A Note on Semidefinite Representable Reformulations for Two Variants of the Trust-Region Subproblem

Sarah Kelly* Yuyuan Ouyang† Boshi Yang‡

Abstract

Motivated by encouraging numerical results in [13], in this note we consider two specific variants of the trust-region subproblem and provide exact semidefinite representable reformulations. The first is over the intersection of two balls; the second is over the intersection of a ball and a special second-order conic representable set. Different from the technique developed in [13], the reformulations in this note are based on partitions of the feasible regions into sub-regions with known lifted convex hulls.

Keywords: Convex hull, Semidefinite programming, Quadratically constrained quadratic programming, Trust-region subproblem

1 Introduction

The trust-region subproblem (TRS) is a classic nonconvex quadratically constrained quadratic program (QCQP) with tractable convex reformulations. The vanilla TRS minimizes a quadratic function over the unit ball and has been well-studied in the literature [14, 23, 25]; for example, it is well known that the TRS has a second-order conic reformulation and can be solved efficiently [16]. Variants of the TRS have also been widely studied in the literature; see, e.g., the studies in [3, 4, 7–11, 16–18, 22, 26–33]. Many of the aforementioned variants have been proven to be polynomial-time solvable to ϵ-optimality [7], but explicit semidefinite representable convex reformulations are known only for a few special cases such as the two-sided generalized TRS [24] and the TRS with non-intersecting linear constraints [11].

*School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, USA. Email: skelly5@g.clemson.edu.

†School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, USA. Email: yuyuano@clemson.edu.

‡School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, USA. Email: boshiy@clemson.edu.

Recently, Eltved and Burer [13] consider the following variant, where a lower bound and an additional second-order cone constraint is added to the TRS:

$$\begin{aligned} \min \quad & \mathbf{x}^T H\mathbf{x} + 2\mathbf{g}^T\mathbf{x} \\ s.t. \quad & r \le \|\mathbf{x}\| \le R \\ & \|\mathbf{x} - \mathbf{c}\| \le \mathbf{b}^T\mathbf{x} - a \end{aligned} \tag{1}$$

Here, $H \in \mathcal{S}^n$ is a real symmetric matrix, $\mathbf{x}, \mathbf{g}, \mathbf{c}, \mathbf{b} \in \mathbb{R}^n$, $a \in \mathbb{R}$, and $r, R \in \mathbb{R}_+$. The major novelty in [13] is that the authors construct nonnegative quartic expressions over the feasible region, which lead to a class of polynomial-time separable valid inequalities in the lifted space. Numerical results show that the valid inequalities are effective in reducing the gap of the strongest semidefinite programming (SDP) relaxations in the literature, especially in low dimensions. The authors also pay particular attention to two special cases of the problem. One case is when $r = 0$ and $\mathbf{b} = \mathbf{0}$, and the other is when $r = a = 0$ and $\mathbf{c} = \mathbf{0}$. The first case is a special case of the widely studied two-trust-region subproblem (TTRS). The second case is worth studying because that when $\mathbf{b} = \mathbf{e}$ (namely, $\mathbf{b}$ is the vector of all ones), its feasible region is a superset of the intersection of the unit ball and the nonnegative orthant, i.e., $\{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1, \mathbf{x} \ge \mathbf{0} \,\}$. Valid inequalities of the set can lead to cuts for the completely positive cone. Based on great numerical results, the authors conjecture that their valid inequalities help build an exact SDP reformulation of the second case when $n = 2$.

Motivated by the aforementioned encouraging numerical results and the conjecture made in [13], in this paper we focus on the two special cases mentioned above with the goal of developing their exact SDP reformulations. To our knowledge, no exact SDP reformulations for such two cases have been discovered in the literature; the nonnegative quartic expression technique and the associated polynomial-time separable valid inequalities in [13] are currently the strongest relaxation approach for solving both problems. Note that although the feasible regions in both cases are convex, the problems are still challenging due to the nonconvexity of the quadratic objective functions. However, with the great numerical results in [13], we are encouraged to think that certain exact SDP reformulation might be formed for these special cases.

Our final research outcome is achieved through a different (but possibly simpler) path from the nonnegative quartic expression approach in [13]. Specifically, the main ingredient of our study in this note is the partitions of the feasible regions and a convex hull result of disjunctions. Note that the idea of partitioning the feasible region of TRS variants has appeared previously in the literature. As far as we are aware of, the idea was first mentioned in [27] for feasible regions defined by two quadratic inequalities with the same Hessian.

In [2], the authors solved quadratic programs with two ball constraints by partitioning the problem into two extended TRS. The idea was also used to develop a branch-and-bound algorithm for quadratic programs with ball and linear constraints in [1]. Most recently, Anstreicher proposed to solve the TTRS by branching the feasible region with eigenvector-based linear constraints [5]. However, all the above results focus solely from the perspective of developing polynomial-time solvable algorithms. The main difference between our paper and all the above results is that we focus not only on globally solving the problems but also on convexification of the problems. In particular, we provide a semidefinite representation of the lifted convex hull of the feasible region. The ability to solve the problems is an immediate consequence of the convexification.

Our contribution in this note is described in detail below. For future reference purposes, we restate the two special cases of (1) that we study in this paper (Without loss of generality, we assume that $R = 1$ in (1)):

$$\begin{aligned} \min \quad & \mathbf{x}^T H \mathbf{x} + 2\mathbf{g}^T \mathbf{x} \\ s.t. \quad & \|\mathbf{x}\| \leq 1 \\ & \|\mathbf{x} - \mathbf{c}\| \leq a \end{aligned} \tag{2}$$

and

$$\begin{aligned} \min \quad & \mathbf{x}^T H \mathbf{x} + 2\mathbf{g}^T \mathbf{x} \\ s.t. \quad & \|\mathbf{x}\| \leq 1 \\ & \|\mathbf{x}\| \leq \mathbf{b}^T \mathbf{x} - a. \end{aligned} \tag{3}$$

In the sequel, the former problem (2) will be referred to as the one with two ball constraints; the latter (3) will be referred to as the one with a ball and a second-order cone representable constraint. For each problem, we provide an exact convex reformulation through a semidefinite representation of the lifted convex hull of the feasible region. The representations are based on partitions of the feasible regions and a convex hull result of disjunctions.

The paper is organized as follows. In Section 2, we introduce the lifted convex hull and related properties. In Section 3, we derive the semidefinite representable reformulations of the two problems. Separation algorithms and computational results are presented in Section 4. The paper is concluded in Section 5.

Notation. We use $\mathcal{S}^n$ and $\mathcal{S}^n_+$ to represent the sets of $n \times n$ real symmetric matrices and real symmetric positive semidefinite matrices, respectively. For $A, B \in \mathcal{S}^n$, the relation

$A \succeq B$ holds if and only if $A - B \in \mathcal{S}^n_+$, and the Frobenious product of A and B is defined as $A \bullet B = \mathrm{tr}(AB)$, where $\mathrm{tr}(\cdot)$ is the trace of a matrix. The Kronecker product of A and B is denoted by $A \otimes B$. For a nonempty set $S \subseteq \mathbb{R}^n$, the convex hull of S is denoted by $\mathrm{conv}(S)$.

2 The lifted convex hull

To begin, consider a general QCQP:

$$\begin{aligned} \inf \quad & \mathbf{x}^T H \mathbf{x} + 2\mathbf{g}^T \mathbf{x} \\ s.t. \quad & \mathbf{x} \in \mathcal{F}, \end{aligned} \tag{4}$$

where $H \in \mathcal{S}^n$ and $\mathcal{F} \subseteq \mathbb{R}^n$ is a nonempty closed set. The problem can be equivalently lifted to

$$\begin{aligned} \inf \quad & H \bullet X + 2\mathbf{g}^T \mathbf{x} \\ s.t. \quad & X = \mathbf{x}\mathbf{x}^T \\ & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{5}$$

with variables $(\mathbf{x}, X)$ in the space of $\mathbb{R}^n \times \mathcal{S}^n$. Moreover, it is shown (e.g. in [12, 21]) that (5) is equivalent to

$$\begin{aligned} \inf \quad & H \bullet X + 2\mathbf{g}^T \mathbf{x} \\ s.t. \quad & (x, X) \in \mathcal{C}(\mathcal{F}), \end{aligned} \tag{6}$$

where

$$\mathcal{C}(\mathcal{F}) := \mathrm{conv}\left\{ (\mathbf{x}, \mathbf{x}\mathbf{x}^T) \mid \mathbf{x} \in \mathcal{F} \right\}. \tag{7}$$

We refer to $\mathcal{C}(\mathcal{F})$ as the lifted convex hull in this paper. The analysis on $\mathcal{C}(\mathcal{F})$ has been critical in several previous QCQP studies in the literature; see, e.g., [9, 20, 31] and the references within. The derivation throughout our paper also relies heavily on the characterization of $\mathcal{C}(\mathcal{F})$. Note that when $\mathcal{F}$ is compact, so is the lifted convex hull $\mathcal{C}(\mathcal{F})$, and optimal solutions of (4)-(6) can be attained.

The main results of this paper are explicit semidefinite representations of $\mathcal{C}(\mathcal{F})$ for (2) and (3). In this section, we introduce preliminary results related to $\mathcal{C}(\mathcal{F})$. The following lemma characterizes the lifted convex hull of the union of two sets.

Lemma 1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two nonempty closed sets in $\mathbb{R}^n$. We have*

$$\begin{aligned}\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) &= \operatorname{conv}(\mathcal{C}(\mathcal{F}_1) \cup \mathcal{C}(\mathcal{F}_2)) \\ &= \left\{ (\mathbf{x}, X) \;\middle|\; \begin{array}{l} \exists\ \lambda \in [0,1],\ (\mathbf{x}_1, X_1) \in \mathcal{C}(\mathcal{F}_1),\ (\mathbf{x}_2, X_2) \in \mathcal{C}(\mathcal{F}_2) \\ \textit{such that } (\mathbf{x}, X) = \lambda(\mathbf{x}_1, X_1) + (1-\lambda)(\mathbf{x}_2, X_2) \end{array} \right\}.\end{aligned}$$

Proof. Denote the last set in the statement by S. We will show that $S \subseteq \operatorname{conv}(\mathcal{C}(\mathcal{F}_1) \cup \mathcal{C}(\mathcal{F}_2)) \subseteq \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) \subseteq S$. The first two inclusions are straightforward. Now, let $(\mathbf{x}, X) \in \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2)$. Consequently, there exist $\mathbf{x}_i \in \mathcal{F}_1 \cup \mathcal{F}_2$ and $\mu_i > 0$, $i = 1, \ldots, k$, such that $\sum_{i=1}^k \mu_i = 1$ and

$$(\mathbf{x}, X) = \sum_{i=1}^k \mu_i \left(\mathbf{x}_i, \mathbf{x}_i \mathbf{x}_i^T\right).$$

Let $I = \{\, i \mid \mathbf{x}_i \in \mathcal{F}_1 \,\}$ and $J = \{\, i \mid \mathbf{x}_i \notin \mathcal{F}_1 \,\}$. If $I = \varnothing$, then $(\mathbf{x}, X) \in \mathcal{C}(\mathcal{F}_2) \subseteq S$ (by choosing $\lambda = 0$ in the definition of S). If $J = \varnothing$, then $(\mathbf{x}, X) \in \mathcal{C}(\mathcal{F}_1) \subseteq S$ (by choosing $\lambda = 1$). If $I \neq \varnothing$ and $J \neq \varnothing$, then for $\lambda = \sum_{i \in I} \mu_i$, we note that

$$(\mathbf{x}, X) = \lambda \underbrace{\left(\sum_{i \in I} \frac{\mu_i}{\lambda} (\mathbf{x}_i, \mathbf{x}_i \mathbf{x}_i^T) \right)}_{\in \mathcal{C}(\mathcal{F}_1)} + (1-\lambda) \underbrace{\left(\sum_{i \in J} \frac{\mu_i}{1-\lambda} (\mathbf{x}_i, \mathbf{x}_i \mathbf{x}_i^T) \right)}_{\in \mathcal{C}(\mathcal{F}_2)} \in S.$$

Therefore, $\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) \subseteq S$ and consequently $\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) = S = \operatorname{conv}(\mathcal{C}(\mathcal{F}_1) \cup \mathcal{C}(\mathcal{F}_2))$. ☐

Remark. *Lemma 1 can also be derived from the perspective of the cone of nonnegative quadratic functions. See [19]. The above proof takes a basic approach without considering propositions of the dual cones.*

The structure of $\mathcal{C}(\mathcal{F})$ has been studied in the literature for specially structured $\mathcal{F}$ (citations). In the following, we list two results related to our derivation in Section 3. The first proposition is related to the intersection of a halfspace and a ball.

Proposition 2 ([10], [27])**.** *For $\mathcal{F} = \{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{c}\| \leq a, \mathbf{p}^T \mathbf{x} + q \geq 0 \,\}$, where $\mathbf{p} \in \mathbb{R}^n$ and $q \in \mathbb{R}$,*

$$\mathcal{C}(\mathcal{F}) = \left\{ (\mathbf{x}, X) \;\middle|\; \begin{array}{l} \operatorname{tr}(X) - 2\mathbf{c}^T \mathbf{x} + \mathbf{c}^T \mathbf{c} - a^2 \leq 0, \\ \|X\mathbf{p} + q\mathbf{x} - \mathbf{c}\mathbf{p}^T \mathbf{x} - q\mathbf{c}\| \leq a(\mathbf{p}^T \mathbf{x} + q),\ X \succeq \mathbf{x}\mathbf{x}^T \end{array} \right\}.$$

We briefly explain the construction of the constraints in $\mathcal{C}(\mathcal{F})$. The first constraint is built by squaring both sides of $\|\mathbf{x} - \mathbf{c}\| \leq a$ and lifting $\mathbf{x}\mathbf{x}^T$ to X. The second constraint is obtained by multiplying both sides of $\|\mathbf{x} - \mathbf{c}\| \leq a$ by the nonnegative quantity $(\mathbf{p}^T \mathbf{x} + q)$ and lifting $\mathbf{x}\mathbf{x}^T$ to X. The last constraint is a semidefinite relaxation of the nonconvex constraint $X = \mathbf{x}\mathbf{x}^T$.

The next proposition considers the intersection of a halfspace and a second-order cone representable set that share certain special structure (in terms of the linear term in their descriptions).

Proposition 3 ([19]). *For $\mathcal{F} = \{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le \mathbf{p}^T\mathbf{x} + q \le s \,\}$, where $\mathbf{p} \in \mathbb{R}^n$ and $q, s \in \mathbb{R}$,*

$$\mathcal{C}(\mathcal{F}) = \left\{ (\mathbf{x}, X) \;\middle|\; \begin{array}{l} (I - \mathbf{p}\mathbf{p}^T) \bullet X - 2q\mathbf{p}^T\mathbf{x} - q^2 \le 0,\ \mathbf{p}^T\mathbf{x} + q - s \le 0, \\ \|X\mathbf{p} + (q-s)\mathbf{x}\| \le -\mathbf{p}\mathbf{p}^T \bullet X + (s - 2q)\mathbf{p}^T\mathbf{x} - q(q-s),\ X \succeq \mathbf{x}\mathbf{x}^T \end{array} \right\}.$$

The construction of the constraints in $\mathcal{C}(\mathcal{F})$ in Proposition 3 is similar to that in Proposition 2. The first constraint is built by squaring both sides of $\|\mathbf{x}\| \le \mathbf{p}^T\mathbf{x} + q$ and lifting $\mathbf{x}\mathbf{x}^T$ to X. The second constraint is directly inherited from $\mathcal{F}$. The third constraint is obtained by multiplying both sides of $\|\mathbf{x}\| \le \mathbf{p}^T\mathbf{x} + q$ by the nonnegative quantity $(s - q - \mathbf{p}^T\mathbf{x})$ and lifting $\mathbf{x}\mathbf{x}^T$ to X.

3 Semidefinite reformulations

In this section, we derive semidefinite reformulations of (2) and (3). The key idea is to partition the feasible region $\mathcal{F}$ of each problem as $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{C}(\mathcal{F}_1)$ and $\mathcal{C}(\mathcal{F}_2)$ are known.

3.1 Two ball constraints

Let $\mathcal{F}_{TB} = \{\, \mathbf{x} \in \mathbb{R}^n \mid \ \|\mathbf{x}\| \le 1,\ \|\mathbf{x} - \mathbf{c}\| \le a \,\}$ be the feasible region of (2). We ignore the trivial cases when $\mathcal{F}_{TB}$ is empty, or not full-dimensional (when $\mathcal{F}_{TB}$ is a singleton), or one ball is contained in the other (when $\mathcal{F}_{TB}$ is a ball and problem (2) reduces to the TRS). In the nontrivial cases, observe that the intersection of $\{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1 \,\}$ and $\{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{c}\| \le a \,\}$ is an $(n-1)$-dimensional sphere. Therefore, the following partition is possible for $\mathcal{F}_{TB}$.

Lemma 4 ([2]). *For the nontrivial cases of problem (2) with two ball constraints, $\mathcal{F}_{TB} = \mathcal{F}_1 \cup \mathcal{F}_2$, where*

$$\mathcal{F}_1 = \{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1,\ -2\mathbf{c}^T\mathbf{x} + (1 + \mathbf{c}^T\mathbf{c} - a^2) \le 0 \,\},$$
$$\mathcal{F}_2 = \{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{c}\| \le a,\ -2\mathbf{c}^T\mathbf{x} + (1 + \mathbf{c}^T\mathbf{c} - a^2) \ge 0 \,\}.$$

With Lemma 1 and Proposition 2, $\mathcal{C}(\mathcal{F}_{TB})$ has the following semidefinite representation.

Proposition 5. *For the nontrivial cases of problem* (2) *with two ball constraints, we have*

$$\mathcal{C}(\mathcal{F}_{TB}) = \left\{ (\mathbf{x}, X) \;\middle|\; \begin{array}{l} \exists\ \lambda \in [0,1],\ (\mathbf{y}_1, Y_1),\ (\mathbf{y}_2, Y_2) \in \mathbb{R}^n \times \mathcal{S}^n \text{ such that} \\ (\mathbf{x}, X) = (\mathbf{y}_1, Y_1) + (\mathbf{y}_2, Y_2), \\ \mathrm{tr}(Y_1) \le \lambda,\ \|2Y_1\mathbf{c} - q\mathbf{y}_1\| \le 2\mathbf{c}^T\mathbf{y}_1 - q\lambda, \\ \mathrm{tr}(Y_2) - 2\mathbf{c}^T\mathbf{y}_2 + (\mathbf{c}^T\mathbf{c} - a^2)(1-\lambda) \le 0, \\ \|2Y_2\mathbf{c} - q\mathbf{y}_2 - 2\mathbf{c}\mathbf{c}^T\mathbf{y}_2 + q\mathbf{c}(1-\lambda)\| \le a(-2\mathbf{c}^T\mathbf{y}_2 + q(1-\lambda)), \\ \begin{pmatrix} \lambda & \mathbf{y}_1^T \\ \mathbf{y}_1 & Y_1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} 1-\lambda & \mathbf{y}_2^T \\ \mathbf{y}_2 & Y_2 \end{pmatrix} \succeq 0 \end{array} \right\},$$

where $q = 1 + \mathbf{c}^T\mathbf{c} - a^2$.

Proof. Denote the set on the right side of the equation by S. Recalling the sets $\mathcal{F}_1$ and $\mathcal{F}_2$ defined in Lemma 4 applying Proposition 2 we have

$$\mathcal{C}(\mathcal{F}_1) = \{ (\mathbf{x}_1, X_1) \mid \mathrm{tr}(X_1) \le 1,\ \|2X_1\mathbf{c} - q\mathbf{x}_1\| \le 2\mathbf{c}^T\mathbf{x}_1 - q,\ X_1 \succeq \mathbf{x}_1\mathbf{x}_1^T \},$$
$$\mathcal{C}(\mathcal{F}_2) = \left\{ (\mathbf{x}_2, X_2) \;\middle|\; \begin{array}{l} \|2X_2\mathbf{c} - q\mathbf{x}_2 - 2\mathbf{c}\mathbf{c}^T\mathbf{x}_2 - q\mathbf{c}\| \le a(-2\mathbf{c}^T\mathbf{x}_2 + q), \\ \mathrm{tr}(X_2) - 2\mathbf{c}^T\mathbf{x}_2 + \mathbf{c}^T\mathbf{c} - a^2 \le 0,\ X_2 \succeq \mathbf{x}_2\mathbf{x}_2^T \end{array} \right\}.$$

Note that for any $\lambda \in (0,1)$, $(\mathbf{x}_1, X_1) \in \mathcal{C}(\mathcal{F}_1)$ if and only if $(\mathbf{y}_1, Y_1) = \lambda(\mathbf{x}_1, X_1)$ satisfies

$$\mathrm{tr}(Y_1) \le \lambda,\ \|2Y_1\mathbf{c} - q\mathbf{y}_1\| \le 2\mathbf{c}^T\mathbf{y}_1 - q\lambda, \text{ and } \lambda Y_1 \succeq \mathbf{y}_1\mathbf{y}_1^T.$$

Similarly, $(\mathbf{x}_2, X_2) \in \mathcal{C}(\mathcal{F}_2)$ if and only if $(\mathbf{y}_2, Y_2) = (1-\lambda)(\mathbf{x}_2, X_2)$ satisfies

$$\begin{array}{l} \|2Y_2\mathbf{c} - q\mathbf{y}_2 - 2\mathbf{c}\mathbf{c}^T\mathbf{y}_2 - (1-\lambda)q\mathbf{c}\| \le a(-2\mathbf{c}^T\mathbf{y}_2 + q(1-\lambda)), \\ \mathrm{tr}(Y_2) - 2\mathbf{c}^T\mathbf{y}_2 + (1-\lambda)(\mathbf{c}^T\mathbf{c} - a^2) \le 0, \text{ and } (1-\lambda)Y_2 \succeq \mathbf{y}_2\mathbf{y}_2^T. \end{array}$$

For any $(\mathbf{x}, X) \in \mathcal{C}(\mathcal{F}_{TB})$, by Lemma 1, there exist $\lambda \in [0,1]$, $(\mathbf{x}_1, X_1) \in \mathcal{C}(\mathcal{F}_1)$ and $(\mathbf{x}_2, X_2) \in \mathcal{C}(\mathcal{F}_2)$ such that $(\mathbf{x}, X) = \lambda(\mathbf{x}_1, X_1) + (1-\lambda)(\mathbf{x}_2, X_2)$. Let

$$(\mathbf{y}_1, Y_1) = \begin{cases} \lambda(\mathbf{x}_1, X_1) & \text{if } \lambda > 0 \\ (\mathbf{0}, 0) & \text{if } \lambda = 0 \end{cases} \quad \text{and} \quad (\mathbf{y}_2, Y_2) = \begin{cases} (1-\lambda)(\mathbf{x}_2, X_2) & \text{if } \lambda < 1 \\ (\mathbf{0}, 0) & \text{if } \lambda = 1. \end{cases}$$

Then it is clear that $(\mathbf{x}, X) \in S$. On the other hand, for any $(\mathbf{x}, X) \in S$, there exists

$(\lambda, \mathbf{y}_1, Y_1, \mathbf{y}_2, Y_2)$ satisfying the constraints in S. Let

$$(\mathbf{x}_1, X_1) = \begin{cases} \frac{1}{\lambda}(\mathbf{y}_1, Y_1) & \text{if } \lambda > 0 \\ \text{any point in } \mathcal{C}(\mathcal{F}_1) & \text{if } \lambda = 0 \end{cases} \text{ and } (\mathbf{x}_2, X_2) = \begin{cases} \frac{1}{1-\lambda}(\mathbf{y}_2, Y_2) & \text{if } \lambda < 1 \\ \text{any point in } \mathcal{C}(\mathcal{F}_2) & \text{if } \lambda = 1. \end{cases}$$

Then it is clear that $(\mathbf{x}, X) \in \mathcal{C}(\mathcal{F}_{TB})$. □

3.2 A ball and a second-order cone representable constraint

Let $\mathcal{F}_{SOC} = \{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1, \|\mathbf{x}\| \le \mathbf{b}^T\mathbf{x} - a \}$ be the feasible region of (3). We make the following assumptions to avoid trivial cases.

Assumption 1. *There exist* $\tilde{\mathbf{x}}, \hat{\mathbf{x}}, \bar{\mathbf{x}} \in \mathbb{R}^n$ *such that*

$$\begin{aligned} \|\tilde{\mathbf{x}}\| \le 1, \ \|\tilde{\mathbf{x}}\| \le \mathbf{b}^T\tilde{\mathbf{x}} - a, \\ \|\bar{\mathbf{x}}\| \le 1, \ \|\bar{\mathbf{x}}\| > \mathbf{b}^T\bar{\mathbf{x}} - a, \\ \|\hat{\mathbf{x}}\| > 1, \ \|\hat{\mathbf{x}}\| \le \mathbf{b}^T\hat{\mathbf{x}} - a. \end{aligned}$$

Assumption 1 guarantees that $\{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1 \}$ and $\{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le \mathbf{b}^T\mathbf{x} - a \}$ intersect and neither of the sets contains the other. When Assumption 1 is violated, $\mathcal{F}_{SOC}$ is either empty, a ball (when problem (3) reduces to the TRS), or a set defined by a second-order cone constraint (when problem (3) can be handled by [19]). It is clear that $\mathcal{F}_{SOC} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\begin{aligned} \mathcal{F}_1 &= \{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1 \le \mathbf{b}^T\mathbf{x} - a \}, \\ \mathcal{F}_2 &= \{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le \mathbf{b}^T\mathbf{x} - a \le 1 \}. \end{aligned} \tag{8}$$

Under Assumption 1, we can show that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonempty.

Lemma 6. *Under Assumption 1, there exists* $\mathbf{x}_0 \in \mathbb{R}^n$ *such that* $\|\mathbf{x}_0\| \le 1 = \mathbf{b}^T\mathbf{x}_0 - a$.

Proof. If $\mathbf{b}^T\tilde{\mathbf{x}} - a = 1$, then $\mathbf{x}_0 = \tilde{\mathbf{x}}$ is such a point. If $\mathbf{b}^T\tilde{\mathbf{x}} - a > 1$, since $\|\bar{\mathbf{x}}\| \le 1$ and $\mathbf{b}^T\bar{\mathbf{x}} - a < 1$, we can choose $\mathbf{x}_0$ as the convex combination of $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$ such that $\mathbf{b}^T\mathbf{x}_0 - a = 1$. If $\mathbf{b}^T\tilde{\mathbf{x}} - a < 1$, since $\|\hat{\mathbf{x}}\| \le \mathbf{b}^T\hat{\mathbf{x}} - a$ and $\mathbf{b}^T\hat{\mathbf{x}} - a > 1$, we can choose $\mathbf{x}_0$ as the convex combination of $\tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}$ such that $\mathbf{b}^T\mathbf{x}_0 - a = 1$. □

With Lemma 1 and Propositions 2 and 3, $\mathcal{C}(\mathcal{F}_{SOC})$ has the following semidefinite representation.

Proposition 7. *Under Assumption 1,*

$$\mathcal{C}(\mathcal{F}_{SOC}) = \left\{ (\mathbf{x}, X) \;\middle|\; \begin{array}{l} \exists\ \lambda \in [0,1],\ (\mathbf{y}_1, Y_1),\ (\mathbf{y}_2, Y_2) \in \mathbb{R}^n \times \mathcal{S}^n \textit{ such that} \\ (\mathbf{x}, X) = (\mathbf{y}_1, Y_1) + (\mathbf{y}_2, Y_2), \\ \operatorname{tr}(Y_1) \le \lambda,\ \ \|Y_1\mathbf{b} - (a+1)\mathbf{y}_1\| \le \mathbf{b}^T\mathbf{y}_1 - (a+1)\lambda, \\ (I - \mathbf{b}\mathbf{b}^T) \bullet Y_2 + 2a\mathbf{b}^T\mathbf{y}_2 - a^2(1-\lambda) \le 0, \\ \mathbf{b}^T\mathbf{y}_2 - (a+1)(1-\lambda) \le 0, \\ \|Y_2\mathbf{b} - (a+1)\mathbf{y}_2\| \le -\mathbf{b}\mathbf{b}^T \bullet Y_2 + (1+2a)\mathbf{b}^T\mathbf{y}_2 - a(1+a)(1-\lambda), \\ \begin{pmatrix} \lambda & \mathbf{y}_1^T \\ \mathbf{y}_1 & Y_1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} 1-\lambda & \mathbf{y}_2^T \\ \mathbf{y}_2 & Y_2 \end{pmatrix} \succeq 0 \end{array} \right\}.$$

Proof. The proof is similar to that of Proposition 5. Denote the set on the right side of the equation by S. Also, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sets defined in (8). By Propositions 2 and 2,

$$\mathcal{C}(\mathcal{F}_1) = \{ (\mathbf{x}_1, X_1) \mid \operatorname{tr}(X_1) \le 1,\ \|2X_1\mathbf{b} - (a+1)\mathbf{x}_1\| \le \mathbf{b}^T\mathbf{x}_1 - a - 1,\ X_1 \succeq \mathbf{x}_1\mathbf{x}_1^T \},$$
$$\mathcal{C}(\mathcal{F}_2) = \left\{ (\mathbf{x}_2, X_2) \;\middle|\; \begin{array}{l} \|X_2\mathbf{b} - (a+1)\mathbf{x}_2\| \le -\mathbf{b}\mathbf{b}^T \bullet X_2 + (2a+1)\mathbf{b}^T\mathbf{x}_2 - a(a+1), \\ (I - \mathbf{b}\mathbf{b}^T) \bullet X_2 + 2a\mathbf{b}^T\mathbf{x}_2 - a^2 \le 0,\ \mathbf{b}^T\mathbf{x}_2 - a - 1 \le 0,\ X_2 \succeq \mathbf{x}_2\mathbf{x}_2^T \end{array} \right\}.$$

For any $\lambda \in (0,1)$, $(\mathbf{x}_1, X_1) \in \mathcal{C}(\mathcal{F}_1)$ if and only if $(\mathbf{y}_1, Y_1) = \lambda(\mathbf{x}_1, X_1)$ satisfies

$$\operatorname{tr}(Y_1) \le \lambda,\ \|Y_1\mathbf{b} - (a+1)\mathbf{y}_1\| \le \mathbf{b}^T\mathbf{y}_1 - (a+1)\lambda,\ \text{and } \lambda Y_1 \succeq \mathbf{y}_1\mathbf{y}_1^T.$$

Similarly, $(\mathbf{x}_2, X_2) \in \mathcal{C}(\mathcal{F}_2)$ if and only if $(\mathbf{y}_2, Y_2) = (1-\lambda)(\mathbf{x}_2, X_2)$ satisfies

$$(I - \mathbf{b}\mathbf{b}^T) \bullet Y_2 + 2a\mathbf{b}^T\mathbf{y}_2 - a^2(1-\lambda) \le 0,\ \mathbf{b}^T\mathbf{y}_2 - (a+1)(1-\lambda) \le 0,$$
$$\|Y_2\mathbf{b} - (a+1)\mathbf{y}_2\| \le -\mathbf{b}\mathbf{b}^T \bullet Y_2 + (2a+1)\mathbf{b}^T\mathbf{y}_2 - a(a+1)(1-\lambda)),\ \text{and } (1-\lambda)Y_2 \succeq \mathbf{y}_2\mathbf{y}_2^T.$$

The rest of the proof is the same as the proof of Proposition 5 with $\mathcal{F}_{TB}$ replaced by $\mathcal{F}_{SOC}$. □

4 Separation and computational results

The convex hull results we derive in Section 3 are presented as projections of convex sets in a higher-dimensional space. Due to the nonlinear feature of the sets, it is challenging to find explicit expressions of the projections in the space of $(\mathbf{x}, X)$. However, one can generate valid inequalities for the lifted convex hull in the $(\mathbf{x}, X)$-space as needed. In particular, for any given $(\hat{\mathbf{x}}, \hat{X})$ not in the lifted convex hull, a separating hyperplane can be found by solving an SDP. We use problem (3) as an example to explain the idea, and the separation problem for (2) can be built in the same manner.

For a given $(\hat{\mathbf{x}}, \hat{X})$, we consider the following separation problem for $\mathcal{C}(\mathcal{F}_{SOC})$.

$$
\begin{aligned}
v^*(\hat{\mathbf{x}}, \hat{X}) := \inf \quad & u_1 + u_2 + u_3 + u_4 + u_5 + w_1 + w_2 \\
s.t. \quad & \begin{pmatrix} \lambda & \mathbf{y}_1^T \\ \mathbf{y}_1 & Y_1 \end{pmatrix} + \begin{pmatrix} \mu & \mathbf{y}_2^T \\ \mathbf{y}_2 & Y_2 \end{pmatrix} = \begin{pmatrix} 1 & \hat{\mathbf{x}}^T \\ \hat{\mathbf{x}} & \hat{X} \end{pmatrix} \\
& \mathrm{tr}(Y_1) \le \lambda + u_1 \\
& \|Y_1 \mathbf{b} - (a+1)\mathbf{y}_1\| \le \mathbf{b}^T \mathbf{y}_1 - (a+1)\lambda + u_2 \\
& (I - \mathbf{b}\mathbf{b}^T) \bullet Y_2 + 2a\mathbf{b}^T \mathbf{y}_2 - a^2 \mu \le u_3 \\
& \|Y_2 \mathbf{b} - (a+1)\mathbf{y}_2\| \le -\mathbf{b}\mathbf{b}^T \bullet Y_2 + (1+2a)\mathbf{b}^T \mathbf{y}_2 - a(1+a)\mu + u_4 \\
& \mathbf{b}^T \mathbf{y}_2 - (a+1)\mu \le u_5 \\
& \begin{pmatrix} \lambda & \mathbf{y}_1^T \\ \mathbf{y}_1 & Y_1 \end{pmatrix} + w_1 I_{n+1} \succeq 0 \\
& \begin{pmatrix} \mu & \mathbf{y}_2^T \\ \mathbf{y}_2 & Y_2 \end{pmatrix} + w_2 I_{n+1} \succeq 0 \\
& \lambda, \mu, u_1, u_2, u_3, u_4, u_5, w_1, w_2 \ge 0
\end{aligned}
\tag{9}
$$

Here, $\mathbf{u} \in \mathbb{R}^5$ and $\mathbf{w} \in \mathbb{R}^2$ are nonnegative artificial variables. Note that $(\hat{\mathbf{x}}, \hat{X}) \in \mathcal{C}(\mathcal{F}_{SOC})$ if and only if $v^*(\hat{\mathbf{x}}, \hat{X}) = 0$. Consider the dual problem of (9). Let $Z \in \mathcal{S}^{n+1}$ be the dual variable associated with the equality constraint. The dual problem of (9) can be represented as

$$
\begin{aligned}
d^*(\hat{\mathbf{x}}, \hat{X}) := \sup \quad & \begin{pmatrix} 1 & \hat{\mathbf{x}}^T \\ \hat{\mathbf{x}} & \hat{X} \end{pmatrix} \bullet Z \\
s.t. \quad & Z \in \mathcal{D},
\end{aligned}
\tag{10}
$$

where $\mathcal{D}$ is a semidefinite representable set independent of the choice of $(\hat{\mathbf{x}}, \hat{X})$. Since (9) clearly satisfies the Slater's condition, strong duality holds and $v^*(\hat{\mathbf{x}}, \hat{X}) = d^*(\hat{\mathbf{x}}, \hat{X})$. For any dual feasible solution $Z \in \mathcal{D}$,

$$
Z \bullet \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & X \end{pmatrix} \le d^*(\mathbf{x}, X) = v^*(\mathbf{x}, X) = 0 \qquad \forall\ (\mathbf{x}, X) \in \mathcal{C}(\mathcal{F}_{SOC}).
$$

That is, any dual solution $Z \in \mathcal{D}$ can be used to generate a valid inequality for $\mathcal{C}(\mathcal{F}_{SOC})$.

Moreover, for any $(\hat{\mathbf{x}}, \hat{X}) \notin \mathcal{C}(\mathcal{F}_{SOC})$, let $\hat{Z}$ be an optimal solution to (10), then

$$\hat{Z} \bullet \begin{pmatrix} 1 & \hat{\mathbf{x}}^T \\ \hat{\mathbf{x}} & \hat{X} \end{pmatrix} = d^*(\hat{\mathbf{x}}, \hat{X}) = v^*(\hat{\mathbf{x}}, \hat{X}) > 0.$$

That is, $\left\{ (\mathbf{x}, X) \;\middle|\; \hat{Z} \bullet \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & X \end{pmatrix} = 0 \right\}$ is a hyperplane strictly separating $(\hat{\mathbf{x}}, \hat{X})$ and $\mathcal{C}(\mathcal{F}_{SOC})$.

The last observation can be used to generate valid inequalities to tighten existing SDP relaxations of (3) in the $(\mathbf{x}, X)$-space. To show its effectiveness, we test the idea on the instances used in Sections 5.2 and 5.3 of [13]. Same as in [13], we consider two SDP relaxations with feasible regions $\mathcal{R}_{\text{shor}}$ and $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$, respectively, where

$$\mathcal{R}_{\text{shor}} := \{ (\mathbf{x}, X) \mid \text{tr}(X) \leq 1,\, \text{tr}(X) \leq \mathbf{b}\mathbf{b}^T \bullet X - 2a\mathbf{b}^T\mathbf{x} + a^2,\, \mathbf{b}^T\mathbf{x} - a \geq 0,\, X \succeq \mathbf{x}\mathbf{x}^T \}$$

is the feasible region of the standard SDP (Shor) relaxation of (3), and $\mathcal{R}_{\text{ksoc}}$ is the set of $(\mathbf{x}, X)$ satisfying the linearized version of the Kronecker product constraint

$$\begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & I_n \end{pmatrix} \otimes \begin{pmatrix} \mathbf{b}^T\mathbf{x} - a & \mathbf{x}^T \\ \mathbf{x} & (\mathbf{b}^T\mathbf{x} - a)I_n \end{pmatrix} \succeq 0.$$

For more details about the Kronecker product constraints, we refer the readers to [4].

Let $(\hat{\mathbf{x}}, \hat{X})$ be an optimal solution to an SDP relaxation of (3), and let $\lambda_1 \geq \ldots \geq \lambda_{n+1}$ denote the eigenvalues of $\begin{pmatrix} 1 & \hat{\mathbf{x}}^T \\ \hat{\mathbf{x}} & \hat{X} \end{pmatrix}$. Following [13], we say that the SDP relaxation is exact if $\frac{\lambda_1}{\lambda_2} > 10^4$, i.e., the matrix $\begin{pmatrix} 1 & \hat{\mathbf{x}}^T \\ \hat{\mathbf{x}} & \hat{X} \end{pmatrix}$ is numerically rank-1. For each tested instance, we solve problem (6) with $\mathcal{C}(\mathcal{F})$ replaced with $\mathcal{R}_{\text{shor}}/\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ to get an initial optimal solution $(\hat{\mathbf{x}}, \hat{X})$. If the relaxation is inexact, we solve the separation problem (9) and generate a valid inequity $\hat{Z} \bullet \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & X \end{pmatrix} \leq 0$. We add the valid inequality to the relaxation and resolve. We repeat the process until the relaxation is exact.

We implement our experiments in Matlab 9.5 (R2018b) using CVX [15] to model the relaxations and MOSEK 9.1 [6] to solve them. We run the instances on an Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz with four cores and 16GB of memory. For each dimension n, we test the 15,000 instances generated in [13][1]. We report the number of instances with an inexact initial ($\mathcal{R}_{\text{shor}}/\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$) relaxation, the maximum and average number of cuts needed to close the gaps, and the percentage of instances whose gaps are closed with a single

[1]The instances can be found at https://github.com/A-Eltved/strengthened_sdr.

n	Inexact initial[2]	Max cuts	Avg cuts	Closed with one cut (%)
2	7744	21	2.42	27.83
3	7634	36	1.99	50.21
4	7733	19	1.59	66.48
5	7704	16	1.40	76.43
6	7584	21	1.29	82.45
7	7648	10	1.22	86.36
8	7614	12	1.18	88.80
9	7566	7	1.13	91.05
10	7552	10	1.11	92.74

Table 1: Cut effectiveness for the $\mathcal{R}_{\text{shor}}$ relaxation of (3).

n	Inexact initial	Max cuts	Avg cuts	Closed with one cut (%)
2	15	3	1.40	73.33
3	50	4	1.64	54.00
4	36	4	1.69	52.78
5	27	6	1.52	62.96
6	15	3	1.40	66.67
7	13	3	1.31	76.92
8	12	2	1.08	91.67
9	5	1	1.00	100
10	5	2	1.20	80

Table 2: Cut effectiveness for the $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ relaxation of (3).

cut. The results are reported in Tables 1 and 2.

For the Shor relaxation, we observe that the gaps of all instances are closed within 36 cuts. For the tighter relaxation with $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$, the gaps of all instances are closed within six cuts. On average, less than two cuts are needed for $n \geq 3$. A majority of the instances require only one cut to close the gaps. We remark here that with respect to time, each separation problem takes about 0.3 seconds to solve.

We conduct similar tests on the two-ball problem (2) and observe similar performance. Compared with (3), more cuts are needed to close the gaps on average, but most instances need no more than two cuts. The results are presented in Tables 3 and 4.

5 Conclusion

In this paper, we derive semidefinite representable reformulations for two variants of the trust-region subproblem. By partitioning the feasible region of each problem, we are able

[2]The number of instances with inexact initial relaxations is slightly different from [13], possibly due to different versions of platforms. Same for the other tables.

n	Inexact initial	Max cuts	Avg cuts	Closed with one cut (%)
2	1403	19	2.45	7.98
3	1285	22	2.48	12.68
4	983	15	2.28	20.04
5	739	11	2.18	22.60
6	508	6	2.11	21.46
7	453	14	2.11	26.93
8	346	9	2.08	27.75
9	292	6	1.96	31.85
10	251	7	2.01	29.08

Table 3: Cut effectiveness for the $\mathcal{R}_{\text{shor}}$ relaxation of (2).

n	Inexact initial	Max cut	Avg cuts	Closed with one cut (%)
2	31	10	2.32	41.94
3	78	9	2.18	41.03
4	62	7	1.89	46.77
5	34	8	2.03	35.29
6	21	4	1.67	47.62
7	16	4	1.81	50.00
8	14	4	1.50	64.29
9	6	2	1.17	83.33
10	4	3	2.00	25.00

Table 4: Cut effectiveness for the $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ relaxation of (2).

to take advantage of its structure to find the convex hull in a lifted space. The derivation is from a disjunctive perspective, and the lifted convex hull is given as a projection of a convex set in a higher dimensional space. To find valid inequalities of the convex hull in the original lifted space, a separation problem is constructed, and computational results show that the generated cuts are effective in closing the relaxation gap for instances considered in the literature.

Acknowledgements This material is based upon Yang's research supported by the Office of Naval Research under Award Number (N00014-20-1-2154) and Ouyang's research supported by the Office of Naval Research under Award Number (N00014-20-1-2089).

References

[1] Temadher A Almaadeed, Saeid Ansary Karbasy, Maziar Salahi, and Abdelouahed Hamdi. On indefinite quadratic optimization over the intersection of balls and linear constraints. *Journal of Optimization Theory and Applications*, 194(1):246–264, 2022.

[2] Saeid Ansary Karbasy and Maziar Salahi. Quadratic optimization with two ball constraints. *Numerical Algebra, Control & Optimization*, 10(2):165, 2020.

[3] Saeid Ansary Karbasy and Maziar Salahi. On the branch and bound algorithm for the extended trust-region subproblem. *Journal of Global Optimization*, pages 1–13, 2021.

[4] Kurt M Anstreicher. Kronecker product constraints with an application to the two-trust-region subproblem. *SIAM Journal on Optimization*, 27(1):368–378, 2017.

[5] Kurt M Anstreicher. Solving two-trust-region subproblems using semidefinite optimization with eigenvector branching. *Journal of Optimization Theory and Applications*, pages 1–17, 2022.

[6] Mosek ApS. Mosek optimization toolbox for matlab. *User's Guide and Reference Manual, Version*, 4, 2019.

[7] Daniel Bienstock and Alexander Michalka. Polynomial solvability of variants of the trust-region subproblem. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 380–390. 2014.

[8] Immanuel M Bomze, Vaithilingam Jeyakumar, and Guoyin Li. Extended trust-region problems with one or two balls: exact copositive and lagrangian relaxations. *Journal of Global Optimization*, 71(3):551–569, 2018.

[9] Samuel Burer. A gentle, geometric introduction to copositive optimization. *Mathematical Programming*, 151(1):89–116, 2015.

[10] Samuel Burer and Kurt M. Anstreicher. Second-order-cone constraints for extended trust-region subproblems. *SIAM Journal on Optimization*, 23(1):432–451, 2013.

[11] Samuel Burer and Boshi Yang. The trust region subproblem with non-intersecting linear constraints. *Mathematical Programming*, 149(1-2):253–264, 2015.

[12] Gabriele Eichfelder and Janez Povh. On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets. *Optimization Letters*, 7(6):1373–1386, 2013.

[13] Anders Eltved and Samuel Burer. Strengthened SDP relaxation for an extended trust region subproblem with an application to optimal power flow. *Mathematical Programming*, pages 1–26, 2022.

[14] Nicholas I. M. Gould, Stefano Lucidi, Massimo Roma, and Philippe L. Toint. Solving the trust-region subproblem using the lanczos method. *SIAM Journal on Optimization*, 9(2):504–525, 1999.

[15] Michael Grant and Stephen Boyd. CVX: Matlab software for disciplined convex programming, version 2.2. http://cvxr.com/cvx, January 2020.

[16] Nam Ho-Nguyen and Fatma Kılınç-Karzan. A second-order cone based approach for solving the trust-region subproblem and its variants. *SIAM Journal on Optimization*, 27(3):1485–1512, 2017.

[17] V. Jeyakumar and G.Y. Li. Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. *Mathematical Programming*, 147(1-2):171–206, 2014.

[18] Rujun Jiang and Duan Li. Novel reformulations and efficient algorithms for the generalized trust region subproblem. *SIAM Journal on Optimization*, 29(2):1603–1633, 2019.

[19] Qingwei Jin, Ye Tian, Zhibin Deng, Shu-Cherng Fang, and Wenxun Xing. Exact computable representation of some second-order cone constrained quadratic programming problems. *Journal of the Operations Research Society of China*, 1(1):107–134, 2013.

[20] Alex Joyce and Boshi Yang. Convex hull results on quadratic programs with non-intersecting constraints. *Optimization Online preprint*, 2021.

[21] Cheng Lu, Shu-Cherng Fang, Qingwei Jin, Zhenbo Wang, and Wenxun Xing. KKT solution and conic relaxation for solving quadratically constrained quadratic programming problems. *SIAM Journal on Optimization*, 21(4):1475–1490, 2011.

[22] Tiago Montanher, Arnold Neumaier, and Ferenc Domes. A computational study of global optimization solvers on two trust region subproblems. *Journal of Global Optimization*, 71(4):915–934, 2018.

[23] Jorge J. Moré and D. C. Sorensen. Computing a trust region step. *SIAM Journal on Scientific and Statistical Computing*, 4(3):553–572, 1983.

[24] Ting Kei Pong and Henry Wolkowicz. The generalized trust region subproblem. *Computational Optimization and Applications*, 58(2):273–322, 2014.

[25] Franz Rendl and Henry Wolkowicz. A semidefinite framework for trust region subproblems with applications to large scale minimization. *Mathematical Programming*, 77(1):273–299, 1997.

[26] Shinsaku Sakaue, Yuji Nakatsukasa, Akiko Takeda, and Satoru Iwata. Solving generalized cdt problems via two-parameter eigenvalues. *SIAM Journal on Optimization*, 26(3):1669–1694, 2016.

[27] Jos F. Sturm and Shuzhong Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operations Research*, 28(2):246–267, 2003.

[28] Alex Wang. *On Quadratically Constrained Quadratic Programs and their Semidefinite Program Relaxations*. PhD thesis, Carnegie Mellon University.

[29] Alex L Wang and Fatma Kılınç-Karzan. The generalized trust region subproblem: solution complexity and convex hull results. *Mathematical Programming*, pages 1–42, 2020.

[30] Shu Wang and Yong Xia. Strong duality for generalized trust region subproblems: S-lemma with interval bounds. *Optimization Letters*, 9(6):1063–1073, 2015.

[31] Boshi Yang, Kurt Anstreicher, and Samuel Burer. Quadratic programs with hollows. *Mathematical Programming*, 170(2):541–553, 2018.

[32] Boshi Yang and Samuel Burer. A two-variable approach to the two-trust-region subproblem. *SIAM Journal on Optimization*, 26(1):661–680, 2016.

[33] Yinyu Ye and Shuzhong Zhang. New results on quadratic minimization. *SIAM Journal on Optimization*, 14(1):245–267, 2003.