfloor),$$

then $\boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) = \boldsymbol{O}$. *In this case,*

$$\begin{aligned}
&\begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix}^{\top} \boldsymbol{M}(\boldsymbol{A}, \boldsymbol{\alpha}) \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix} \ge 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2}\\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) + 2\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})^{\top}\boldsymbol{v} \ge 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2}\\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) - 2\|\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})\|_2 \ge 0\\
&\iff \begin{pmatrix} M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha})\\ 2\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})\end{pmatrix} \in \mathbb{L}^{n_2}.
\end{aligned}$$

In particular, when $k = 0$, *i.e.,* $\alpha_{21} = \alpha_{22} = 0$, *as* $\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})$ *is also zero, the above second-order cone constraint reduces to the nonnegative constraint* $M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) \ge 0$.

Finally, we show that the size of some semidefinite constraints can be reduced by 1. If $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \alpha_{21}, \alpha_{22}) \in \mathbb{I}^{\mathrm{rk}}_{=r+2}$ *satisfies* $\alpha_{21} = \alpha_{22} \ne 0$, *we have* $\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha}) = \boldsymbol{0}$. *Then,*

$$\begin{aligned}
&\begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix}^{\top} \boldsymbol{M}(\boldsymbol{A}, \boldsymbol{\alpha}) \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix} \ge 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2}\\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) + \boldsymbol{v}^{\top}\boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha})\boldsymbol{v} \ge 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2}\\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) + \lambda_{\min}(\boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha})) \ge 0\\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha})\boldsymbol{I}_{n_2-1} + \boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) \in \mathbb{S}^{n_2-1}_{+},
\end{aligned}$$

where $\lambda_{\min}(\cdot)$ *denotes the minimum eigenvalue for an input matrix.*

4.2 Outer-approximation hierarchy

Unlike the case of inner-approximation hierarchies, an outer-approximation hierarchy for $\mathcal{COP}^{\mathrm{rk},m}$ always induces that for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Proposition 4.10. *Let* $\{\mathcal{O}^{\mathrm{rk},m}_r\}_r$ *be a sequence such that each* $\mathcal{O}^{\mathrm{rk},m}_r$ *is a closed convex cone, and the sequence satisfies the following two conditions:*

(i) $\mathcal{O}^{\mathrm{rk},m}_{r+1} \subseteq \mathcal{O}^{\mathrm{rk},m}_r$ *for all* $r \in \mathbb{N}$.

(ii) $\mathcal{COP}^{\mathrm{rk},m} = \bigcap_{r=0}^{\infty} \mathcal{O}^{\mathrm{rk},m}_r$.

In the following, the notation "$\mathcal{O}_r^{\mathrm{rk},m} \downarrow \mathcal{COP}^{\mathrm{rk},m}$" is used to represent the two conditions as in the case of inner-approximation hierarchies. We fix a set $\mathfrak{F}$ with $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$. Let

$$\mathcal{O}_r^{n,m}(\mathbb{E}_+) := \bigcap_{(c_1,\ldots,c_{\mathrm{rk}})\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\ldots,c_{\mathrm{rk}}) \in \mathcal{O}_r^{\mathrm{rk},m}\}. \tag{23}$$

Then, each $\mathcal{O}_r^{n,m}(\mathbb{E}_+)$ is a closed convex cone and $\mathcal{O}_r^{n,m}(\mathbb{E}_+) \downarrow \mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Proof. Because $\mathcal{O}_r^{n,m}(\mathbb{E}_+)$ can easily be shown to be a closed convex cone and $\mathcal{O}_{r+1}^{n,m}(\mathbb{E}_+) \subseteq \mathcal{O}_r^{n,m}(\mathbb{E}_+)$ for all $r \in \mathbb{N}$ in the same manner as Proposition 4.7, we prove only $\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{r=0}^{\infty} \mathcal{O}_r^{n,m}(\mathbb{E}_+)$. The "$\subseteq$" part follows from Proposition 4.5. To prove the "$\supseteq$" part, let $\mathcal{A} \in \bigcap_{r=0}^{\infty} \mathcal{O}_r^{n,m}(\mathbb{E}_+)$. Then, $\mathcal{A}(c_1,\ldots,c_{\mathrm{rk}}) \in \mathcal{O}_r^{\mathrm{rk},m}$ for all $r \in \mathbb{N}$ and $(c_1,\ldots,c_{\mathrm{rk}}) \in \mathfrak{F}$. Because $r \in \mathbb{N}$ is arbitrary, we have $\mathcal{A}(c_1,\ldots,c_{\mathrm{rk}}) \in \mathcal{COP}^{\mathrm{rk},m}$. In addition, because $(c_1,\ldots,c_{\mathrm{rk}}) \in \mathfrak{F}$ is also arbitrary, we have $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. □

As in Proposition 4.7, the outer-approximation hierarchy induces a semi-infinite conic constraint. However, as with the inner-approximation hierarchy, when the symmetric cone $\mathbb{E}_+$ is the direct product of a nonnegative orthant and second-order cone and $m = 2$, if we choose an appropriate polyhedral outer-approximation hierarchy as $\{\mathcal{O}_r^{\mathrm{rk},2}\}_r$ (written as $\{\mathcal{O}_r^{\mathrm{rk}}\}_r$ hereafter), each $\mathcal{O}_r^{n,2}(\mathbb{E}_+)$ (written as $\mathcal{O}_r^{n}(\mathbb{E}_+)$ hereafter) can be represented by finitely many semidefinite constraints.

Let $\mathbb{E}$ be the same Euclidean Jordan algebra with the induced symmetric cone $\mathbb{R}_+^{n_1} \times \mathbb{L}^{n_2}$ as defined in Sect. 4.1. The polyhedral outer-approximation hierarchy $\{\mathcal{O}_r^{\mathrm{rk}}\}_r$ we use is based on a discretization of the standard simplex and written as

$$\mathcal{O}_r^{\mathrm{rk}} = \bigcap_{\boldsymbol{x}\in\delta_r^{\mathrm{rk}-1}} \{\boldsymbol{A} \in \mathbb{S}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \geq 0\}, \tag{24}$$

where $\delta_r^{\mathrm{rk}-1}$ is a finite subset of $\Delta_=^{\mathrm{rk}-1}$ for each $r \in \mathbb{N}$. This type of outer-approximation hierarchy includes, for example, that given by Yıldırım [52]. The outer-approximation hierarchy (23) induced by the set (19) and polyhedral outer-approximation hierarchy (24) is

$$\mathcal{O}_r^{n}(\mathbb{E}_+) = \bigcap_{\boldsymbol{v}\in S^{n_2-2}} \bigcap_{\boldsymbol{x}\in\delta_r^{\mathrm{rk}-1}} \{\boldsymbol{A} \in \mathbb{S}^n \mid f(\boldsymbol{x};\boldsymbol{A},\boldsymbol{v}) \geq 0\},$$

where $f(\boldsymbol{x};\boldsymbol{A},\boldsymbol{v})$ is defined as (20). Let

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) := \sum_{\boldsymbol{\alpha}\in\mathbb{I}_{=2}^{\mathrm{rk}}} \frac{1}{\boldsymbol{\alpha}!}\boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Then,

$$\begin{aligned}
\mathcal{O}_r^n(\mathbb{E}_+) &= \bigcap_{\boldsymbol{x}\in\delta_r^{\mathrm{rk}-1}} \bigcap_{\boldsymbol{v}\in S^{n_2-2}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix}^\top \boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix} \geq 0 \right\} \\
&= \bigcap_{\boldsymbol{x}\in\delta_r^{\mathrm{rk}-1}} \{\boldsymbol{A}\in\mathbb{S}^n \mid \boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) \in \mathcal{COP}(\partial\mathbb{L}^{n_2})\} \\
&= \bigcap_{\boldsymbol{x}\in\delta_r^{\mathrm{rk}-1}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{array}{l} \text{There exists } t_{\boldsymbol{x}}\in\mathbb{R} \text{ such that} \\ \boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) - t_{\boldsymbol{x}} \begin{pmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{I}_{n_2-1}\end{pmatrix} \in \mathbb{S}_+^{n_2} \end{array} \right\}. \qquad (25)
\end{aligned}$$

In summary, $\mathcal{O}_r^n(\mathbb{E}_+)$ can be described by the $|\delta_r^{\mathrm{rk}-1}|$ semidefinite constraints of size n_2. In particular, if we use the outer-approximation hierarchy $\{\mathcal{O}_r^{\mathrm{rk}}\}_r$ proposed by Yıldırım [52], then $\delta_r^{\mathrm{rk}-1}$ is given by

$$\delta_r^{\mathrm{rk}-1} = \bigcup_{k=0}^{r} \{\boldsymbol{x}\in\Delta_=^{\mathrm{rk}-1} \mid (k+2)\boldsymbol{x}\in\mathbb{N}^{\mathrm{rk}}\}, \qquad (26)$$

and $|\delta_r^{\mathrm{rk}-1}|$ is bounded by $\mathrm{rk}^2(\frac{\mathrm{rk}^{r+1}-1}{\mathrm{rk}-1})$, which is polynomial in rk for every fixed $r\in\mathbb{N}$ (see [52, Eq. (10)]). We call the sequence $\{\mathcal{O}_r^n(\mathbb{E}_+)\}_r$ obtained by exploiting the hierarchy given by Yıldırım [52] the Yıldırım-type outer-approximation hierarchy.

Remark 4.11. *As with the inner-approximation hierarchy, we can make the expression* (25) *more concise. Let* $\boldsymbol{N}(\boldsymbol{x},\boldsymbol{A})$ *be partitioned as follows:*

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) = \begin{pmatrix} N^{(11)}(\boldsymbol{x},\boldsymbol{A}) & \boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A})^\top \\ \boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A}) & \boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A}) \end{pmatrix},$$

where $N^{(11)}(\boldsymbol{x},\boldsymbol{A})\in\mathbb{R}$, $\boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A})\in\mathbb{R}^{n_2-1}$, *and* $\boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A})\in\mathbb{S}^{n_2-1}$ *and* $\boldsymbol{x} = (\boldsymbol{x}_1, x_{21}, x_{22})\in\mathbb{R}^{\mathrm{rk}}$. *Then, we have*

$$\begin{aligned}
N^{(11)}(\boldsymbol{x},\boldsymbol{A}) &= 4\boldsymbol{x}_1^\top\boldsymbol{A}^{(11)}\boldsymbol{x}_1 + 4(x_{21}+x_{22})\boldsymbol{x}_1^\top\boldsymbol{A}^{(121)} + (x_{21}+x_{22})^2A^{(2121)} \\
\boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A}) &= 2(x_{21}-x_{22})\boldsymbol{A}^{(122)}\boldsymbol{x}_1 + (x_{21}^2-x_{22}^2)\boldsymbol{A}^{(2122)} \\
\boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A}) &= (x_{21}-x_{22})^2\boldsymbol{A}^{(2222)}.
\end{aligned}$$

First, the number of constraints in (25) *may be reduced. Suppose that* $\delta_r^{\mathrm{rk}-1}$ *is permutation-invariant in the sense of Definition 4.2. (Note that* $\mathcal{S}^{\mathrm{rk},1}=\mathbb{R}^{\mathrm{rk}}$*.) For example,* (26) *is permutation-invariant. For* $\boldsymbol{x}=(\boldsymbol{x}_1,x_{21},x_{22})\in\delta_r^{\mathrm{rk}-1}$, *let* $\widetilde{\boldsymbol{x}} \coloneqq (\boldsymbol{x}, x_{22}, x_{21})$, *then* $\widetilde{\boldsymbol{x}}\in\delta_r^{\mathrm{rk}-1}$ *because* $\delta_r^{\mathrm{rk}-1}$ *is permutation-invariant. Given that* $N^{(11)}(\widetilde{\boldsymbol{x}},\boldsymbol{A}) = N^{(11)}(\boldsymbol{x},\boldsymbol{A})$, $\boldsymbol{N}^{(21)}(\widetilde{\boldsymbol{x}},\boldsymbol{A}) = -\boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A})$, *and* $\boldsymbol{N}^{(22)}(\widetilde{\boldsymbol{x}},\boldsymbol{A}) = \boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A})$ *for each* $\boldsymbol{A}\in\mathbb{S}^n$, *taking the intersection with respect to* $\boldsymbol{x}\in\delta_r^{\mathrm{rk}-1}$ *with* $x_{21}\leq x_{22}$ *in* (25) *is sufficient for the same reason as for Remark 4.9.*

Second, some semidefinite constraints in (25) *can be written as nonnegative constraints. If* $\boldsymbol{x} = (\boldsymbol{x}_1, x_{21}, x_{22}) \in \delta_r^{\mathrm{rk}-1}$ *satisfies* $x_{21} = x_{22}$*, then we have* $\boldsymbol{N}^{(21)}(\boldsymbol{x}, \boldsymbol{A}) = \boldsymbol{0}$ *and* $\boldsymbol{N}^{(22)}(\boldsymbol{x}, \boldsymbol{A}) = \boldsymbol{O}$*. Therefore, the constraint*

$$\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{A}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \textit{ for all } \boldsymbol{v} \in S^{n_2-2}$$

reduces to $N^{(11)}(\boldsymbol{x}, \boldsymbol{A}) \geq 0$.

5 Comparison with other approximation hierarchies

The previous sections provide new approximation hierarchies applicable to the COP cone $\mathcal{COP}(\mathbb{K})$ over the cone $\mathbb{K} = \mathbb{R}^{n_1} \times \mathbb{L}^{n_2}$. For such $\mathbb{K}$, those given by Zuluaga et al. [54] and Lasserre [32] are also applicable to $\mathcal{COP}(\mathbb{K})$. In this section, we compare the proposed approximation hierarchies for $\mathcal{COP}(\mathbb{K})$ with those existing theoretically. In the following, we set $n \coloneqq n_1 + n_2$ as in Sect. 4 and reindex $(1, \ldots, n)$ as $(11, \ldots, 1n_1, 21, \ldots, 2n_2)$, i.e., $1i \coloneqq i$ for $i = 1, \ldots, n_1$ and $2i \coloneqq n_1 + i$ for $i = 1, \ldots, n_2$. Section 6 uses this notation.

First, the cone $\mathbb{K} = \mathbb{R}^{n_1} \times \mathbb{L}^{n_2}$ can be represented as a semialgebraic set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \phi_i^{(1)}(\boldsymbol{x}) \coloneqq x_{1i} \geq 0 \ (i = 1, \ldots, n_1), \\ \phi_{n_1+1}^{(1)}(\boldsymbol{x}) \coloneqq x_{21} \geq 0, \\ \phi_{n_1+2}^{(1)}(\boldsymbol{x}) \coloneqq \boldsymbol{e}^\top \boldsymbol{x} \geq 0, \\ \phi^{(2)}(\boldsymbol{x}) \coloneqq x_{21}^2 - \sum_{i=2}^{n_2} x_{2i}^2 \geq 0 \end{array} \right\}, \tag{27}$$

where $\boldsymbol{e} \coloneqq (\boldsymbol{1}_{n_1+1}, \boldsymbol{0}_{n_2-1}) \in \mathrm{int}(\mathbb{K})$; thus, the inner-approximation hierarchy given by Zuluaga et al. [54] is applicable to $\mathcal{COP}(\mathbb{K})$. Note that the inequality $\boldsymbol{e}^\top \boldsymbol{x} \geq 0$ in (27) is redundant but necessary to deriving the hierarchy (see [54, Assumption 1]). The inner-approximation hierarchy for $\mathcal{COP}(\mathbb{K})$ is summarized as follows:

Theorem 5.1 ([54, Proposition 17]). *Let*

$$E^{n,m}(\mathbb{K}) \coloneqq \mathrm{conv} \left\{ \psi^2 \prod_{j=1}^k \phi_{i_j} \;\middle|\; \begin{array}{l} k \in \mathbb{N},\ m - \sum_{j=1}^k \deg(\phi_{i_j}) \in \mathbb{N} \textit{ is even}, \\ \psi \in H^{n,(m - \sum_{j=1}^k \deg(\phi_{i_j}))/2}, \\ \phi_{i_j} \in \{\phi_1^{(1)}, \ldots, \phi_{n_1+2}^{(1)}, \phi^{(2)}\} \ (j = 1, \ldots, k) \end{array} \right\} \tag{28}$$

and $\mathcal{K}_{\mathrm{ZVP},r}(\mathbb{K}) \coloneqq \{\boldsymbol{A} \in \mathbb{S}^n \mid (\boldsymbol{e}^\top \boldsymbol{x})^r \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} \in E^{n,r+2}(\mathbb{K})\}$ *for each* $r \in \mathbb{N}$*. Then, the sequence* $\{\mathcal{K}_{\mathrm{ZVP},r}(\mathbb{K})\}_r$ *satisfies* $\mathcal{K}_{\mathrm{ZVP},r}(\mathbb{K}) \uparrow \mathcal{COP}(\mathbb{K})$.

In the following, we call the sequence $\{\mathcal{K}_{\mathrm{ZVP},r}(\mathbb{K})\}_r$ the ZVP-type inner-approximation hierarchy. Although the representation (28) of the set $E^{n,m}(\mathbb{K})$ is somewhat abstract, we can represent it recursively.

Lemma 5.2. *If $m = 2k$ for some $k \in \mathbb{N}$, then*

$$\begin{aligned} &E^{n,m}(\mathbb{K}) \\ &= \operatorname{conv}\left(\Sigma^{n,2k} \cup \left\{\psi\phi_i^{(1)} \;\middle|\; \begin{array}{l} i = 1, \dots, n_1 + 2, \\ \psi \in E^{n,2k-1}(\mathbb{K}) \end{array}\right\} \cup \{\psi\phi_1^{(2)} \mid \psi \in E^{n,2k-2}(\mathbb{K})\}\right) \\ &= \left\{\psi^{(0)} + \sum_{i=1}^{n_1+2} \psi_i^{(1)}\phi_i^{(1)} + \psi^{(2)}\phi^{(2)} \;\middle|\; \begin{array}{l} \psi^{(0)} \in \Sigma^{n,2k}, \\ \psi_i^{(1)} \in E^{n,2k-1}(\mathbb{K}), \\ \psi^{(2)} \in E^{n,2k-2}(\mathbb{K}) \end{array}\right\}. \end{aligned}$$

If $m = 2k + 1$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} E^{n,m}(\mathbb{K}) &= \operatorname{conv}\left(\left\{\psi\phi_i^{(1)} \;\middle|\; \begin{array}{l} i = 1, \dots, n_1 + 2, \\ \psi \in E^{n,2k}(\mathbb{K}) \end{array}\right\} \cup \{\psi\phi_1^{(2)} \mid \psi \in E^{n,2k-1}(\mathbb{K})\}\right) \\ &= \left\{\sum_{i=1}^{n_1+2} \psi_i^{(1)}\phi_i^{(1)} + \psi^{(2)}\phi^{(2)} \;\middle|\; \begin{array}{l} \psi_i^{(1)} \in E^{n,2k}(\mathbb{K}), \\ \psi^{(2)} \in E^{n,2k-1}(\mathbb{K}) \end{array}\right\}. \end{aligned}$$

From Lemma 5.2, we note that each $\mathcal{K}_{\mathrm{ZVP},r}(\mathbb{K})$ can be described by semidefinite constraints. More precisely, the size and number of the semidefinite constraints that define the set $E^{n,m}(\mathbb{K})$ can be calculated.

Proposition 5.3. *Let*

$$a_m := \frac{1}{\sqrt{\mathrm{rk}^2 + 4}} \left\{\left(\frac{\mathrm{rk} + \sqrt{\mathrm{rk}^2 + 4}}{2}\right)^{m+1} - \left(\frac{\mathrm{rk} - \sqrt{\mathrm{rk}^2 + 4}}{2}\right)^{m+1}\right\}$$

for each $m \in \mathbb{N}$. Note that a_m is the mth term of the recurrence $a_{m+2} = \mathrm{rk}\, a_{m+1} + a_m$ with initial conditions $a_0 = 1$ and $a_1 = \mathrm{rk}$ and is of order $O(n_1^m)$. If $m = 2k$ for some $k \in \mathbb{N}$, then $E^{n,m}(\mathbb{K})$ is described by a_{2i} semidefinite constraints of size $|\mathbb{I}^n_{=k-i}|$ ($i = 0, \dots, k$). If $m = 2k + 1$ for some $k \in \mathbb{N}$, then $E^{n,m}(\mathbb{K})$ is described by a_{2i+1} semidefinite constraints of size $|\mathbb{I}^n_{=k-i}|$ ($i = 0, \dots, k$).

Second, we introduce the outer-approximation hierarchy given by Lasserre [32]. Let $\Delta(\mathbb{K}) := \{\boldsymbol{x} \in \mathbb{K} \mid \boldsymbol{e}^\top \boldsymbol{x} \le 1\}$, which is a compact set of $\mathbb{R}^n$. Note that $\boldsymbol{A} \in \mathcal{COP}(\mathbb{K})$ if and only if $\boldsymbol{A} \in \mathcal{COP}(\Delta(\mathbb{K}))$ for each $\boldsymbol{A} \in \mathbb{S}^n$ with a slight abuse of notation. Let ν be the finite Borel measure uniformly supported on $\Delta(\mathbb{K})$, i.e., $\nu(B) := \int_B 1_{\Delta(\mathbb{K})} d\boldsymbol{x}$ for each B in the Borel σ-algebra of $\mathbb{R}^n$, where $1_{\Delta(\mathbb{K})}$ is the indicator function of $\Delta(\mathbb{K})$, and the notation $d\boldsymbol{x}$ represents the Lebesgue measure. Then, the moment $\boldsymbol{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ of the measure ν satisfies

$$\begin{aligned} y_{\boldsymbol{\alpha}} &:= \int_{\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\alpha}} d\nu \\ &= \begin{cases} \frac{2\boldsymbol{\alpha}_1!\left(\sum_{i=1}^{n_2} \alpha_{2i} + n_2 - 1\right)!\prod_{i=2}^{n_2} \Gamma(\beta_{2i})}{\left(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1\right)(n + |\boldsymbol{\alpha}|)!\Gamma(\sum_{i=2}^{n_2} \beta_{2i})} & \text{(if all of } \alpha_{22}, \dots, \alpha_{2n_n} \text{ are even)}, \\ 0 & \text{(if some of } \alpha_{22}, \dots, \alpha_{2n_n} \text{ are odd)} \end{cases} \end{aligned} \tag{29}$$

Table 1: Comparison of the size and number of semidefinite constraints defining the rth level of each approximation hierarchy $\{\mathcal{K}_r\}_r$ for $\mathcal{COP}(\mathbb{K})$

Type	Direction	Number	Size
dP (proposed)	Inner	$\lvert\mathbb{I}^{\mathrm{rk}}_{=r+2}\rvert$	n_2
ZVP	Inner	Includes at most rk semidefinite constraints of size $\lvert\mathbb{I}^{n}_{=\lfloor \frac{r}{2}\rfloor+1}\rvert$ (see Proposition 5.3 for details)	
NN (proposed)	Inner	1	$\lvert\mathbb{I}^{n}_{=r+2}\rvert$
Yıldırım (proposed)	Outer	$\lvert\delta_r^{\mathrm{rk}-1}\rvert$ (see (26))	n_2
Lasserre	Outer	1	$\lvert\mathbb{I}^{n}_{\leq r}\rvert$

for each $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \alpha_{21}, \dots, \alpha_{2n_2}) \in \mathbb{N}^n$, where $\Gamma(\cdot)$ denotes the gamma function and $\beta_{2i} \coloneqq (\alpha_{2i}+1)/2$ for $i = 2, \dots, n_2$. See Appendix A for the calculation of (29). Using the moment, the outer-approximation hierarchy for $\mathcal{COP}(\mathbb{K})$ given by Lasserre [32] can be constructed, which we call the Lasserre-type outer-approximation hierarchy.

Theorem 5.4 ([32, Sect. 2.4])**.** *For each $r \in \mathbb{N}$, we define*

$$\mathcal{K}_{\mathrm{L},r}(\mathbb{K}) \coloneqq \left\{ \boldsymbol{A} \in \mathbb{S}^n \;\middle|\; \boldsymbol{M}_r(f_{\boldsymbol{A}}\boldsymbol{y}) \in \mathbb{S}_+^{\mathbb{I}^n_{\leq r}} \right\},$$

where $\boldsymbol{M}_r(f_{\boldsymbol{A}}\boldsymbol{y})$ is the symmetric matrix with the $(\boldsymbol{\alpha},\boldsymbol{\beta})$th element $\sum_{i,j=1}^n A_{ij} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{e}_i+\boldsymbol{e}_j}$ for each $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{I}^n_{\leq r}$. Then, the sequence $\{\mathcal{K}_{\mathrm{L},r}(\mathbb{K})\}_r$ satisfies $\mathcal{K}_{\mathrm{L},r}(\mathbb{K}) \downarrow \mathcal{COP}(\mathbb{K})$.

The above discussion indicates that the three (dP-, ZVP-, and NN-type) inner-approximation hierarchies and two (Yıldırım- and Lasserre-type) outer-approximation hierarchies for $\mathcal{COP}(\mathbb{K})$ are basically described by semidefinite constraints. Table 1 summarizes the approximation hierarchies, from which we can observe the characteristics of each approximation hierarchy. In particular, the dP- and Yıldırım-type approximation hierarchies have features that differ from those of the ZVP-, NN-, and Lasserre-type approximation hierarchies.

First, the number of semidefinite constraints defining the dP- and Yıldırım-type approximation hierarchies is exponential in r but depends only on n_1 and not on n_2 because $\mathrm{rk} = n_1 + 2$. In addition, their size is linear in n_2. Thus, they would not be affected much by the increase in n_2, and n_1 determines to extent to which depth parameter r can be computationally increased. Conversely, the other approximation hierarchies include semidefinite constraints whose maximum size is exponential in r and dependent on $n = n_1 + n_2$. Thus, they would be considerably affected by the increase in n_2 as well as in n_1. The numerical experiment conducted in Sect. 6 demonstrates this theoretical comparison.

Second, the dP- and Yıldırım-type approximation hierarchies are defined by multiple but small semidefinite constraints, which means that linear conic programming over these hierarchies can be reformulated as SDP with a block diagonal matrix structure.

In this case, we can conduct some of the operations in the primal-dual interior-point methods independently for each block [18], thereby reducing the computational and spatial complexity.

6 Numerical experiments

In this section, we consider the following COPP problem with the COP cone $\mathcal{COP}(\mathbb{K})$ over $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \mathbb{L}^{n_2}$:

$$\begin{aligned} \underset{y,\boldsymbol{S}}{\text{maximize}} \quad & y \\ \text{subject to} \quad & y\boldsymbol{E}_n + \boldsymbol{S} = \boldsymbol{C}, \\ & \boldsymbol{S} \in \mathcal{COP}(\mathbb{K}). \end{aligned} \tag{30}$$

Note that the dual problem of (30) is

$$\begin{aligned} \underset{\boldsymbol{X}}{\text{minimize}} \quad & \langle \boldsymbol{C}, \boldsymbol{X} \rangle \\ \text{subject to} \quad & \langle \boldsymbol{E}_n, \boldsymbol{X} \rangle = 1, \\ & \boldsymbol{X} \in \mathcal{CP}(\mathbb{K}). \end{aligned} \tag{31}$$

Both (30) and its dual problem (31) satisfy Slater's condition when $\boldsymbol{C}$ is a symmetric positive definite matrix; thus, (30) is ideal in a sense.

Lemma 6.1. *Both problems* (30) *and* (31) *satisfy Slater's condition, i.e., they have a feasible interior solution if* $\boldsymbol{C}$ *is a symmetric positive definite matrix.*

Proof. Let $y_0 \coloneqq 0$ and $\boldsymbol{S}_0 \coloneqq \boldsymbol{C}$. Then, $(y_0, \boldsymbol{S}_0)$ is a feasible interior solution of (30). Next, let

$$\begin{aligned} \boldsymbol{X}'_0 &\coloneqq \begin{pmatrix} \boldsymbol{E}_{n_1} + \boldsymbol{I}_{n_1} & \boldsymbol{1}_{n_1} & \boldsymbol{O}_{n_1 \times (n_2-1)} \\ \boldsymbol{1}_{n_1}^\top & 2n_2 + 1 & \boldsymbol{0}_{n_2-1}^\top \\ \boldsymbol{O}_{(n_2-1)\times n_1} & \boldsymbol{0}_{n_2-1} & 2\boldsymbol{I}_{n_2-1} \end{pmatrix}, \\ \boldsymbol{X}_0 &\coloneqq \frac{1}{n_1^2 + 3n_1 + 4n_2 - 1} \boldsymbol{X}'_0 \end{aligned}$$

and we prove that $\boldsymbol{X}_0$ is a feasible interior solution of problem (31). Let

$$\begin{aligned} \boldsymbol{u}_i^{(1)} &\coloneqq (0, \dots, 0, \underbrace{1}_{i\text{th}}, 0 \dots, 0) \in \mathbb{K} && (i = 11, \dots, 1n_1), \\ \boldsymbol{u}_i^{(2)} &\coloneqq (\boldsymbol{0}_{n_1}, 1, 0, \dots, 0, \underbrace{1}_{i\text{th}}, 0, \dots, 0) \in \mathbb{K} && (i = 22, \dots, 2n_2), \\ \boldsymbol{u}_i^{(3)} &\coloneqq (\boldsymbol{0}_{n_1}, 1, 0, \dots, 0, \underbrace{-1}_{i\text{th}}, 0, \dots, 0) \in \mathbb{K} && (i = 22, \dots, 2n_2), \\ \boldsymbol{u}^{(4)} &\coloneqq (\boldsymbol{1}_{n_1+1}, \boldsymbol{0}_{n_2-1}) \in \operatorname{int}(\mathbb{K}). \end{aligned}$$

Then, they span $\mathbb{R}^n$ as each $\boldsymbol{x} \in \mathbb{R}^n$ can be written as

$$\boldsymbol{x} = \sum_{i=11}^{1n_1} x_i \boldsymbol{u}_i^{(1)} + \frac{x_{21}}{2}(\boldsymbol{u}_{22}^{(2)} + \boldsymbol{u}_{22}^{(3)}) + \sum_{i=22}^{2n_2} \frac{x_i}{2}(\boldsymbol{u}_i^{(2)} - \boldsymbol{u}_i^{(3)}).$$

Therefore, it follows from [21, Theorem 3.3] that

$$\begin{aligned}\boldsymbol{X}_0' = &\sum_{i=11}^{1n_1} \boldsymbol{u}_i^{(1)}(\boldsymbol{u}_i^{(1)})^\top + \sum_{i=22}^{2n_2} \boldsymbol{u}_i^{(2)}(\boldsymbol{u}_i^{(2)})^\top + \sum_{i=22}^{2n_2} \boldsymbol{u}_i^{(3)}(\boldsymbol{u}_i^{(3)})^\top + \boldsymbol{u}^{(4)}(\boldsymbol{u}^{(4)})^\top \\ &\in \operatorname{int} \mathcal{CP}(\mathbb{K}).\end{aligned}$$

Given that $\langle \boldsymbol{E}_n, \boldsymbol{X}_0' \rangle = n_1^2 + 3n_1 + 4n_2 - 1 > 0$, we obtain $\langle \boldsymbol{E}_n, \boldsymbol{X}_0 \rangle = 1$ and $\boldsymbol{X}_0 \in \operatorname{int} \mathcal{CP}(\mathbb{K})$. □

All experiments in this section were conducted on a computer with an Intel Core i5-8279U 2.40 GHz CPU and 16 GB of memory. The modeling language YALMIP [34] (version 20210331), the MOSEK solver [36] (version 9.3.3), and MATLAB (R2022a), were used to solve optimization problems. Based on Lemma 6.1, a coefficient matrix $\boldsymbol{C}$ in problem (30) was randomly generated such that it was symmetric positive definite. We measured three types of time when solving the optimization problems:

- preparetime: Time taken before calling YALMIP commands `optimize` or `solvesos`.
- yalmiptime: Time between calling the above commands and beginning to solve an optimization problem in MOSEK.
- solvertime: Time required to solve an optimization problem in MOSEK.

We defined the total time as the sum of the three types of time, and the calculation was considered invalid when the total time exceeded 7200 s.

6.1 Comparison of approximation hierarchies

Table 1 lists five approximation hierarchies for $\mathcal{COP}(\mathbb{K})$ we have introduced. Here, we solve optimization problems obtained by replacing the COP cone $\mathcal{COP}(\mathbb{K})$ in (30) with the approximation hierarchies. For convenience, we hereafter call such problems dP-type approximation problems (of depth r), for example. The YALMIP command `optimize` was used when solving dP-, Yıldırım-, and Lasserre-type approximation problems, and `solvesos` was used when solving ZVP- and NN-type approximation problems. For each approximation hierarchy, we continuously increased the parameter r that decides the depth of the hierarchy until the total time exceeded 7200 s. When solving dP- and Yıldırım-type approximation problems, the more concise expressions mentioned in Remarks 4.9 and 4.11 were adopted. (Section 6.2 investigates their numerical effect.) The pair (n_1, n_2) was set to $(20, 5)$, $(5, 20)$, and $(5, 25)$.

Table 2: Optimal values (optv) of the optimization problems obtained by replacing the COP cone $\mathcal{COP}(\mathbb{R}^{20}_{+} \times \mathbb{L}^{5})$ in (30) with each approximation hierarchy, as well as the solver time (solt) and total time (tott) required to solve them. All values are rounded to the second decimal place. The asterisk * indicates that the total time exceeded 7200 s. Because more than 7200 s were required to solve the dP-type approximation problem of depth 6, only the results up to depth $r \leq 5$ are shown.

	dP			ZVP			NN			Yıldırım			Lasserre		
r	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott
0	$-\infty$	0.00	0.72	1.52	0.04	4.20	1.52	364.56	373.31	6.09	0.01	0.48	$+\infty$	0.00	0.30
1	$-\infty$	0.02	3.24	1.52	4.09	234.49	*	*	*	5.14	0.06	3.53	$+\infty$	0.00	24.85
2	-0.22	0.16	18.87	*	*	*	*	*	*	4.42	0.36	23.99	$+\infty$	0.10	4110.63
3	0.08	1.85	106.15	*	*	*	*	*	*	3.90	2.09	151.11	*	*	*
4	0.18	32.60	693.39	*	*	*	*	*	*	3.49	12.62	1200.10	*	*	*
5	0.26	513.54	6357.99	*	*	*	*	*	*	*	*	*	*	*	*

Table 3: Optimal values (optv) of the optimization problems obtained by replacing the COP cone $\mathcal{COP}(\mathbb{R}^5_+ \times \mathbb{L}^{20})$ in (30) with each approximation hierarchy, as well as the solver time (solt) and total time (tott) required to solve them. All values are rounded to the second decimal place. The asterisk * indicates that the total time exceeded 7200 s. Because more than 7200 s were required to solve the dP-type approximation problem of depth 17, only results up to depth $r \leq 16$ are shown.

	dP			ZVP			NN			Yıldırım			Lasserre		
r	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott
0	$-\infty$	0.00	0.23	1.79	0.03	1.17	1.79	270.90	280.22	2.25	0.01	0.18	$+\infty$	0.00	0.35
1	$-\infty$	0.02	0.26	1.79	0.30	4.88	*	*	*	2.25	0.03	0.36	$+\infty$	0.00	18.29
2	$-\infty$	0.05	0.62	1.79	569.21	751.10	*	*	*	2.18	0.09	0.84	$+\infty$	0.11	2781.12
3	$-\infty$	0.15	1.32	*	*	*	*	*	*	2.10	0.23	2.06	*	*	*
4	-0.11	0.37	2.77	*	*	*	*	*	*	2.02	0.52	4.42	*	*	*
5	0.40	0.65	4.95	*	*	*	*	*	*	1.95	1.05	10.19	*	*	*
6	0.91	1.30	9.33	*	*	*	*	*	*	1.87	2.07	22.01	*	*	*
7	1.19	2.12	17.04	*	*	*	*	*	*	1.81	3.94	49.55	*	*	*
8	1.30	3.89	32.91	*	*	*	*	*	*	1.79	7.33	110.76	*	*	*
9	1.37	6.23	62.80	*	*	*	*	*	*	1.79	13.33	261.56	*	*	*
10	1.42	9.46	118.91	*	*	*	*	*	*	1.79	21.98	603.44	*	*	*
11	1.45	15.59	230.01	*	*	*	*	*	*	1.79	40.07	1491.77	*	*	*
12	1.47	22.72	440.28	*	*	*	*	*	*	1.79	63.83	3266.76	*	*	*
13	1.48	38.53	889.31	*	*	*	*	*	*	1.79	99.29	7158.94	*	*	*
14	1.49	49.62	1642.49	*	*	*	*	*	*	*	*	*	*	*	*
15	1.50	69.87	2987.17	*	*	*	*	*	*	*	*	*	*	*	*
16	1.52	101.21	5274.74	*	*	*	*	*	*	*	*	*	*	*	*

Table 4: Optimal values (optv) of the optimization problems obtained by replacing the COP cone $\mathcal{COP}(\mathbb{R}^5_+ \times \mathbb{L}^{25})$ in (30) with each approximation hierarchy, as well as the solver time (solt) and total time (tott) required to solve them. All values are rounded to the second decimal place. The asterisk * indicates that the total time exceeded 7200 s. Because more than 7200 s were required to solve the dP-type approximation problem of depth 16, only results up to depth $r \leq 15$ are shown.

	dP			ZVP			NN			Yıldırım			Lasserre		
r	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott
0	$-\infty$	0.02	1.14	1.83	0.06	1.35	1.83	3864.82	3904.27	2.23	0.01	0.18	$+\infty$	0.00	0.33
1	$-\infty$	0.02	0.47	1.83	0.47	5.59	*	*	*	2.23	0.05	0.38	$+\infty$	0.00	37.90
2	$-\infty$	0.09	0.76	1.83	6126.23	6587.04	*	*	*	2.18	0.12	0.89	*	*	*
3	$-\infty$	0.17	1.55	*	*	*	*	*	*	2.13	0.38	2.30	*	*	*
4	$-\infty$	0.45	3.85	*	*	*	*	*	*	2.06	0.68	5.00	*	*	*
5	0.25	1.24	6.70	*	*	*	*	*	*	1.98	1.65	12.36	*	*	*
6	0.95	2.56	13.70	*	*	*	*	*	*	1.92	3.18	28.62	*	*	*
7	1.22	4.02	25.25	*	*	*	*	*	*	1.87	5.71	69.53	*	*	*
8	1.33	5.73	48.05	*	*	*	*	*	*	1.84	12.62	173.91	*	*	*
9	1.41	11.50	102.73	*	*	*	*	*	*	1.83	22.40	431.75	*	*	*
10	1.46	15.19	199.37	*	*	*	*	*	*	1.83	38.45	1076.99	*	*	*
11	1.49	23.34	400.18	*	*	*	*	*	*	1.83	64.26	2689.87	*	*	*
12	1.52	39.72	780.18	*	*	*	*	*	*	1.83	94.82	6229.90	*	*	*
13	1.53	57.08	1571.46	*	*	*	*	*	*	*	*	*	*	*	*
14	1.54	81.10	2922.42	*	*	*	*	*	*	*	*	*	*	*	*
15	1.55	113.98	5315.38	*	*	*	*	*	*	*	*	*	*	*	*

Tables 2, 3, and 4 show the results of $(n_1, n_2) = (20, 5)$, $(5, 20)$, and $(5, 25)$, respectively. These tables report the solver and total time because some of the approximation hierarchies spent most of the total time before beginning to solve optimization problems in MOSEK. Although we used YALMIP for convenience, the total time would be substantially reduced if we implemented these approximation hierarchies directly.

As shown in Tables 3 and 4, the optimal values of the ZVP- and NN-type inner-approximation problems agree with those of the Yıldırım-type outer-approximation problems, which implies that the ZVP-, NN-, and Yıldırım-type approximation hierarchies almost approach the optimal value of the original COPP problem (30) for $(n_1, n_2) = (5, 20)$ and $(5, 25)$.

Although all Lasserre-type outer-approximation problems were considered unbounded, this would result from the moments (29) taking infinitesimal values. Indeed, the MOSEK solver provided a warning about treating nearly zero elements, and we determined that $y_{\boldsymbol{\alpha}}$ is approximately 3.52×10^{-37} for $(n_1, n_2) = (5, 25)$ and $\boldsymbol{\alpha} = 2\boldsymbol{e}_1 \in \mathbb{R}^{30}$, for instance. Hence, the Lasserre-type outer-approximation hierarchy is numerically unstable, whereas the Yıldırım-type outer-approximation hierarchy is numerically stable.

The results of this numerical experiment support the theoretical comparison mentioned in Sect. 5. As shown in Tables 3 and 4, the dP- and Yıldırım-type approximation hierarchies are not affected much by the increase in n_2. The solver and total time required to solve the dP- and Yıldırım-type approximation problems with $(n_1, n_2) = (5, 25)$ is less than twice as long as those with $(n_1, n_2) = (5, 20)$ regardless of r, except for $r = 0$. Conversely, the others are considerably affected by the increase in n_2. For example, the solver and total time required to solve the NN-type inner-approximation problem with $(n_1, n_2) = (5, 25)$ of depth 0 is more than ten times as long as those with $(n_1, n_2) = (5, 20)$. In addition, although the increase for $r = 0, 1$ is mild, those required to solve the ZVP-type inner-approximation problem with $(n_1, n_2) = (5, 25)$ of depth 2 is also approximately ten times as long as those with $(n_1, n_2) = (5, 20)$. [*1]

As shown in Tables 2 and 3, the dP- and Yıldırım-type approximation hierarchies are considerably affected by the increase in n_1. The solver and total time required to solve the dP- and Yıldırım-type approximation problems with $(n_1, n_2) = (20, 5)$ is longer than those with $(n_1, n_2) = (5, 20)$ for each r, and the difference rapidly increases with r. Because $n = n_1 + n_2$ is the same for the two pairs and because n_2 in the pair $(n_1, n_2) = (20, 5)$ is smaller than that in the pair $(n_1, n_2) = (5, 20)$, we can conclude that the increase in the required time results from the increase in n_1. Conversely, if n_1 is small, we can increase depth parameter r and, in this case, the Yıldırım-type outer-approximation hierarchy may approach a nearly optimal value of the COPP problems.

[*1] This is because, as shown in Table 1, the maximum size of the semidefinite constraints defining the ZVP-type inner-approximation hierarchy increases from $|\mathbb{I}^n_{=1}| = n$ to $|\mathbb{I}^n_{=2}| = n(n+1)/2$ when r increases from 1 to 2.

Finally, the ZVP- and NN-type inner-approximation hierarchies provided much tighter bounds than that of the dP-type in all cases, and, as mentioned above, the two hierarchies are guaranteed to approach nearly optimal values even at a depth of 0 for $(n_1, n_2) = (5, 20)$ and $(5, 25)$. Moreover, the time required to solve the ZVP-type inner-approximation problems of depth 0 is much faster than that for the NN-type ones. Therefore, from a practical perspective, using the zeroth level of the ZVP-type inner-approximation hierarchy may be preferable if aiming to obtain a reasonable lower bound of the optimal value of the original problem (30).

6.2 Effect of more concise expressions of dP- and Yıldırım-type approximation hierarchies

In this subsection, we investigate the numerical effect of the more concise expressions of the dP- and Yıldırım-type approximation hierarchies mentioned in Remarks 4.9 and 4.11. The dP- and Yıldırım-type approximation hierarchies without the more concise expressions are provided by (21) and (25), respectively. Except for the differences in the expressions of these approximation hierarchies, the experimental settings were the same as those in Sect. 6.1.

Table 5 provides the results of $(n_1, n_2) = (5, 25)$. The information of the optimal values is omitted because those of the dP- and Yıldırım-type approximation problems without the more concise expressions are the same as those provided theoretically (and numerically). The effect of the more concise expressions is significant, and the solver and total time required to solve the dP- and Yıldırım-type approximation problems with the more concise expressions are shorter than those without, except for the total time of solving the dP-type inner-approximation problem of depth 0. Note that the effect was also confirmed for $(n_1, n_2) = (20, 5)$.

7 Conclusion

In this study, we provided approximation hierarchies for the COP cone over a symmetric cone and compared them with existing approximation hierarchies. We first provided the NN-type inner-approximation hierarchy. Its strength is that the hierarchy permits an SOS representation for a general symmetric cone. We then provided the dP- and Yıldırım-type approximation hierarchies for the COP matrix cone over the direct product of a nonnegative orthant and second-order cone by exploiting those for the usual COP cone provided by de Klerk and Pasechnik [11] and Yıldırım [52]. Remarkably, they are not affected much by the increase in the size of the second-order cone, unlike the NN-type and existing approximation hierarchies. Combining the proposed approximation hierarchies with those existing, we obtained nearly optimal values of COPP problems when the size of the nonnegative orthant is small.

Table 5: Solver time (solt) and total time (tott) required to solve the optimization problems obtained by replacing the COP cone $\mathcal{COP}(\mathbb{R}^5_+ \times \mathbb{L}^{25})$ in (30) with the dP- and Yıldırım-type approximation hierarchies without the more concise expressions. The "solt ratio" (resp. "tott ratio") column lists values obtained by dividing "solt" (resp. "tott") in this table (the case of not making the expressions more concise) by that in Table 4 (the case of making the expressions more concise). The asterisk * indicates that the total time exceeded 7200 s, and the symbol $+\infty$ in the columns of "solt ratio" and "tott ratio" indicates that the total time in this table exceeded 7200 s, whereas that in Table 4 did not. The ratios that are more than 1 are in bold. All values are rounded to the second decimal place.

	dP				Yıldırım			
r	solt	solt ratio	tott	tott ratio	solt	solt ratio	tott	tott ratio
0	0.03	**1.23**	0.71	0.62	0.04	**2.97**	0.27	**1.48**
1	0.08	**3.63**	0.77	**1.64**	0.11	**2.34**	0.78	**2.06**
2	0.26	**2.95**	1.82	**2.40**	0.30	**2.58**	2.03	**2.27**
3	0.57	**3.34**	4.47	**2.88**	0.98	**2.60**	5.65	**2.46**
4	1.17	**2.57**	7.45	**1.93**	1.96	**2.86**	13.37	**2.67**
5	2.34	**1.89**	15.04	**2.25**	4.44	**2.70**	36.01	**2.91**
6	5.17	**2.02**	33.12	**2.42**	8.57	**2.70**	92.24	**3.22**
7	7.81	**1.94**	65.96	**2.61**	15.44	**2.71**	258.37	**3.72**
8	12.25	**2.14**	143.70	**2.99**	32.80	**2.60**	700.01	**4.03**
9	19.66	**1.71**	304.94	**2.97**	57.39	**2.56**	1893.38	**4.39**
10	34.48	**2.27**	690.06	**3.46**	95.71	**2.49**	4710.17	**4.37**
11	50.02	**2.14**	1465.11	**3.66**	*	$+\infty$	*	$+\infty$
12	71.78	**1.81**	2943.20	**3.77**	*	$+\infty$	*	$+\infty$
13	108.49	**1.90**	5723.38	**3.64**	*	*	*	*
14	*	$+\infty$	*	$+\infty$	*	*	*	*
15	*	$+\infty$	*	$+\infty$	*	*	*	*

Unfortunately, the infinity of a set of Jordan frames is guaranteed to be solved in Sects. 4.1 and 4.2 only when the symmetric cone is the direct product of a nonnegative orthant and second-order cone and $m = 2$. However, by replacing a set $\mathfrak{F}$ of Jordan frames appearing in (23) with its finite subset $\mathfrak{F}_r$ such that the union $\bigcup_r \mathfrak{F}_r$ is dense in $\mathfrak{F}$, we obtain an outer-approximation hierarchy implementable on a computer for the COP cone over a general symmetric cone.

Finally, questions arise concerning the inclusion among the dP-, ZVP-, and NN-type inner-approximation hierarchies. In the numerical experiment conducted in Sect. 6.1, the ZVP- and NN-type inner-approximation hierarchies provided considerably tighter bounds than the dP-type. In the case in which the symmetric cone is a nonnegative orthant, the dP-type inner-approximation hierarchy is well known [11] to be included in the ZVP- and NN-type ones, which agree with that provided by Parrilo [39]. Investigating whether the inclusion also holds where the symmetric cone is the direct product of a nonnegative orthant and second-order cone would be interesting.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

Appendix A Calculation of (29)

Note that we can represent the set $\Delta(\mathbb{K})$ as

$$\left\{ \boldsymbol{x} = (\boldsymbol{x}_1, x_{21}, \dots, x_{2n_2}) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (\boldsymbol{x}_1, x_{21}) \in \Delta_{\leq}^{n_1+1}, \\ (x_{22}, \dots, x_{2n_2}) \in B^{n_2-1}(x_{21}) \end{array} \right\},$$

where

$$\begin{aligned} \Delta_{\leq}^{n_1+1} &\coloneqq \{\boldsymbol{z} \in \mathbb{R}_{+}^{n_1+1} \mid \boldsymbol{z}^\top \mathbf{1} \leq 1\}, \\ B^{n_2-1}(x_{21}) &\coloneqq \{\boldsymbol{z} \in \mathbb{R}^{n_2-1} \mid \|\boldsymbol{z}\|_2 \leq x_{21}\}. \end{aligned}$$

Then, for $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \alpha_{21}, \dots, \alpha_{2n_2}) \in \mathbb{N}^n$, it follows that

$$\begin{aligned} y_{\boldsymbol{\alpha}} &= \int_{\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\alpha}} d\nu \\ &= \int_{\Delta(\mathbb{K})} \boldsymbol{x}^{\boldsymbol{\alpha}} d\boldsymbol{x} \\ &= \int_{\Delta_{\leq}^{n_1+1}} \boldsymbol{x}_1^{\boldsymbol{\alpha}_1} x_{21}^{\alpha_{21}} \left(\int_{B^{n_2-1}(x_{21})} x_{22}^{\alpha_{22}} \cdots x_{2n_2}^{\alpha_{2n_2}} dx_{22} \cdots dx_{2n_2} \right) d\boldsymbol{x}_1 dx_{21}. \end{aligned}$$

From [17], we note that

$$\begin{aligned} &\int_{B^{n_2-1}(x_{21})} x_{22}^{\alpha_{22}} \cdots x_{2n_2}^{\alpha_{2n_2}} dx_{22} \cdots dx_{2n_2} \\ &= \begin{cases} \frac{2 \prod_{i=2}^{n_2} \Gamma(\beta_{2i}) x_{21}^{\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1}}{\left(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1\right) \Gamma(\sum_{i=2}^{n_2} \beta_{2i})} & \text{(if all of } \alpha_{22}, \dots, \alpha_{2n_n} \text{ are even)}, \\ 0 & \text{(if some of } \alpha_{22}, \dots, \alpha_{2n_n} \text{ are odd)}. \end{cases} \end{aligned} \tag{32}$$

As (32) implies that $y_{\boldsymbol{\alpha}} = 0$ where some of $\alpha_{22}, \ldots, \alpha_{2n_n}$ are odd, considering the case in which all $\alpha_{22}, \ldots, \alpha_{2n_n}$ are even is sufficient. In this case, we have

$$
\begin{aligned}
y_{\boldsymbol{\alpha}} &= \frac{2 \prod_{i=2}^{n_2} \Gamma(\beta_{2i})}{(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1)\, \Gamma(\sum_{i=2}^{n_2} \beta_{2i})} \int_{\Delta_{\leq}^{n_1+1}} \boldsymbol{x}_1^{\boldsymbol{\alpha}_1} x_{21}^{\sum_{i=1}^{n_2} \alpha_{2i} + n_2 - 1} d\boldsymbol{x}_1 dx_{21} \\
&= \frac{2\boldsymbol{\alpha}_1!\, (\sum_{i=1}^{n_2} \alpha_{2i} + n_2 - 1)\,!\, \prod_{i=2}^{n_2} \Gamma(\beta_{2i})}{(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1)\, (n + |\boldsymbol{\alpha}|)!\, \Gamma(\sum_{i=2}^{n_2} \beta_{2i})}.
\end{aligned}
$$

See [22], for example, to obtain the last equation.

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