A Branch and Bound Algorithm for Biobjective Mixed Integer Quadratic Programs*

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Abstract. Multiobjective quadratic programs (MOQPs) are appealing since convex quadratic programs have elegant mathematical properties and model important applications. Adding mixed-integer variables extends their applicability while the resulting programs become global optimization problems. We design and implement a branch and bound (BB) algorithm for biobjective mixed-integer quadratic programs (BOMIQPs). In contrast to the existing algorithms in which the Pareto set is approximated, the proposed algorithm provides the exact Pareto set in closed form. The algorithm relies on three fundamental modules of the BB scheme: solving node problems, branching, and fathoming; and a newly developed module of set dominance. Continuous relaxations of the BOMIQP are solved at the BB nodes with the mpLCP method that provides exact efficient solutions to MOQPs. The branching module is extended to be applicable to BOMIQPs and is integrated with the mpLCP method. Selected fathoming rules are implemented in a new way to account for the properties of the Pareto set of the BOMIQP. In the module of set dominance, Pareto sets are compared under incomplete information to yield the resulting nondominated set and eventually produce the Pareto set of the BOMIQP. Numerical results are provided.

Key words. Biobjective mixed integer optimization, Parametric linear complementarity problem, Branching, Fathoming, Set dominance, Polynomial equations.

AMS subject classifications. 90C29, 90C20, 90C11

1. Introduction.

Multiobjective programs (MOPs) with mixed-integer variables have recently been the objects of numerous studies since they model decision-making problems arising in many areas of human activity and their Pareto sets have interesting mathematical properties. Consequently, the development of algorithms for computing these sets has been the main goal of those studies. Multiobjective mixed-integer nonlinear programs make up a class of MOPs for which various algorithms have already been developed. Branch and bound (BB) methods to approximate the Pareto set for such problems are proposed in [8, 9, 27], while the biobjective case is examined in [10]. Some authors go further and assume nonconvexity of the functions on top of variable integrality. In [23], a BB type algorithm is proposed for nonconvex MOPs with continuous or mixed-integer variables, while in [7] the general case of bounded objective functions and disconnected feasible sets is addressed.

In this paper we continue the research direction to compute the Pareto set for the convex quadratic case. Multiobjective quadratic programs (MOQPs) are appealing since they have elegant mathematical properties and model important applications such as regression analysis, portfolio optimization, predictive control, and others. We design and implement a BB algorithm for biobjective mixed-integer quadratic programs (BOMIQPs). In contrast to the studies above, in which the Pareto set is approximated or represented by computing specific points, the proposed algorithm provides the exact Pareto set in closed form. We emphasize

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that there have been so far three classes of MOPs whose exact Pareto set can be computed in closed form, namely, multiobjective linear programs (e.g., [28, 20]) including the case of mixed-integer variables ([3]), and multiobjective quadratic programs (refer to [17] for a review). BOMIQPs now join this "elite" group of MOPs as the current paper appears to be the first study making it possible.

The algorithm relies on three fundamental modules of the BB scheme: solving node problems, branching, and fathoming; and a newly developed module of set dominance. Continuous relaxations of the BOMIQP are solved at the BB nodes with the recently developed mpLCP method, which solves MOQPs as multiparametric linear complementarity problems (mpLCPs) and provides the exact efficient solutions to MOQPs in parametric form [2]. The branching module is extended to be applicable to BOMIQPs and is integrated with the mpLCP method. Selected fathoming rules are implemented in a new way to account for the properties of the Pareto set of the BOMIQP. In the module of set dominance, Pareto sets are compared under incomplete information to yield the resulting nondominated set and eventually produce the Pareto set of the BOMIQP.

The paper consists of eight sections and an appendix. In Section 2, the BOMIQP is formulated and accompanied by related auxiliary biobjective quadratic programs (BOQPs), and the methods for solving these BOQPs are reviewed in the context of the BB algorithm. An overview and the modules of the algorithm are presented in Sections 3-6, while the complete algorithm and numerical results are given in Section 7. The paper is concluded in Section 8. In the appendix, some theoretical results, the details of solving an example BOMIQP with the BB algorithm, and figures in support of the algorithm are included.

2. Preliminaries.

We begin with notation and define nondominated points in an arbitrary set. We then formulate the BOMIQP, define the related concepts and present auxiliary optimization problems and solution methods that are needed for the BB algorithm.

Let $n, p \in \mathbb{N}$ and $0 < p < n$. Let $\mathbb{R}^n, \mathbb{R}^p \subset \mathbb{R}^n$ be Euclidean vector spaces, and $\mathbb{Z}^{n-p} \subset \mathbb{R}^n$ be the set of all integer vectors. For $\mathbf{y}^1, \mathbf{y}^2 \in \mathbb{R}^2$ the following binary relations are defined: $\mathbf{y}^1 \leqq \mathbf{y}^2$ if and only if $y^1_k \le y^2_k$ for $k = 1, 2$; $\mathbf{y}^1 \leq \mathbf{y}^2$ if and only if $y^1_k \le y^2_k$ for $k = 1, 2$ and $\mathbf{y}^1 \neq \mathbf{y}^2$; $\mathbf{y}^1 < \mathbf{y}^2$ if and only if $y^1_k < y^2_k$ for $k = 1, 2$.

Definition 2.1. *Let $S \subset \mathbb{R}^2$. A point $\boldsymbol{y}^1 \in S$ is said to dominate a point $\boldsymbol{y}^2 \in S$ provided $\boldsymbol{y}^1 \leq \boldsymbol{y}^2$. A point $\boldsymbol{y}^1 \in S$ is said to be nondominated provided there does not exist $\boldsymbol{y}^2 \in S$ such that $\boldsymbol{y}^2 \leq \boldsymbol{y}^1$. Let S_N denote the set of nondominated points in S and $N(\cdot)$ denote the operator on a set such that $S_N = N(S)$.*

We define $\mathbb{R}^2_{\geqq} = \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y} \geqq \mathbf{0}\}$ and the sets $\mathbb{R}^2_{\geq}, \mathbb{R}^2_{>}$ are defined accordingly. We also define $\mathbb{R}^2_{\substack{\geqq\\ \leqq}} = \{\mathbf{y} \in \mathbb{R}^2 : y_1 \ge 0, y_2 \le 0\}$, $\mathbb{R}^2_{\substack{\geq\\ \leq}} = \mathbb{R}^2_{\substack{\geqq\\ \leqq}} \setminus \{\mathbf{0}\}$ and $\mathbb{R}^2_{\gtrless} = \{\mathbf{y} \in \mathbb{R}^2 : y_1 > 0, y_2 < 0\}$. For a set $S \subseteq \mathbb{R}^2$ we define $S_{\geqq} = S + \mathbb{R}^2_{\geqq}$. The sets $S_{\geq}$ and $S_{>}$ are defined accordingly. In addition, $bd(S)$ and $|S|$ denote the boundary and cardinality of S, respectively.

Consider the BOMIQP:

$$
\begin{array}{llll}
(2.1) & \mathcal{P}: & \min & \mathbf{f}(\mathbf{x}) = \left[f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q_1 \mathbf{x} + \boldsymbol{p}_1^T \mathbf{x},\ \ f_2(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q_2 \mathbf{x} + \boldsymbol{p}_2^T \mathbf{x}\right] \\
 & & \text{s.t.} & \mathbf{x} \in \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^p \times \mathbb{Z}^{n-p} : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\},
\end{array}
$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^2$, $Q_1, Q_2 \in \mathbb{R}^{n \times n}$, $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$. The spaces $\mathbb{R}^p \times \mathbb{Z}^{n-p}$ and $\mathbb{R}^2$ are referred to as the decision space and the objective space to BOMIQP

(2.1), respectively. Problem $\mathcal{P}$ (2.1), which includes p continuous and $n-p$ integer variables, is referred to as the original problem. We make the following assumptions.

Assumption 1. 1. *The set $\mathcal{X}$ is nonempty and compact.*
2. *The function f_i is strictly convex, i.e., matrix $Q_i, i = 1, 2,$ is positive definite (PD).*
3. *Each integer variable is bounded, i.e., $a_{l_i} \leq x_i \leq a_{u_i}$, for some $a_{l_i}, a_{u_i} \in \mathbb{Z}, i = p+1, \ldots, n$.*

Based on Assumption 1, the following uniqueness property is immediate.

Proposition 2.2. *There exists a unique optimal solution to* $\min\{f_i(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T Q_i \boldsymbol{x} + \boldsymbol{p}_i^T \boldsymbol{x}, \boldsymbol{x} \in \mathcal{X}\}$ *for* $i = 1, 2$.

For BOMIQP (2.1), the outcome (attainable) set, $\mathcal{Y}$, in the objective space is defined as

$$\mathcal{Y} = \mathbf{f}(\mathcal{X}) = \left\{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y} = (f_1(\mathbf{x}), f_2(\mathbf{x})), \mathbf{x} \in \mathcal{X}\right\}.$$

The elements of $\mathcal{Y}$ are referred to as outcome or criterion vectors. Solving BOMIQP (2.1) is defined as finding its efficient solutions in $\mathcal{X}$ and Pareto outcomes in $\mathcal{Y}$.

Definition 2.3. *A feasible solution $\boldsymbol{x}^1 \in \mathcal{X}$ is said to be an efficient solution to BOMIQP* (2.1) *provided there does not exist $\boldsymbol{x}^2 \in \mathcal{X}$ such that $\boldsymbol{f}(\boldsymbol{x}^2) \leq \boldsymbol{f}(\boldsymbol{x}^1)$. The outcome $\boldsymbol{y}^1 = \boldsymbol{f}(\boldsymbol{x}^1)$ is said to be a Pareto (or nondominated) outcome in $\mathcal{Y}$. The sets of efficient solutions and Pareto outcomes to* (2.1) *are denoted by $\mathcal{X}_E$ and $\mathcal{Y}_P$, respectively.*

For (2.1), the notions of Pareto and nondominated outcomes in $\mathcal{Y}$ are used interchangeably because $\mathcal{Y}_P = N(\mathcal{Y})$. For (2.1), we also define the ideal point $\mathbf{y}^I = (y_1^I, y_2^I) \in \mathbb{R}^2$, where $y_k^I := \min_{\mathbf{x}\in\mathcal{X}} f_k(x) = \min_{\mathbf{y}\in\mathcal{Y}} \mathbf{y}_k$ for $k = 1, 2$, and the nadir point $\mathbf{y}^{\mathcal{N}} = (y_1^{\mathcal{N}}, y_2^{\mathcal{N}}) \in \mathbb{R}^2$, where $y_k^{\mathcal{N}} := \max_{\mathbf{x}\in\mathcal{X}_E} f_k(x) = \max_{\mathbf{y}\in\mathcal{Y}_P} \mathbf{y}_k$ for $k = 1, 2$.

In the next section we formulate auxiliary BOQPs for which we define the efficient and Pareto sets maintaining the notation of Definition 2.3. For each BOQP we denote the efficient set and the Pareto set, respectively, by attaching the subscripts E and P to the symbol denoting this problem's feasible set and this problem's image of the feasible set, respectively.

2.1. Auxiliary Biobjective Quadratic Programs.

Given BOMIQP (2.1), four BOQPs are introduced. Each of these four problems has the same two quadratic objective functions but a different feasible set that is a subset or a superset of the set $\mathcal{X}$.

Consider first Problem $\tilde{\mathcal{P}}$ being the continuous relaxation of BOMIQP (2.1),

$$\text{(2.2)} \quad \begin{aligned} \tilde{\mathcal{P}} : \min \quad & \mathbf{f}(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \tilde{\mathcal{X}} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\}, \end{aligned}$$

which is referred to as the relaxed BOMIQP. Let $\tilde{\mathcal{Y}} = \mathbf{f}(\tilde{\mathcal{X}})$ denote the outcome set of $\tilde{\mathcal{P}}$.

The second problem we introduce is the so-called slice problem $\mathcal{P}(\bar{\mathbf{x}})$, which is also a continuous BOQP obtained by fixing all the integer variables at some feasible values. Let $\bar{\mathbf{x}} \in X \cap (\mathbb{R}^p_{\geqq} \times \mathbb{Z}^{n-p}_{\geqq})$, then the slice problem of BOMIQP (2.1) can be written as

$$\text{(2.3)} \quad \begin{aligned} \mathcal{P}(\bar{\mathbf{x}}) : \min \quad & \mathbf{f}(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}(\bar{\mathbf{x}}) = \{\mathbf{x} \in \mathbb{R}^p \times \mathbb{Z}^{n-p} : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}, x_i = \bar{x}_i, i = p+1, \ldots, n\}. \end{aligned}$$

The set $\mathcal{X}(\bar{\mathbf{x}})$ is referred to as a slice of the set $\mathcal{X}$. The slice problem is equivalent to a leaf node problem in the BB tree, because all integer variables have been fixed and no branching is necessary. Let $\mathcal{Y}(\bar{x}) = \mathbf{f}(\mathcal{X}(\bar{\mathbf{x}}))$ denote the outcome set of $\mathcal{P}(\bar{\mathbf{x}})$. Figure 5a [1] depicts the objective space of three slice problems and reveals that (2.1) is a global optimization problem whose Pareto set, $\mathcal{Y}_P$, may be a nonconvex and disconnected curve that is neither open nor closed. This set can be constructed as the nondominated set of the union of the Pareto sets of all the slice problems. Consequently, a naive method to compute $\mathcal{Y}_P$ is to solve all the slice problems and perform this construction. The goal of this work is to design an algorithm to compute this Pareto set in a more efficient way.

The BB tree recursively subdivides the feasible set and creates new BOMIQPs. The third problem, which is the BOMIQP associated with a node s of the BB tree, can be written as

$$\text{(2.4)} \qquad \begin{array}{ll} \mathcal{P}^s : \min & \mathbf{f}(\mathbf{x}) \\ \qquad \text{s.t.} & \mathbf{x} \in \mathcal{X}^s = \{\mathbf{x} \in \mathbb{R}^p \times \mathbb{Z}^{n-p} : A^s\mathbf{x} \leqq \mathbf{b}^s, \mathbf{x} \geqq \mathbf{0}\}. \end{array}$$

and is referred to as the node problem $\mathcal{P}^s$. The matrix A^s and vector b^s are obtained by augmenting the original linear constraints with additional constraints in the form of bounds on integer variables according to a branching rule.

Although $\mathcal{P}^s$ is associated with a node s, the continuous relaxation, $\tilde{\mathcal{P}}^s$, of this problem is the BOQP that is repetitively solved and therefore plays the role of the engine of the BB algorithm. $\tilde{\mathcal{P}}^s$ is the fourth problem we consider and formulate as follows:

$$\text{(2.5)} \qquad \begin{array}{ll} \tilde{\mathcal{P}}^s : \min & \mathbf{f}(\mathbf{x}) \\ \qquad \text{s.t.} & \mathbf{x} \in \tilde{\mathcal{X}}^s = \{\mathbf{x} \in \mathbb{R}^n : A^s\mathbf{x} \leqq \mathbf{b}^s, \mathbf{x} \geqq \mathbf{0}\}. \end{array}$$

$\tilde{\mathcal{P}}^s$ is referred to as the relaxed node problem. Let $\tilde{\mathcal{Y}}^s = \mathbf{f}(\tilde{\mathcal{X}}^s)$, $\tilde{\mathcal{X}}^s_E$ and $\tilde{\mathcal{Y}}^s_P = N(\tilde{\mathcal{Y}}^s_P)$ be the outcome set, efficient set, and Pareto set of $\tilde{\mathcal{P}}^s$, respectively. The set $\tilde{\mathcal{Y}}^s$ is $\mathbb{R}^2_{\geq}$-convex, i.e., the set $\tilde{\mathcal{Y}}^s + \mathbb{R}^2_{\geq}$ is convex [17]. The set, $\tilde{\mathcal{Y}}^s_P$, is a continuous strictly convex curve with the end points $\tilde{\mathbf{y}}^{s1} = \big(\tilde{y}^{s1}_1 = f_1(\tilde{\mathbf{x}}^{s1}),\ \tilde{y}^{s1}_2 = f_2(\tilde{\mathbf{x}}^{s1})\big)$ and $\tilde{\mathbf{y}}^{s2} = \big(\tilde{y}^{s2}_1 = f_1(\tilde{\mathbf{x}}^{s2}), \tilde{y}^{s2}_2 = f_2(\tilde{\mathbf{x}}^{s2})\big)$, where $\tilde{\mathbf{x}}^{si} = argmin_{\mathbf{x}\in\tilde{\mathcal{X}}^s} f_i(\mathbf{x})$ for $i = 1, 2$ (see Figure 5b). Let $\tilde{\mathbf{y}}^{sI}$ and $\tilde{\mathbf{y}}^{s\mathcal{N}}$ be the ideal and nadir points of the relaxed node s problem, respectively, where $\tilde{\mathbf{y}}^{sI} = (\tilde{y}^{sI}_1 = \tilde{y}^{s1}_1, \tilde{y}^{sI}_2 = \tilde{y}^{s2}_2), \tilde{\mathbf{y}}^{s\mathcal{N}} = (\tilde{y}^{s\mathcal{N}}_1 = \tilde{y}^{s2}_1, \tilde{y}^{s\mathcal{N}}_2 = \tilde{y}^{s1}_2)$.

For $\tilde{\mathcal{P}}^s$ we define additional sets in the objective space $\mathbb{R}^2$. The set

$$\text{(2.6)} \qquad T^s = \{\lambda_1\tilde{\mathbf{y}}^{s1} + \lambda_2\tilde{\mathbf{y}}^{s2} + \lambda_3\tilde{\mathbf{y}}^{sI} : \sum_{i=1}^{3}\lambda_i = 1, \lambda_i \geq 0 \text{ for all } i = 1, 2, 3\}$$

creates a closed triangle in $\mathbb{R}^2$. Two subsets in $\mathbb{R}^2$ for $\kappa = s, l$ are also defined

$$\text{(2.7)} \qquad \begin{aligned} C^{\kappa W} &= (\{\tilde{\mathbf{y}}^{\kappa 1}\}^{<} \setminus \{\tilde{\mathbf{y}}^{\kappa I}\}^{\leqq}) \cup L(\tilde{\mathbf{y}}^{\kappa 1}, \tilde{\mathbf{y}}^{\kappa I}) \\ C^{\kappa S} &= (\{\tilde{\mathbf{y}}^{\kappa 2}\}^{<} \setminus \{\tilde{\mathbf{y}}^{\kappa I}\}^{\leqq}) \cup L(\tilde{\mathbf{y}}^{\kappa 2}, \tilde{\mathbf{y}}^{\kappa I}), \end{aligned}$$

where $L(\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{sI})$ and $L(\tilde{\mathbf{y}}^{s2}, \tilde{\mathbf{y}}^{sI})$ are open line segments joining the ideal point with each end point of the Pareto set (see Figure 5b).

2.2. Solving Biobjective Quadratic Programs. The traditional approach to solve MOPs is by scalarization in which the problem is reformulated into a single objective program whose optimal solution is efficient to the original problem. In the BB algorithm we employ three well-established scalarizations to solve problems $\mathcal{P}$ and $\tilde{\mathcal{P}}^s$. This section can safely be skipped by the reader who is familiar with scalarization techniques in multiobjective optimization.

[1] Figures 3-28 are included in the Appendix.

2.2.1. The Weighted-Sum Problem associated with BOMIQP(2.1). Consider the following weighted-sum problem associated with BOMIQP(2.1).

$$\mathcal{P}(\lambda): \quad \min \quad \tfrac{1}{2}\mathbf{x}^T Q(\lambda)\mathbf{x} + \boldsymbol{p}(\lambda)^T\mathbf{x} \qquad \text{s.t.} \quad \mathbf{x} \in \mathcal{X}, \tag{2.8}$$

where $\lambda \in [0,1]$, $Q(\lambda) = \lambda Q_i + (1-\lambda)Q_j$ and $\boldsymbol{p}(\lambda) = \lambda \boldsymbol{p}_i + (1-\lambda)\boldsymbol{p}_j, i, j = 1, 2, i \neq j$.

Proposition 2.4. *[14] Let $\lambda \in [0,1]$. If $\check{\boldsymbol{x}} = \check{\boldsymbol{x}}(\lambda)$ is an optimal solution to problem $\mathcal{P}(\lambda)$ (2.8), then $\check{\boldsymbol{x}}$ is efficient to BOMIQP(2.1).*

In the BB algorithm, problem (2.8) is solved for a fixed set of values of the parameter λ to obtain an initial set of Pareto points of BOMIQP(2.1). For a fixed weight, problem (2.8) is a single objective mixed integer quadratic program (SOMIQP) and commercial software such as GUROBI can solve it. We emphasize that these SOMIQPs are solved only at the initialization of the BB algorithm.

2.2.2. The Weighted-Sum Problem associated with BOQP(2.5). In the algorithm, the relaxed weighted-sum problem, $\tilde{\mathcal{P}}^s(\lambda)$, which is associated with problem (2.5), is solved

$$\tilde{\mathcal{P}}^s(\lambda): \min \quad \frac{1}{2}\mathbf{x}^T Q(\lambda)\mathbf{x} + \boldsymbol{p}(\lambda)^T\mathbf{x} \qquad \text{s.t.} \quad \mathbf{x} \in \tilde{\mathcal{X}}^s. \tag{2.9}$$

Analogously to Proposition 2.4, an optimal solution to problem $\tilde{\mathcal{P}}^s(\lambda)$ (2.9) is efficient to $\tilde{\mathcal{P}}^s$ (2.5). Problem (2.9) is solved in two different ways: for a fixed parameter $\lambda \in [0,1]$ and as a parametric optimization problem. In the parametric case, an optimal solution to (2.9) is a function of λ. Despite being a convex problem, (2.9) is challenging because it has a parameter in the quadratic term of the objective function. Methods to solve (2.9) for its optimal parametric solutions include the spLCP method [30], the mpLCP method [2] with a *MATLAB* implementation available [1], and the algorithms in [21]. All these methods rely on the well-known reformulation of the KKT optimality conditions for quadratic programs into an LCP problem [4], which converts (2.9) into a single-parametric linear complementarity problem (spLCP). The parameter space $[0,1]$ is partitioned into invariancy intervals over which the solutions to the spLCP are computed providing also an optimal solution to (2.9). Based on the comparisons in [17, 18], the mpLCP method emerges as the most universal method to solve MOQPs and is used as a solver for (2.9) in the BB algorithm.

2.2.3. The Achievement Function Problem associated with BOQP (2.5). An achievement function can be used to scalarize BOQP (2.5) [31]. Consider the following problem that is formulated for (2.5) with a reference point $\mathbf{y}^R = (y_1^R, y_2^R) \in \mathbb{R}^2$:

$$\tilde{\mathcal{P}}(\mathbf{y}^R): \quad \min_{\mathbf{x}} \max_{k=1,2} \quad \{f_k(\mathbf{x}) - y_k^R\} \qquad \text{s.t.} \quad \mathbf{x} \in \tilde{\mathcal{X}}^s. \tag{2.10}$$

Theorem 2.5. *[31] Let $\hat{\boldsymbol{x}} \in \tilde{\mathcal{X}}^s$. If $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{y}^R)$ is an optimal solution to problem $\tilde{\mathcal{P}}(\boldsymbol{y}^R)$ (2.10), then $\hat{\boldsymbol{x}}$ is efficient to $\tilde{\mathcal{P}}^s$ (2.5).*

$\tilde{\mathcal{P}}(\mathbf{y}^R)$, known as the achievement function problem, is used in the fathoming and set dominance modules of the BB algorithm to check the location of a reference point $\mathbf{y}^R$ with respect to the Pareto set of a relaxed node problem. $\tilde{\mathcal{P}}(\mathbf{y}^R)$ is a SOQP and commercial software such as *MATLAB* can solve it. The following result is immediate based on the strict convexity assumption and Theorem 2.5.

Theorem 2.6. *Let $\boldsymbol{y}^R \in \mathbb{R}^2$. Let $\hat{\boldsymbol{x}}^s(\boldsymbol{y}^R)$ be an optimal solution to (2.10) and $\hat{\boldsymbol{y}}^s(\boldsymbol{y}^R) = \big(f_1(\hat{\boldsymbol{x}}^s(\boldsymbol{y}^R)), f_2(\hat{\boldsymbol{x}}^s(\boldsymbol{y}^R))\big) = (\hat{y}_1^s, \hat{y}_2^s)$, and let $\tilde{\mathcal{Y}}_P^s$ be the Pareto set of problem (2.5).*

(i) If $\hat{\boldsymbol{y}}^s(\boldsymbol{y}^R) \geq \boldsymbol{y}^R$ then $\boldsymbol{y}^R \in \tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{>}$

(ii) If $\hat{\boldsymbol{y}}^s(\boldsymbol{y}^R) \leqq \boldsymbol{y}^R$ then $\boldsymbol{y}^R \in \tilde{\mathcal{Y}}_P^s + \mathbb{R}^2_{\geqq}$

(iii) If $\hat{y}_1^s < y_1^R$ and $\hat{y}_2^s > y_1^R$ then $\boldsymbol{y}^R \in \tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{\gtrless}$

(iv) If $\hat{y}_1^s > y_1^R$ and $\hat{y}_2^s < y_1^R$ then $\boldsymbol{y}^R \in \tilde{\mathcal{Y}}_P^s + \mathbb{R}^2_{\gtrless}$.

2.2.4. The ϵ-Constraint Problem associated with BOQP (2.5). Another commonly used scalarization technique to solve MOQPs is the $\boldsymbol{\epsilon}$-constraint problem [15], which we apply to BOQP (2.5). The i^{th} $\boldsymbol{\epsilon}$-constraint problem, $i = 1, 2$, is formulated as:

$$\begin{aligned} \tilde{\mathcal{P}}(\epsilon_j): \quad & \min \quad f_i(\mathbf{x}) \\ & \text{s.t.} \quad f_j(\mathbf{x}) \leq \epsilon_j \quad j = \{1,2\},\ j \neq i \\ & \qquad\ \mathbf{x} \in \tilde{\mathcal{X}}^s. \end{aligned} \tag{2.11}$$

Theorem 2.7. *[15] Let $\epsilon_j \in \mathbb{R}$ be fixed and $\hat{\boldsymbol{x}} \in \tilde{\mathcal{X}}^s(\epsilon_j) = \{\boldsymbol{x} \in \tilde{\mathcal{X}}^s : f_j(\boldsymbol{x}) \leq \epsilon_j, j = \{1,2\},\ j \neq i\} \neq \emptyset$. If $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\epsilon_j)$ is an optimal solution to problem $\tilde{\mathcal{P}}(\epsilon_j)$ (2.11), then $\hat{\boldsymbol{x}}$ is efficient to problem $\tilde{\mathcal{P}}^s$ (2.5).*

Let $\hat{\mathbf{y}}(\epsilon_j)$ denote the image of $\hat{\mathbf{x}}(\epsilon_j)$ in the objective space of BOQP (2.5). If at optimality of $\tilde{\mathcal{P}}(\epsilon_j)$ (2.11) the ϵ-constraint is active, i.e., $f_j(\hat{\mathbf{x}}(\epsilon_j)) = \epsilon_j$, then $\hat{\mathbf{y}}(\epsilon_j) = (f_1(\hat{\mathbf{x}}(\epsilon_j)), f_2(\hat{\mathbf{x}}(\epsilon_j)))$.

$\tilde{\mathcal{P}}(\epsilon_j)$ (2.11) is used in the set dominance module to discard the points of a Pareto set that are dominated by another Pareto set. The parameter ϵ is selected as a coordinate of an end point of one of the two Pareto sets. For $i, j = \{1,2\}$, $i \neq j$ and $\nu, \omega = \{s, l\}, \nu \neq \omega$, we formulate a single objective quadratically constrained quadratic program

$$\begin{aligned} \tilde{\mathcal{P}}(\tilde{y}_j^{\nu j}): \quad & \min \quad f_i(\mathbf{x}) \\ & \text{s.t.} \quad f_j(\mathbf{x}) \leq \tilde{y}_j^{\nu j} \quad j = \{1,2\},\ j \neq i \\ & \qquad\ \mathbf{x} \in \tilde{\mathcal{X}}^\omega, \end{aligned} \tag{2.12}$$

where $\tilde{y}_j^{\nu j}$ is the j^{th} coordinate of the j^{th} end point of the ν^{th} node problem. Commercial software such as *MATLAB* can solve problem $\tilde{\mathcal{P}}(\tilde{y}_j^{\nu j})$. Let $\hat{\mathbf{x}}(\tilde{y}_j^{\nu j})$ denote an optimal solution to (2.12) and $\hat{\mathbf{y}}(\tilde{y}_j^{\nu j})$ denote its image in the objective space of BOQP (2.5). The following proposition shows how to retrieve the weight λ to be used in problem $\tilde{\mathcal{P}}^s(\lambda)$ (2.9) and obtain the same efficient solution that is obtained by solving $\tilde{\mathcal{P}}(\tilde{y}_j^{\nu j})$(2.12).

Proposition 2.8. *[19] Let $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\tilde{y}_j^{\nu j})$ be an optimal solution to $\tilde{\mathcal{P}}(\tilde{y}_j^{\nu j})$(2.12) and $\hat{u} = \hat{u}(\tilde{y}_j^{\nu j}) > 0$ be the Lagrange multiplier associated with the constraint $f_j(\boldsymbol{x}) \leq \tilde{y}_j^{\nu j}$. Then $\hat{\boldsymbol{x}}$ is also an optimal solution to problem $\tilde{\mathcal{P}}^s(\lambda)$ (2.9) for $\lambda = \frac{\hat{u}}{\hat{u}+1}$.*

Having prepared the foundations for the development of the BB algorithm, in the next section we give its overview and then focus on its modules.

3. Algorithm Overview.

The following definitions and notations are used to develop the BB algorithm for BOMIQP (2.1).

1. Let $\mathcal{Y}_a \subset \mathcal{Y}$ be the set of points and curves found so far by the BB algorithm as candidates to be the elements in the Pareto set, $\mathcal{Y}_P$, for (2.1). $\mathcal{Y}_a$ is referred to as the incumbent set.
2. Let $\mathcal{X}_a \subset \mathcal{X}$ be the set of preimages of the points in $\mathcal{Y}_a$.
3. Let $\mathcal{Y}_P^0 \subset \mathcal{Y}_P$ denote an initial subset of $\mathcal{Y}_P$ and $\mathcal{X}_a^0 \subset \mathcal{X}_E$ be the set of preimages of the points in $\mathcal{Y}_P^0$.

Items 4, 5, and 6 below pertain to points $\mathbf{y}^i = (y_1^i, y_2^i) \in \mathcal{Y}_a$ and $\mathbf{y}^j = (y_1^j, y_2^j) \in \mathcal{Y}_a$ such that $y_1^i < y_1^j$ and $y_2^i > y_2^j$.

4. The point $\mathbf{y}^{i,j} = (y_1^j, y_2^i)$ is said to be the nadir point and the point $\check{\mathbf{y}}^{i,j} = (y_1^i, y_2^j)$ is said to be the ideal point implied by the points $\mathbf{y}^i, \mathbf{y}^j \in \mathcal{Y}_a$. The set of all nadir points generated by selected points in $\mathcal{Y}_a$ is denoted by $\mathcal{Y}^{\mathcal{N}}$.
5. Points $\mathbf{y}^i, \mathbf{y}^j \in \mathcal{Y}_a$ are said to be adjacent in $\mathcal{Y}_a$ if their nadir point $\mathbf{y}^{i,j}$ and ideal point $\check{\mathbf{y}}^{i,j}$ satisfy $\left(\{\mathbf{y}^{i,j}\}^{<}\right) \cap \left(\{\check{\mathbf{y}}^{i,j}\}^{>}\right) \cap \mathcal{Y}_a = \emptyset$.
6. A nadir point $\mathbf{y}^{i,j} \in \mathcal{Y}^{\mathcal{N}}$ is said to be adjacent if it is implied by two adjacent points $\mathbf{y}^i, \mathbf{y}^j \in \mathcal{Y}_a$.
7. Let $\mathcal{Y}_a^s \subset \mathcal{Y}_a$ be defined as $\mathcal{Y}_a^s = \mathcal{Y}_a \cap (T^s - \mathbb{R}^2_{\geqq})$.
8. Let $\mathbf{y}^{i,j} \in \mathcal{Y}^{\mathcal{N}}$. The set R^s is defined as the rectangle spanned between $\tilde{\mathbf{y}}^{sI}$ and $\mathbf{y}^{i,j}$, $R^s = \{\tilde{\mathbf{y}}^{sI}\}^{>} \cap \{\mathbf{y}^{i,j}\}^{\leqq}$.

The algorithm consists of the initialization and the main step. The following information is available after the initialization step has been completed: (i) An initial subset of Pareto points, $\mathcal{Y}_P^0$, that are computed by solving the weighted sum problem (2.8) with a set of predetermined weights. Then $\mathcal{Y}_a = \mathcal{Y}_P^0$ at the initialization, and $\mathcal{Y}_P^0 \subset \mathcal{Y}_a \cap \mathcal{Y}_P$ during the execution of the algorithm. (ii) The set of nadir points implied by the adjacent Pareto points in $\mathcal{Y}_P^0$.

At every main step of the algorithm the relaxed BOQP (2.5) associated with a node s is solved for its efficient set, $\tilde{\mathcal{X}}_E^s = \{\mathbf{x}^s(\lambda) \in \mathbb{R}^n : \lambda \in [\lambda', \lambda'']\}_{0 \leq \lambda', \lambda'' \leq 1}$, which is as a collection of parametric efficient solution functions with the associated invariancy intervals. The Pareto set $\tilde{\mathcal{Y}}_P^s = \mathbf{f}(\tilde{\mathcal{X}}_E^s)$ (or its subset $\mathbf{y}^s(\lambda) = \mathbf{f}(\mathbf{x}^s(\lambda))$), is a strictly convex curve (or a strictly convex subcurve) that is available parametrically in the form $(f_1(\mathbf{x}(\lambda)), f_2(\mathbf{x}(\lambda)))$ for $\lambda \in [0, 1]$ (or for $\lambda \in [\lambda', \lambda'']$). Both sets, $\tilde{\mathcal{X}}_E^s$ and $\tilde{\mathcal{Y}}_P^s$, are stored. The coordinates of specific points in $\tilde{\mathcal{Y}}_P^s$: (i) the end points $\tilde{\mathbf{y}}^{s1} = \mathbf{f}(\mathbf{x}^s(1))$ and $\tilde{\mathbf{y}}^{s2} = \mathbf{f}(\mathbf{x}^s(0))$; (ii) the points $\tilde{\mathbf{y}}^s(\lambda') = \mathbf{f}(\mathbf{x}^s(\lambda'))$ and $\tilde{\mathbf{y}}^s(\lambda'') = \mathbf{f}(\mathbf{x}^s(\lambda''))$ associated with the end points of the invariancy intervals $[\lambda', \lambda'']$ for $0 < \lambda', \lambda'' < 1$, are actively used during the execution of the algorithm.

The algorithm proceeds differently depending on the properties of $\tilde{\mathcal{X}}_E^s$ and $\tilde{\mathcal{Y}}_P^s$. If the entire set $\tilde{\mathcal{X}}_E^s$ is feasible for (2.1), i.e., $\tilde{\mathcal{X}}_E^s \subset \mathcal{X}_E$, then its image, $\tilde{\mathcal{Y}}_P^s$, is added to $\mathcal{Y}_a$ and the resulting nondominated set is computed , i.e., $\mathcal{Y}_a = N(\mathcal{Y}_a \cup \tilde{\mathcal{Y}}_P^s)$. This latter step is performed in the set dominance module. Additionally, if $\tilde{\mathcal{Y}}_P^s$ satisfies certain conditions executed in the fathoming module, then node s is fathomed. The set $\mathcal{Y}^{\mathcal{N}}$ of all nadir points generated by selected points in $\mathcal{Y}_a$ is used in this module.

In a similar way, if an efficient solution function $\mathbf{x}^s(\lambda) \in \tilde{\mathcal{X}}_E^s$ for some $\lambda \in [\lambda', \lambda'']$ is feasible for (2.1), i.e., $\mathbf{x}^s(\lambda) \in \mathcal{X}_E$ for $\lambda \in [\lambda', \lambda'']$, then its image, $\mathbf{y}^s(\lambda) = \mathbf{f}(\mathbf{x}^s(\lambda))$ for $\lambda \in [\lambda', \lambda'']$, is added to $\mathcal{Y}_a$ and the resulting nondominated set is computed in the set dominance module.

If there exists an interval $[\lambda', \lambda'']$ in the collection such that the corresponding parametric solution is not feasible for (2.1), then the algorithm initiates the branching module.

The incumbent set $\mathcal{Y}_a$ is a union of points and strictly convex curves. The objective space images of the newly obtained efficient solutions to relaxed node problems that are feasible for (2.1) are added to $\mathcal{Y}_a$ while this set remains nondominated. Because of the nondominance test, an interval $[\lambda', \lambda'']$ associated with the subcurve $\mathbf{y}(\lambda)$ for $\lambda \in [\lambda', \lambda'']$ may be partitioned into subintervals such that a subcurve $\mathbf{y}(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R]$ is nondominated in $\mathcal{Y}_a$, where λ^L and λ^R are the parameter values associated with the end points of the subcurve that passed the test. Consequently, we have $\mathcal{Y}_a = \mathcal{Y}_P^0 \cup \{\mathbf{y}(\lambda) \in \mathbb{R}^2 : \lambda \in [\lambda^L, \lambda^R]\}_{0 \leq \lambda^L, \lambda^R \leq 1}$ and $\mathcal{Y}_a = N(\mathcal{Y}_a)$. As the algorithm progresses, $\mathcal{Y}_a$ keeps changing. As new curves or points are added, some curves, subcurves, or points that have been elements of $\mathcal{Y}_a$ so far may be dropped.

In the fathoming module, a subset $\bar{\mathcal{Y}}_a \subseteq \mathcal{Y}_a$ containing only specific points in $\mathcal{Y}_a$ is used because the information about only these points is sufficient to run this module. We define

$$\bar{\mathcal{Y}}_a = \mathcal{Y}_P^0 \cup \{\mathbf{y}(\lambda) \in \mathbb{R}^2 : \lambda = \lambda^L, \lambda^R\}_{0 \leq \lambda^L, \lambda^R \leq 1} \text{ and } \bar{\mathcal{Y}}_a = N(\bar{\mathcal{Y}}_a). \tag{3.1}$$

In the subsequent sections, the three modules of the algorithm are presented in detail.

4. Branching. In the branching module, at the parent node s, the mpLCP method solves BOQP (2.5) for the efficient set, $\tilde{\mathcal{X}}_E^s$. If this set is not feasible to BOMIQP (2.1), i.e., there exists an interval $[\lambda', \lambda'']$ and the corresponding parametric solution with an integer variable $x_i(\lambda)$, for some $i \in \{p+1, \ldots, n\}$, that is equal to a function of λ for $\lambda \in [\lambda', \lambda'']$, then this variable is selected for branching. The range of values this variable takes for $\lambda \in [\lambda', \lambda'']$ is then computed. Let $\bar{x}_i^{min}$ and $\bar{x}_i^{max}$ be the smallest and largest value the variable $x_i(\lambda)$ respectively assumes for $\lambda \in [\lambda', \lambda'']$. To obtain these bounds we solve $\bar{x}_i^{min} = \min_\lambda\{x_i(\lambda) : \lambda \in [\lambda', \lambda'']\}$ and $\bar{x}_i^{max} = \max_\lambda\{x_i(\lambda) : \lambda \in [\lambda', \lambda'']\}$. Since $x_i(\lambda)$ are rational functions of λ, a specialized solution method for polynomial fractional optimization should be used [11]. However, because adopting that method into the BB algorithm requires further research, in this first implementation, these optimization problems are solved by discretization. The interval $[\lambda', \lambda'']$ is discretized, the values of $x_i(\lambda)$ are computed for each value of λ, and the minimum and the maximum values, $\bar{x}_i^{min}$ and $\bar{x}_i^{max}$, are found.

Let $\phi_i' = \lfloor \bar{x}_i^{min} \rfloor$ and $\phi_i'' = \lceil \bar{x}_i^{max} \rceil$. Then the constraints of problem $\mathcal{P}^s(\lambda)$(2.9) are extended with bounds on the integer variable x_i, and two new node problems, $\mathcal{P}^{s+1}(\lambda)$ and $\mathcal{P}^{s+2}(\lambda)$, are created and solved for each integer $\phi_i \in [\phi_i', \phi_i'']$.

$$\begin{aligned} \mathcal{P}^{s+1}(\lambda) : \min \quad & \frac{1}{2}\mathbf{x}^T Q(\lambda)\mathbf{x} + \boldsymbol{p}(\lambda)^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \tilde{\mathcal{X}}^s = \{\mathbf{x} \in \mathbb{R}^n : A^s\mathbf{x} \leqq \mathbf{b}^s, x_i \leq \phi_i, \mathbf{x} \geqq \mathbf{0}\} \\ & \lambda \in [0,1], \end{aligned} \tag{4.1}$$

$$\begin{aligned} \mathcal{P}^{s+2}(\lambda) : \min \quad & \frac{1}{2}\mathbf{x}^T Q(\lambda)\mathbf{x} + \boldsymbol{p}(\lambda)^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \tilde{\mathcal{X}}^s = \{\mathbf{x} \in \mathbb{R}^n : A^s\mathbf{x} \leqq \mathbf{b}^s, x_i \geq \phi_i + 1, \mathbf{x} \geqq \mathbf{0}\} \\ & \lambda \in [0,1]. \end{aligned} \tag{4.2}$$

The first new node problem has the constraint $x_i \leq \phi_i'$, while the last new node problem has the constraint $x_i \geq \phi_i''$. Using these bounds on the integer variables that have not assumed integer values we ensure that no efficient solution to BOMIQP (2.1) is excluded.

This procedure is applied to every invariancy interval in the collection that carries the infeasibility to the original BOMIQP. Once all such invariancy intervals associated with a node problem have been processed, the efficient set of another node problem is checked for feasibility and if needed, the above process is repeated. Note that a newly generated node problem may be identical to a previously obtained node problem. If so, such a new node problems is discarded. The branching process is started at the root node with problem (2.2) and may continue until leaf nodes have been reached with slice problem (2.3).

5. Fathoming.

The goal of the fathoming module is to decide whether a node problem can be discarded. A fathoming rule gives a condition for discarding a subset of the feasible set that contains no efficient points to (2.1). Until the termination of the BB algorithm, one may not know whether an efficient solution to a node problem, which is feasible to (2.1), is also efficient to (2.1). Regardless of whether that solution is efficient or not, one can still use the feasibility of that solution for dominance purposes.

Fathoming rules make use of the feasibility or infeasibility of a node problem and of dominance between bound sets. The first fathoming rule uses infeasibility. If node problem (2.5) is infeasible, then the corresponding mixed-integer problem (2.4) has no efficient solutions, i.e., if $\tilde{\mathcal{X}}^s = \emptyset$ then $\mathcal{X}^s = \emptyset$. The second fathoming rule is based on the slice problem. If the node problem is a slice problem, then this node problem can be fathomed. If these two rules are not applicable, the fathoming rules based on bound sets become relevant.

5.1. Bound Sets.

In the biobjective case, the bound sets are subsets of the objective space $\mathbb{R}^2$ and determine a region within which the Pareto set to BOMIQP (2.1) is located. In the literature, different bound sets have been proposed and generally presented as pairs (UB, LB^s) where $UB \subseteq \mathbb{R}^2$ is the upper bound set and $LB^s \subseteq \mathbb{R}^2$ is the lower bound sets at node s [13, 5, 26]. Having examined the bound sets introduced in the literature, we select the pair in which the upper bound set is first proposed in [6], while the pair is also used in [5, 8, 26, 3].

$$UB = \mathcal{Y}^{\geqq} \text{ and } LB^s = \tilde{\mathcal{Y}}_P^{s\geqq} \text{ where } \tilde{\mathcal{Y}}_P^{s\geqq} = \tilde{\mathcal{Y}}_P^{s} + \mathbb{R}^2_{\geqq}. \tag{5.1}$$

A general sufficient condition for a node s to be fathomed is that that node's lower bound set does not contain the Pareto points of BOMIQP (2.1) [5]. In our work, using the sets in (5.1) we apply the equivalent condition [3]:

$$\text{If } LB^s \subset UB, \text{then node } s \text{ can be fathomed.} \tag{5.2}$$

In the next section, we discuss how to make a fathoming decision based on bound sets (5.1). Since obtaining the lower bound set using $\tilde{\mathcal{Y}}_P^s$ is a challenging task, we intend to make a fathoming decision without obtaining the complete $\tilde{\mathcal{Y}}_P^s$.

5.2. Practical Fathoming Rules.

We develop fathoming rules based on the condition in (5.2), which, in the context of the BOMIQP, can be written as $\tilde{\mathcal{Y}}_P^{s\geqq} \subset \mathcal{Y}_a^{\geqq}$, or equivalently, $\tilde{\mathcal{Y}}_P^{s\geqq} \subset \mathcal{Y}_a^{s\geqq}$ meaning that each point in $\tilde{\mathcal{Y}}_P^{s}$ is dominated by at least one point in $\mathcal{Y}_a^s$ [3]. Recall that $\bar{\mathcal{Y}}_a \subseteq \mathcal{Y}_a$ is a set of points containing the initial Pareto points in $\mathcal{Y}_P^0$ and the end points of the strictly convex curves and subcurves stored in $\mathcal{Y}_a$. Given $\bar{\mathcal{Y}}_a$, we define the set $\bar{\mathcal{Y}}_a^s \subseteq \mathcal{Y}_a^s$:

$$\bar{\mathcal{Y}}_a^s = \bar{\mathcal{Y}}_a \cap (T^s - \mathbb{R}^2_{\geqq}) \tag{5.3}$$

and use it for fathoming. Let $\mathbf{y}^i = (y_1^i, y_2^i) \in \bar{\mathcal{Y}}_a^s \cap C^{sW}$ and $\mathbf{y}^j = (y_1^j, y_2^j) \in \bar{\mathcal{Y}}_a^s \cap C^{sS}$. The points $\mathbf{y}^i$ and $\mathbf{y}^j$ are said to be the closest points to the point $\tilde{\mathbf{y}}^{sI} = (\tilde{y}_1^{s1}, \tilde{y}_2^{s2})$ if $y_1^i = \underset{y_1:\mathbf{y}\in(\bar{\mathcal{Y}}_a^s\cap C^{sW})}{\arg\min} \big(\tilde{y}_1^{s1} - y_1\big)$ and $y_2^j = \underset{y_2:\mathbf{y}\in(\bar{\mathcal{Y}}_a^s\cap C^{sS})}{\arg\min} \big(\tilde{y}_2^{s2} - y_2\big)$. Since $\bar{\mathcal{Y}}_a^s$ contains a finite number of points, we have the fathoming rule:

$$\text{(5.4)} \quad \text{If } \tilde{\mathcal{Y}}_P^s \subset \left(\bigcup_{\mathbf{y} \in \bar{\mathcal{Y}}_a^s} \{\mathbf{y}\}^{\geqq} \right), \text{ then node } s \text{ can be fathomed.}$$

Based on this rule, two practical fathoming rules are now introduced.

Rule 1: If there exists $\mathbf{y}^i \in \bar{\mathcal{Y}}_a^s$ such that $\mathbf{y}^i \in \{\tilde{\mathbf{y}}^{sI}\}^{\leqq}$, then node s can be fathomed.

In Rule 1, $\tilde{\mathcal{Y}}_P^s \subset \{\mathbf{y}^i\}^{\geqq} \subset \left(\bigcup_{\mathbf{y} \in \bar{\mathcal{Y}}_a^s} \{\mathbf{y}\}^{\geqq} \right)$ (see Figure 6a). Note that if Rule 1 does not hold and $\bar{\mathcal{Y}}_a^s \cap C^{sW} = \emptyset$ or $\bar{\mathcal{Y}}_a^s \cap C^{sS} = \emptyset$, then node s cannot be fathomed (see Figure 7).

Rule 2 is constructed using the following information. Assume $\mathcal{Y}^{\mathcal{N}} \cap T^s \neq \emptyset$. Let the nadir point $\mathbf{y}^{i,j} \in \mathcal{Y}^{\mathcal{N}}$ be implied by the points $\mathbf{y}^i \in \bar{\mathcal{Y}}_a^s \cap C^{sW}$ and $\mathbf{y}^j \in \bar{\mathcal{Y}}_a^s \cap C^{sS}$ that are the closest points to the point $\tilde{\mathbf{y}}^{sI}$. Note that these closest points $\mathbf{y}^i$ and $\mathbf{y}^j$ do not have to be adjacent because there might be other points $\mathbf{y} \in \bar{\mathcal{Y}}_a^s$ located in R^s. We examine the adjacent nadir points $\mathbf{y}^{\kappa,\eta}$ in R^s implied by the adjacent points $\mathbf{y}^{\kappa}, \mathbf{y}^{\eta} \in \left(\bar{\mathcal{Y}}_a^s \cap R^s\right) \cup \{\mathbf{y}^i\} \cup \{\mathbf{y}^j\}$. Note that the nadir point $\mathbf{y}^{i,j}$ is also included in R^s. Using these nadir points in R^s, Rule 2 can be written as follows (see Figure 8).

Rule 2: If $\mathbf{y}^{\kappa,\eta} \in \left(\tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{\geqq}\right)$ for all nadir points $\mathbf{y}^{\kappa,\eta} \in R^s$, then node s can be fathomed.

However, if there is at least one nadir point such that $\mathbf{y}^{\kappa,\eta} \in \left(\tilde{\mathcal{Y}}_P^s + \mathbb{R}^2_{>}\right)$ then node s cannot be fathomed. Implementing Rule 2 is not straightforward as the complete set $\tilde{\mathcal{Y}}_P^s$ is not available. To check the location of the nadir points with respect to $\tilde{\mathcal{Y}}_P^s$, we solve the achievement function problem (2.10) $\tilde{\mathcal{P}}(\mathbf{y}^R)$ with $\mathbf{y}^R = \mathbf{y}^{\kappa,\eta}$, where $\mathbf{y}^{\kappa,\eta}$ is a nadir point in R^s. A fathoming decision is then made based on the following proposition.

Proposition 5.1. *Let $\boldsymbol{y}^i = (y_1^i, y_2^i) \in \bar{\mathcal{Y}}_a^s \cap C^{sW}$ and $\boldsymbol{y}^j = (y_1^j, y_2^j) \in \bar{\mathcal{Y}}_a^s \cap C^{sS}$ be the points that are the closest to the point $\tilde{\boldsymbol{y}}^{sI}$, $\boldsymbol{y}^{\kappa,\eta} = (y_1^{\kappa,\eta}, y_2^{\kappa,\eta}) \in R^s$ be a nadir point, $\hat{\boldsymbol{x}}^s(\boldsymbol{y}^R)$ be an optimal solution to (2.10) $\tilde{\mathcal{P}}(\boldsymbol{y}^R)$ with $\boldsymbol{y}^R = \boldsymbol{y}^{\kappa,\eta}$, and $\hat{\boldsymbol{y}}^s(\boldsymbol{y}^R) = \left(f_1(\hat{\boldsymbol{x}}^s(\boldsymbol{y}^R)), f_2(\hat{\boldsymbol{x}}^s(\boldsymbol{y}^R))\right)$. If $\hat{\boldsymbol{y}}^s(\boldsymbol{y}^R) \in \{\boldsymbol{y}^{\kappa,\eta}\}^{\geq}$ for all $\boldsymbol{y}^{\kappa,\eta} \in R^s$ then node s can be fathomed.*

Proof. Assume $\hat{\mathbf{y}}^s(\mathbf{y}^R) \in \{\mathbf{y}^{\kappa,\eta}\}^{\geq}$ for all $\mathbf{y}^{\kappa,\eta} \in R^s$, then by Theorem 2.6(i) $\mathbf{y}^{\kappa,\eta} \in \tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{>}$ for all $\mathbf{y}^{\kappa,\eta} \in R^s$. Therefore,

$$\text{(5.5)} \quad \tilde{\mathcal{Y}}_P^s \subset \left(\{\mathbf{y}^i\}^{\geqq} \cup \{\mathbf{y}^j\}^{\geqq} \cup \bigcup_{\mathbf{y} \in \bar{\mathcal{Y}}_a^s \cap R^s} \{\mathbf{y}\}^{\geqq} \cup \bigcup_{\mathbf{y}^{\kappa,\eta} \in R^s} \{\mathbf{y}^{\kappa,\eta}\}^{\geq} \right).$$

Note that

$$\text{(5.6)} \quad \left(\bigcup_{\mathbf{y}^{\kappa,\eta} \in R^s} \{\mathbf{y}^{\kappa,\eta}\}^{\geq} \right) \subset \left(\{\mathbf{y}^i\}^{\geqq} \cup \{\mathbf{y}^j\}^{\geqq} \cup \bigcup_{\mathbf{y} \in \bar{\mathcal{Y}}_a^s \cap R^s} \{\mathbf{y}\}^{\geqq} \right).$$

From (5.5) and (5.6) we have $\tilde{\mathcal{Y}}_P^s \subset \left(\{\mathbf{y}^i\}^{\geqq} \cup \{\mathbf{y}^j\}^{\geqq} \cup \bigcup_{\mathbf{y} \in \bar{\mathcal{Y}}_a^s \cap R^s} \{\mathbf{y}\}^{\geqq} \right)$, and then $\tilde{\mathcal{Y}}_P^s \subset \bigcup_{\mathbf{y} \in \bar{\mathcal{Y}}_a^s} \{\mathbf{y}\}^{\geqq}$. Hence by the rule in (5.4), node s can be fathomed. ■

Based on Proposition 5.1, problem (2.10) has to be solved with each nadir point $\mathbf{y}^{\kappa,\eta} \in R^s$. Consequently, there are many but a finite number of optimization problems to be solved for one node. Although these are single objective problems, solving them may be time-consuming. The following strategy may allow one to speed up this process by solving fewer problems. Consider again the points $\mathbf{y}^i \in \bar{\mathcal{Y}}_a^s \cap C^{sW}$ and $\mathbf{y}^j \in \bar{\mathcal{Y}}_a^s \cap C^{sS}$ that are the closest points to the point $\tilde{\mathbf{y}}^{sI}$, and the nadir point $\mathbf{y}^{i,j}$ implied by these two points. If $\mathbf{y}^{i,j} \in T^s$, then we check whether $\mathbf{y}^{i,j} \in \left(\tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{\geqq}\right)$. If this holds, then without considering the other nadir points we can conclude that node s can be fathomed (see Figure 6b). To check whether $\mathbf{y}^{i,j} \in \left(\tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{\geqq}\right)$, we solve achievement function problem (2.10) with $\mathbf{y}^R = \mathbf{y}^{i,j}$. Otherwise, if $\mathbf{y}^{i,j} \notin T^s$ or $\mathbf{y}^{i,j} \notin \left(\tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_{>}\right)$, the process cannot be speeded up and problem (2.10) has to be solved with each nadir point in R^s.

The steps of the resulting fathoming module are given in Algorithm 5.1. The points in $\mathcal{Y}_a^s$ and the points $\tilde{\mathbf{y}}^{s1}$ and $\tilde{\mathbf{y}}^{s2}$ in $\tilde{\mathcal{Y}}_P^s$ are the input to this module. The adjacent nadir points implied by the points in $\bar{\mathcal{Y}}_a^s$, the ideal point $\tilde{\mathbf{y}}^{sI}$, and the sets T^s, C^{sW}, C^{sS}, R^s are computed. A sequence of conditions are checked. First, by Rule 1, if there exists a point $\mathbf{y}^i$ in $\bar{\mathcal{Y}}_a^s$ such that $\mathbf{y}^i \in \{\tilde{\mathbf{y}}^{sI}\}^{\leqq}$, then node s can be fathomed. Otherwise, the sets C^{sW} or C^{sS} are checked whether they contain points in $\bar{\mathcal{Y}}_a^s$. If $\bar{\mathcal{Y}}_a^s \cap C^{sW} = \emptyset$ or $\bar{\mathcal{Y}}_a^s \cap C^{sS} = \emptyset$, then node s cannot be fathomed (lines 3-6). Figures 7a and 7b depict this case. If $\bar{\mathcal{Y}}_a^s \cap C^{sW} \neq \emptyset$ and $\bar{\mathcal{Y}}_a^s \cap C^{sS} \neq \emptyset$, then the points in $\bar{\mathcal{Y}}_a^s \cap C^{sW}$ and $\bar{\mathcal{Y}}_a^s \cap C^{sS}$ that are the closest to $\tilde{\mathbf{y}}^{sI}$ are found. Let $\mathbf{y}^i \in \bar{\mathcal{Y}}_a^s \cap C^{sW}$ and $\mathbf{y}^j \in \bar{\mathcal{Y}}_a^s \cap C^{sS}$ be the closest points to $\tilde{\mathbf{y}}^{sI}$. The nadir point $\mathbf{y}^{i,j} = (y_1^{i,j}, y_2^{i,j})$ implied by $\mathbf{y}^i$ and $\mathbf{y}^j$ is computed and its location with respect to the set T^s is checked. If $\mathbf{y}^{i,j} \in T^s$, the achievement function problem (2.10) $\tilde{\mathcal{P}}(\mathbf{y}^R)$ is solved with $\mathbf{y}^R = \mathbf{y}^{i,j}$. Let $\hat{\mathbf{y}}^s(\mathbf{y}^R)$ be the image of an optimal solution to problem (2.10). If $\hat{\mathbf{y}}^s(\mathbf{y}^R) > \mathbf{y}^{i,j}$, then node s can be fathomed (lines 8-13). If this condition does not hold or $\mathbf{y}^{i,j} \notin T^s$ (see Figures 9a and 9b), problem (2.10) is solved with all adjacent nadir points $\mathbf{y}^{\kappa,\eta} = (y_1^{\kappa,\eta}, y_2^{\kappa,\eta}) \in R^s$ and the condition $\hat{\mathbf{y}}^s(\mathbf{y}^R) > \mathbf{y}^{\kappa,\eta}$ for all $\mathbf{y}^{\kappa,\eta} \in R^s$ is checked, where $\hat{\mathbf{y}}^s(\mathbf{y}^R)$ is the image of an optimal solution to problem (2.10) for a given nadir point. If this condition holds for every nadir point, then by Rule 2, node s can be fathomed. Otherwise, node s cannot be fathomed (lines 15-29) (see Figure 10).

6. Dominance Between Sets.

The goal of this section is to address the dominance between two Pareto sets, $\tilde{\mathcal{Y}}_P^l$ and $\tilde{\mathcal{Y}}_P^s$ in $\mathbb{R}^2$, that are associated with the corresponding relaxed node problems as given in (2.5). We start with general definitions pertaining to two sets S_1 and S_2 in $\mathbb{R}^2$. Assume $(S_i)_N \neq \emptyset$ for $i = 1, 2$. Following [32], the following definitions of (non)dominance for sets are used.

Definition 6.1. *Let S_1 and S_2 be two nonempty sets in $\mathbb{R}^2$. S_1 is said to (strictly, weakly) dominate $S_2 \subset \mathbb{R}^2$, or equivalently, S_2 is said to be (strictly, weakly) dominated by S_1, denoted by $S_1(<, \leqq) \leq S_2$, provided for each point $\boldsymbol{y}^2 \in S_2$ there exists a point $\boldsymbol{y}^1 \in S_1$, such that $\boldsymbol{y}^1(<, \leqq) \leq \boldsymbol{y}^2$.*

For two Pareto sets $\tilde{\mathcal{Y}}_P^l, \tilde{\mathcal{Y}}_P^s$, Figures 11a and 11b depict the dominance and the weak dominance, respectively, while Figure 12 depicts the strict dominance.

Definition 6.2. *Let S_1 and S_2 be two nonempty sets in $\mathbb{R}^2$. S_1 is said to be nondominated*

Algorithm 5.1 Fathoming procedure.

INPUT: All points in $\bar{\mathcal{Y}}^s_a$, $\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{s2} \in \tilde{\mathcal{Y}}^s_P$
OBTAIN: Adjacent nadir points implied by points in $\bar{\mathcal{Y}}^s_a$, $\tilde{\mathbf{y}}^{sI}$, T^s, C^{sW}, C^{sS}, R^s
if $\exists\ \mathbf{y}^i \in \bar{\mathcal{Y}}^s_a$ s.t. $\mathbf{y}^i \in \{\tilde{\mathbf{y}}^{sI}\}^{\leqq}$ **then**
    Fathom
else if $\bar{\mathcal{Y}}^s_a \cap C^{sW} = \emptyset$ or $\bar{\mathcal{Y}}^s_a \cap C^{sS} = \emptyset$ **then**
    Not fathom
else
    Find $\mathbf{y}^i \in \bar{\mathcal{Y}}^s_a \cap C^{sW}$ and $\mathbf{y}^j \in \bar{\mathcal{Y}}^s_a \cap C^{sS}$ that are the closest points to point $\tilde{\mathbf{y}}^{sI}$
    Obtain $\mathbf{y}^{i,j}$
    if $\mathbf{y}^{i,j} \in T^s$ **then**
        Solve $\tilde{\mathcal{P}}(\mathbf{y}^R)$ (2.10) with $\mathbf{y}^R = \mathbf{y}^{i,j}$ and obtain $\hat{\mathbf{y}}^s(\mathbf{y}^R) \in \tilde{\mathcal{Y}}^s_p$
        if $\hat{\mathbf{y}}^s(\mathbf{y}^R) > \mathbf{y}^{i,j}$ **then**
            Fathom
        else
            Find all adjacent nadir points $\mathbf{y}^{\kappa,\eta} \in R^s$
            Solve $\tilde{\mathcal{P}}(\mathbf{y}^R)$ (2.10) with $\mathbf{y}^R = \mathbf{y}^{\kappa,\eta}$ and obtain $\hat{\mathbf{y}}^s(\mathbf{y}^R)$
            if $\hat{\mathbf{y}}^s(\mathbf{y}^R) > \mathbf{y}^{\kappa,\eta}\ \forall\ \mathbf{y}^{\kappa,\eta}$ **then**
                Fathom
            else
                Not fathom
            end if
        end if
    else
        Find all adjacent nadir points $\mathbf{y}^{\kappa,\eta} \in R^s$
        Solve $\tilde{\mathcal{P}}(\mathbf{y}^R)$ (2.10) with $\mathbf{y}^R = \mathbf{y}^{\kappa,\eta}$ and obtain $\hat{\mathbf{y}}^s(\mathbf{y}^R)$
        if $\hat{\mathbf{y}}^s(\mathbf{y}^R) > \mathbf{y}^{\kappa,\eta}\ \forall\ \mathbf{y}^{\kappa,\eta}$ **then**
            Fathom
        else
            Not fathom
        end if
    end if
end if
OUTPUT: Fathoming Decision

with respect to S_2 provided there does not exists a point $\boldsymbol{y}^2 \in S_2$ such that $\boldsymbol{y}^2 \leq \boldsymbol{y}^1$ for each $\boldsymbol{y}^1 \in S_1$.

For our purposes we introduce a definition of partial dominance between two sets.

Definition 6.3. *Let S_1 and S_2 be two nonempty sets in $\mathbb{R}^2$. $S_1 \subset \mathbb{R}^2$ is said to partially (weakly) dominate $S_2 \subset \mathbb{R}^2$, or equivalently, S_2 is said to be partially (weakly) dominated by S_1, denoted by $S_1 (\leqq_{\mathfrak{p}}) \leq_{\mathfrak{p}} S_2$, provided there exists a nonempty subset $S_2' \subset S_2$ such that*

$S_1(\leqq) \leq S_2'$ *and the subset* $S_2 \setminus S_2'$ *is nondominated with respect to* S_1.

The partially strictly dominated sets are not defined as if there exists a subset $S_2' \subset S_2$ such that $S_1 < S_2'$, then there does not exist a subset $S_2 \setminus S_2'$ that is nondominated with respect to S_1. For two Pareto sets $\tilde{\mathcal{Y}}_P^l, \tilde{\mathcal{Y}}_P^s$, Figures 13 and 14 depict the partial dominance and partial weak dominance, respectively. The subsets that are associated with the partially dominated sets and nondominated are defined as follows.

Definition 6.4. *Let* $S_1, S_2 \subset \mathbb{R}^2$, *and* $(S_1 \cup S_2)_N \neq \emptyset$. *A set* $S_{iN} \subset S_i$ *is called a nondominated subset of* S_i *for* $i = 1, 2$ *provided* $S_{iN} = (S_1 \cup S_2)_N \cap (S_i)_N$.

Based on these definitions the following properties hold.

Proposition 6.5. *Let* $S \subset \mathbb{R}^2$, $S_N \neq \emptyset$ *and be externally stable [12]. Then* $S_N \leqq S$.

Proof. By Definition 6.1 we show for each $\mathbf{y} \in S$ there exists $\mathbf{y}^1 \in S_N$ such that $\mathbf{y}^1 \leqq \mathbf{y}$. Since $\mathbf{y} \in S$, we consider two cases: (1) Let $\mathbf{y} \in S_N$. Then $\mathbf{y} \leqq \mathbf{y}$. (2) Let $\mathbf{y} \in S \setminus S_N$. Then since S_N is externally stable, there exists $\mathbf{y}^1 \in S_N$ such that $\mathbf{y}^1 \leqq \mathbf{y}$. Hence $S_N \leqq S$. ■

Proposition 6.6. *Let* $S_1, S_2 \subset \mathbb{R}^2$, $(S_i + \mathbb{R}^2_{\geqq})_N \neq \emptyset$ *and be externally stable for* $i = 1, 2$.

(i) The set S_1 *dominates the set* S_2, $S_1 \leq S_2$, *if and only if* $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{\geq}$.

(ii) The set S_1 *weakly dominates the set* S_2, $S_1 \leqq S_2$, *if and only if* $S_2 + \mathbb{R}^2_{\geqq} \subseteq S_1 + \mathbb{R}^2_{\geqq}$.

(iii) The set S_1 *strictly dominates the set* S_2, $S_1 < S_2$, *if and only if* $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{>}$.

Proof. The proof is included in the Appendix. ■

6.1. Dominance Between Two Pareto Sets: Theory. We now analyze the mutual location of two Pareto sets, $\tilde{\mathcal{Y}}_P^l$ and $\tilde{\mathcal{Y}}_P^s$, in $\mathbb{R}^2$ to conclude about the (partial) dominance between them. We assume that each Pareto set is a strictly convex curve for which only limited information is available in the form of the end points $\tilde{\mathbf{y}}^{\kappa 1}$ and $\tilde{\mathbf{y}}^{\kappa 2}$, and the ideal point $\tilde{\mathbf{y}}^{\kappa I} \notin \tilde{\mathcal{Y}}_P^{\kappa}$, $\kappa = l, s, l \neq s$. We therefore define the triangles T^κ, $\kappa = l, s, l \neq s$, as given in (2.6). The presented results address numerous mutual locations of two Pareto sets in the objective space.

Proposition 6.7. *Let* $T^l \cap T^s = \emptyset$. *The triangle* T^l *(strictly) dominates the triangle* T^s, $T^l(<) \leq T^s$, *if and only if the Pareto set* $\tilde{\mathcal{Y}}_P^l$ *(strictly) dominates the Pareto set* $\tilde{\mathcal{Y}}_P^s$, $\tilde{\mathcal{Y}}_P^l(<) \leq \tilde{\mathcal{Y}}_P^s$.

Proof. $\Rightarrow$ Assume $T^l(<) \leq T^s$. Then for each $\mathbf{y}^s \in T^s$ there exists $\mathbf{y}^l \in T^l$ such that $\mathbf{y}^l(<) \leq \mathbf{y}^s$. We have $\tilde{\mathcal{Y}}_P^l \subset T^l$ and $\tilde{\mathcal{Y}}_P^s \subset T^s$ and therefore for each $\tilde{\mathbf{y}}^s \in \tilde{\mathcal{Y}}_P^s$ there exists $\tilde{\mathbf{y}}^l \in \tilde{\mathcal{Y}}_P^l$ such that $\tilde{\mathbf{y}}^l(<) \leq \tilde{\mathbf{y}}^s$. Thus, by definition $\tilde{\mathcal{Y}}_P^l(<) \leq \tilde{\mathcal{Y}}_P^s$.
$\Leftarrow$ Assume $\tilde{\mathcal{Y}}_P^l(<) \leq \tilde{\mathcal{Y}}_P^s$. Then from Proposition 6.6 $(\tilde{\mathcal{Y}}_P^s + \mathbb{R}^2_{\geqq} \subset \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_{>})$ $\tilde{\mathcal{Y}}_P^s + \mathbb{R}^2_{\geqq} \subset \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_{\geq}$. Because $T^l \cap T^s = \emptyset$, we have $(T^s + \mathbb{R}^2_{\geqq} \subset T^l + \mathbb{R}^2_{>})$ $T^s + \mathbb{R}^2_{\geqq} \subset T^l + \mathbb{R}^2_{\geq}$. Then from Proposition 6.6((iii)) (i) we have $T^l(<) \leq T^s$. ■

In the following propositions, the cases conceiving the possible mutual locations of $\tilde{\mathcal{Y}}_P^l$ and $\tilde{\mathcal{Y}}_P^s$ are listed. Although the statements are immediate and therefore not proved, they are helpful in the development of a set dominance procedure.

Proposition 6.8. *Let* $\tilde{\mathcal{Y}}_P^l$, $\tilde{\mathcal{Y}}_P^s$ *be the Pareto sets of two instances of BOQP (2.5).*

(i) If $\tilde{\mathbf{y}}^{sI} \in \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_{\geqq}$, *then* $\tilde{\mathcal{Y}}_P^l \leq \tilde{\mathcal{Y}}_P^s$ *(Figure 12).*

(ii) If $\tilde{\mathbf{y}}^{s2} \in C^{lW}$ *or* $\tilde{\mathbf{y}}^{s1} \in C^{lS}$, *then* $\tilde{\mathcal{Y}}_P^s \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^l$ *(Figure 13).*

Proposition 6.9. *Let $\tilde{\mathcal{Y}}^l_P$, $\tilde{\mathcal{Y}}^s_P$ be the Pareto sets of two instances of BOQP* (2.5). *If one of the following holds*

(i) $\tilde{\boldsymbol{y}}^{si} \in \tilde{\mathcal{Y}}^l_P - \mathbb{R}^2_>$ *and* $\tilde{\boldsymbol{y}}^{sj} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_>$ *for* $i, j \in \{1, 2\}, i \neq j$ *(Figure 14),*

(ii) $\tilde{\boldsymbol{y}}^{sI} \in C^{lW}(C^{lS})$, $\tilde{\boldsymbol{y}}^{s2}(\tilde{\boldsymbol{y}}^{s1}) \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_>$ *and* $\tilde{\boldsymbol{y}}^{l1}(\tilde{\boldsymbol{y}}^{l2}) \in \tilde{\mathcal{Y}}^s_P + \mathbb{R}^2_>$ *(Figure 15),*

(iii) $\tilde{\boldsymbol{y}}^{sI} \in C^{lW}(C^{lS})$, $\tilde{\boldsymbol{y}}^{s1} \in \tilde{\boldsymbol{y}}^{l1} - \mathbb{R}^2_{\gtrless}(\tilde{\boldsymbol{y}}^{s2} \in \tilde{\boldsymbol{y}}^{l2} + \mathbb{R}^2_{\gtrless})$, $\tilde{\boldsymbol{y}}^{s2}(\tilde{\boldsymbol{y}}^{s1}) \in \tilde{\mathcal{Y}}^l_P - \mathbb{R}^2_>$ *and* $\tilde{\boldsymbol{y}}^{l1}(\tilde{\boldsymbol{y}}^{l2}) \in \tilde{\mathcal{Y}}^s_P - \mathbb{R}^2_>$ *(Figure 16),*

then the Pareto sets intersect and each Pareto set is partially weakly dominated by the other one.

The following two propositions cover special cases. Proposition 6.10 addresses a special case of Proposition 6.9 (i).

Proposition 6.10. *Let $\tilde{\mathcal{Y}}^l_P$, $\tilde{\mathcal{Y}}^s_P$ be the Pareto sets of two instances of BOQP* (2.5). *If* $\tilde{\boldsymbol{y}}^{si} \in \tilde{\mathcal{Y}}^l_P$, $\tilde{\boldsymbol{y}}^{sj} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_>$ *and* $\left(\tilde{\mathcal{Y}}^s_P \setminus \tilde{\boldsymbol{y}}^{si}\right) \cap \tilde{\mathcal{Y}}^l_P = \emptyset$ *for* $i, j \in \{1, 2\}, i \neq j$, *then* $\tilde{\mathcal{Y}}^l_P \leqq \tilde{\mathcal{Y}}^s_P$ *(Figure 17).*

Proposition 6.11. *Let $\tilde{\mathcal{Y}}^l_P$, $\tilde{\mathcal{Y}}^s_P$ be the Pareto sets of two instances of BOQP* (2.5). *If* $\tilde{\boldsymbol{y}}^{s1} \in bd\left(\tilde{\boldsymbol{y}}^{l2} + \mathbb{R}^2_{\substack{\geq\\<}}\right)\left(\tilde{\boldsymbol{y}}^{s2} \in bd\left(\tilde{\boldsymbol{y}}^{l1} - \mathbb{R}^2_{\substack{\geq\\<}}\right)\right)$, *then* $\tilde{\mathcal{Y}}^l_P \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}^s_P$ *(Figure 18).*

6.2. Dominance Between Two Pareto Sets: Methodology.

In this section, we develop a set dominance procedure to make a dominance decision between two Pareto sets and compute the resulting nondominated set. We first give an overview of this procedure and then present its components.

6.2.1. Overview.

A parametric representation of each efficient set in the form of rational functions is available from the mpLCP method before the set dominance procedure is started. One could use this representation and solve the resulting polynomial equations to find the intersection points between two Pareto sets or decide there is none. However, solving polynomial equations is computationally costly while finding the intersection points is not sufficient to determine the nondominated set resulting from two Pareto sets. Additionally, there are many cases of mutual location of these sets with no intersection points when solving the polynomial equations is obviously unnecessary. Keeping this mind, in the proposed dominance procedure we postpone solving the polynomial equations and first investigate the mutual location of the Pareto sets. When the existence of the intersection point is confirmed or at least is highly probable, we solve the polynomial equations.

The inputs to the procedure are $\tilde{\mathbf{y}}^{l1}, \tilde{\mathbf{y}}^{l2} \in \tilde{\mathcal{Y}}^l_P$ and $\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{s2} \in \tilde{\mathcal{Y}}^s_P$. We then obtain the associated ideal points $\tilde{\mathbf{y}}^{lI}$, $\tilde{\mathbf{y}}^{sI}$, and the triangles T^l and T^s. In this description, we assume the Pareto set $\tilde{\mathcal{Y}}^l_P$ has already been stored in the set $\mathcal{Y}_a$, while the Pareto set $\tilde{\mathcal{Y}}^s_P$ is being introduced to $\mathcal{Y}_a$.

We use the achievement function problem (2.10) to check the location of a point with respect to the Pareto set of (2.5) and the $\epsilon-$constraint problem (2.12) to recognize the nondominated subsets to store. If the Pareto sets are partially dominated, then only their nondominated subsets are stored in $\mathcal{Y}_a$ and the $\epsilon-$constraint problem (2.12) is used to identify these subsets. In general, the nondominated set to store is the nondominated set of the union of the two original Pareto sets. Let $\tilde{\mathcal{Y}}^{sl}_N$ denote the nondominated set obtained after applying the set dominance procedure to Pareto sets $\tilde{\mathcal{Y}}^s_P$ and $\tilde{\mathcal{Y}}^l_P$. Then $\tilde{\mathcal{Y}}^{sl}_N = N(\tilde{\mathcal{Y}}^s_P \cup \tilde{\mathcal{Y}}^l_P)$.

The set dominance procedure leads to one of the three decisions, one Pareto set (strictly, weakly) dominates the other, or one Pareto set partially (weakly) dominates the other, or both Pareto sets are nondominated. The nondominated set resulting from the first decision is $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^{\kappa}$ with $\kappa = s$ or $\kappa = l$. The second decision is made either with $\tilde{\mathcal{Y}}_P^{l} \cap \tilde{\mathcal{Y}}_P^{s} = \emptyset$ or $\tilde{\mathcal{Y}}_P^{l} \cap \tilde{\mathcal{Y}}_P^{s} \neq \emptyset$. The nondominated set associated with the latter decision is $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^{l} \cup \tilde{\mathcal{Y}}_P^{s}$.

To report the subset of the Pareto set to store, the following notation is used. For any two points $\mathbf{y}^1$, $\mathbf{y}^2$ and the associated Pareto set $\tilde{\mathcal{Y}}_P^{\kappa}$, $\kappa = s, l$, let $\mathscr{C}^{\kappa}[\mathbf{y}^1, \mathbf{y}^2] \subseteq \tilde{\mathcal{Y}}_P^{\kappa}$ denote the closed strictly convex curve and $\mathscr{C}^{\kappa}(\mathbf{y}^1, \mathbf{y}^2) \subseteq \tilde{\mathcal{Y}}_P^{\kappa}$ denote the open strictly convex curve from $\mathbf{y}^1$ to $\mathbf{y}^2$. Also, let $\mathscr{C}^{\kappa}(\mathbf{y}^1, \mathbf{y}^2] = \mathscr{C}^{\kappa}[\mathbf{y}^1, \mathbf{y}^2] \setminus \{\mathbf{y}^1\}$ and $\mathscr{C}^{\kappa}[\mathbf{y}^1, \mathbf{y}^2) = \mathscr{C}^{\kappa}[\mathbf{y}^1, \mathbf{y}^2] \setminus \{\mathbf{y}^2\}$ such that $\mathscr{C}^{\kappa}(\mathbf{y}^1, \mathbf{y}^2], \mathscr{C}^{\kappa}[\mathbf{y}^1, \mathbf{y}^2) \subseteq \tilde{\mathcal{Y}}_P^{\kappa}$.

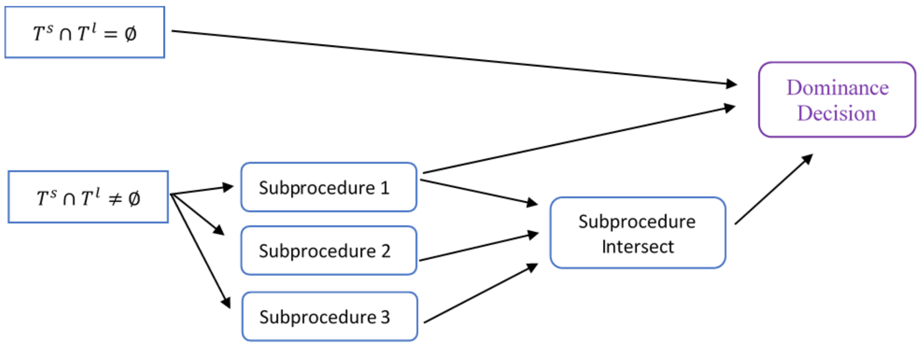

Figure 1: Flowchart for the set dominance procedure

The set dominance procedure consists of four subprocedures as depicted in Figure 1 and starts with checking whether the triangles T^s and T^l intersect or not. If the triangles do not intersect, then a dominance decision is made. Otherwise, the mutual locations of the end points of the Pareto sets $\tilde{\mathcal{Y}}_P^{s}$ and $\tilde{\mathcal{Y}}_P^{l}$ are examined in Subprocedures 1-3 that are independently initiated and rely on solving the achievement function problem. In some cases, a dominance decision can be made directly from Subprocedure 1. Otherwise, the three subprocedures continue to Subprocedure Intersect to check for and compute intersection points between two Pareto sets. In all cases at the end of the entire process, a dominance decision is made and the resulting nondominated set is computed based on solving the ϵ-constraint problem.

6.2.2. Subprocedures. In this section, the subprocedures of the set dominance procedure given in Figure 1 are presented in detail. These subprocedures are leading to several different cases based on the locations of the end points of two Pareto sets. In each case, the locations of end points, the associated dominance decision, and the nodminated set $\mathcal{Y}_N^{sl}$ are discussed.

1. Let $T^l \cap T^s = \emptyset$.
   I. If $T^s \in \{\tilde{\mathbf{y}}^{l1}\} - \mathbb{R}^2_{\geqq}$ or $T^s \in \{\tilde{\mathbf{y}}^{l2}\} + \mathbb{R}^2_{\geqq}$, then both $\tilde{\mathcal{Y}}_P^{s}$ and $\tilde{\mathcal{Y}}_P^{l}$ are nondominated. Hence $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^{l} \cup \tilde{\mathcal{Y}}_P^{s}$. Figure 19 depicts this case. In particular, if $\tilde{\mathbf{y}}^{s2} \in bd\left(\tilde{\mathbf{y}}^{l1} - \mathbb{R}^2_{\geqq}\right)$ $\left(\tilde{\mathbf{y}}^{s1} \in bd\left(\tilde{\mathbf{y}}^{l2} + \mathbb{R}^2_{\geqq}\right)\right)$, then by Proposition 6.11, $\tilde{\mathcal{Y}}_P^{l} \leqq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^{s}$ and $\tilde{\mathcal{Y}}_N^{sl} = (\tilde{\mathcal{Y}}_P^{s} \setminus \tilde{\mathbf{y}}^{s2}) \cup \tilde{\mathcal{Y}}_P^{l}$ $\left(\tilde{\mathcal{Y}}_N^{sl} = (\tilde{\mathcal{Y}}_P^{s} \setminus \tilde{\mathbf{y}}^{s1}) \cup \tilde{\mathcal{Y}}_P^{l}\right)$. Figure 18 depicts this case.

II. Otherwise, if $T^s \subset T^l + \mathbb{R}^2_{\geq}$ ($T^l \subset T^s + \mathbb{R}^2_{\geq}$), then $T^l \leq T^s$ ($T^s \leq T^l$), and by Proposition 6.7, $\tilde{\mathcal{Y}}^l_P \leq \tilde{\mathcal{Y}}^s_P$ ($\tilde{\mathcal{Y}}^s_P \leq \tilde{\mathcal{Y}}^l_P$). Then $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^l_P$ ($\tilde{\mathcal{Y}}^s_P$) and $\tilde{\mathcal{Y}}^s_P$ ($\tilde{\mathcal{Y}}^l_P$) is discarded. Figure 20 depicts this case.

III. If the above conditions do not hold, then by Proposition 6.8(ii), $\tilde{\mathcal{Y}}^s_P \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}^l_P$. Figure 13 depicts this case. The complete Pareto set $\tilde{\mathcal{Y}}^s_P$ and a subset of the Pareto set $\tilde{\mathcal{Y}}^l_P$ are stored as the nondominated set. To obtain this subset of $\tilde{\mathcal{Y}}^l_P$, $\mathcal{P}(\tilde{y}^{s2}_2)$ (2.12) is solved and the point $\hat{\mathbf{y}}(\tilde{y}^{s2}_2)$ is obtained. Then $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^s_P \cup \mathscr{C}^l(\hat{\mathbf{y}}(\tilde{y}^{s2}_2), \tilde{\mathbf{y}}^{l2}]$. Alternatively, $\mathcal{P}(\tilde{y}^{s1}_1)$ (2.12) is solved and $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^s_P \cup \mathscr{C}^l[\tilde{\mathbf{y}}^{l1}, \hat{\mathbf{y}}(\tilde{y}^{s1}_1))$ is stored.

2. Let $T^l \cap T^s \neq \emptyset$.

I. Without loss of generality, first check whether $T^s \subset T^l$. If this holds, apply Subprocedure 1.

II. Otherwise, check whether $\tilde{\mathbf{y}}^{sI} \in T^l$. If this does not hold, continue to Subprocedure 2. If this condition holds, check whether $\tilde{\mathbf{y}}^{sI} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{\geqq}$ by solving problem (2.10) with point $\mathbf{y}^R = \tilde{\mathbf{y}}^{sI}$. If $\tilde{\mathbf{y}}^{sI} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{\geqq}$, then by Proposition 6.8(i), $\tilde{\mathcal{Y}}^l_P \leq \tilde{\mathcal{Y}}^s_P$. Figure 12 depicts this case. Then $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^l_P$ and $\tilde{\mathcal{Y}}^s_P$ is discarded. Otherwise, apply Subprocedure 3.

Subprocedure 1: Let $T^s \subset T^l$.

Solving problem (2.10) with $\mathbf{y}^R = \tilde{\mathbf{y}}^{sI}$, first check whether $\tilde{\mathbf{y}}^{sI} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{\geqq}$. If this holds, by Proposition 6.8(i), $\tilde{\mathcal{Y}}^l_P \leq \tilde{\mathcal{Y}}^s_P$. Then $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^l_P$ and $\tilde{\mathcal{Y}}^s_P$ is discarded. Figure 21 depicts this case. Else, check whether $\tilde{\mathbf{y}}^{si} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{>}$ for $i = 1, 2$. If both $\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{s2} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{>}$, then continue directly to Subprocedure Intersect. Figure 22 depicts this case. If an intersect point does not exist, that is, $\tilde{\mathcal{Y}}^l_P \cap \tilde{\mathcal{Y}}^s_P = \emptyset$, then $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^l_P$ (Figure 22a). Otherwise, report the set $\tilde{\mathcal{Y}}^{sl}_N$ accordingly with the intersection points (Figure 22b).

If $\tilde{\mathbf{y}}^{si} \in \tilde{\mathcal{Y}}^l_P - \mathbb{R}^2_{>}$ and $\tilde{\mathbf{y}}^{sj} \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{>}$ for i $\neq j$, then by Proposition 6.9(i), $\tilde{\mathcal{Y}}^l_P$ and $\tilde{\mathcal{Y}}^s_P$ intersect and each Pareto set is partially weakly dominated by the other one. That is, $\tilde{\mathcal{Y}}^l_P \leqq_{\mathfrak{p}} \tilde{\mathcal{Y}}^s_P$ and $\tilde{\mathcal{Y}}^s_P \leqq_{\mathfrak{p}} \tilde{\mathcal{Y}}^l_P$. Figure 14 depicts this case. Problem $\mathcal{P}(\tilde{y}^{s1}_1)$(2.12) or $\mathcal{P}(\tilde{y}^{s2}_2)$(2.12) is solved to identify the nondominated subsets of $\tilde{\mathcal{Y}}^l_P$. Subprocedure Intersect is used to find an intersection point. Otherwise, that is, if both $\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{s2} \in \tilde{\mathcal{Y}}^l_P - \mathbb{R}^2_{>}$, then continue directly to Subprocedure Intersect. Figure 23 depicts this case. If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}^l_P \cap \tilde{\mathcal{Y}}^s_P = \emptyset$, then $\tilde{\mathcal{Y}}^s_P \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}^l_P$. For the case depicted in Figure 23a, problems $\mathcal{P}(\tilde{y}^{s1}_1)$(2.12) and $\mathcal{P}(\tilde{y}^{s2}_2)$(2.12) are solved and the points $\hat{\mathbf{y}}(\tilde{y}^{s1}_1)$, $\hat{\mathbf{y}}(\tilde{y}^{s2}_2)$ in $\tilde{\mathcal{Y}}^l_P$ are obtained, respectively. Then $\tilde{\mathcal{Y}}^{sl}_N = \mathscr{C}^l[\tilde{\mathbf{y}}^{l1}, \hat{\mathbf{y}}(\tilde{y}^{s1}_1)) \cup \tilde{\mathcal{Y}}^s_P \cup \mathscr{C}^l(\hat{\mathbf{y}}(\tilde{y}^{s2}_2), \tilde{\mathbf{y}}^{l2}]$. If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}^l_P \cap \tilde{\mathcal{Y}}^s_P \neq \emptyset$, then $\tilde{\mathcal{Y}}^s_P \leqq_{\mathfrak{p}} \tilde{\mathcal{Y}}^l_P$ and $\tilde{\mathcal{Y}}^l_P \leqq_{\mathfrak{p}} \tilde{\mathcal{Y}}^s_P$. For the case depicted in Figure 23b, problems $\mathcal{P}(\tilde{y}^{s1}_1)$(2.12) and $\mathcal{P}(\tilde{y}^{s2}_2)$(2.12) are solved and the set $\tilde{\mathcal{Y}}^{sl}_N$ is reported accordingly with the two intersection points.

Subprocedure 2: Let $T^s \not\subset T^l$ and $\tilde{\mathbf{y}}^{sI} \notin T^l$.

I. Let $\tilde{\mathbf{y}}^{sI}, \tilde{\mathbf{y}}^{s1} \in C^{lW}$ ($\tilde{\mathbf{y}}^{sI}\tilde{\mathbf{y}}^{s2} \in C^{lS}$).

(i) If $\tilde{\mathbf{y}}^{s2}(\tilde{\mathbf{y}}^{s1}) \in \tilde{\mathcal{Y}}^l_P + \mathbb{R}^2_{>}$, then by Proposition 6.9(i), the Pareto sets intersect and each Pareto set is partially weakly dominated by the other one. Figure 24 depicts this case. Subprocedure Intersect is used to find the intersection point;

(ii) else, that is, $\tilde{\mathbf{y}}^{s2}(\tilde{\mathbf{y}}^{s1}) \in \tilde{\mathcal{Y}}^l_P - \mathbb{R}^2_{>}$, continue to Subprocedure Intersect. Figure 25 depicts this case. If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}^l_P \cap \tilde{\mathcal{Y}}^s_P = \emptyset$, then $\tilde{\mathcal{Y}}^s_P \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}^l_P$. For the case depicted in Figure 25a, problem $\mathcal{P}(\tilde{y}^{s2}_2)$(2.12) is solved and point $\hat{\mathbf{y}}(\tilde{y}^{s2}_2)$ in

$\tilde{\mathcal{Y}}_P^l$ is obtained. Then $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^s \cup \mathscr{C}^l(\hat{\mathbf{y}}(\tilde{y}_2^{s2}), \tilde{\mathbf{y}}^{l2}]$. If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$, then each Pareto set is partially weakly dominated by the other one. For the case depicted in Figure 25b, problem $\mathcal{P}(\tilde{y}_2^{s2})$(2.12) is solved and the set $\tilde{\mathcal{Y}}_N^{sl}$ is reported accordingly with the two intersection points.

II. Let $\tilde{\mathbf{y}}^{sI} \in C^{lW}$ ($\tilde{\mathbf{y}}^{sI} \in C^{lS}$) and $\tilde{\mathbf{y}}^{s1} \in \tilde{\mathcal{Y}}_P^l - \mathbb{R}^2_{\gtrless}(\tilde{\mathbf{y}}^{s2} \in \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_{\gtrless})$.

(i) If $\left(\tilde{\mathbf{y}}^{l1}(\tilde{\mathbf{y}}^{l2}) \in \tilde{\mathcal{Y}}_P^s + \mathbb{R}^2_> \text{ and } \tilde{\mathbf{y}}^{s2}(\tilde{\mathbf{y}}^{s1}) \in \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_>\right)$ or $\left(\tilde{\mathbf{y}}^{l1}(\tilde{\mathbf{y}}^{l2}) \in \tilde{\mathcal{Y}}_P^s - \mathbb{R}^2_> \text{ and } \tilde{\mathbf{y}}^{s2}(\tilde{\mathbf{y}}^{s1}) \in \tilde{\mathcal{Y}}_P^l - \mathbb{R}^2_>\right)$, then by Propositions 6.9(ii)(iii), the Pareto sets intersect and each Pareto set is partially dominated by the other one. Figures 15 and 16 depict these cases. Subprocedure Intersect is used to find the intersection points. For the case depicted in Figure 16a, problems $\mathcal{P}(\tilde{y}_1^{l1})$(2.12) and $\mathcal{P}(\tilde{y}_2^{s2})$(2.12) are solved, while for the case depicted in Figure 16b, $\mathcal{P}(\tilde{y}_1^{s1})$(2.12) and $\mathcal{P}(\tilde{y}_2^{l2})$(2.12) are solved to identify the nondominated subsets of $\tilde{\mathcal{Y}}_P^l$;

(ii) else continue to Subprocedure Intersect. Figures 26 and 27 depict some of these cases. If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$, and
(i) $\tilde{\mathcal{Y}}_P^s \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^l$, then for the case depicted in Figure 26a, problem $\mathcal{P}(\tilde{y}_2^{s2})$(2.12) is solved and point $\hat{\mathbf{y}}(\tilde{y}_2^{s2})$ in $\tilde{\mathcal{Y}}_P^l$ is obtained. Then $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^s \cup \mathscr{C}^l(\hat{\mathbf{y}}(\tilde{y}_2^{s2}), \tilde{\mathbf{y}}^{l2}]$.
(ii) $\tilde{\mathcal{Y}}_P^l \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^s$, then for the case depicted in Figure 27a, problem $\mathcal{P}(\tilde{y}_1^{l1})$(2.12) is solved and point $\hat{\mathbf{y}}(\tilde{y}_1^{l1})$ in $\tilde{\mathcal{Y}}_P^s$ is obtained. Then $\tilde{\mathcal{Y}}_N^{sl} = \mathscr{C}^s[\tilde{\mathbf{y}}^{s1}, \hat{\mathbf{y}}(\tilde{y}_1^{l1})) \cup \tilde{\mathcal{Y}}_P^l$.
If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$ (Figures 26b, 27b), then each Pareto set is partially weakly dominated by the other one. Problem $\mathcal{P}(\tilde{y}_1^{l1})$(2.12) or $\mathcal{P}(\tilde{y}_2^{s2})$(2.12) is solved and the set $\tilde{\mathcal{Y}}_N^{sl}$ is reported with the intersection points.

Subprocedure 3: Let $T^s \not\subset T^l$, $\tilde{\mathbf{y}}^{sI} \in T^l$ and $\tilde{\mathbf{y}}^{sI} \in \tilde{\mathcal{Y}}_P^l - \mathbb{R}^2_>$.

Check the locations of $\tilde{\mathbf{y}}^{s1}$ and $\tilde{\mathbf{y}}^{s2}$ with respect to $\tilde{\mathcal{Y}}_P^l$ by solving problem (2.10) with $\mathbf{y}^R = \tilde{\mathbf{y}}^{s1}$ and $\mathbf{y}^R = \tilde{\mathbf{y}}^{s2}$. If both conditions $\tilde{\mathbf{y}}^{si} \in \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_>$ for $i = 1, 2$ are satisfied, then proceed directly to Subprocedure Intersect. Figure 28 depicts this case. If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$, then $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^l$ (Figure 28a). If Subprocedure Intersect concludes $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$, then each Pareto set is partially weakly dominated by the other one (Figure 28b) and the set $\tilde{\mathcal{Y}}_N^{sl}$ is reported accordingly with the intersection points. Otherwise, i.e., if $\tilde{\mathbf{y}}^{si} \in \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_>$ for $i \in \{1, 2\}$, then by Proposition 6.9 (i) the Pareto sets intersect and each Pareto set is partially weakly dominated by the other one. Subprocedure Intersect is used to find the intersection point.The case in which both points satisfy $\tilde{\mathbf{y}}^{si} \in \tilde{\mathcal{Y}}_P^l - \mathbb{R}^2_>$ for $i = 1, 2$ is addressed in Subprocedure 1.

When the dominance decision taken in Subprocedures 1-3 implies that two Pareto sets intersect or are likely to intersect, Subprocedure Intersect is called.

Subprocedure Intersect: This subprocedure is used to compute the intersection points between two Pareto sets. When this subprocedure is initiated, the existence of these points may be unknown. Each of the two Pareto sets, $\tilde{\mathcal{Y}}_P^s$ and $\tilde{\mathcal{Y}}_P^l$, being the input to this suprocedure, is available parametrically in the form $(f_1(\tilde{\mathbf{x}}(\lambda)), f_2(\tilde{\mathbf{x}}(\lambda)))$ for $\lambda \in [0, 1]$, where $\tilde{\mathbf{x}}(\lambda)$ is the parametric optimal solution to problem (2.9) and is provided by the mpLCP method for each invariancy interval in $[0, 1]$. Since an intersection point may be located in any invariancy interval of each Pareto set, all pairs of invariancy intervals shall be checked. Let $[\lambda_1^l, \lambda_2^l] \subseteq [0, 1]$

and $[\lambda_1^s, \lambda_2^s] \subseteq [0,1]$ be two invariancy intervals found for $\tilde{\mathcal{Y}}_P^l$ and $\tilde{\mathcal{Y}}_P^s$, respectively. For the pair $([\lambda_1^l; \lambda_2^l], [\lambda_1^s; \lambda_2^s])$, the following system of two polynomial equations is solved to identify the parameter values, $\lambda^l \in [\lambda_1^l; \lambda_2^l]$ and $\lambda^s \in [\lambda_1^s; \lambda_2^s]$, that determine the intersection point(s).

$$\begin{aligned} &f_i(\tilde{\mathbf{x}}(\lambda^s)) - f_i(\tilde{\mathbf{x}}(\lambda^l)) = 0 \quad i =, 1, 2 \\ &\lambda^s \in [\lambda_1^s, \lambda_2^s],\ \lambda^l \in [\lambda_1^l, \lambda_2^l] \end{aligned} \tag{6.1}$$

Let $(\hat{\lambda}^s, \hat{\lambda}^l)$ be a solution to system (6.1). Then the intersection point is given by $\tilde{\mathbf{y}}^{int} = \Big(f_1(\tilde{\mathbf{x}}(\hat{\lambda^\kappa})), f_2(\tilde{\mathbf{x}}(\hat{\lambda^\kappa}))\Big)$ for $\kappa = s, l$. Note that one invariancy interval may contain more than one intersection point or even infinitely many intersection points if the two curves (partially) coincide. The polynomial equation solver *roots* in *MATLAB* is used to solve (6.1). While the solutions to (6.1) can be real or complex numbers, only the real solutions are reported. If a real solution to (6.1) is not found or all solutions found are complex numbers for the examined pair of invariancy intervals, we conclude that the Pareto sets do not intersect in that pair [29].

With the intersection points found from Subprocedure Intersect, the nondominated set $\tilde{\mathcal{Y}}_N^{sl}$ can be constructed. For example, consider the case depicted in Figure 24 which has one intersection point and let $\tilde{\mathbf{y}}^{int}$ denote that point. Then $\tilde{\mathcal{Y}}_N^{sl} = \mathscr{C}^s[\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{int}] \cup \mathscr{C}^l[\tilde{\mathbf{y}}^{int}, \tilde{\mathbf{y}}^{l2}]$.

7. Complete BB Algorithm.

The four modules presented in the prior sections are now integrated into a BB Algorithm 7.1 that computes efficient solutions and Pareto points to BOMIQP (2.1). The algorithm is presented in the form of pseudo-code, its properties and complexity are discussed, an example BOMIQP is solved, and numerical results on test instances are presented. The details of solving the example are given in the Appendix.

7.1. Algorithm 7.1 .

The data of problem $\mathcal{P}$ is the input to Algorithm 7.1. Four sets are initiated: the set S contains all new node problems, the set Λ^ℓ contains all invariancy intervals associated with the solution to node ℓ problem, and the incumbent sets, $\mathcal{X}_a$ and $\mathcal{Y}_a$, in the decision and objective space respectively, as defined in Section 3 (line 2).

The BB algorithm begins by computing an initial set of efficient and Pareto solutions, $\mathcal{X}_E^0$ and $\mathcal{Y}_P^0$, as described in Section 3 (line 3). At the root node 0, all integer variables are relaxed and problem $\tilde{\mathcal{P}}^0$ is solved for the parametric sets $\tilde{\mathcal{X}}_E^0$ and $\tilde{\mathcal{Y}}_P^0$ (line 5). If the set $\tilde{\mathcal{X}}_E^0$ is feasible to $\mathcal{P}$, then $\mathcal{P}$ has been solved: $\mathcal{X}_E = \tilde{\mathcal{X}}_E^0$ and $\mathcal{Y}_P = \tilde{\mathcal{Y}}_P^0$ (lines 6-7).

Else, all the invariancy intervals of $\tilde{\mathcal{P}}^0$ are added to Λ^0 and examined one at a time (lines 10-19). Consider an invariancy interval $[\lambda', \lambda'']$, and the associated solution $\mathbf{x}(\lambda)$ and its image $\mathbf{y}(\lambda)$ for $\lambda \in [\lambda', \lambda'']$. If this solution is feasible to $\mathcal{P}$, then $\mathbf{y}(\lambda)$ is added to the set $\mathcal{Y}_a$ that is updated in the set dominance module to satisfy $\mathcal{Y}_a = N(\mathcal{Y}_a)$. Due to this update, $\mathcal{Y}_a$ may contain $\mathbf{y}(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R] \subseteq [\lambda', \lambda'']$. The set $\mathcal{X}_a$ is then updated accordingly to contain the preimage of the current $\mathcal{Y}_a$. Otherwise, the branching is performed, i.e., node problems are created for all variables $\mathbf{x}_i(\lambda) \notin \mathbb{Z}$, $i = \{p+1, \ldots, n\}$ and added to the set S. If there are more than one candidate for a branching variable, one can select a variable based on the index of that variable choosing the one with the lowest (or the largest) index.

In the main step, a node problem, say node s problem from the set S, is solved for $\tilde{\mathcal{X}}_E^s$ and $\tilde{\mathcal{Y}}_P^s$ (line 22). The fathoming is applied next (lines 23-33). If the node is fathomed due to infeasibility or based on rule (5.4), then it is deleted from S. Rule (5.4) is implemented using Rule 1 and Rule 2 (see Section 5). If the node is fathomed due to integer feasibility, then $\mathbf{y}^s(\lambda)$ for $\lambda \in [0,1]$ is added to the set $\mathcal{Y}_a$ that is updated in the set dominance module

Algorithm 7.1 The branch-and-bound algorithm for BOMIQPs.

1: INPUT: Problem $\mathcal{P}$
2: $S = \emptyset$, $\Lambda^{\ell} = \emptyset$, $\mathcal{X}_a = \emptyset$, $\mathcal{Y}_a = \emptyset$
3: Calculate $\mathcal{X}_E^0$ and $\mathcal{Y}_P^0$ (solve (2.8))
4: $\mathcal{X}_a \leftarrow \mathcal{X}_E^0$, $\mathcal{Y}_a \leftarrow \mathcal{Y}_P^0$
5: Compute $\tilde{\mathcal{X}}_E^0$ and $\tilde{\mathcal{Y}}_P^0$ (solve (2.2))
6: **if** $\mathbf{x}_i(\lambda) \in \mathbb{Z}\ \forall\ i = p+1, \ldots, n,\ \lambda \in [0, 1]$ **then**
7:     $\mathcal{X}_E = \tilde{\mathcal{X}}_E^0$, $\mathcal{Y}_P = \tilde{\mathcal{Y}}_P^0$
8: **else**
9:     Add all invariancy intervals to Λ^0
10:     **while** $\Lambda^0 \neq \emptyset$ **do**
11:         Select an invariancy interval $[\lambda', \lambda''] \subseteq \Lambda^0$
12:         **if** $\mathbf{x}_i(\lambda) \in \mathbb{Z}\ \forall\ i = p+1, \ldots, n,\ \lambda \in [\lambda', \lambda'']$ **then**
13:             $\mathcal{Y}_a \leftarrow \mathbf{y}(\lambda)$, $\mathcal{Y}_a = N(\mathcal{Y}_a)$
14:             $\mathcal{X}_a \leftarrow \mathbf{x}(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R] \subseteq [\lambda', \lambda'']$
15:         **else**
16:             Branch: Create node problems on each $x_i(\lambda) \in \tilde{\mathcal{X}}_E^0$ for $i = \{p+1, \ldots, n\}$ s.t. $x_i(\lambda) \notin \mathbb{Z}$
17:             Add new node problems to S
18:         **end if**
19:         Delete the invariancy interval
20:     **end while**
21:     **while** $S \neq \emptyset$ **do**
22:         Select problem $\tilde{\mathcal{P}}^s$ from S and compute $\tilde{\mathcal{X}}_E^s$, $\tilde{\mathcal{Y}}_P^s$
23:         **if** $\tilde{\mathcal{P}}^s$ infeasible **then**
24:             Delete $\tilde{\mathcal{P}}^s$ from S
25:             Goto line 21
26:         **else if** node fathomed due to rule (5.4) **then**
27:             Delete $\tilde{\mathcal{P}}^s$ from S
28:             Goto line 21
29:         **else if** $\mathbf{x}_i^s(\lambda) \in \mathbb{Z},\ \forall\ i = p+1, \ldots, n,\ \lambda \in [0, 1]$ **then**
30:             $\mathcal{Y}_a \leftarrow \mathbf{y}^s(\lambda)$, $\mathcal{Y}_a = N(\mathcal{Y}_a)$
31:             $\mathcal{X}_a \leftarrow \mathbf{x}^s(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R] \subseteq [0, 1]$
32:             Delete $\tilde{\mathcal{P}}^s$ from S
33:             Goto line 21
34:         **else**
35:             Add all invariancy intervals to Λ^s
36:             **while** $\Lambda^s \neq \emptyset$ **do**
37:                 Select an invariancy interval from $[\lambda', \lambda''] \subseteq \Lambda^s$
38:                 **if** $\mathbf{x}_i^s(\lambda) \in \mathbb{Z}\ \forall\ i = p+1, \ldots, n,\ \lambda \in [\lambda', \lambda'']$ **then**
39:                     $\mathcal{Y}_a \leftarrow \mathbf{y}^s(\lambda)$, $\mathcal{Y}_a = N(\mathcal{Y}_a)$
40:                     $\mathcal{X}_a \leftarrow \mathbf{x}^s(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R] \subseteq [\lambda', \lambda'']$
41:                 **else**
42:                     Branch: Create node problems on each $x_i^s(\lambda) \in \tilde{\mathcal{X}}_E^s$ for $i = \{p+1, \ldots, n\}$ s.t. $x_i^s(\lambda) \notin \mathbb{Z}$
43:                     Add new node problems to S
44:                 **end if**
45:                 Delete the invariancy interval
46:             **end while**
47:         **end if**
48:     **end while**
49: **end if**
50: OUTPUT: $\mathcal{X}_E = \mathcal{X}_a$ and $\mathcal{Y}_P = \mathcal{Y}_a$

to satisfy $\mathcal{Y}_a = N(\mathcal{Y}_a)$. Due to this update, $\mathcal{Y}_a$ may contain $\mathbf{y}^s(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R] \subseteq [0, 1]$. The set $\mathcal{X}_a$ is then updated accordingly to contain the preimage of the current $\mathcal{Y}_a$ and the node is deleted from S.

If the node is not fathomed, all the invariancy intervals of $\tilde{\mathcal{P}}^s$ are added to Λ^s and examined one at a time (lines 35-45). Consider an invariancy interval $[\lambda', \lambda'']$, and the associated solution $\mathbf{x}^s(\lambda)$ and its image $\mathbf{y}^s(\lambda)$ for $\lambda \in [\lambda', \lambda'']$. If this solution is feasible to $\mathcal{P}$, then $\mathbf{y}^s(\lambda)$ is added to the set $\mathcal{Y}_a$ that is updated in the set dominance module to satisfy $\mathcal{Y}_a = N(\mathcal{Y}_a)$. Due to this update, $\mathcal{Y}_a$ may contain $\mathbf{y}^s(\lambda)$ for $\lambda \in [\lambda^L, \lambda^R] \subseteq [\lambda', \lambda'']$. The set $\mathcal{X}_a$ is then updated accordingly to contain the preimage of the current $\mathcal{Y}_a$.

Else, the branching is performed on fractional variables (line 42). The current node is deleted from S and the new node problems are added to S (lines 43-45). This process is repeated until all the node problems in S have been examined. At termination, the current sets $\mathcal{X}_a$ and $\mathcal{Y}_a$ yield the solution sets, $\mathcal{X}_E$ and $\mathcal{Y}_P$, to $\mathcal{P}$ respectively.

7.2. Properties of Algorithm 7.1. We now prove that all Pareto points to BOMIQP (2.1) are computed upon termination of Algorithm 7.1.

Theorem 7.1. *Upon termination, the BB Algorithm 7.1 returns the complete Pareto set for $\mathcal{P}$ (2.1), i.e., $\mathcal{Y}_P = \mathcal{Y}_a$.*

Proof. The proof is based on the properties of the three main modules of Algorithm 7.1, the branching, fathoming and set dominance, which are responsible for performing the steps of the BB scheme. During the execution of the algorithm, the incumbent sets $\mathcal{Y}_a$ and $\mathcal{X}_a$ are dynamically updated by adding elements that are feasible to problem $\mathcal{P}$. These elements are computed by solving the node problems that are created by the branching and fathoming modules. At the initialization of the algorithm, an initial set of of efficient solutions, $\mathcal{X}_E^0$, and their images, $\mathcal{Y}_P^0$, are computed and stored in the incumbent sets $\mathcal{X}_a$ and $\mathcal{Y}_a$, respectively.

Note that the root node 0 is a special node of the BB tree. The following discussion addresses the main step of the algorithm in which an arbitrary node s (including node 0) of the BB tree is examined. At node s, problem $\tilde{\mathcal{P}}^s$ is solved and the fathoming module is applied. If the node problem is infeasible or the set $\tilde{\mathcal{Y}}_P^s$ satisfies the condition in rule (5.4), then this node can be fathomed. This guarantees that the infeasible solutions or the solutions dominated by the incumbent set are excluded from the search. If the set (or a subset of) $\tilde{\mathcal{X}}_E^s$ is feasible to $\mathcal{P}$, then $\tilde{\mathcal{Y}}_P^s$ (or a subset of $\tilde{\mathcal{Y}}_P^s$) is added to $\mathcal{Y}_a$ so that $\mathcal{Y}_a$ remains nondominated. $\tilde{\mathcal{X}}_E^s$ is added to $\mathcal{X}_a$ that is updated to be the preimage of the current $\mathcal{Y}_a$. This guarantees that no efficient solutions or the associated Pareto points are excluded from the search.

If the node is not fathomed, the branching module is applied to all invariancy intervals whose integer variables $x_i, i \in \{p+1, \ldots, n\}$, have fractional values. While this module creates more node problems, some of them are new and some others may have been obtained earlier. Only the new node problems are considered. The branching and fathoming procedures are repeated until all the node problems have been examined. Since every node is generated through branching, no feasible solutions to $\mathcal{P}$ are eliminated during the search. Each time new elements are added to $\mathcal{Y}_a$, the set dominance module is executed to filter and discard the dominated points in $\mathcal{Y}_a$ and keep $\mathcal{Y}_a = N(\mathcal{Y}_a)$.

The three main modules guarantee that only the Pareto points to $\mathcal{P}$ remain in $\mathcal{Y}_a$ and that $\mathcal{Y}_a$ contains all Pareto points to $\mathcal{P}$. The associated set $\mathcal{X}_a$ is also updated accordingly to contain only the efficient solutions to $\mathcal{P}$. ■

The complexity of Algorithm 7.1 originates mainly from solving three types of single objective quadratic programs (QPs). First, the mpLCP method employed at the nodes as the solver for $\tilde{\mathcal{P}}^s$ solves parametric QPs and primarily contributes to the complexity. The number of invariancy intervals is exponential in $(n+m)$, where n and m are the numbers of variables and constraints in $\tilde{\mathcal{P}}^s$ respectively, and a polynomial-time algorithm can never be developed to solve the node problem [2]. The total number of node problems can be determined before the algorithm runs based on the number of slice problems, while the actual number of these problems being solved results from the branching module and remains unknown. Second, mixed integer QPs are solved to compute an initial set of Pareto points and efficient solutions at the initialization of Algorithm 7.1, and it can take an exponential time to solve them [24]. The number of these problems is decided by the user. Third, convex quadratically constrained QPs (QCQPs), whose number is unknown at the beginning of the algorithm, are solved in the fathoming and set dominance modules. Convex QCQPs are solvable in polynomial time [22]. Additionally, polynomial fractional optimization problems, which are solvable with algorithms of limited complexity only for certain cases [11, 25], are solved in the branching module and systems of two polynomial equations are solved in the set dominance module. Solving all these problems contributes to the total run time of Algorithm 7.1.

7.3. Example. Consider the following BOMIQP with one integer variable.

$$(7.1)\quad \begin{array}{lll} \mathcal{P}: & min & \mathbf{f}(\mathbf{x}) = \left[f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q_1 \mathbf{x} + \boldsymbol{p}_1^T \mathbf{x}, f_2(\mathbf{x}) \frac{1}{2}\mathbf{x}^T Q_2 \mathbf{x} + \boldsymbol{p}_2^T \mathbf{x}\right] \\ & s.t. & \mathbf{x} \in \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 \times \mathbb{Z} : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}, x_2 \in \mathbb{Z}, x_2 \leq 4\}, \end{array}$$

where

$$Q_1 = \begin{bmatrix} 6 & -6 & 6 \\ -6 & 14 & -10 \\ 6 & -10 & 8 \end{bmatrix}, \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 16 & -2 \\ 2 & -2 & 4 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Algorithm 7.1 computes the efficient and Pareto sets for this BOMIQP. Table 1 presents this efficient set being the union of four subsets originating from three node problems. The parameters $\lambda_i, i = 1, 2, 3$ are the weights used to solve each of these problems. The efficient solutions are associated with a single invariancy interval in the first two problems having the integer variable x_2 assume the values of 0 and 1, but with two invariancy intervals in the third problem with $x_2 = 2$.

Figure 2 depicts the Pareto set being a disconnected nonconvex curve consisting of convex subcurves. Two of the three subcurves are neither open nor closed in agreement with the invariancy intervals. The reader is referred to the Appendix for more details on the example.

7.4. Numerical experiments. Algorithm 7.1 is implemented in the MATLAB programming language and numerical experiments are performed. This implementation follows the

Table 1: Efficient set, $\mathcal{X}_E$, for BOMIQP (7.1)

$$\hat{\mathbf{x}}(\lambda_1) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{1-\lambda_1}{2\lambda_1+1} \\ x_2 = 0 \\ x_3 = 0 \end{array} \quad \text{for } \lambda_1 \in [0,1] \right\}$$

$$\hat{\mathbf{x}}(\lambda_2) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{3}{2\lambda_2+1} - 5\lambda_2 + 3 \\ x_2 = 1 \\ x_3 = 5\lambda_2 - 3 \end{array} \quad \text{for } \lambda_2 \in (0.85,1] \right\}$$

$$\hat{\mathbf{x}}(\lambda_3) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{-16\lambda_3^2+\lambda_3+9}{2\lambda_3+1} \\ x_2 = 2 \\ x_3 = 8\lambda_3 - 4 \end{array} \quad \text{for } \lambda_3 \in (0.7317, 0.7819] \right\}$$

$$\hat{\mathbf{x}}(\lambda_3) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = 0 \\ x_2 = 2 \\ x_3 = \frac{9\lambda_3+1}{2(\lambda_3+1)} \end{array} \quad \text{for } \lambda_3 \in [0.7819,1] \right\}$$

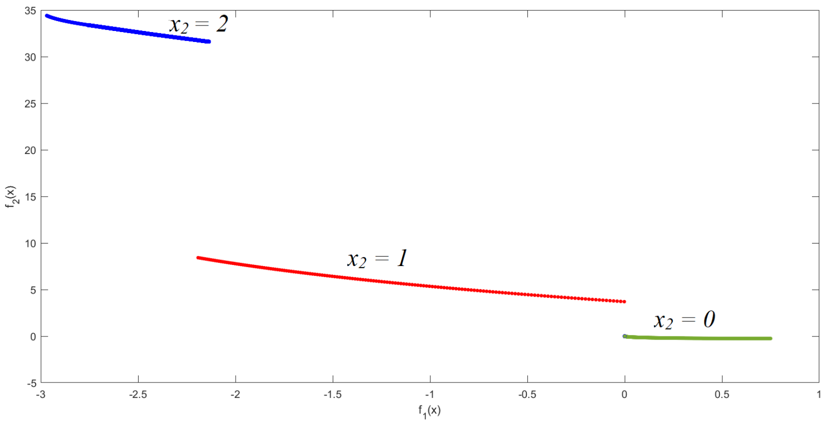

Figure 2: Pareto set, $\mathcal{Y}_P$, for BOMIQP (7.1)

pseudo-code of Algorithm 7.1 and uses the node problem solver implemented by Adelgren [1].

A set of randomly generated strictly convex BOMIQPs are solved. The tests have been performed on a Lenovo Ideapad FLEX 4 with a 256 GB SSD storage, 6th Generation Intel Core i5-6200U, 2.30GHz, 2401 Mhz, 2 Cores, 4 Logical Processors and 8GB memory.

The obtained numerical results are summarized in Table 2. In the columns from the first to the last one the following items are displayed: the dimension of the decision space, number of integer or binary variables, number of instances, average number of invariancy intervals ($\mathcal{II}s$) in the root node problem, average number of nodes in the BB tree without the root node, average number of invariancy intervals examined, and CPU time for solving the instances. This time aggregates the total CPU time it takes to solve an instance.

n	$n-p$	no. of instances	Average no. of $\mathcal{II}s$ in root node	Average no. of nodes	Average no. of $\mathcal{II}s$ examined	Average time (seconds)
2	1	5	1.4	3.2	2.8	87.7
3	1	5	2.6	3.8	7.4	137.9
	2	5	3.0	5.4	8.4	303.0
4	1	5	2.6	5.3	10.2	1002.3
	2	5	2.2	6.2	14.0	1479.2
5	1	3	2.3	8.3	18.3	3075.3
	2	3	2.6	8.6	22.3	4243.6
6	1	3	3.0	10.3	23.6	4521.3
	2	3	3.3	11.6	25.6	6135.3
7	1	3	3.0	11.3	30.6	8835.6
	2	3	2.0	12.3	31.6	9343.6
8	1	3	2.0	11.3	33.3	10343.3
	2	3	2.3	11.3	29.6	14445.3

Table 2: Summary of the results for BOMIQP instances solved with Algorithm 7.1

Since the implementation of the mpLCP method is recent and still at the stage of being rudimentary, it is sensitive to adding branching constraints. Therefore only small-sized instances with one or two binary or integer variables and up to eight variables total have been solved. One can observe that as the number of integer variables increases, the total computation time also increases as expected. Because solving the node problems takes a big portion of the total computational time, as the number of node problems increases, the run time of the algorithm also increases. The total number of the node problems seems related to the number of the examined invariancy intervals. Since BOMIQPs have previously been unsolved, there are no instances in the literature to use, and there is no other algorithm to compare with.

8. Conclusion.

We have developed the first algorithm to compute the exact Pareto set of BOMIQP (2.1). The algorithm computes two solution sets in parametric form: the Pareto set in the objective space and its pre-image, the efficient set, in the decision space. Since (2.1) is a global optimization problem, the algorithm follows a BB scheme. The branching module is integrated with the mpLCP method, a state-of-the-art algorithm employed as a node problem solver. The new practical fathoming rules introduced in the fathoming module are based on the bound sets established in a multiobjective setting. The new set domination module filters the incumbent set to become the Pareto set of (2.1) at termination of the algorithm.

The appealing and valuable feature of providing the exact solution sets to BOMIQPs is contrasted with the exponential complexity of the algorithm, which is reflected in an increase in computational time with an increase of the number of integer variables as well as node problems even for BOMIQPs with up to eight variables.

This work immediately opens up numerous avenues for future research. It is desirable to make the mpLCP solver more stable when solving the node problems that are encountered along the branches of the BB tree. Adopting methods to solve fractional programs will enhance the performance of the branching module, while using a more robust solver for polynomial equations will improve the set dominance module. More numerical studies are needed to determine the tradeoff in the set dominance module between the solving of the polynomial equations to determine intersection points between two Pareto sets and the solving of single objective quadratic programs to examine their mutual location. In general, since the proposed algorithm relies on solving four types of optimization problems as well as systems of polynomial equations, improvements in each of the five directions will affect not only the presented algorithm performance but will also advance polynomial, mixed-integer, integer, and fractional optimization.

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Appendix for the manuscript entitled
"A Branch and Bound Algorithm for Biobjective Mixed Integer Quadratic Programs"
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Appendix A. Supporting Information. This appendix consists of three sections and contains additional information in support of the theory and methodology presented in the paper. Some theory to accompany Section 6 and the steps of the BB algorithm on BOMIQP example (7.1) are presented in the first two sections respectively. The last section contains figures illustrating certain concepts defined in Section 2, the fathoming rules presented in Section 5, and the dominance procedure described in Section 6.

A.1. Results from Section 6. Proposition 6.6 is quoted below as Proposition A.1 and the proof follows.

Proposition A.1. *Let $S_1, S_2 \subset \mathbb{R}^2$, $(S_i + \mathbb{R}^2_{\geqq})_N \neq \emptyset$ and be externally stable for $i = 1, 2$.*
(i) The set S_1 dominates the set S_2, $S_1 \leq S_2$, if and only if $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{\geq}$.
(ii) The set S_1 weakly dominates the set S_2, $S_1 \leqq S_2$, if and only if $S_2 + \mathbb{R}^2_{\geqq} \subseteq S_1 + \mathbb{R}^2_{\geqq}$.
(iii) The set S_1 strictly dominates the set S_2, $S_1 < S_2$, if and only if $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{>}$.

Proof. $(i) \Rightarrow$ Assume $S_1 \leq S_2$. By Definition 6.1, for each $\mathbf{y}^2 \in S_2$ there exists $\mathbf{y}^1 \in S_1$ such that $\mathbf{y}^1 \leq \mathbf{y}^2$. Let $\mathbf{z}^2 \in S_2 + \mathbb{R}^2_{\geqq}$ such that $\mathbf{z}^2 = \mathbf{y}^2 + \mathbf{d}^2$ where $\mathbf{y}^2 \in S_2$ and $\mathbf{d}^2 \geqq \mathbf{0}$. We have $\mathbf{y}^1 + \mathbf{d}^2 \leq \mathbf{y}^2 + \mathbf{d}^2 = \mathbf{z}^2$, which implies $\mathbf{z}^2 = \mathbf{y}^1 + \mathbf{d}^2 + \bar{\mathbf{d}}^2$ where $\bar{\mathbf{d}}^2 \geq \mathbf{0}$. Then $\mathbf{z}^2 = \mathbf{y}^1 + \mathbf{d}^1$, where $\mathbf{d}^1 = \mathbf{d}^2 + \bar{\mathbf{d}}^2 \geq \mathbf{0}$. Then $\mathbf{z}^2 \in S_1 + \mathbb{R}^2_{\geq}$ and hence $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{\geq}$. Note that by Definition 6.1, $S_2 + \mathbb{R}^2_{\geqq} \neq S_1 + \mathbb{R}^2_{\geq}$.
$\Leftarrow$ Assume $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{\geq}$. By Proposition 6.5, $(S_1 + \mathbb{R}^2_{\geqq})_N \leqq S_1 + \mathbb{R}^2_{\geqq}$, or equivalently, for each $\mathbf{y} \in S_1 + \mathbb{R}^2_{\geqq}$ there exists $\mathbf{y}^1 \in (S_1 + \mathbb{R}^2_{\geqq})_N$ such that $\mathbf{y}^1 \leqq \mathbf{y}$. Then for each $\mathbf{y} \in S_1 + \mathbb{R}^2_{\geq}$ there exists $\mathbf{y}^1 \in (S_1 + \mathbb{R}^2_{\geqq})_N$ such that $\mathbf{y}^1 \leq \mathbf{y}$. By Proposition 2.3 in [12], we have $(S_1 + \mathbb{R}^2_{\geqq})_N = (S_1)_N$. Then for each $\mathbf{y} \in S_1 + \mathbb{R}^2_{\geq}$ there exists $\mathbf{y}^1 \in (S_1)_N$ such that $\mathbf{y}^1 \leq \mathbf{y}$. Because $S_2 + \mathbb{R}^2_{\geqq} \subset S_1 + \mathbb{R}^2_{\geq}$, for each $\mathbf{y} \in S_2 + \mathbb{R}^2_{\geqq}$ there exists $\mathbf{y}^1 \in (S_1)_N$ such that $\mathbf{y}^1 \leq \mathbf{y}$. Because $S_2 \subset S_2 + \mathbb{R}^2_{\geqq}$, for each $\mathbf{y} \in S_2$ there exists $\mathbf{y}^1 \in (S_1)_N$ such that $\mathbf{y}^1 \leq \mathbf{y}$. Since $(S_1)_N \subseteq S_1$, for each $\mathbf{y} \in S_2$ there exists $\mathbf{y}^1 \in S_1$ such that $\mathbf{y}^1 \leq \mathbf{y}$ and thus $S_1 \leq S_2$.
(ii) and (iii) The proofs are similar to the proof of part (i) and therefore omitted. ■

Proposition A.2 is included as an additional result, which is associated with the efficient and Pareto sets of relaxed node problems of the same branch in the BB tree. However, this proposition also pertains to the case of a general multiobjective optimization problem with the feasible set reduced to a subset.

Proposition A.2. *Let $\tilde{\mathcal{P}}^s, \tilde{\mathcal{P}}^l$ be two relaxed node problems (2.5) such that $\tilde{\mathcal{X}}^l \subseteq \tilde{\mathcal{X}}^s$.*
1. *If $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l = \emptyset$, then (i) $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l_E = \emptyset$, (ii) $\tilde{\mathcal{Y}}^s_P \leq \tilde{\mathcal{Y}}^l_P$.*
2. *If $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l \neq \emptyset$, (i) then $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l \subseteq \tilde{\mathcal{X}}^l_E$, (ii) and $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l \subset \tilde{\mathcal{X}}^l_E$, then $\tilde{\mathcal{Y}}^s_P \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}^l_P$.*

Proof. 1. (i) By definition $\tilde{\mathcal{X}}^l_E \subseteq \tilde{\mathcal{X}}^l$, but $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l = \emptyset$, therefore $\tilde{\mathcal{X}}^s_E \cap \tilde{\mathcal{X}}^l_E = \emptyset$.
(ii) From (i), if $\mathbf{x} \in \tilde{\mathcal{X}}^l_E$, then $\mathbf{x} \notin \tilde{\mathcal{X}}^s_E$. By Definition 2.3, there exists $\mathbf{x}^1 \in \tilde{\mathcal{X}}^s$ such that $\mathbf{f}(\mathbf{x}^1) \leq \mathbf{f}(\mathbf{x})$ and then there exists $\mathbf{x}^2 \in \tilde{\mathcal{X}}^s_E$ such that $\mathbf{f}(\mathbf{x}^2) \leq \mathbf{f}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}^2) = \mathbf{y}^2 \in$

$\tilde{\mathcal{Y}}_P^s$. Since $\mathbf{x} \in \tilde{\mathcal{X}}_E^l$, $\mathbf{f}(\mathbf{x}) = \mathbf{y} \in \tilde{\mathcal{Y}}_P^l$ and therefore $\mathbf{y}^2 \leq \mathbf{y}$. Since $\mathbf{x} \in \tilde{\mathcal{X}}_E^l$ is arbitrary, so is $\mathbf{y} \in \tilde{\mathcal{Y}}_P^l$. Then for each $\mathbf{y} \in \tilde{\mathcal{Y}}_P^l$ there exists $\mathbf{y}^2 \in \tilde{\mathcal{Y}}_P^s$ such that $\mathbf{y}^2 \leq \mathbf{y}$. Therefore by Definition 6.1, $\tilde{\mathcal{Y}}_P^s \leq \tilde{\mathcal{Y}}_P^l$.

2. (i) By contradiction, assume $\mathbf{x} \in \tilde{\mathcal{X}}_E^s \cap \tilde{\mathcal{X}}^l$ and $\mathbf{x} \notin \tilde{\mathcal{X}}_E^l$. The former implies $\mathbf{x} \in \tilde{\mathcal{X}}_E^s$ and $\mathbf{x} \in \tilde{\mathcal{X}}^l$. The latter implies either (a) $\mathbf{x} \notin \tilde{\mathcal{X}}^l$, which is a contradiction, or (b) $\mathbf{x} \in \tilde{\mathcal{X}}^l$ and there exists $\mathbf{x}^1 \in \tilde{\mathcal{X}}_E^l$ such that $\mathbf{f}(\mathbf{x}^1) \leq \mathbf{f}(\mathbf{x})$. Then $\mathbf{x}^1 \in \tilde{\mathcal{X}}^l$ and hence $\mathbf{x}^1 \in \tilde{\mathcal{X}}^s$. This implies $\mathbf{x} \notin \tilde{\mathcal{X}}_E^s$, a contradiction.

   (ii) We need to show $\tilde{\mathcal{Y}}_P^s \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^l$. By Definition 6.3, we show there exists a nonempty subset $\tilde{\mathcal{Y}}_P^{l1} \subset \tilde{\mathcal{Y}}_P^l$ such that (a) $\tilde{\mathcal{Y}}_P^s \leq \tilde{\mathcal{Y}}_P^{l1}$; and (b) $\tilde{\mathcal{Y}}_P^l \setminus \tilde{\mathcal{Y}}_P^{l1}$ is nondominated with respect to $\tilde{\mathcal{Y}}_P^s$. By Definition 6.1, part (a) becomes that for each $\mathbf{y}^1 \in \tilde{\mathcal{Y}}_P^{l1}$ there exists $\mathbf{y} \in \tilde{\mathcal{Y}}_P^s$ such that $\mathbf{y} \leq \mathbf{y}^1$. By Definition 6.2, part (b) becomes that there does not exist $\mathbf{y} \in \tilde{\mathcal{Y}}_P^s$ such that $\mathbf{y} \leq \mathbf{y}'$ for each $\mathbf{y}' \in \tilde{\mathcal{Y}}_P^l \setminus \tilde{\mathcal{Y}}_P^{l1}$. Below we continue part (a) and (b) separately.

   (a) From (i), if $\mathbf{x} \in \tilde{\mathcal{X}}_E^s \cap \tilde{\mathcal{X}}^l$ then $\mathbf{x} \in \tilde{\mathcal{X}}_E^l$, and therefore if $\mathbf{f}(\mathbf{x}) = \mathbf{y} \in \tilde{\mathcal{Y}}_P^s \cap \tilde{\mathcal{Y}}^l$ then $\mathbf{y} \in \tilde{\mathcal{Y}}_P^l$. Because of the strict containment in the assumption, (a1) there exists $\mathbf{x}^1 \in \tilde{\mathcal{X}}_E^l$ such that $\mathbf{x}^1 \notin \tilde{\mathcal{X}}_E^s \cap \tilde{\mathcal{X}}^l$; and (a2) we can define $\tilde{\mathcal{Y}}_P^{l1} = \tilde{\mathcal{Y}}_P^l \setminus (\tilde{\mathcal{Y}}_P^s \cap \tilde{\mathcal{Y}}_P^l) \neq \emptyset$ and $\mathbf{f}(\mathbf{x}^1) = \mathbf{y}^1 \in \tilde{\mathcal{Y}}_P^{l1}$. In (a1), the former implies $\mathbf{x}^1 \in \tilde{\mathcal{X}}^l$ which contradicts the latter. We are left with $\mathbf{x}^1 \notin \tilde{\mathcal{X}}_E^s$ meaning there exists $\mathbf{x}^2 \in \tilde{\mathcal{X}}^s$ such that $\mathbf{f}(\mathbf{x}^2) \leq \mathbf{f}(\mathbf{x}^1)$. Then there also exists $\mathbf{x}^3 \in \tilde{\mathcal{X}}_E^s$ such that $\mathbf{f}(\mathbf{x}^3) \leq \mathbf{f}(\mathbf{x}^1)$, which using (a2) can be written that there exists $\mathbf{f}(\mathbf{x}^3) = \mathbf{y}^3 \in \tilde{\mathcal{Y}}_P^s$ such that $\mathbf{y}^3 \leq \mathbf{y}^1$. Since $\mathbf{x}^1 \in \tilde{X}_E^l \setminus (\tilde{X}_E^s \cap \tilde{X}^l)$ is arbitrary, so is $\mathbf{y}^1 \in \tilde{\mathcal{Y}}_P^{l1}$. Therefore, for all $\mathbf{y}^1 \in \tilde{\mathcal{Y}}_P^{l1}$ there exists $\mathbf{y}^3 \in \tilde{\mathcal{Y}}_P^s$ such that $\mathbf{y}^3 \leq \mathbf{y}^1$.

   (b) Using the identities of relative complements [16] in the set theory, $\tilde{\mathcal{Y}}_P^l \setminus \tilde{\mathcal{Y}}_P^{l1} = \tilde{\mathcal{Y}}_P^l \setminus (\tilde{\mathcal{Y}}_P^l \setminus (\tilde{\mathcal{Y}}_P^s \cap \tilde{\mathcal{Y}}_P^l)) = \tilde{\mathcal{Y}}_P^s \cap \tilde{\mathcal{Y}}_P^l$. Then, by Definition 2.3, there does not exist a point $\mathbf{y} \in \tilde{\mathcal{Y}}_P^s$ such that $\mathbf{y} \leq \mathbf{y}'$ for each $\mathbf{y}' \in \tilde{\mathcal{Y}}_P^s \cap \tilde{\mathcal{Y}}_P^l$, as desired. ■

A.2. Example. The BB algorithm relies on the initialization and four modules: solving a node problem, branching, fathoming, and a dominance procedure. Each of these five procedures is applied to an example BOMIQP for illustration. Consider the BOMIQP (7.1) with one integer variable.

$$\text{(A.1)} \qquad \begin{array}{lll} \mathcal{P}: & \min & \mathbf{f}(\mathbf{x}) = \left[f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q_1 \mathbf{x} + \boldsymbol{p}_1^T \mathbf{x}, f_2(\mathbf{x}) \frac{1}{2}\mathbf{x}^T Q_2 \mathbf{x} + \boldsymbol{p}_2^T \mathbf{x}\right] \\ & s.t. & \mathbf{x} \in \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 \times \mathbb{Z} : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}, x_2 \in \mathbb{Z}, x_2 \leq 4\}, \end{array}$$

where

$$Q_1 = \begin{bmatrix} 6 & -6 & 6 \\ -6 & 14 & -10 \\ 6 & -10 & 8 \end{bmatrix}, \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 16 & -2 \\ 2 & -2 & 4 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Initialization The set $\mathcal{Y}_P^0$ for $\lambda = 0, 0.1, 0.2, \cdots, 1$ where $|\mathcal{Y}_P^0| = 11$, is first computed and is depicted in Figure 3. The sets $\mathcal{Y}_a = \mathcal{Y}_P^0$ and $\mathcal{X}_a = \mathcal{X}_E^0$ are initialized.

Solving the root node problem The relaxed BOQP of (A.1), which is the root node problem of the BB tree, assumes the form:

$$\text{(A.2)} \qquad \begin{array}{lll} \tilde{\mathcal{P}}^0: & \min & \mathbf{f}(\mathbf{x}) \\ & s.t. & \mathbf{x} \in \tilde{\mathcal{X}} = \tilde{\mathcal{X}}^0 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\}. \end{array}$$

The weighted-sum problem associated with (A.2) is formulated

$$\tilde{\mathcal{P}}^0(\lambda): \quad \min \quad \lambda f_1(\mathbf{x}) + (1-\lambda) f_2(\mathbf{x})$$
$$\text{s.t.} \quad \mathbf{x} \in \tilde{\mathcal{X}}^1 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\}, \tag{A.3}$$
$$\lambda \in [0,1]$$

and solved with the mpLCP method that provides the optimal solution functions and the associated invariancy intervals. At optimality of (A.3), the parameter space is partitioned into three invariancy intervals. These intervals and the optimal solution functions, which are also efficient solutions to (A.2), are given in Table 3.

Table 3: Efficient set for $\tilde{\mathcal{P}}^0(\lambda)$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{3\lambda^2-4\lambda+10}{-3\lambda^2+11\lambda+4} \\ x_2 = \frac{3\lambda^2+3\lambda+3}{-3\lambda^2+11\lambda+4} \\ x_3 = 0 \end{array} \text{ for } \lambda \in [0, 0.6747] \right\}$$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{4\lambda^3-\lambda-14\lambda^2+8}{-6\lambda^3+5\lambda+1} \\ x_2 = \frac{12\lambda^3-7\lambda^2-\lambda+5}{-6\lambda^3+5\lambda+1} \\ x_3 = \frac{3\lambda^3+34\lambda^2-8\lambda-11}{3(-6\lambda^3+5\lambda+1)} \end{array} \text{ for } \lambda \in [0.6747, 0.8182] \right\}$$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = 0 \\ x_2 = \frac{8\lambda+3\lambda^2+1}{3(-6\lambda^2+2\lambda+5)} \\ x_3 = \frac{7\lambda^2+15\lambda-7}{3(-6\lambda^2+2\lambda+5)} \end{array} \text{ for } \lambda \in [0.8182, 1] \right\}$$

Branching Consider first the solutions in the first invariancy interval, $[0, 0.6747]$. To obtain the range of the values for the integer variable x_2 on the associated interval, the following polynomial fractional programs are solved: $x_2^{min} = \min_\lambda\{x_2 = \frac{8\lambda+3\lambda^2+1}{3(-6\lambda^2+2\lambda+5)} : \lambda \in [0, 0.6747]\}$ and $x_2^{max} = \max_\lambda\{x_2 = \frac{8\lambda+3\lambda^2+1}{3(-6\lambda^2+2\lambda+5)} : \lambda \in [0, 0.6747]\}$. Using discretization of the intervals, $x_2^{min} = 0.5902$ and $x_2^{max} = 0.75$ are obtained and therefore $x_2 \in [0.5902, 0.75]$. Then $[\phi_2', \phi_2'']$ $= [0, 1]$. With this range of x_2, two new node problems are created. The first one, $\tilde{\mathcal{P}}^1(\lambda)$, with the feasible set $\tilde{\mathcal{X}}^1 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \leq 0, \mathbf{x} \geqq \mathbf{0}\}$, and the second one, $\tilde{\mathcal{P}}^2(\lambda)$, with the feasible set $\tilde{\mathcal{X}}^2 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \geq 1, \mathbf{x} \geqq \mathbf{0}\}$. Both problems are solved with the mpLCP method. The solution to $\tilde{\mathcal{P}}^1(\lambda)$ is given in Table 4 while the solution to $\tilde{\mathcal{P}}^2(\lambda)$ is given in Table 5.

Table 4: Efficient set for $\tilde{\mathcal{P}}^1(\lambda)$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{1-\lambda}{2\lambda+1} \\ x_2 = 0 \\ x_3 = 0 \end{array} \quad \text{for } \lambda \in [0,1] \right\}$$

In Table 4, $x_2 = 0$ and hence the efficient solutions to $\tilde{\mathcal{P}}^1(\lambda)$ are feasible to $\mathcal{P}$. These efficient solutions and the associated Pareto outcomes are saved in sets $\mathcal{X}_a$ and $\mathcal{Y}_a$, respectively. Then the set dominance procedure is applied to satisfy the condition $\mathcal{Y}_a = N(\mathcal{Y}_a)$ and the set$\mathcal{X}_a$ is updated accordingly to contain the preimage of the current $\mathcal{Y}_a$. Based on the feasibility, this node is fathomed. Since this node is a leaf node, branching cannot continue along this branch.

Table 5: Efficient set for $\tilde{\mathcal{P}}^2(\lambda)$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{3}{2\lambda+1} \\ x_2 = 1 \\ x_3 = 0 \end{array} \quad \text{for } \lambda \in [0,0.6] \right\}$$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{3}{2\lambda+1} - (5\lambda - 3) \\ x_2 = 1 \\ x_3 = 5\lambda - 3 \end{array} \quad \text{for } \lambda \in [0.6,0.7974] \right\}$$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = \frac{4\lambda^3-14\lambda^2-\lambda+8}{-6\lambda^3+5\lambda+1} \\ x_2 = \frac{12\lambda^3-7\lambda^2-\lambda+5}{-6\lambda^3+5\lambda+1} \\ x_3 = \frac{3\lambda^3-34\lambda^2-8\lambda+11}{3(-6\lambda^3+5\lambda+1)} \end{array} \quad \text{for } \lambda \in [0.7974,0.8182] \right\}$$

$$\hat{\mathbf{x}}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{array}{l} x_1 = 0 \\ x_2 = \frac{8\lambda^2+3\lambda+1}{3(-6\lambda^2+2\lambda+5)} \\ x_3 = \frac{7\lambda^2+15\lambda-7}{3(-6\lambda^2+2\lambda+5)} \end{array} \quad \text{for } \lambda \in [0.8182,1] \right\}$$

The solutions to $\tilde{\mathcal{P}}^2(\lambda)$ in Table 5 are now examined. In the first two invariancy intervals $x_2 = 1$ and hence the associated efficient solutions to $\tilde{\mathcal{P}}^2(\lambda)$ are feasible to problem $\mathcal{P}$. These efficient solutions and the associated Pareto outcomes are saved in sets $\mathcal{X}_a$ and $\mathcal{Y}_a$, respectively. Then the set dominance procedure is applied to satisfy the condition $\mathcal{Y}_a = N(\mathcal{Y}_a)$ and the set $\mathcal{X}_a$ is updated accordingly to contain the preimage of the current $\mathcal{Y}_a$. Branching is applied to the third and fourth invariancy intervals and is summarized in Table 6.

Table 6: Branching for invariancy intervals 3 and 4 in $\tilde{\mathcal{P}}^2(\lambda)$

	Branching process	
Invariancy intervals $[\lambda', \lambda'']$	$[0.7974, 0.8182]$	$[0.8182, 1.0000]$
x_2	$\frac{12\lambda^3-7\lambda^2-\lambda+5}{-6\lambda^3+5\lambda+1}$	$\frac{8\lambda^2+3\lambda+1}{3(-6\lambda^2+2\lambda+5)}$
x_2^{min}	1.0000	1.1858
x_2^{max}	1.1651	3.9861
New child node problem constraints	$x_2 \leq 1,\ x_2 \geq 2$	$x_2 \leq 1,\ x_2 \geq 2$ $x_2 \leq 2,\ x_2 \geq 3$ $x_2 \leq 3,\ x_2 \geq 4$

In Table 6, the second row shows the invariancy intervals $[\lambda', \lambda'']$; the third row shows the solution function x_2; the third and fourth rows show the minimum and maximum values assumed by $x_2(\lambda)$ in each invariancy interval; the last row shows the pairs of branching constraints that are generated for $x_2 \in [x_2^{min}, x_2^{max}]$. The eight child node problems reduce to six since two pairs of problems have identical constraints.

Going back to $\tilde{\mathcal{P}}^0(\lambda)$, the first invariancy interval in Table 3 has been explored. The second and third invariancy intervals, $[0.6747, 0.8182]$ and $[0.8182, 1]$, are now examined and a summary is given in Table 7.

Table 7: Branching for invariancy intervals 2 and 3 in $\tilde{\mathcal{P}}^0(\lambda)$

	Branching process	
Invariancy intervals $[\lambda', \lambda'']$	$[0.6747, 0.8182]$	$[0.8182, 1.0000]$
x_2	$\frac{12\lambda^3-7\lambda^2-\lambda+5}{-6\lambda^3+5\lambda+1}$	$\frac{8\lambda+3\lambda^2+1}{3(-6\lambda^2+2\lambda+5)}$
x_2^{min}	0.6355	1.0986
x_2^{max}	1.1210	3.9180
New child node problem constraints	$x_2 \leq 0,\ x_2 \geq 1$ $x_2 \leq 1,\ x_2 \geq 2$	$x_2 \leq 1,\ x_2 \geq 2$ $x_2 \leq 2,\ x_2 \geq 3$ $x_2 \leq 3,\ x_2 \geq 4$

In this table there are ten child node problems. The problems with the constraints $x_2 \leq 0$ and $x_2 \geq 1$ for $\lambda \in [0.6747, 0.8182]$ have already been added to the BB tree as problems $\tilde{\mathcal{P}}^1(\lambda)$ and $\tilde{\mathcal{P}}^2(\lambda)$. The remaining eight problems are identical to those in Table 6. Based on Tables 6 and 7 and to avoid duplication, the following six new node problems are formulated:

$\tilde{\mathcal{P}}^3(\lambda)$ with the feasible set $\tilde{\mathcal{X}}^3 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \leq 1, \mathbf{x} \geqq \mathbf{0}\}$;

$\tilde{\mathcal{P}}^4(\lambda)$ with $\tilde{\mathcal{X}}^4 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \geq 2, \mathbf{x} \geqq \mathbf{0}\}$;

$\tilde{\mathcal{P}}^5(\lambda)$ with $\tilde{\mathcal{X}}^5 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \leq 2, \mathbf{x} \geqq \mathbf{0}\}$;

$\tilde{\mathcal{P}}^6(\lambda)$ with $\tilde{\mathcal{X}}^6 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \geq 3, \mathbf{x} \geqq \mathbf{0}\}$;

$\tilde{\mathcal{P}}^7(\lambda)$ with $\tilde{\mathcal{X}}^7 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \leq 3, \mathbf{x} \geqq \mathbf{0}\}$;

$\tilde{\mathcal{P}}^8(\lambda)$ with $\tilde{\mathcal{X}}^8 = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} \leqq \mathbf{b}, x_2 \geq 4, \mathbf{x} \geqq \mathbf{0}\}$.

When needed, other node problems are generated with this branching procedure.

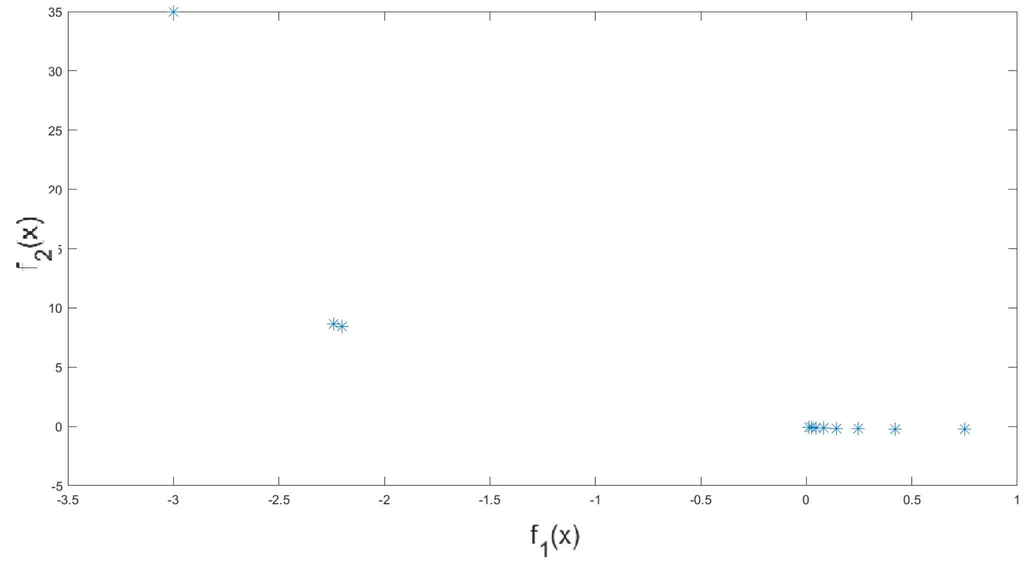


Figure 3: Initial set of Pareto points, $\mathcal{Y}_P^0$, for BOMIQP (A.1) obtained by solving $\tilde{\mathcal{P}}^0(\lambda)$ for 11 fixed values of λ

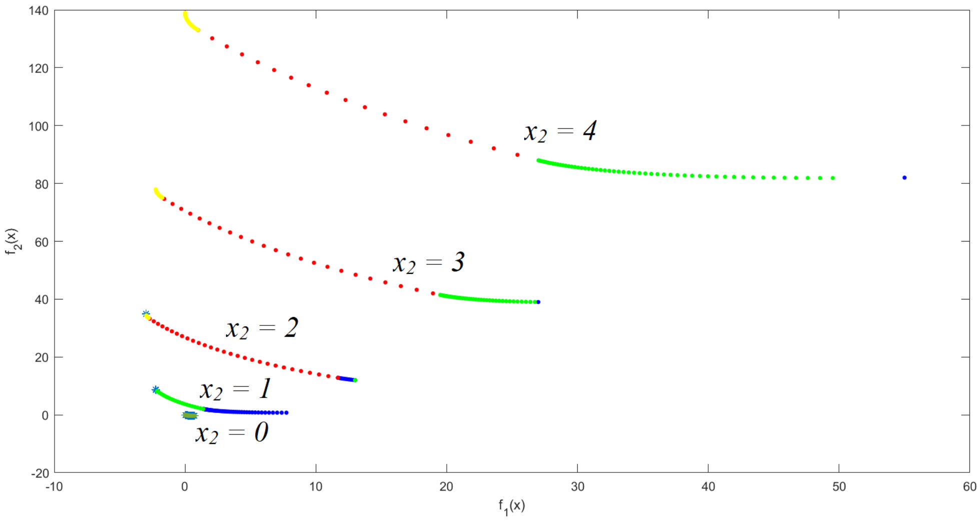


Figure 4: $\mathcal{Y}_P^0$ and the Pareto sets of five leaf node problems for BOMIQP (A.1).

Set Dominance The goal of this module is to add the Pareto sets of the node problems to $\mathcal{Y}_a$ such that $\mathcal{Y}_a = N(\mathcal{Y}_a)$.

Note that the solution to the node problem $\tilde{\mathcal{P}}^1(\lambda)$ is feasible for problem $\mathcal{P}$. We illustrate the set dominance procedure with this node. Initially, $\mathcal{Y}_a = \mathcal{Y}_P^0$ and $\mathcal{X}_a = \mathcal{X}_a^0$. By solving $\tilde{\mathcal{P}}^1(\lambda)$ we obtain $\tilde{\mathcal{Y}}_P^1$, the Pareto set to $\tilde{\mathcal{P}}^1$. The ideal point $\tilde{\mathbf{y}}^{1I} = (0, -0.25)$ for $\tilde{\mathcal{P}}^1$ is computed and the triangle T^1 with vertices $\tilde{\mathbf{y}}^{11} = (0, 0)$, $\tilde{\mathbf{y}}^{12} = (0.75, -0.25)$, and $\tilde{\mathbf{y}}^{1I} = (0, -0.25)$ is constructed. Here $\tilde{\mathbf{y}}^{11}$ and $\tilde{\mathbf{y}}^{12}$ are the end points of $\tilde{\mathcal{Y}}_P^1$ computed for $\lambda = 1$ and $\lambda = 0$, respectively.

Based on Figure 3, three clusters of the Pareto points in $\mathcal{Y}_P^0$ can be identified. These clusters are associated with the integer solutions $x_2 = 0$ (southeast cluster), $x_2 = 1$ (middle cluster) and $x_2 = 3$ (northwest cluster). Let $\mathcal{Y}_P^{SE}, \mathcal{Y}_P^M, \mathcal{Y}_P^{NW} \subset \mathcal{Y}_P^0$ denote the sets of points that are contained in the southeast, middle, and northwest cluster, respectively. We have $\mathcal{Y}_a = \mathcal{Y}_P^0 = \mathcal{Y}_P^{SE} \cup \mathcal{Y}_P^M \cup \mathcal{Y}_P^{NW}$. To add $\tilde{\mathcal{Y}}_P^1$ to $\mathcal{Y}_a$ and have $\mathcal{Y}_a = N(\mathcal{Y}_a)$, the location of $\tilde{\mathcal{Y}}_P^1$ and T^1 with respect to the current $\mathcal{Y}_a$ is examined. By solving $\tilde{\mathcal{P}}(\mathbf{y}^R)$ (2.10) with $\mathbf{y}^R = \mathbf{y}$ for all $\mathbf{y} \in \mathcal{Y}_P^{SE}$, we obtain that all $\mathbf{y} \in \mathcal{Y}_P^{SE}$ are in $\tilde{\mathcal{Y}}_P^1$. By solving $\tilde{\mathcal{P}}(\mathbf{y}^R)$ (2.10) with $\mathbf{y}^R = \mathbf{y}$ for all $\mathbf{y} \in \mathcal{Y}_P^M$, and later for all $\mathbf{y} \in \mathcal{Y}_P^{NW}$, we discover that the points in $\mathcal{Y}_P^M, \mathcal{Y}_P^{NW}$, and $\tilde{\mathcal{Y}}_P^1$ are all nondominated. Therefore, $\mathcal{Y}_a$ is updated as $\mathcal{Y}_a = \mathcal{Y}_P^M \cup \mathcal{Y}_P^{NW} \cup \tilde{\mathcal{Y}}_P^1$, and the set $\mathcal{X}_a$ is updated accordingly.

When needed, the set dominance procedure is executed to filter the Pareto sets of the other node problems.

Fathoming The bound set based fathoming is applied to the node problems that are not discarded due to the fathoming rules relying on these problems' feasibility or infeasibility (cf. Section 5). In the example, having applied the set dominance procedure to $\tilde{\mathcal{P}}^1$, we have $\mathcal{Y}_a = \mathcal{Y}_P^M \cup \mathcal{Y}_P^{NW} \cup \tilde{\mathcal{Y}}_P^1$. The fathoming makes use of the set $\bar{\mathcal{Y}}_a$, the set of points containing the initial Pareto points in $\mathcal{Y}_P^0$ and the end points of the strictly convex curves and subcurves stored in $\mathcal{Y}_a$, that is $\bar{\mathcal{Y}}_a = \{(-3.0, 35.0), (-2.24, 8.87)$,(-2.20, 8.46) ,$(0, 0), (0.75, -0.25)\}$.

We illustrate the fathoming procedure with the node problem $\tilde{\mathcal{P}}^2$. By solving $\tilde{\mathcal{P}}^2(\lambda)$ we obtain $\tilde{\mathcal{Y}}_P^2$, the Pareto set to $\tilde{\mathcal{P}}^2$. The ideal point $\tilde{\mathbf{y}}^{2I} = (-3, 0.75)$ for $\tilde{\mathcal{P}}^2$ is computed and the triangle T^2 with vertices $\tilde{\mathbf{y}}^{21} = (-3, 35)$, $\tilde{\mathbf{y}}^{22} = (7.75, 0.75)$, and $\tilde{\mathbf{y}}^{2I} = (-3, 0.75)$ is constructed. Here $\tilde{\mathbf{y}}^{21}$ and $\tilde{\mathbf{y}}^{22}$ are the end points of $\tilde{\mathcal{Y}}_P^2$ for $\lambda = 1$ and $\lambda = 0$, respectively. Then the set $\bar{\mathcal{Y}}_a^2 = \bar{\mathcal{Y}}_a \cap (T^2 - \mathbb{R}^2_{\geqq})$ is obtained. Now Rule 1 (cf. Section 5) is checked. Since there does not exist a point $\mathbf{y}^i \in \bar{\mathcal{Y}}_a^2$ such that $\mathbf{y}^i \in \{\tilde{\mathbf{y}}^{2I}\}^{\leqq}$, Rule 1 does not hold. Then the conditions $\bar{\mathcal{Y}}_a^2 \cap C^{2W} = \emptyset$ and $\mathcal{Y}_a^2 \cap C^{2S} = \emptyset$ are checked. We have $\mathcal{Y}_a^2 \cap C^{2W} = \emptyset$. Therefore, node 1 cannot be fathomed.

When needed, the fathoming procedure is executed to decide whether a node problem can be discarded or not.

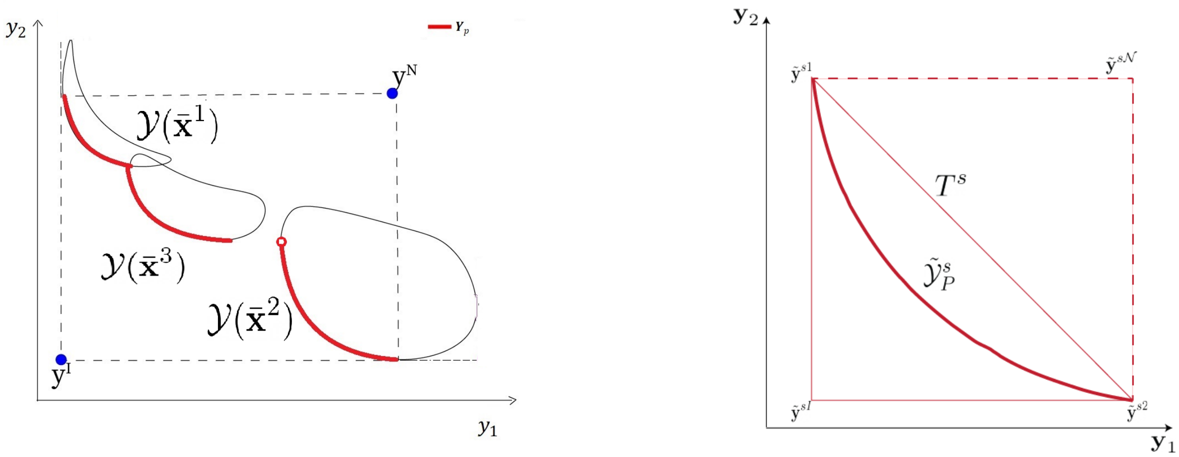
Termination. Having examined all the invariancy intervals for all the nodes that required investigation, the algorithm provides $\mathcal{Y}_P = \mathcal{Y}_a$ and $\mathcal{X}_E = \mathcal{X}_a$ as given in Figure 2 and Table 1 respectively.

Leaf Node Problems We also provide additional information on the leaf node problems for example (A.1) to better illustrate the structure of the final Pareto set, $\mathcal{Y}_P$. Figure 4 depicts the Pareto curves of five node problems solved for the integer variable $x_2 = 0, 1, 2, 3, 4$. Each of the curves for $x_2 = 2, 3, 4$ is composed of three subcurves while the curve for $x_2 = 1$ is composed of two subcurves. Each subcurve is associated with a different invariancy interval. The Pareto set for $x_2 = 0$ is associated with a single invariancy interval. Note that some Pareto points computed during the initialization are located on the Pareto sets for $x_2 = 0, 1, 2$. In particular, the entire Pareto set for $x_2 = 0$ has been computed during the initialization.

The final Pareto set, $\mathcal{Y}_P$, is composed of a nondominated subset of the Pareto set for $x_2 = 2$, a nondominated subset of the Pareto set for $x_2 = 1$, and the entire Pareto set for $x_2 = 0$.

A.3. Figures.

A.3.1. Figures Illustrating Preliminaries. .

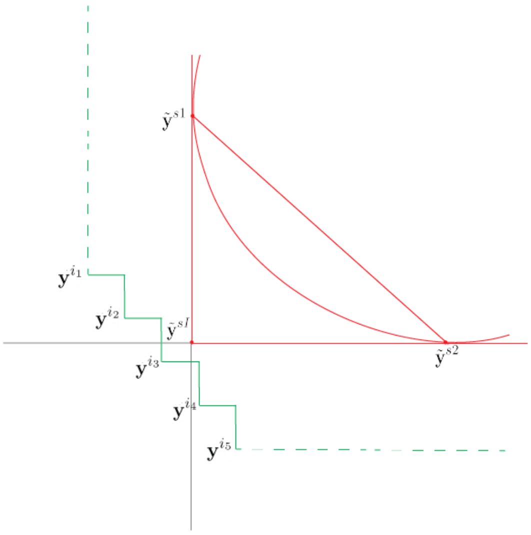


(a) Outcome sets $\mathcal{Y}(\bar{x}^i), i = 1, 2, 3$; ideal point, nadir point and Pareto set for BOMIQP (2.1).

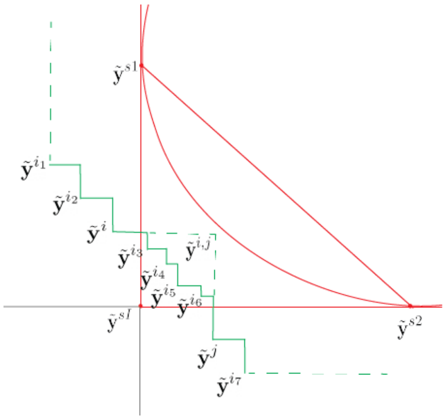
(b) Pareto set with end points, ideal point, nadir point, triangle T^s for $\tilde{\mathcal{P}}^s$ (2.5).

Figure 5: Objective Space and Pareto Sets of $\mathcal{P}$, and Pareto Sets of $\tilde{\mathcal{P}}^s$.

A.3.2. Figures Illustrating the Fathoming Rules. .

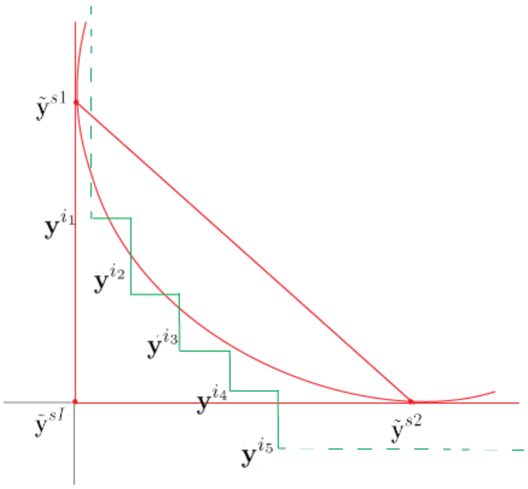


(a) Rule 1: node s can be fathomed.

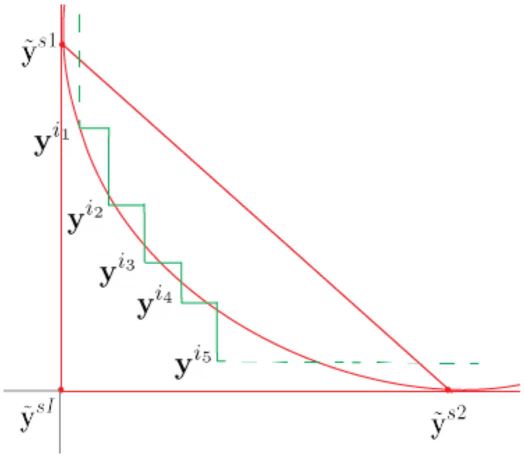


(b) The case which requires to solve only one achievement function problem (2.10) to make a fathoming decision.

Figure 6: Fathoming Rules.



(a)



(b)

Figure 7: Node s cannot be fathomed since $\bar{\mathcal{Y}}^s_a \cap C^{sW} = \emptyset$ or $\bar{\mathcal{Y}}^s_a \cap C^{sS} = \emptyset$.

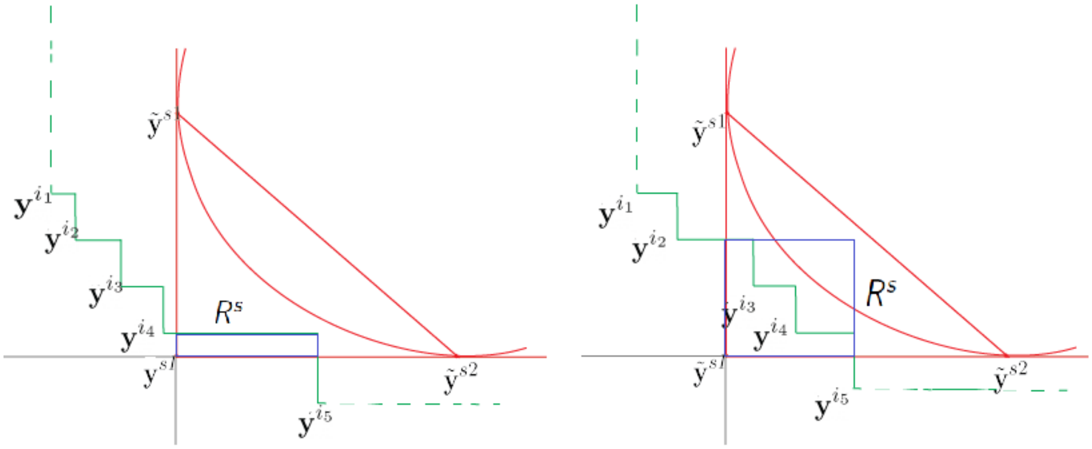
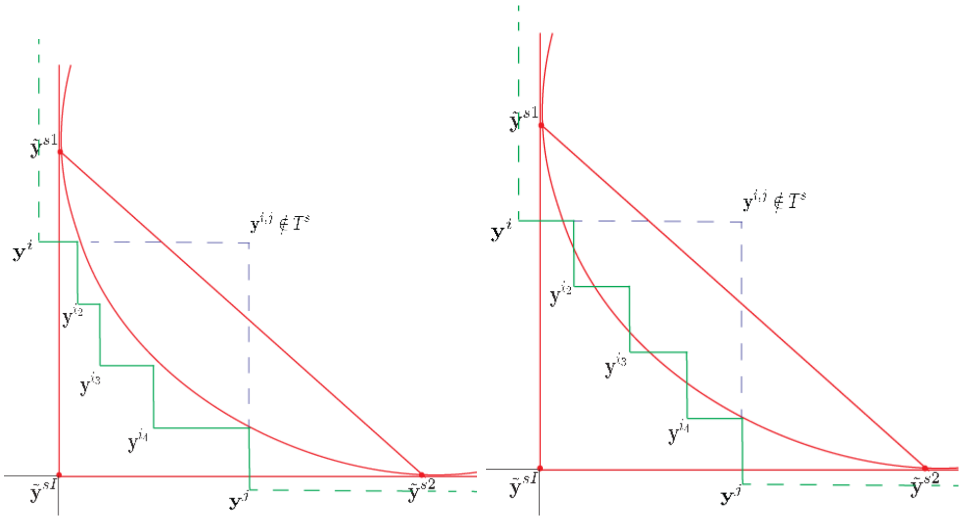


Figure 8: Fathoming Rule 2: node s can be fathomed.



(a) Node s can be fathomed (b) Node s cannot be fathomed

Figure 9: A fathoming decision is not immediate when the nadir point implied by the closest nondominated points to $\tilde{\mathbf{y}}^{sI}$ is not in T^s.

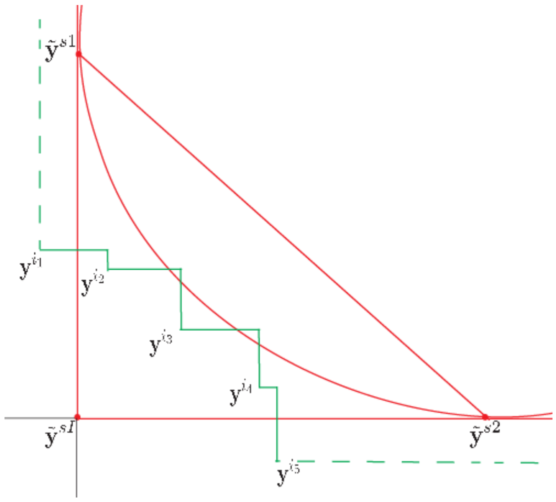
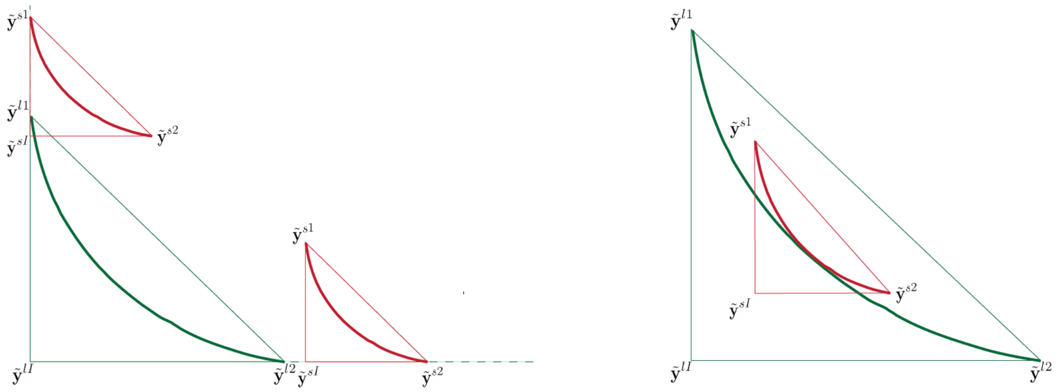


Figure 10: Node s cannot be fathomed since $\hat{\mathbf{y}}^s(\mathbf{y}^R) \not\geq \mathbf{y}^{\kappa,\eta}$ for at least one $\mathbf{y}^{\kappa,\eta}$.

A.3.3. Figures Illustrating the Set Dominance Procedure. .



(a) $\tilde{\mathcal{Y}}^l_P \leq \tilde{\mathcal{Y}}^s_P$. (b) $\tilde{\mathcal{Y}}^l_P \leqq \tilde{\mathcal{Y}}^s_P$.

Figure 11: Dominance and weak dominance between two Pareto sets.

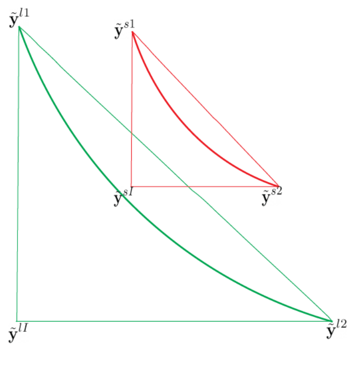


Figure 12: Strict dominance between sets: $\tilde{\mathcal{Y}}^l_P < \tilde{\mathcal{Y}}^s_P$.

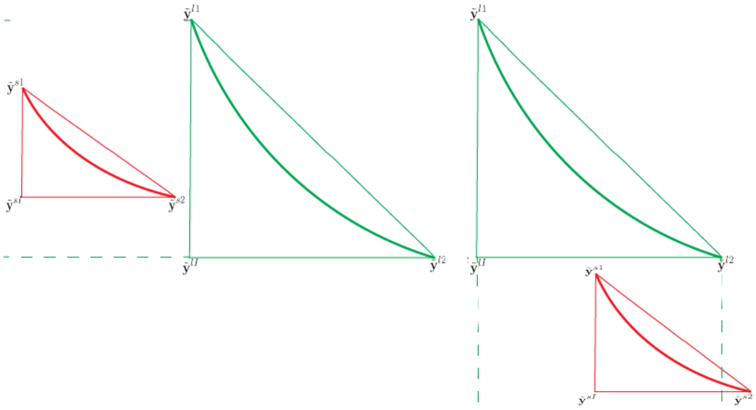


Figure 13: Two cases of Proposition 6.8(ii): $\tilde{\mathcal{Y}}_P^s \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^l$.

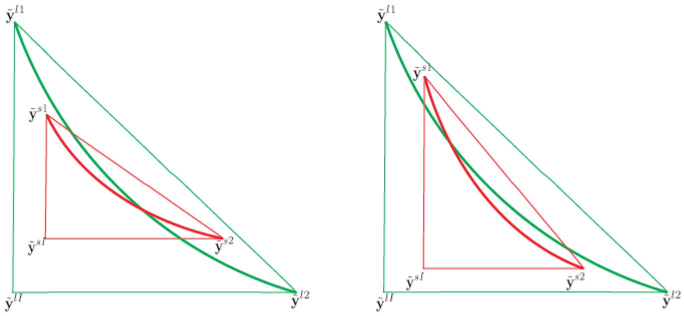
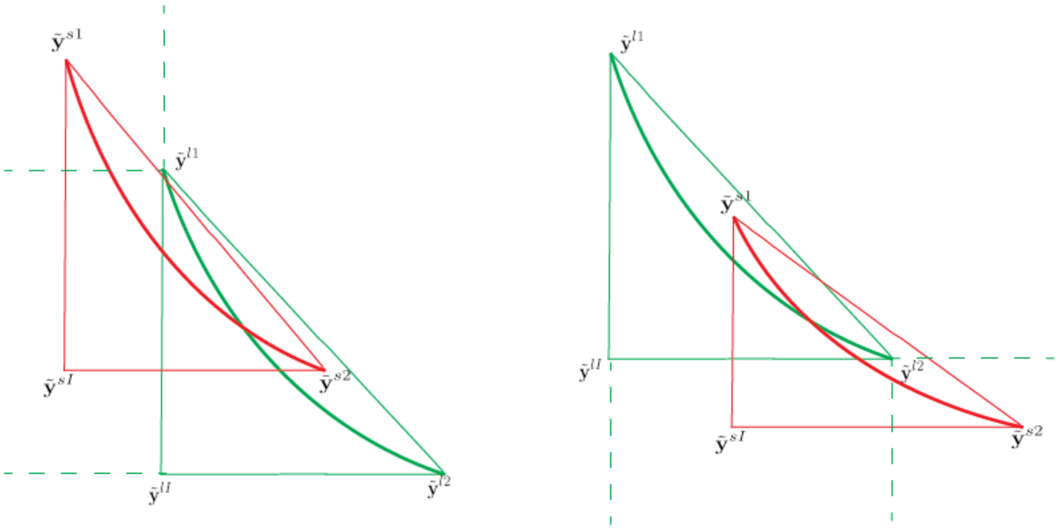
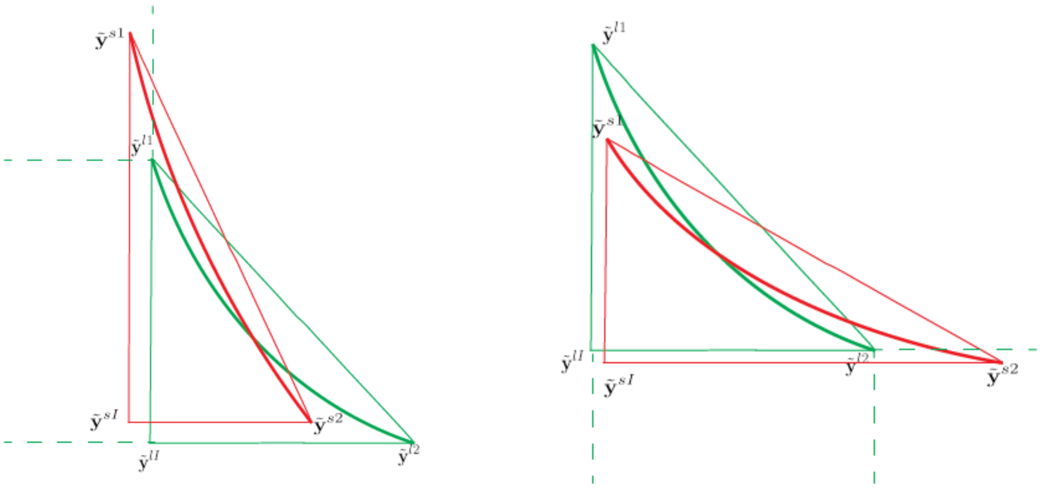


Figure 14: Two cases of Proposition 6.9(i): Pareto sets intersect, $\tilde{\mathcal{Y}}_P^s \nleqq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^l$ and $\tilde{\mathcal{Y}}_P^l \nleqq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^s$.



(a) $\tilde{\mathbf{y}}^{sI} \in C^{lW}$ and $\tilde{\mathbf{y}}^{l1} \in \tilde{\mathcal{Y}}^s_P + \mathbb{R}^2_>$ (b) $\tilde{\mathbf{y}}^{sI} \in C^{lS}$ and $\tilde{\mathbf{y}}^{l2} \in \tilde{\mathcal{Y}}^s_P + \mathbb{R}^2_>$

Figure 15: Two cases of Proposition 6.9(ii): Pareto sets intersect.



(a) $\tilde{\mathbf{y}}^{sI} \in C^{lW}$ and $\tilde{\mathbf{y}}^{l1} \in \tilde{\mathcal{Y}}^s_P - \mathbb{R}^2_>$ (b) $\tilde{\mathbf{y}}^{sI} \in C^{lS}$ and $\tilde{\mathbf{y}}^{l2} \in \tilde{\mathcal{Y}}^s_P - \mathbb{R}^2_>$

Figure 16: Two cases of Proposition 6.9(iii): Pareto sets intersect.

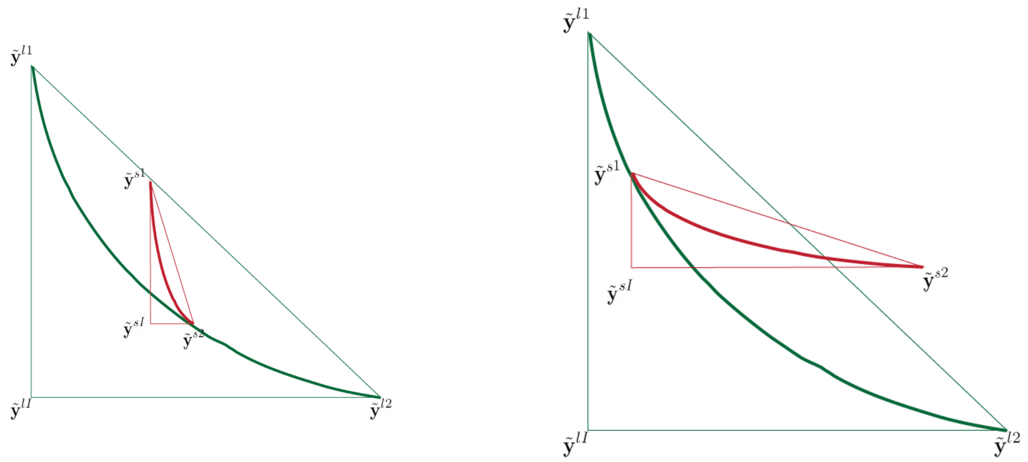


Figure 17: Proposition 6.10: $\tilde{\mathcal{Y}}_P^l \leqq \tilde{\mathcal{Y}}_P^s$.

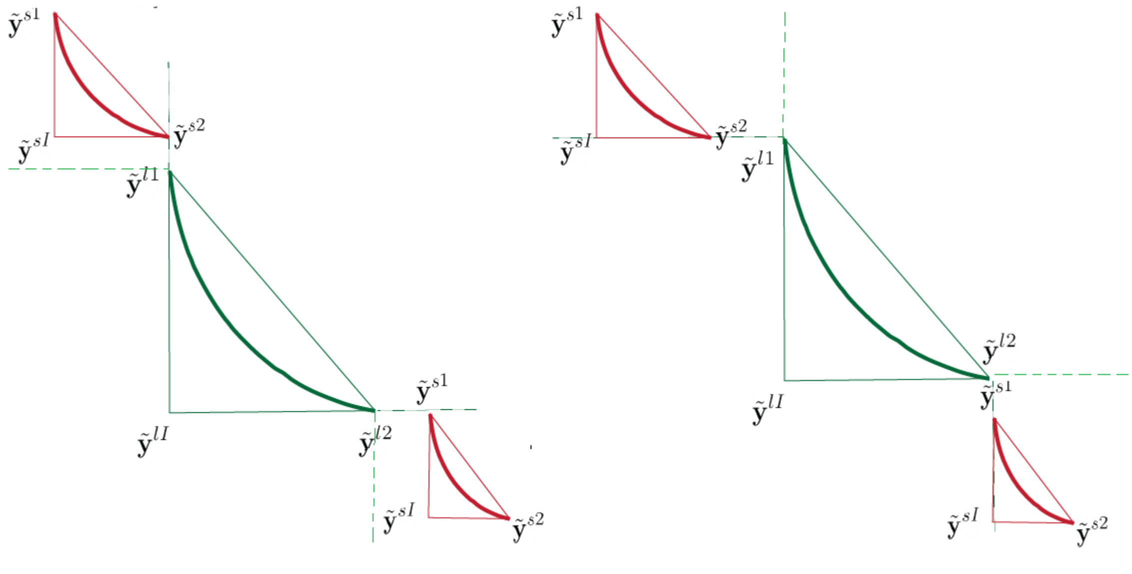
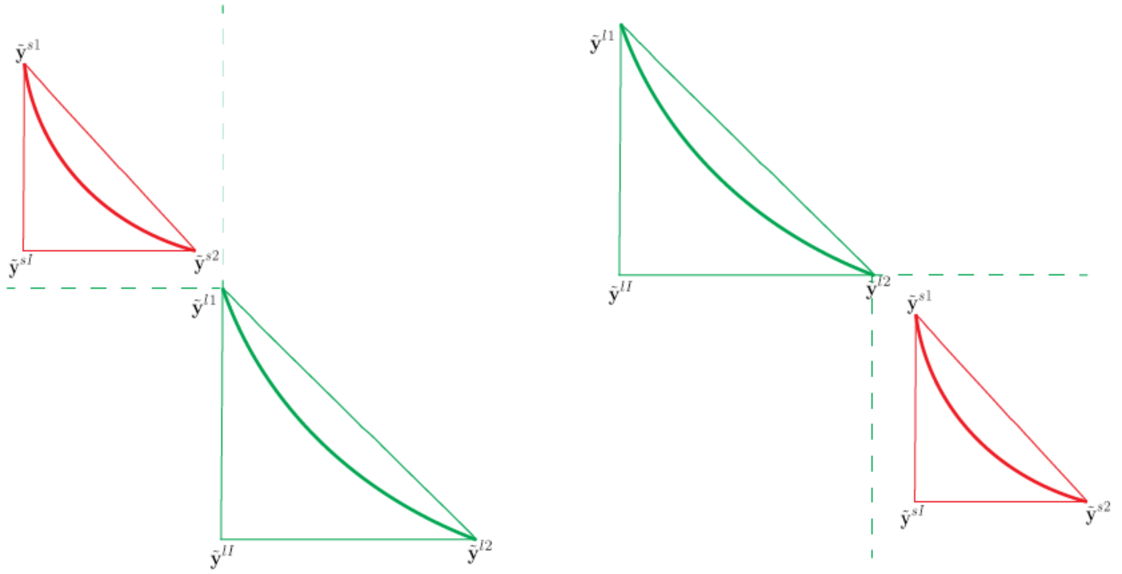
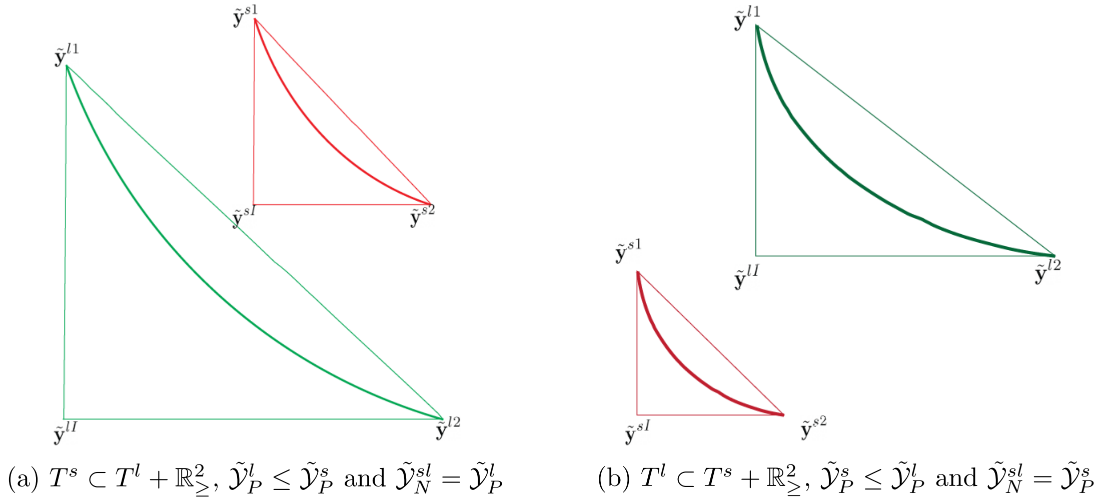


Figure 18: Proposition 6.11: $\tilde{\mathcal{Y}}_P^l \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^s$.



(a) $T^s \in \{\tilde{\mathbf{y}}^{l1}\} - \mathbb{R}^2_{\gtrless}$ (b) $T^s \in \{\tilde{\mathbf{y}}^{l2}\} + \mathbb{R}^2_{\gtrless}$

Figure 19: Two cases for $T^l \cap T^s = \emptyset$ and $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^s_P \cup \tilde{\mathcal{Y}}^l_P$.



(a) $T^s \subset T^l + \mathbb{R}^2_{\geqq}$, $\tilde{\mathcal{Y}}^l_P \leq \tilde{\mathcal{Y}}^s_P$ and $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^l_P$ (b) $T^l \subset T^s + \mathbb{R}^2_{\geqq}$, $\tilde{\mathcal{Y}}^s_P \leq \tilde{\mathcal{Y}}^l_P$ and $\tilde{\mathcal{Y}}^{sl}_N = \tilde{\mathcal{Y}}^s_P$

Figure 20: One Pareto set dominates the other.

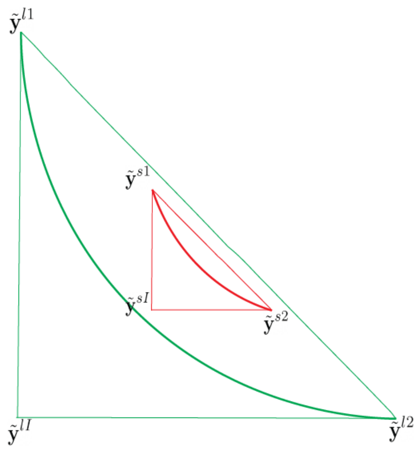
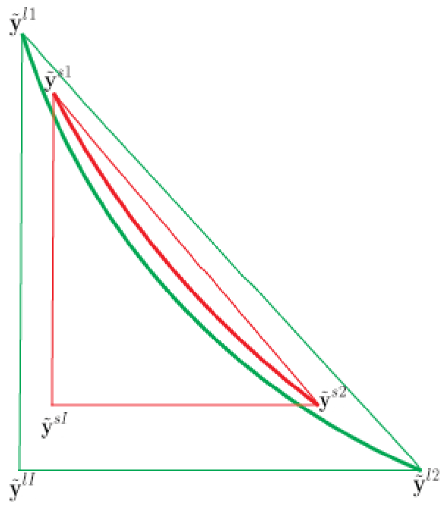
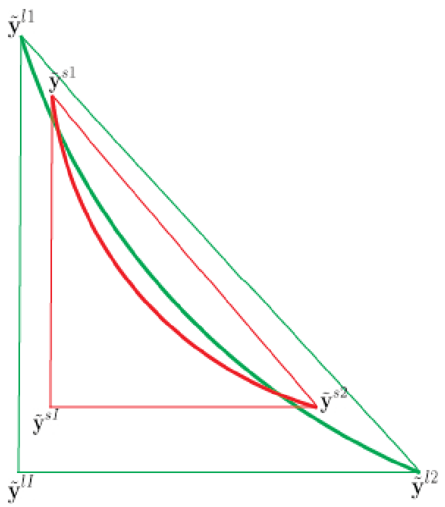


Figure 21: Subprocedure 1: $T^s \subset T^l$, $\tilde{\mathcal{Y}}_P^l \leq \tilde{\mathcal{Y}}_P^s$ and $\tilde{\mathcal{Y}}_N^{sl} = \tilde{\mathcal{Y}}_P^l$.

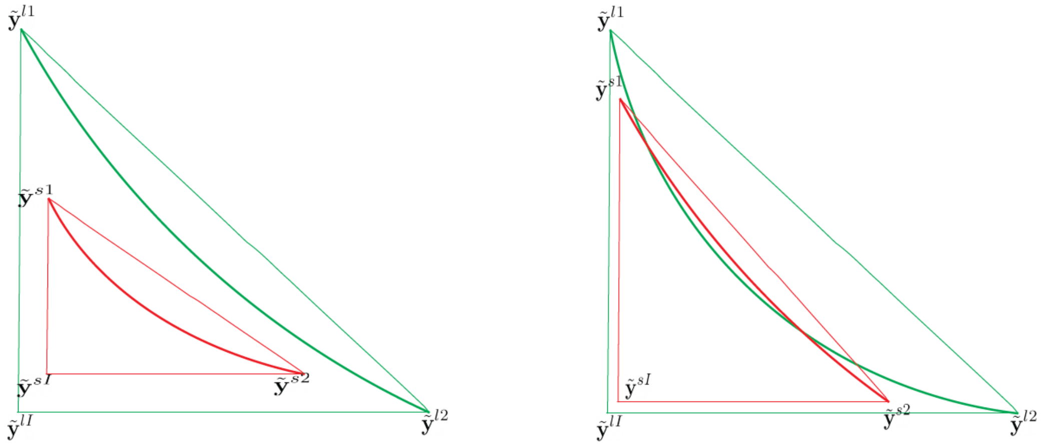


(a) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$



(b) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$

Figure 22: Subprocedure 1.



(a) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$ (b) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$

Figure 23: Subprocedure 1.

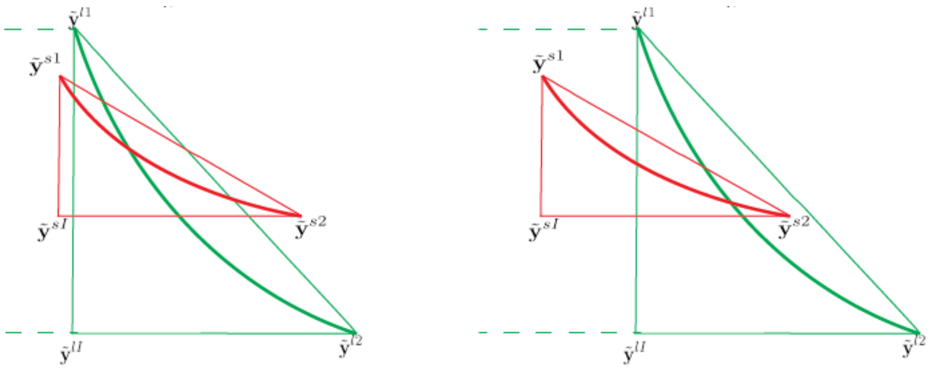
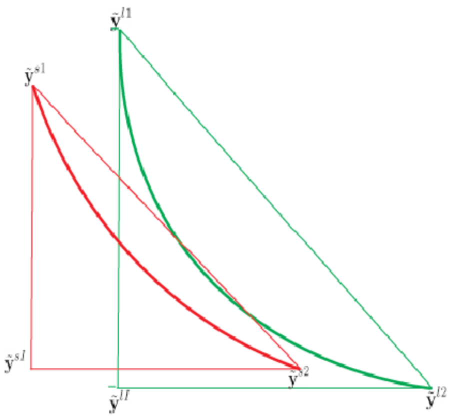
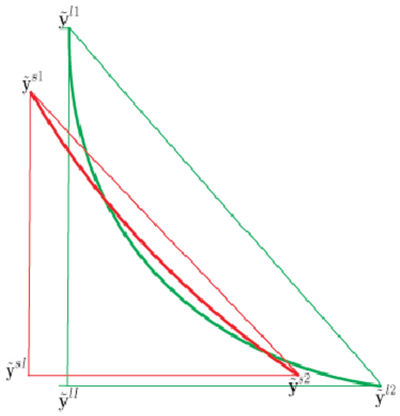


Figure 24: Subprocedure 2: $T^s \not\subset T^l$, $\tilde{\mathbf{y}}^{sI} \notin T^l$, $\tilde{\mathcal{Y}}_P^l \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^s$ and $\tilde{\mathcal{Y}}_P^s \leq_{\mathfrak{p}} \tilde{\mathcal{Y}}_P^l$.

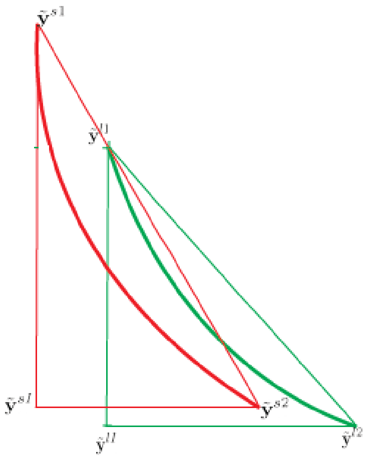


(a) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$

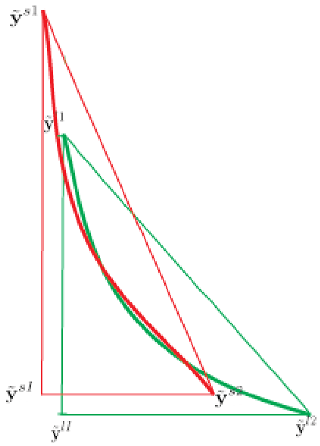


(b) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$

Figure 25: Subprocedure 2.

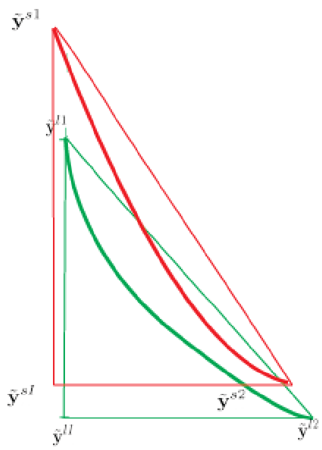


(a) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$

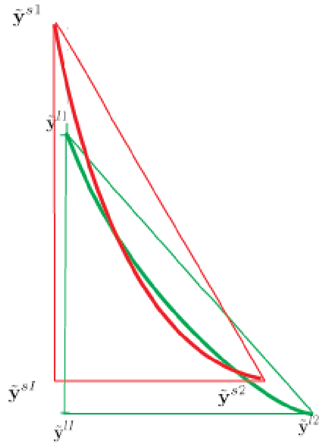


(b) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$

Figure 26: Subprocedure 2.

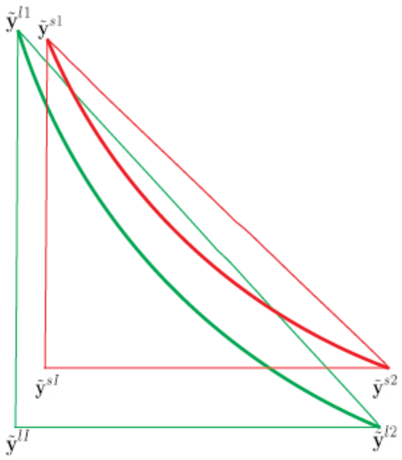


(a) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$

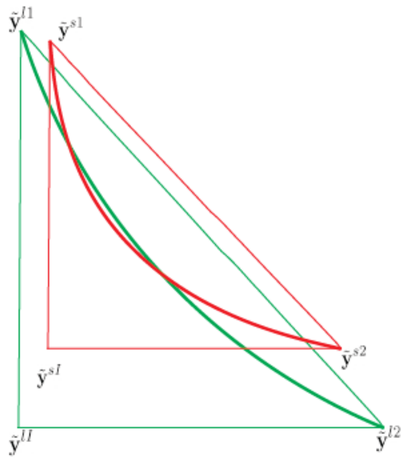


(b) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$

Figure 27: Subprocedure 2.



(a) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s = \emptyset$



(b) $\tilde{\mathcal{Y}}_P^l \cap \tilde{\mathcal{Y}}_P^s \neq \emptyset$

Figure 28: Subprocedure 3: $\tilde{\mathbf{y}}^{s1}, \tilde{\mathbf{y}}^{s2} \in \tilde{\mathcal{Y}}_P^l + \mathbb{R}^2_{>}$.