Recognition of Facets for Knapsack Polytope is D^p-complete

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Abstract. D^p is a complexity class that is the class of all languages that are the intersection of a language in $\mathcal{NP}$ and a language in co-$\mathcal{NP}$, as coined by Papadimitriou and Yannakakis. In this paper, we will establish that, recognizing a facet for the knapsack polytope is D^p-complete, as conjectured by Hartvigsen and Zemel [10] in 1992. Moreover, we show that the recognition problem of a supporting hyperplane for the knapsack polytope and the exact knapsack problem are both D^p-complete, and the membership problem of knapsack polytope is $\mathcal{NP}$-complete.

Keywords: Knapsack polytope · D^p-complete · Computational complexity · Facet

1 Introduction

The polyhedral approach to solve combinatorial optimization problems has been used extensively over the last few decades. Typically, many important combinatorial optimization problems call for the optimization of a linear function over certain discrete set of vectors. By having a complete description of the convex hull of this discrete set, the combinatorial optimization problem can be equivalent solved as a linear program (LP). Traditionally, the convex hull of such discrete set associated with some of the well-studied combinatorial problems such as *travelling salesman problem (TSP), clique problem, knapsack problem* is called *TSP polytope, clique polytope* and *knapsack polytope.* The characterization of the facets of these polytopes is one of the main subjects in *polyhedral combinatorics* (see, e.g., [18]), and has been under constant attack over 70 years. From the mixed-integer program (MIP) perspective, obtaining facet-defining inequalities or strong valid inequalities for the polytope is also very critical for the branch-and-cut procedure, and there has been a very large body of literature aimed at generating facet-defining inequalities for certain combinatorial polytopes. However, the non-redundant linear inequalities describing the combinatorial polytope are generally very hard to obtain. Karp and Papadimitriou [11] showed that, unless $\mathcal{NP}$=co-$\mathcal{NP}$, otherwise there does not exist a computational tractable description by linear inequalities of the polyhedron associated with any $\mathcal{NP}$-complete combinatorial optimization problem.

As most combinatorial optimization problems are indeed $\mathcal{NP}$-hard, and it is not computationally tractable to generate all the facet-defining inequalities for the associated polytopes, in view of the computational complexity of these problems, it is natural to ask about the complexity questions concerning the facets or valid inequalities of the associated polytopes.

From the computational point of view, in practice, an important aspect of measuring the facet-defining inequalities (or valid inequalities) complexity is the computational effort needed to generate separating inequalities in the process of cutting-plane method or branch-and-cut. For a given family of valid inequalities to a particular MIP problem, the associated *separation problem* is defined as: let x^* be a feasible solution to the LP relaxation of the MIP, does there exist an inequality from the family that is violated by x^*? The separation problem for several families of valid inequalities for the knapsack problem has been shown to be $\mathcal{NP}$-complete, including cover inequalities [12], lifted cover inequalities [9], extended cover inequalities [5], (1, k)-configuration inequalities [5], and weight inequalities [5]. With respect to general MIP problem, Letchford et al. [13] showed that the separation problem for $\{0, 1/2\}$-cuts is strongly $\mathcal{NP}$-hard and Eisenbrand [7] showed that the separation problem for Chvátal-Gomory cuts is $\mathcal{NP}$-complete. In another recent paper [4] for studying valid inequalities of complementarity knapsack problem, Del Pia et al. also showed that, the separation problem associated with the inequalities defined in [3] is $\mathcal{NP}$-complete.

In 1982, a more theoretical and fundamental perspective of viewing the complexity of valid inequalities of combinatorial polytopes is provided in a series papers by Karp, Papadimitriou and Yannakakis. For the decision problem *TSP FACETS*, the problem of recognizing whether a given inequality is a facet of the TSP polytope, indirect information concerning the complexity of this problem was obtained in [11]. Karp and Papadimitriou showed that, if TSP FACETS $\in \mathcal{NP}$, then $\mathcal{NP}$=co-$\mathcal{NP}$. To provide a more general complexity class for the decision problem of recognizing whether an inequality is a facet of a particular polytope, in a seminal paper by Papadimitriou and Yannakakis [16], they introduced a new complexity class that is the class of all languages that are the intersection of a language in $\mathcal{NP}$ and a language in co-$\mathcal{NP}$, and they coined it as D^p. It is important to note that, here $\mathcal{NP}\cap$ co-$\mathcal{NP}$ is a proper sub-class of D^p. D^p is a natural niche for many important classes of problems. For instance, as the motivation problem in [16], TSP FACETS is in D^p. This is because, a facet-defining inequality to a polytope P is essentially a valid inequality that holds at equality at $\dim(P)$ affinely independent points in P. So in order to determine whether an inequality is facet-defining to a polytope, it is equivalent to deciding if this inequality is valid to the polytope (co-$\mathcal{NP}$ problem), and there exist $\dim(P)$ affinely independent points in P that satisfy the inequality with equality ($\mathcal{NP}$ problem). Moreover, some of the hardest problems in graph theory concern *critical graphs*, which are the graphs that do not have certain property but the deletion (sometimes addition) of an edge (or node) creates a graph with the property. Then the *critical problem*, the problem of recognizing if a given graph is a critical graph, is also naturally in D^p. This is because verifying if the given graph does not have certain property is in co-$\mathcal{NP}$, while verifying if the deletion (or addition) of an edge (or node) creates a graph with the property is in $\mathcal{NP}$.

With this more general complexity class D^p and some of the problems that are in D^p, it is therefore an interesting question to know whether these problems are complete for D^p. In [1], Cai and Meyer showed that the *graph minimal 3-colorability problem* is D^p-complete, which is the problem of deciding if a given graph is not colorable with 3 colors, but deleting any node from this graph results in a 3-colorable graph. For the *exact-k-colorability problem*, the problem of determining whether the chromatic number of a given graph is exactly k, Wagner [19] showed that *exact-7-colorability problem* is D^p-complete, and further asked if it is still D^p-complete to solve *exact-4-colorability problem*. This question was later answered positively by Rothe [17]. Regarding the recognition problem of facet to a polytope, in the original paper of Papadimitriou and Yannakakis [16], they showed that the recognition of a facet to the clique polytope is D^p-complete, and conjectured the same hardness for TSP FACETS. This conjecture was later resolved by Papadimitriou and Wolfe [15]. When studying the complexity of lifted inequalities for the knapsack problem, along with some other interesting results, Hartvigsen and Zemel [10] showed that recognizing valid inequalities for knapsack polytope (KP) is co-$\mathcal{NP}$-complete, and conjectured that recognizing facets for knapsack polytope is D^p-complete. It is one of the main goals of this paper to resolve this conjecture.

The structure of this paper is as follows: In Section 2, along with a few auxiliary D^p-complete results, we establish that the recognition problem of a supporting hyperplane for the knapsack polytope is D^p-complete. In Section 3, we will prove the main result of this paper, which is recognizing facets for knapsack polytope is also D^p-complete. Lastly, in Section **??**, we show that the problem of recognizing if a given point is in a given knapsack polytope is $\mathcal{NP}$-complete.

Notation. For an integer n we set $[n] := \{1, 2, \dots, n\}$. For a vector $x \in \mathbb{R}^n$ and $S \subseteq [n]$, we set $x(S) := \sum_{i \in S} x_i$.

2 KP Supporting Hyperplane Problem

In [16], Papadimitriou and Yannakakis showed that the *TSP supporting hyperplane problem*, which is the problem of deciding if a given inequality with integer coefficients provides a supporting hyperplane to the given TSP polytope, is D^p-complete. In this section, we are going to extend the same completeness result to the following ***Knapsack polytope (KP) supporting hyperplane problem***: given an inequality $\alpha^\top x \leq \beta$ with $\alpha \in \mathbb{Z}^n$ and a KP $\operatorname{conv}\left(\left\{x \in \{0,1\}^n : a^\top x \leq b\right\}\right)$, is this inequality valid to the KP and the corresponding hyperplane has nonempty intersection with the KP?

Before we proceeding to the proof of the main result in this section, first, we introduce the following two D^p problems that were previously defined in literature.

3-SAT-UNSAT: Given two boolean formulae F and F' in 3-CNF, is it true that F is satisfiable whereas F' is not not? Specifically, here we call *3-UNSAT* the problem of determining whether a boolean formula in 3-CNF is

unsatisfiable, and call *3-SAT* the problem of determining whether a boolean formula in 3-CNF is satisfiable or not.
Minimal unsatisfiability (MU) problem: Given a boolean formula, is it true that it is unsatisfiable, but removing **any** clause makes it satisfiable?

Both of these problems have been shown to be D^p-complete.

Theorem 1 (Lemma 1 [16]). *3-SAT-UNSAT is D^p-complete.*

Here we should remark that, in the original paper [16], it was briefly shown that SAT-UNSAT (two boolean formulae in the instance are not necessarily in 3-CNF) is D^p-complete, but this above theorem follows immediately from the standard reduction from SAT to 3-SAT. In fact, the proof to the following theorem in [15] was based on the reduction from Theorem 1.

Theorem 2 (Theorem 1 [15]). *MU is D^p-complete.*

Now we define two auxiliary new problems, which are slight variants of the MU problem and *subset sum problem.*

Restricted minimal unsatisfiability (RMU) problem: Given a boolean formula G and one of its clause c, is it true that G is unsatisfiable, but removing **such** clause c makes it satisfiable? We will also use tuple (G, c) to denote one particular instance of RMU.
Critical subset sum (CSS) problem: Given $(w_1, \ldots, w_n) \in \mathbb{Z}_+^n$ and a target-sum t, is it true that there exists a subset $S \subseteq [n]$ such that $w(S) = t-1$, but does not exist subset $T \subseteq [n]$ such that $w(T) = t$? We will use $(w_1, \ldots, w_n; t)$ to denote one particular instance of CSS.

The proof outline for the main result in this section is as follows: (i) we will first establish that 3-SAT-UNSAT is reducible to RMU problem, (ii) and then show that RMU problem is further reducible to CSS problem, (iii) lastly we show CSS problem is reducible to KP supporting hyperplane problem. Here we remark that all reductions mentioned in this paper refer to the polynomial time many-one reduction, or Karp reduction.

The proof of the first reduction is almost identical to the proof of Theorem 1 in [15]. Specifically, we will proceed the proof in two stages. First, we shall prove that both 3-SAT and 3-UNSAT are reducible to RMU problem, and then we show that there exists a reduction from two instances of RMU to one instance, such that the answer to the single instance is "yes" if and only if the answers to the original two instances are both "yes". To avoid redundancy, here we only give sketches for the proofs, for complete details we refer interested readers to the elegant proof of Theorem 1 in [15].

Lemma 1. *3-UNSAT is reducible to RMU.*

Proof (Proof sketch). Let F be an instance of 3-UNSAT. Using the reduction technique in the proof of Lemma 1 in [15], we can generate a new formula $G = c_1 \wedge c_2 \wedge \ldots \wedge c_n$, which is guaranteed to be minimal (that is, removing any clause results in a satisfiable formula), yet G is satisfiable if and only if F is. Consider the following instance of RMU problem: Given a boolean formula G and one of its clause c_1, is it true that G is unsatisfiable, but $c_2 \wedge \ldots \wedge c_n$ is satisfiable? By the above properties of G, we know that F is unsatisfiable if and only if the RMU instance has "yes" answer, so 3-UNSAT can be reduced to RMU problem. □

Lemma 2. *3-SAT is reducible to RMU.*

Proof (Proof sketch). Let F be an instance of 3-SAT and let G' be the obtained corresponding new formula in the proof of Lemma 2 in [15], which is obtained from adding an extra clause c to the obtained formula G in Lemma 1 [15]. In other words, $G' = G \wedge c$. Consider the instance of RMU problem: Given a boolean formula $G' = G \wedge c$ and one of its clause c, is it true that G' is unsatisfiable but G is satisfiable? Here G' is guaranteed to be unsatisfiable, and G is satisfiable if and only if F is. Therefore, we know that F is satisfiable if and only if the above RMU instance has a "yes" answer, so 3-SAT can be reduced to RMU. □

Lemma 3. *There is a polynomial time many-one reduction from two instances (F_1, c_1) and (F_2, c_2) of RMU problem to one instance (G, c) of RMU problem, such that (G, c) has a "yes" answer if and only if both (F_1, c_1) and (F_2, c_2) have "yes" answers.*

Proof. Let $F_1 = c_{1,1} \wedge \ldots c_{1,m}$ together with one of its clause $c_{1,1}$ be the first instance of RMU, and $F_2 = c_{2,1} \wedge \ldots \wedge c_{2,n}$ together with one of its clause $c_{2,1}$ be the second instance or RMU. Then for each possible pair of clauses, one from F_1 and one from F_2, add to G a clause which is the disjunct of the pair of clauses. In other words:

$$G = \bigwedge_{i \in [m], j \in [n]} (c_{1,i} \vee c_{2,j}) = \left(\bigwedge_{i \in [m]} c_{1,i} \right) \vee \left(\bigwedge_{j \in [n]} c_{2,j} \right) = F_1 \vee F_2. \tag{1}$$

Let $c := (c_{1,1} \vee c_{2,1})$. Next, we want to show that, (G, c) has a "yes" answer if and only if both $(F_1, c_{1,1})$ and $(F_2, c_{2,1})$ have "yes" answers. Denote $F_1' = c_{1,2} \wedge \ldots c_{1,m}$ and $F_2' = c_{2,2} \wedge \ldots \wedge c_{2,n}$.

First, assuming (G, c) has a "yes" answer: G is unsatisfiable while the boolean formula $\bigwedge_{i \in [m], j \in [n], (i,j) \neq (1,1)} (c_{1,i} \vee c_{2,j})$ is satisfiable. From (1), we know that both F_1 and F_2 are unsatisfiable. Moreover, there is:

$$\begin{aligned} \bigwedge_{i \in [m], j \in [n], (i,j) \neq (1,1)} (c_{1,i} \vee c_{2,j}) &= \left(c_{1,1} \vee \left(\bigwedge_{j=2}^{n} c_{2,j} \right) \right) \wedge \left(\left(\bigwedge_{i=2}^{m} c_{1,i} \right) \vee c_{2,1} \right) \bigwedge_{\substack{i=2,\ldots,m, \\ j=2,\ldots,n}} (c_{1,i} \vee c_{2,j}) \\ &= (c_{1,1} \vee F_2') \wedge (F_1' \vee c_{2,1}) \wedge (F_1' \vee F_2'). \end{aligned} \tag{2}$$

Since $\bigwedge_{i \in [m], j \in [n], (i,j) \neq (1,1)} (c_{1,i} \vee c_{2,j})$ is satisfiable, if F_1' is unsatisfiable, then from the above (2), we must have $c_{2,1} \wedge F_2' = F_2$ being satisfiable, which gives the contradiction. Similarly, we can also show that F_2' is satisfiable. Therefore, we have shown that both $(F_1, c_{1,1})$ and $(F_2, c_{2,1})$ have "yes" answers.

Now we assume that both $(F_1, c_{1,1})$ and $(F_2, c_{2,1})$ have "yes" answers: F_1 and F_2 are unsatisfiable and F_1' and F_2' are satisfiable. From (1), we know that G is unsatisfiable, and from (2), we know that boolean formula $\bigwedge_{i\in[m],j\in[n],(i,j)\neq(1,1)}(c_{1,i}\vee c_{2,j})$ is satisfiable. Hence (G, c) has a "yes" answer, and the lemma follows. □

From the above 3 lemmas, the D^p-completeness of RMU problem naturally follows.

Theorem 3. *RMU problem is D^p-complete.*

Proof. The above Lemma 1, Lemma 2 and Lemma 3 implies that 3-SAT-UNSAT is reducible to RMU problem. Then this theorem follows from Theorem 1. □

Next, using the standard reduction from *SAT* to *subset sum problem*, we are going to prove the following theorem, which will also play a crucial rule in the next section. Here SAT denotes the *boolean satisfiability problem*.

Theorem 4. *CSS problem is D^p-complete.*

Proof. By Theorem 3, it suffices to establish that RMU is reducible to CSS. Let $(G, c) = (c_0 \wedge c_1 \wedge \ldots \wedge c_{m-1}, c_0)$ be an instance of RMU problem, here boolean formula G has n variables $x_1, \ldots, x_n$, and clause c_j has k_j literals, for $j = 0, \ldots, m-1$. We will define our CSS instance using a very large base B, and will write numbers as $\sum_{i=0}^{m+n-1} a_i B^i$ for some integers $a_i < B$. Here we take $B = 2\max_{j=0,\ldots,m-1} k_j$, this will make sure the addition among our numbers will not cause a carry.

Now we are going to construct the numbers within the instance of CSS problem. Written in base B, the digits $i = 0, \ldots, m-1$ will correspond to the m clauses, and the digits $i = m, \ldots, m+n-1$ will correspond to the n variables. For each $j \in [n]$, we will have two numbers w_j and w_{j+n} corresponding to the variable x_j being set as true or false: we set the $(m+j-1)$-th digit of both w_j and w_{j+n} to be 1, set the i-th digit of w_j to be 1 if and only if literal x_j appears in clause c_i, and set the i-th digit of w_{j+n} to be 1 if and only if literal $\bar{x}_j$ appears in c_i. In other words, the number corresponding to literal x_j is $w_j = B^{m+j-1} + \sum_{i:x_j\in c_i} B^i$, while the number corresponding to literal $\bar{x}_j$ is $w_{j+n} = B^{m+j-1} + \sum_{i:\bar{x}_j\in c_i} B^i$.

Next, we add $k_j - 1$ copies of the number B^j for each clause $c_j, j = 0, \ldots, m-1$. Here we denote $w_{j,\ell} = B^j$ for $j = 0, \ldots, m-1$ and $\ell \in [k_j - 1]$. Lastly, we define the target-sum value $t := \sum_{i=0}^{m-1} k_i B^i + \sum_{i=1}^{n} B^{m+i-1}$. Therefore, we have constructed an instance for CSS:

$$(z_1, \ldots, z_N; t) := (w_i \;\forall i \in [n], w_{i+n} \;\forall i \in [n], w_{j,\ell} \;\forall j = 0, \ldots, m-1 \text{ and } \ell \in [k_j - 1]; t).$$

This instance has $N := 2n + \sum_{j=0}^{m-1}(k_j - 1)$ integer numbers, and has polynomial encoding size with respect to the input size of the boolean formula G. To complete the proof of this theorem, we just need to establish: Instance (G, c) of RMU problem has a "yes" answer if and only if instance $(z_1, \ldots, z_N; t)$ of CSS problem has a "yes" answer.

Claim 1. G is satisfiable if and only if there exists $S \subseteq [N]$ such that $z(S) = t$.

Proof of claim. Suppose we have a satisfying assignment for the boolean formula G. Then, for each variable x_i, if it is set as true in the satisfying assignment, we add number w_i, otherwise we add number w_{n+i}. Since we use exactly one of w_i and w_{n+i}, so the summation will have 1 in the $(m + i - 1)$-th digit, for each $i \in [n]$. Denote the summation as $\sum_{i=0}^{m-1} a_i B^i + \sum_{i=1}^{n} B^{m+i-1}$. Here we have $1 \leq a_i \leq k_i$ for $i = 0, \ldots, m - 1$. $a_i \geq 1$ is because the assignment satisfies the formula G, so at least one of the numbers added has a 1 in the i-th digit. $a_i \leq k_i$ is because clause c_i has exactly k_i literals. Note that $B > k_i$, so there will be no carries. Lastly, to make this sum to exactly t, we add $k_i - a_i$ copies of the number B^i for all $i = 0, \ldots, m - 1$, and we end up with a subset $S \subseteq [N]$ such that $z(S) = t$.

Next, we need to prove the other direction. First of all, easy to observe that for any subset we may add, there will never be a carry in any digit, because here base B is set as a large number. Therefore, in order to get a total summation of t, in every digit, the summation of that digits of every selected numbers should be exactly the corresponding digit of t. Now, consider digit $i = m, \ldots, m+n-1$. This digit in t is 1, and the two numbers that have a 1 in this digit are w_i and w_{i+n}, so in order to sum to t, we must select exactly one of the numbers w_i and w_{i+n}. Within the selected numbers $\{z_i\}_{i \in S}$, for any $i \in [n]$, if w_i is selected, then assign true value to variable x_i, if w_{i+n} is selected, then assign false value to variable x_i. Finally, we want to show that this truth assignment satisfies the formula G. Since $z(S) = t = \sum_{i=0}^{m-1} k_i B^i + \sum_{i=1}^{n} B^{m+i-1}$, and there are at most $k_i - 1$ copies of number B^i in $\{z_i\}_{i \in S}$, we know that under the above truth assignment, for each $i = 0, \ldots, m-1$, the i-th digit of the sum of selected numbers in $\{w_i\}_{i \in [n]} \cup \{w_{i+n}\}_{i \in [n]}$ must be at least 1. By the construction of numbers w_i and w_{i+n}, this is equivalent of saying that all clauses of G are satisfied by the truth assignment. $\diamond$

Denote $G' := c_1 \wedge \ldots \wedge c_{m-1}$, which is the formula obtained from G by deleting clause c_0. Similarly, we have the following claim.

Claim 2. G' is satisfiable if and only if there exists $S \subseteq [N]$ such that $z(S) = t - 1$.

Proof of claim. The proof is analogous to that of the above claim. Here it suffices to notice that: $t - 1 = (k_0 - 1) + \sum_{i=1}^{m-1} k_i B^i + \sum_{i=1}^{n} B^{m+i-1}$, and there are $k_i - 1$ copies of number B^i for all $i = 0, 1, \ldots, m - 1$. So for any selection of numbers in $\{z_i\}_{i \in [N]}$, by adding at most extra $k_0 - 1$ number 1's, the 0-th digit of the summation number can always be the same as the 0-th digit of $t - 1$. The remaining argument is identical to the above claim. $\diamond$

Lastly, notice that by definition, instance (G, c) of RMU problem has a "yes" answer if and only if G is unsatisfiable while G' is satisfiable, and instance $(z_1, \ldots, z_N; t)$ of CSS problem has a "yes" answer if and only if there exists $S \subseteq [N]$ such that $z(S) = t - 1$, but does not exist subset $T \subseteq [N]$ such that $z(T) = t$. We have completed the proof from the above two claims. $\square$

The above theorem has an immediate corollary. *Exact KP* problem asks: given a n-dimensional vector c, a knapsack constraint $a^\mathsf{T} x \leq b$ and an integer L, is it true that $\max\{c^\mathsf{T} x : a^\mathsf{T} x \leq b, x \in \{0,1\}^n\} = L$?

Corollary 1. *Exact knapsack problem is D^p-complete.*

Proof. Let $(w_1, \ldots, w_n; t)$ be an instance of CSS problem. Then this instance has "yes" answer if and only if $\max\{w^\mathsf{T} x : w^\mathsf{T} x \leq t, x \in \{0,1\}^n\} = t - 1$, which is a "yes" answer to a particular exact KP problem. $\square$

We are finally in the position to prove the main theorem of this section.

Theorem 5. *KP supporting hyperplane problem is D^p-complete.*

Proof. By Theorem 4, it suffices to establish that CSS is reducible to KP supporting hyperplane problem. Given an instance $(w_1, \ldots, w_n; t)$ of CSS problem. Now consider the following KP supporting hyperplane problem: given an inequality $\sum_{i=1}^n w_i x_i \leq t - 1$ and a KP $\operatorname{conv}(\{x \in \{0,1\}^n : \sum_{i=1}^n w_i x_i \leq t\})$, is this inequality valid to the KP and the corresponding hyperplane has nonempty intersection with the KP? It is easy to see that, there is a "yes" answer to this KP supporting hyperplane problem if and only if $\sum_{i=1}^n w_i x_i = t$ has no solution over $x \in \{0,1\}^n$, but there exists $x^* \in \{0,1\}^n$ such that $\sum_{i=1}^n w_i x_i^* = t - 1$. This is equivalent of saying the CSS problem $(w_1, \ldots, w_n; t)$ has a "yes" answer. $\square$

3 KP FACET

As an original motivational problem to study the D^p complexity class, Papadimitriou and Yannakakis [16] proposed the TSP FACET problem, which is about recognizing if an inequality defines a facet of the TSP polytope. In this section, we will study the following *KP FACET* problem, which can be seen as a stronger version of the KP supporting hyperplane problem in the last section: given an inequality $\alpha^\mathsf{T} x \leq \beta$ with $\alpha \in \mathbb{Z}^n$ and a KP $\operatorname{conv}\left(\left\{x \in \{0,1\}^n : a^\mathsf{T} x \leq b\right\}\right)$, is this inequality a **facet-defining** inequality to the KP? Specifically, we will show that, same as TSP FACET, here KP FACET is also D^p-complete, which answers a 30 year old conjecture by Hartvigsen and Zemel in [10].

First of all, we describe the elegant example constructed by Gu [8], which is a simply revision from Fibonacci sequence. Let r be a given positive integer, then vector $f \in \mathbb{Z}_+^{2r+1}$ is defined as follows:

$$f_1 = f_2 = f_3 = 1, \quad f_i = f_{i-2} + f_{i-1} \ \forall i = 4, \ldots, 2r + 1. \tag{3}$$

Note that this sequence was also utilized in [2] when constructing a hard instance for sequential lifting of cover inequality. The idea of incorporating the sequence f into the reduction technique that we will use later to prove the main result, is also motivated by the constructive example in [2].

For this particular vector f, we have the following easy observation, which can be verified easily by induction.

Observation 1 *For $j = 3, \ldots, 2r+1$, $f_j = \sum_{i=1}^{j-2} f_i$.*

Observation 2 *For $j = 3, \ldots, 2r+1$, $\frac{\sqrt{2}-1}{4}\sqrt{2}^j \leq f_j \leq 2^j$.*

The next lemma is also important for later proof.

Lemma 4 (Lemma 4.1 [2]). *Let f be defined as in (3) where $r \geq 1$ is given. For any $\tau \in \mathbb{Z}_+$ satisfying $0 \leq \tau \leq \sum_{i=1}^{2r+1} f_i$, there exists a subset $S \subseteq [2r+1]$ such that $f(S) = \tau$.*

We are now ready to present the main result of this paper.

Theorem 6. *KP FACET is D^p-complete.*

Proof. It suffices for us to show that CSS is reducible to KP FACET, since by Theorem 4, CSS problem is D^p-complete. Let $(w_1, \ldots, w_n; t)$ be an instance of CSS problem: is it true that there exists $S \subseteq [n]$ such that $w(S) = t-1$, but there does not exist $T \subseteq [n]$ such that $w(T) = t$? Without loss of generality, here we assume that $w_i \leq t-1$ for any $i \in [n]$.

Next, we are going to construct one instance of KP FACET. Let $L = w([n])$, $r = \lceil \log_2(30L+20) - 1 \rceil$, and:

$$a_i = \begin{cases} tf_i, & \text{for } i = 1, \ldots, 2r+1, \\ t(2L+1)+1, & \text{for } i = 2r+2, \\ w_{i-2r-2}(t+1), & \text{for } i = 2r+3, \ldots, 2r+n+2, \\ tf_{2r+1} + t^2 + t(2L+2) + 1, & \text{for } i = 2r+n+3, \\ t+1, & \text{for } i = 2r+n+4. \end{cases} \tag{4}$$

$$b = t\sum_{i=1}^{2r+1} f_i + t^2 + t(2L+2) + 1, \tag{5}$$

$$\alpha_i = \begin{cases} f_i, & \text{for } i = 1, \ldots, 2r+1, \\ 2L+2, & \text{for } i = 2r+2, \\ w_{i-2r-2}, & \text{for } i = 2r+3, \ldots, 2r+n+2, \\ f_{2r+1} + t + 2L + 1, & \text{for } i = 2r+n+3, \\ 0, & \text{for } i = 2r+n+4. \end{cases} \tag{6}$$

$$\beta = \sum_{i=1}^{2r+1} f_i + t + 2L + 1. \tag{7}$$

Here $N := 2r + n + 4$ is the dimension of the vectors we construct. Then we have the following instance of KP FACET: given an inequality $\alpha^\top x \le \beta$ and a KP $\text{conv}(\{x \in \{0,1\}^N : a^\top x \le b\})$, is this inequality facet-defining to the KP? Easy to verify that, the input size of this instance of KP FACET is polynomial of that of the CSS instance. To complete the proof of this theorem, we are going to show: there is a "yes" answer to the CSS problem $(w_1, \ldots, w_n; t)$ if and only if $\alpha^\top x \le \beta$ is a facet-defining inequality to the KP $\text{conv}(\{x \in \{0,1\}^N : a^\top x \le b\})$.

First of all, we have the following claims.

Claim 3. $\sum_{i=1}^{2r} f_i > 3L + 2$.

Proof of claim. The claim follows from: $\sum_{i=1}^{2r} f_i = f_{2r+2} \ge \frac{\sqrt{2}-1}{4}\sqrt{2}^{2r+2} > 2^{r+1}/10 \ge 2^{\log_2(30L+20)}/10 = 3L + 2$, where the first equality is from Observation 1, the second inequality is from Observation 2 and the last inequality is from definition of r. ⋄

Claim 4. Inequality $\sum_{i=1}^{2r+1} \alpha_i x_i \le \sum_{i=1}^{2r} f_i$ is a facet-defining inequality for KP $\text{conv}(\{x \in \{0,1\}^{2r+1} : \sum_{i=1}^{2r+1} a_i x_i \le \sum_{i=1}^{2r} t f_i\})$.

Proof of claim. By our definition in (4)-(7), here both inequalities $\sum_{i=1}^{2r+1} \alpha_i x_i \le \sum_{i=1}^{2r} f_i$ and $\sum_{i=1}^{2r+1} a_i x_i \le \sum_{i=1}^{2r} t f_i$ are identical to $\sum_{i=1}^{2r+1} f_i x_i \le \sum_{i=1}^{2r} f_i$. We proceed by induction. When $r = 1$, the claim is: $x_1 + x_2 + x_3 \le 2$ is facet-defining for $\text{conv}(\{x \in \{0,1\}^3 : x_1 + x_2 + x_3 \le 2\})$, which is obviously true. Assume that this claim is true when $r = R - 1$ for some $R \ge 2$: $\sum_{i=1}^{2R-1} f_i x_i \le \sum_{i=1}^{2R-2} f_i$ is a facet-defining inequality for $\text{conv}(\{x \in \{0,1\}^{2R-1} : \sum_{i=1}^{2R-1} f_i x_i \le \sum_{i=1}^{2R-2} f_i\})$. So there exists $v_1, \ldots, v_{2R-1}$ affinely independent binary points in $\{0,1\}^{2R-1}$, satisfying $\sum_{i=1}^{2R-1} f_i x_i \le \sum_{i=1}^{2R-2} f_i$ at equality. Denote $(v_1, 0, 1)$ to be the binary point in $\{0,1\}^{2R+1}$ obtained from v_1 by adding two new components with value 0 and 1. Simply we have $(v_2, 0, 1), \ldots, (v_{2R-1}, 0, 1)$. Easy to verify that, for any $j \in [2R-1]$, $(v_j, 0, 1)$ satisfies $\sum_{i=1}^{2R+1} f_i x_i \le \sum_{i=1}^{2R} f_i$ at equality, because $f_{2R+1} = f_{2R} + f_{2R-1}$. Denote $p = (0, \ldots, 0, 1) \in \{0,1\}^{2R+1}$, then $\sum_{i=1}^{2R+1} f_i p_i = \sum_{i=1}^{2R} f_i$. Denote $q = (0, \ldots, 0, 1, 1) \in \{0,1\}^{2R+1}$, then $\sum_{i=1}^{2R+1} f_i q_i = f_{2R} + f_{2R+1} = \sum_{i=1}^{2R} f_i$. Here the last equality is from Observation 1. Therefore, we have obtained the following $2R + 1$ binary points in $\{0,1\}^{2R+1}$: $(v_1, 0, 1), \ldots, (v_{2R-1}, 0, 1), p, q$, where $v_1, \ldots, v_{2R-1}$ are affinely independent in $\{0,1\}^{2R-1}$. Easy to see that these $2R+1$ binary points are affinely independent in $\{0,1\}^{2R+1}$, and satisfy $\sum_{i=1}^{2R+1} f_i x_i \le \sum_{i=1}^{2R} f_i$ at equality. ⋄

Claim 5. Inequality $\sum_{i=1}^{2r+2} \alpha_i x_i \le \sum_{i=1}^{2r} f_i$ is a facet-defining inequality for KP $\text{conv}(\{x \in \{0,1\}^{2r+2} : \sum_{i=1}^{2r+2} a_i x_i \le \sum_{i=1}^{2r} t f_i\})$.

Proof of claim. First, let's verify that $\sum_{i=1}^{2r+2} \alpha_i x_i \le \sum_{i=1}^{2r} f_i$ is valid to such KP. When $x_{2r+2} = 0$, it is trivially valid. When $x_{2r+2} = 1$, knapsack constraint implies $\sum_{i=1}^{2r+1} t f_i x_i \le \sum_{i=1}^{2r} t f_i - t(2L+1) - 1$. So $\sum_{i=1}^{2r+1} f_i x_i \le \sum_{i=1}^{2r} f_i - 2L - 2$, which means $\sum_{i=1}^{2r+1} f_i x_i + (2L+2) x_{2r+2} \le \sum_{i=1}^{2r} f_i$ is a valid inequality. Secondly, from the last Claim 4, it suffices to show there exists a binary point

$x^* \in \{0,1\}^{2r+2}$ with $x^*_{2r+2} = 1$, such that $\sum_{i=1}^{2r+1} a_i x^*_i + a_{2r+2} \leq \sum_{i=1}^{2r} tf_i$ while $\sum_{i=1}^{2r+1} \alpha_i x^*_i + \alpha_{2r+2} = \sum_{i=1}^{2r} f_i$. By definition of a in (4) and α in (6), it suffices to find binary point x^*, such that $\sum_{i=1}^{2r+1} f_i x^*_i = \sum_{i=1}^{2r} f_i - 2L - 2$. From the above Claim 3, $\sum_{i=1}^{2r} f_i - 2L - 2 \geq L$. By Lemma 4, we know such binary point x^* must exist. ⋄

Claim 6. Inequality $\sum_{i=1}^{2r+n+2} \alpha_i x_i \leq \sum_{i=1}^{2r} f_i$ is a facet-defining inequality for KP $\operatorname{conv}(\{x \in \{0,1\}^{2r+n+2} : \sum_{i=1}^{2r+n+2} a_i x_i \leq \sum_{i=1}^{2r} tf_i\})$.

Proof of claim. By definition of a in (4) and α in (6), we are going to prove that, inequality

$$\sum_{i=1}^{2r+1} f_i x_i + (2L+2)\, x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_{i-2r-2} x_i \leq \sum_{i=1}^{2r} f_i \tag{8}$$

is a facet-defining inequality for KP given by knapsack constraint in $\{0,1\}^{2r+n+2}$:

$$\sum_{i=1}^{2r+1} tf_i x_i + (t(2L+1)+1)\, x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} (t+1) w_{i-2r-2} x_i \leq \sum_{i=1}^{2r} tf_i. \tag{9}$$

First of all, let's verify that inequality (8) is indeed valid for the KP defined from constraint (9). For any binary point $x \in \{0,1\}^{2r+n+2}$, knapsack constraint (9) implies that

$$\sum_{i=1}^{2r+1} f_i x_i \leq \sum_{i=1}^{2r} f_i - (2L+1) x_{2r+2} - \sum_{i=2r+3}^{2r+n+2} w_{i-2r-2} x_i - \left\lceil \frac{x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_{i-2r-2} x_i}{t} \right\rceil.$$

Hence

$$\begin{aligned} &\sum_{i=1}^{2r+1} f_i x_i + (2L+2)\, x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_{i-2r-2} x_i \\ &\leq \sum_{i=1}^{2r} f_i + x_{2r+2} - \left\lceil \frac{x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_{i-2r-2} x_i}{t} \right\rceil \leq \sum_{i=1}^{2r} f_i. \end{aligned}$$

To complete the proof, it suffices to show that there exist $2r+2+n$ affinely independent binary points satisfying knapsack constraint (9), on which (8) holds at equality. From the last Claim 5, we can find $2r+2$ affinely independent binary points $v_1, \ldots, v_{2r+2}$ in $\{0,1\}^{2r+2}$ satisfying $\sum_{i=1}^{2r+1} f_i x_i + (2L+2)\, x_{2r+2} = \sum_{i=1}^{2r} f_i$ and $\sum_{i=1}^{2r+1} tf_i x_i + (t(2L+1)+1)\, x_{2r+2} \leq \sum_{i=1}^{2r} tf_i$. Hence $(v_1, 0, \ldots, 0), \ldots, (v_{2r+2}, 0, \ldots, 0)$ are affinely independent in $\{0,1\}^{2r+n+2}$, satisfy knapsack constraint (9), and satisfy (8) at equality. Now, for each $i \in [n]$, consider $\sum_{i=1}^{2r} f_i - 2L - 2 - w_i$. By Claim 3, we know that $\sum_{i=1}^{2r} f_i - 2L - 2 - w_i \geq 0$. So from Lemma 4, we can find x^* with $x^*_{2r+2} = x^*_{2r+2+i} = 1, x^*_{2r+2+j} = 0$ for any $j \in [n] \setminus \{i\}$, and $\sum_{i=1}^{2r+1} f_i x^*_i = \sum_{i=1}^{2r} f_i - 2L - 2 - w_i$. Easy to verify, because $w_i \leq t - 1$, here

x^* satisfies the knapsack constraint (9) and satisfies (8) at equality. Therefore, we have found in total $2r+n+2$ binary points that satisfy knapsack constraint (9), and satisfy (8) at equality. Moreover, these $2r+n+2$ points are obviously affinely independent. $\diamond$

Now, we are ready to prove the validity of the reduction: there is a "yes" answer to the CSS problem $(w_1, \ldots, w_n; t)$ if and only if $\alpha^\mathsf{T} x \leq \beta$ is a facet-defining inequality to the KP $\operatorname{conv}(\{x \in \{0,1\}^N : a^\mathsf{T} x \leq b\})$.

First of all, let's verify that $w(S) \neq t$ for any subset $S \subseteq [n]$ if and only if, inequality $\alpha^\mathsf{T} x \leq \beta$ is valid to the KP defined by $a^\mathsf{T} x \leq b$. In other words, $w(S) \neq t$ for any subset $S \subseteq [n]$ if and only if for any $\bar{x} \in \{0,1\}^N$ with $a^\mathsf{T} \bar{x} \leq b$, there is $\alpha^\mathsf{T} \bar{x} \leq \beta$. We are going to discuss according to the values of $\bar{x}_{N-1}$ and $\bar{x}_N$.

(a) $\bar{x}_{N-1} = 1, \bar{x}_N = 0$. In this case, $a^\mathsf{T} x \leq b$ is the same as $\sum_{i=1}^{2r+n+2} a_i x_i \leq \sum_{i=1}^{2r} t f_i$, and $\alpha^\mathsf{T} x \leq \beta$ is the same as $\sum_{i=1}^{2r+n+2} \alpha_i x_i \leq \sum_{i=1}^{2r} f_i$. From Claim 6, we have $\alpha^\mathsf{T} \bar{x} \leq \beta$ is always satisfied.

(b) $\bar{x}_{N-1} = 1, \bar{x}_N = 1$. From $a^\mathsf{T} \bar{x} \leq b$, we have:

$$\sum_{i=1}^{2r+1} t f_i \bar{x}_i + (t(2L+1)+1)\, \bar{x}_{2r+2} + (t+1) \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r} t f_i - t - 1.$$

Since $\sum_{i=1}^{2r+1} f_i \bar{x}_i \in \mathbb{Z}$, this implies that

$$\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r} f_i - 1 - (2L+1)\bar{x}_{2r+2} - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - \left\lceil \frac{1 + \bar{x}_{2r+2} + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right\rceil.$$

Hence:

$$\begin{aligned} \alpha^\mathsf{T} \bar{x} &= \sum_{i=1}^{2r+1} f_i \bar{x}_i + (2L+2)\bar{x}_{i+2r+2} + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} + f_{2r+1} + t + 2L + 1 \\ &\leq \sum_{i=1}^{2r+1} f_i + \bar{x}_{2r+2} + t + 2L - \left\lceil \frac{1 + \bar{x}_{2r+2} + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right\rceil \\ &\leq \sum_{i=1}^{2r+1} f_i + t + 2L = \beta - 1. \end{aligned} \tag{10}$$

(c) $\bar{x}_{N-1} = 0, \bar{x}_N = 0$. In this case, if $\bar{x}_{2r+2} = 0$, then $\alpha^\mathsf{T} \bar{x} \leq \sum_{i=1}^{2r+1} f_i + \sum_{i=1}^{n} w_i < \beta$. So we assume that $\bar{x}_{2r+2} = 1$. Then from $a^\mathsf{T} \bar{x} \leq b$, we have:

$$\sum_{i=1}^{2r+1} t f_i \bar{x}_i + (t+1) \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r+1} t f_i + t^2 + t.$$

This implies

$$\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t + 1 - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - \left\lceil \frac{\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right\rceil. \quad (11)$$

Now, based on the value of $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}$, we have the following 3 cases. If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq t-1$, then $\alpha^\mathsf{T} \bar{x} \leq \sum_{i=1}^{2r+1} f_i + 2L + 2 + t - 1 = \beta$. If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} = t$, then consider the new point $\hat{x} \in \{0,1\}^N$ with $\hat{x}_i = 1$ for $i = 1, \ldots, 2r+2, \hat{x}_i = \bar{x}_i$ for $i = 2r+3, \ldots, 2r+n+2, \hat{x}_{N-1} = \hat{x}_N = 0$. Here $a^\mathsf{T} \hat{x} = \sum_{i=1}^{2r+1} t f_i + t(2L+1)+1+t(t+1) = b$, and $\alpha^\mathsf{T} \hat{x} = \sum_{i=1}^{2r+1} f_i + 2L+2+t = \beta+1$. So here $\hat{x}$ does not satisfy the inequality $\alpha^\mathsf{T} x \leq \beta$. If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \geq t+1$, then (11) implies that $\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - 1$, so $\alpha^\mathsf{T} \bar{x} = \sum_{i=1}^{2r+1} f_i \bar{x}_i + 2L + 2 + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \beta$. Therefore, $\alpha^\mathsf{T} x \leq \beta$ is valid for any binary point $\bar{x}$ satisfying $a^\mathsf{T} \bar{x} \leq b$ and $\bar{x}_{N-1} = \bar{x}_N = 0$ if and only if, there does not exist subset $S \subseteq [n]$ such that $w(S) = t$.

(d) $\bar{x}_{N-1} = 0, \bar{x}_N = 1$. In this case, if $\bar{x}_{2r+2} = 0$, then $\alpha^\mathsf{T} \bar{x} \leq \sum_{i=1}^{2r+1} f_i + \sum_{i=1}^{n} w_i < \beta$. Assuming $\bar{x}_{2r+2} = 1$. $a^\mathsf{T} \bar{x} \leq b$ is equivalent of saying

$$\sum_{i=1}^{2r+1} t f_i \bar{x}_i + (t+1) \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r+1} t f_i + t^2 - 1.$$

This implies

$$\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - \left\lceil \frac{1 + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right\rceil. \quad (12)$$

If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq t-1$, then $\alpha^\mathsf{T} \bar{x} \leq \sum_{i=1}^{2r+1} f_i + 2L + 2 + t - 1 = \beta$. If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \geq t$, then (12) yields that $\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - 2$. So $\alpha^\mathsf{T} \bar{x} = \sum_{i=1}^{2r+1} f_i \bar{x}_i + 2L+2+\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \beta - 1$.

From the discussion in the above 4 cases, we have known that, for any binary point $\bar{x} \in \{x \in \{0,1\}^N : a^\mathsf{T} x \leq b\}$, if $\bar{x}_{N-1} + \bar{x}_N \geq 1$, then $\alpha^\mathsf{T} \bar{x} \leq \beta$ always holds. If $\bar{x}_{N-1} = \bar{x}_N = 0$, then $\alpha^\mathsf{T} \bar{x} \leq \beta$ if and only if there does not exist subset $S \subseteq [n]$ such that $w(S) = t$. We have thus concluded that, $w(S) \neq t$ for any subset $S \subseteq [n]$ if and only if, inequality $\alpha^\mathsf{T} x \leq \beta$ is valid to $\{x \in \{0,1\}^N : a^\mathsf{T} x \leq b\}$.

Lastly, we want to show that, there exist N affinely independent points in $\{x \in \{0,1\}^N : a^\mathsf{T} x \leq b, \alpha^\mathsf{T} x = \beta\}$ if and only if, there exists subset $S \subseteq [n]$ such that $w(S) = t - 1$.

From Claim 6, there are $v_1, \ldots, v_{N-2} \in \{0,1\}^{N-2}$ that are affinely independent, they satisfy $\sum_{i=1}^{N-2} a_i x_i \leq \sum_{i=1}^{2r} t f_i$ and $\sum_{i=1}^{N-2} \alpha_i x_i = \sum_{i=1}^{2r} f_i$. Hence points $(v_1, 1, 0), \ldots, (v_{N-2}, 1, 0) \in \{0,1\}^N$ are affinely independent, they satisfy

$$\sum_{i=1}^{N} a_i x_i \leq \sum_{i=1}^{2r} t f_i + a_{N-1} = \sum_{i=1}^{2r+1} t f_i + t^2 + t(2L+2) + 1 = b$$

and

$$\sum_{i=1}^{N} \alpha_i x_i = \sum_{i=1}^{2r} f_i + \alpha_{N-1} = \sum_{i=1}^{2r+1} f_i + t + 2L + 1 = \beta.$$

If there exists $x^* \in \{0,1\}^n$ such that $\sum_{i=1}^n w_i x_i^* = t-1$, then we can construct another two points p, q as follows:

$$p_i = \begin{cases} 1, & \text{for } i = 1, \ldots, 2r+1, \\ 1, & \text{for } i = 2r+2, \\ x^*_{i-2r-2}, & \text{for } i = 2r+3, \ldots, 2r+n+2, \\ 0, & \text{for } i = 2r+n+3, \\ 0, & \text{for } i = 2r+n+4. \end{cases} \qquad q = p + e_N. \tag{13}$$

Here e_N is the unit vector in $\mathbb{R}^N$ with the N-th component being 1. Notice that

$$a^\mathsf{T} p \le a^\mathsf{T} q = \sum_{i=1}^{2r+1} t f_i + t(2L+1) + 1 + \sum_{i=1}^{n} (t+1) w_i x_i^* + t + 1 = b,$$

$$\alpha^\mathsf{T} p = \alpha^\mathsf{T} q = \sum_{i=1}^{2r+1} f_i + 2L + 2 + \sum_{i=1}^{n} w_i x_i^* = \sum_{i=1}^{2r+1} f_i + t + 2L + 1 = \beta,$$

so both points p and q satisfy $a^\mathsf{T} x \le b$ and $\alpha^\mathsf{T} x = \beta$. It's easy to check that these N points $(v_1, 1, 0), \ldots, (v_{N-2}, 1, 0), p, q$ are affinely independent points in $\{x \in \{0,1\}^N : a^\mathsf{T} x \le b, \alpha^\mathsf{T} x = \beta\}$.

On the other hand, assume that there exist N affinely independent points in $\{x \in \{0,1\}^N : a^\mathsf{T} x \le b, \alpha^\mathsf{T} x = \beta\}$. Then there must exist a point $p^* \in \{x \in \{0,1\}^N : a^\mathsf{T} x \le b, \alpha^\mathsf{T} x = \beta\}$ with $p^*_N = 1$, since otherwise $\{x \in \{0,1\}^N : a^\mathsf{T} x \le b, \alpha^\mathsf{T} x = \beta\}$ will be contained in the hyperplane given by $x_N = 0$, which violates the assumption before. Furthermore, here $p^*_{N-1} = 0$, because if $p^*_{N-1} = 1$, then point p^* falls into the case (b) above, in which case $\alpha^\mathsf{T} p^* \le \beta - 1$, contradicts to the assumption of $\alpha^\mathsf{T} p^* = \beta$. Hence $p^*_{N-1} = 0, p^*_N = 1$, and p^* falls into the case (d) above. Follow the same argument there, in order for $\alpha^\mathsf{T} p^* = \beta$, we must have $p^*_{2r+2} = 1$ and $\sum_{i=1}^n w_i p^*_{i+2r+2} \le t-1$. If $\sum_{i=1}^n w_i p^*_{i+2r+2} \le t-2$, then $\alpha^\mathsf{T} p^* \le \sum_{i=1}^{2r+1} f_i + 2L + 2 + \sum_{i=1}^n w_i p^*_{i+2r+2} \le \sum_{i=1}^{2r+1} f_i + t + 2L = \beta - 1$. Therefore, $\sum_{i=1}^n w_i p^*_{i+2r+2} = t-1$, which means there exists subset $S \subseteq [n]$ such that $w(S) = t-1$.

All in all, we have established that:

(i) There does not exist $S \subseteq [n]$ such that $w(S) = t$ if and only if, inequality $\alpha^\mathsf{T} x \le \beta$ is valid to the KP defined by $a^\mathsf{T} x \le b$;
(ii) there exist N affinely independent points in $\{x \in \{0,1\}^N : a^\mathsf{T} x \le b, \alpha^\mathsf{T} x = \beta\}$ if and only if, there exists subset $S \subseteq [n]$ such that $w(S) = t-1$.

By definition of facet-defining inequality, we have successfully shown that, there is a "yes" answer to the CSS problem $(w_1, \ldots, w_n; t)$ if and only if $\alpha^\mathsf{T} x \le \beta$ is a facet-defining inequality to the KP $\text{conv}(\{x \in \{0,1\}^N : a^\mathsf{T} x \le b\})$. □

4 KP Membership Problem

Generally speaking, the *membership problem* asks the following question: given an element x and a set S, is x contained in this set? Here element x and set S can be any object. As trivial as this problem might seem to be, in some cases this decision problem can be rather hard to answer. As a selective list of examples in literature, Murty and Kabadi [14] showed that, the membership problem for the copositive cone, that is deciding whether or not a given matrix is in the copositive cone, is a co-$\mathcal{NP}$-complete problem. Dickinson and Gijben [6] also showed that the membership problem for the completely positive cone is $\mathcal{NP}$-hard. Moreover, it is worth mentioning that, for any separation problem: "given point x^*, does there exist an inequality from a given cutting-plane family that is violated by x^*?" it can be essentially seen as a membership problem: "given a point x^* and the cutting-plane closure defined by intersecting all inequalities from the family, is x^* contained in this closure?" In combinatorial optimization, the membership problem therein normally takes the form of: given a point p and a combinatorial polytope, is p contained in this polytope? Here the polytope is defined as the convex hull of integer (or binary) points satisfying certain properties, since a linear description of the polytope will trivialize the membership problem. For any polytope arose from combinatorial optimization that is defined as the convex hull of certain set of binary points, e.g., TSP polytope, clique polytope, matching polytope etc., the corresponding membership problem is obviously in $\mathcal{NP}$, since if point p in this polytope, then by Carathédory's theorem there exist at most $d+1$ binary points that p can be written as the convex combination of, here d is the dimension of the polytope. Papadimitriou and Yannakakis [16] showed that, the membership problem of TSP polytope is in fact $\mathcal{NP}$-complete. The next theorem gives an analogous result for KP. Here recall the well-known *partition problem*: given $(w_1, \ldots, w_n) \in \mathbb{Z}^n_+$, does there exist a subset $S \subseteq [n]$, such that $w(S) = w([n] \setminus S)$?

Theorem 7. *The membership problem of KP is $\mathcal{NP}$-complete.*

Proof. Let $(a_1, \ldots, a_n)$ be an input to an instance of the partition problem, let $x^* := (1/2, \ldots, 1/2)$ and the knapsack constraint given by $a^\mathsf{T} x \leq a([n])/2$. Since partition problem is $\mathcal{NP}$-complete, we only have to show: there exists $S \subseteq [n]$ with $a(S) = a([n] \setminus S)$ if and only if $x^* \in \operatorname{conv}\left(\left\{x \in \{0,1\}^n : a^\mathsf{T} x \leq a([n])/2\right\}\right)$.

If there exists $S \subseteq [n]$ with $a(S) = a([n]\setminus S)$, then $a(S) = a([n]\setminus S) = a([n])/2$, so $\chi^S, \chi^{[n]\setminus S} \in \left\{x \in \{0,1\}^n : a^\mathsf{T} x \leq a([n])/2\right\}$. Hence

$$x^* = 1/2\chi^S + 1/2\chi^{[n]\setminus S} \in \operatorname{conv}\left(\left\{x \in \{0,1\}^n : a^\mathsf{T} x \leq a([n])/2\right\}\right).$$

On the other hand, if $x^* \in \operatorname{conv}\left(\left\{x \in \{0,1\}^n : a^\mathsf{T} x \leq a([n])/2\right\}\right)$, since here $a^\mathsf{T} x^* = a([n])/2$, we know x^* can be written as the convex combination of some points in $\{0,1\}^n$ which all satisfy $a^\mathsf{T} x = a([n])/2$. The support of any one of these binary points will serve as a yes-certificate to the partition problem. □

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