A polynomial-size extended formulation for the multilinear polytope of beta-acyclic hypergraphs

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Abstract

We consider the multilinear polytope defined as the convex hull of the set of binary points z, satisfying a collection of equations of the form $z_e = \prod_{v \in e} z_v$ for all $e \in E$. The complexity of the facial structure of the multilinear polytope is closely related to the acyclicity degree of the underlying hypergraph. We obtain a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs, hence characterizing the acyclic hypergraphs for which such a formulation can be constructed.

Key words: Binary polynomial optimization; Multilinear polytope; Hypergraph acyclicity; polynomial-size extended formulation

1 Introduction

Binary polynomial optimization, i.e., the problem of finding a binary point maximizing a polynomial function, is a fundamental NP-hard problem in discrete optimization with a wide range of applications across science and engineering. To formally define this problem, we employ a hypergraph representation scheme introduced in [13]. A *hypergraph* G is a pair (V, E), where V is a finite set of nodes and E is a set of subsets of V, called the edges of G. Throughout this paper we consider hypergraphs without loops or parallel edges, in which case E is a set of subsets of V of cardinality at least two. Moreover, the *rank* of a hypergraph G is the maximum cardinality of any edge in E. With any hypergraph $G = (V, E)$, we associate the following binary polynomial optimization problem:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{e \in E} c_e \prod_{v \in e} z_v \\ \text{s.t.} \quad & z_v \in \{0,1\} \qquad \forall v \in V, \end{aligned} \tag{BP}$$

where without loss of generality we assume $c_e \neq 0$ for all $e \in E$. Following a common practice in nonconvex optimization, we then proceed with linearizing the objective function by introducing new variables for each product term to obtain an equivalent reformulation of Problem BP in a lifted space:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{e \in E} c_e z_e \\ \text{s.t.} \quad & z_e = \prod_{v \in e} z_v \qquad \forall e \in E \\ & z_v \in \{0,1\} \qquad \forall v \in V. \end{aligned} \tag{LBP}$$

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1.1 The multilinear polytope and hypergraph acyclicity

To solve Problem LBP efficiently using polyhedral techniques, it is essential to understand the facial structure of the polyhedral convex hull of its feasible region. To this end, in the same vein as [13], we define the *multilinear set* as

$$\mathcal{S}_G = \Big\{ z \in \{0,1\}^{V+E} : z_e = \prod_{v \in e} z_v, \ \forall e \in E \Big\},$$

and we refer to its convex hull as the *multilinear polytope* and denote it by MP_G. A simple polyhedral relaxation of $\mathcal{S}_G$ can be obtained by replacing each term $z_e = \prod_{v \in e} z_v$ by its convex hull over the unit hypercube:

$$\mathrm{MP}^{\mathrm{LP}}_G = \Big\{ z : z_v \leq 1, \forall v \in V, \ z_e \geq 0, \ z_e \geq \sum_{v \in e} z_v - |e| + 1, \forall e \in E, \ z_e \leq z_v, \forall e \in E, \forall v \in e \Big\}.$$

The above relaxation is often referred to as the *standard linearization* and has been used extensively in the literature [8]. In the special case with $r = 2$; i.e., when all product terms in $\mathcal{S}_G$ are products of two variables, the multilinear polytope coincides with the well-known Boolean quadric polytope BQP_G [25]. Padberg [25] proves that BQP_G coincides with its standard linearization if and only if the graph G is acyclic. Hence it is natural to ask whether the multilinear polytope of acyclic hypergraphs has a simple structure as well. Unlike graphs, the notions of cycles and acyclicity in hypergraphs are not unique. The most well-known types of acyclic hypergraphs, in increasing order of generality, are Berge-acyclic, γ-acyclic, β-acyclic, and α-acyclic hypergraphs [2,4,19,20]. In the following, we present a brief review of the literature on the mutlilinear polytope of acyclic hypergraphs.

In [5,14], the authors prove that $\mathrm{MP}_G = \mathrm{MP}^{\mathrm{LP}}_G$ if and only if the hypergraph G is Berge-acyclic. In [14], the authors introduce *flower inequalities*, a class of facet-defining inequalities for the multilinear polytope, and show that the polytope obtained by adding all such inequalities to $\mathrm{MP}^{\mathrm{LP}}_G$ coincides with MP_G if and only if the hypergraph G is γ-acyclic. While the multilinear polytope of γ-acyclic hypergraphs may contain exponentially many facets, a polynomial-size extended formulation of MP_G is implicit in [14]. [1] Subsequently, in [16], the authors introduce *running intersection inequalities*, a class of facet-defining inequalities for the multilinear polytope that serve as a generalization of flower inequalities. The authors prove that for kite-free β-acyclic hypergraphs, a class that lies between γ-acyclic and β-acyclic hypergraphs, the polytope obtained by adding all running intersection inequalities to $\mathrm{MP}^{\mathrm{LP}}_G$ coincides with MP_G, and it admits a polynomial-size extended formulation. At the other end of the spectrum, in [11,12], the authors prove that Problem BP is strongly NP-hard over α-acyclic hypergraphs. This result implies that, unless P = NP, one cannot construct a polynomial-size extended formulation for the multilinear polytope of α-acyclic hypergraphs. See [3,6,9,10,17,18,21,22,24,28] for further results regarding polyhedral relaxations of multilinear sets.

Hence, to this date, there remains one class of acyclic hypergraphs for which we do not know whether it is possible to obtain a polynomial-size extended formulation: the class of β-acyclic hypergraphs. In [11,12], the authors present a strongly polynomial time algorithm to solve Problem BP over β-acyclic hypergraphs. While this result settles the algorithmic complexity of Problem BP over acyclic hypergraphs, it does not address the complexity of the extended formulation. Indeed, it is well-known that there exist polytopes over which one can optimize any linear function in strongly polynomial time, yet they do not admit any polynomial-size extended formulation (see, e.g., [26]).

We should remark that it is possible and in fact highly plausible that there exists a family of hypergraphs between α-acyclic and β-acyclic hypergraphs for which one can obtain a polynomial-size

[1] By *polynomial-size extended formulation,* we mean that the size of the system of linear inequalities, as defined in [27], is polynomial in the number of nodes and edges of G, which is a stronger notion than asking for a polynomial number of variables and inequalities.

extended formulation of the multilinear polytope. However, our focus in this paper is to characterize the known classes of acyclic hypergraphs for which it is possible to construct a polynomial-size extended formulation.

1.2 Our contribution

In this paper, we present a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs. Recall that a *β-cycle* of length t, for some $t \geq 3$, is a sequence $v_1, e_1, v_2, e_2, \dots, v_t, e_t, v_1$ such that $v_1, v_2, \dots, v_t$ are distinct nodes, $e_1, e_2, \dots, e_t$ are distinct edges, and v_i belongs to e_{i-1}, e_i and no other e_j, for all $i = 1, \dots, t$, where we define $e_0 := e_t$. A hypergraph is called *β-acyclic* if it does not contain any β-cycle. The following statement summarizes our main result regarding the existence of a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs:

Theorem 1. *Let $G = (V, E)$ be a β-acyclic hypergraph of rank r. Then there exists an polynomial-size extended formulation of MP_G comprising of at most $(3r-4)|V| + 4|E|$ inequalities, with at most $(r-2)|V|$ extended variables. The system is explicitly given in Theorem 7.*

It is important to note that the standard linearization of a rank r hypergraph $G = (V, E)$ consists of $2|V| + (r+2)|E|$ inequalities. It is well-understood that $\mathrm{MP}_G^{\mathrm{LP}}$ often leads to very weak relaxations of MP_G for β-acyclic hypergraphs. Theorem 1 implies that while the proposed extended formulation for MP_G contains $(r-2)|V|$ additional variables, it has fewer inequalities than the standard linearization for β-acyclic hypergraphs with $|E| \geq 3|V|$. We should also remark that the inequalities defining our proposed extended formulation are very sparse; that is, they contain at most four variables with non-zero coefficients; a feature that is highly beneficial from a computational perspective.

Our construction relies on the key concept of nest points of hypergraphs. A node $v \in V$ is a *nest point* of G if the set of the edges of G containing v is totally ordered. In other words, the edges in E containing v can be ordered so that $e_1 \subset e_2 \subset \cdots \subset e_k$. It is simple to see that nest points can be found in polynomial time. We define the hypergraph obtained from $G = (V, E)$ by *removing* a node $v \in V$ as $G - v := (V', E')$, where $V' := V \setminus \{v\}$ and $E' := \{e \setminus \{v\} : e \in E,\ |e \setminus \{v\}| \geq 2\}$. A *nest point sequence* of length s for some $s \leq |V|$ of G is an ordering $v_1, \dots, v_s$ of s distinct nodes of G, such that v_1 is a nest point of G, v_2 is a nest point of $G - v_1$, and so on, until v_s is a nest point of $G - v_1 - \cdots - v_{s-1}$. We can write this condition compactly as v_i is a nest point of $G - v_1 - \cdots - v_{i-1}$, for $i = 1, \dots, s$, where we make the slight abuse of notation $G - v_1 - \cdots - v_0 = G$. We then use the following characterization of β-acyclic hypergraphs, in terms of nest points:

Theorem 2 ([19])**.** *A hypergraph G is β-acyclic if and only if after removing recursively a nest point, until one is found, we obtain the empty hypergraph $(\emptyset, \emptyset)$.*

From Theorem 2 it follows that a hypergraph is β-acyclic if and only if it has a nest point sequence of length $|V|$. In fact, our approach to prove Theorem 1 can be used to obtain extended formulations for the multilinear polytope of more general hypergraphs containing β-cycles; namely, hypergraphs containing a nest point sequence of length s for some $1 \leq s \leq |V|$:

Theorem 3. *Let $G = (V, E)$ be a hypergraph of rank r, and let $v_1, \dots, v_s$ be a nest point sequence of G. Then an extended formulation of MP_G is given by a description of $MP_{G-v_1-\cdots-v_s}$, together with a system of at most $|V| + 2|E| + 4rs$ linear inequalities, including at most $(r-2)s$ extended variables. The system is characterized in Theorem 6.*

To prove Theorems 1 and 3, we present, in Theorem 4, a new sufficient condition for decomposability of multilinear sets that is of independent interest.

A natural question is whether it is possible to characterize the multilinear polytope of β-acyclic hypergraphs in the original space of variables. We argue that for a β-acyclic hypergraph G, an explicit description of MP_G in the original space does not have desirable numerical properties, as this polytope may contain *very dense* facet-defining inequalities. To demonstrate this property, in Proposition 1 we present a family of β-acyclic hypergraphs $G = (V, E)$ whose multilinear polytope consists of facet-defining inequalities containing $|E|$ variables with non-zero coefficients. It is well-understood that the addition of such dense inequalities as cutting planes to an LP relaxation in a branch-and-cut solver often leads to increased CPU times. Finally, as a byproduct of our convex hull characterizations, we present a new class of sparse valid inequalities for the multilinear polytope in the original space, which serve as a generalization of running intersection inequalities [16]. These inequalities can be incorporated in branch-and-cut based global solvers to improve the quality of existing relaxations for nonconvex problems whose factorable reformulations contain multilinear sets [17, 23].

Outline. The remainder of this paper is organized as follows. In Section 2, we present a sufficient condition for decomposability of multilinear sets that enables us to decompose multilinear sets of hypergraphs with nest points to simpler multilinear sets (see Theorem 4). In Section 3, we consider a special type of hypergraphs obtained as a result of decomposing hypergraphs with nest points, and characterize its multilinear polytope using a direct approach (see Theorem 5). In Section 4, by combining the results of Sections 2 and 3, we describe the mulilinear poyltope of hypergraphs with nest points in terms of multilinear polytopes of simpler hypergraphs (see Theorems 3 and 6). Subsequently, we obtain a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs (see Theorems 1 and 7). In Section 5, we elaborate on the complexity of the multilinear polytope of β-acyclic hypergraphs in the original space. We conclude by presenting a new class of sparse valid inequalities for the multilinear polytope of general hypergraphs.

2 Decomposability of multilinear sets

In this section, we present a new sufficient condition for decomposability of multilinear sets that we will use to obtain our extended formulations for the multilinear polytope of hypergraphs with nest points.

Consider hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cap V_2 \neq \emptyset$. We denote by $G_1 \cap G_2$ the hypergraph $(V_1 \cap V_2, E_1 \cap E_2)$ and by $G_1 \cup G_2$ the hypergraph $(V_1 \cup V_2, E_1 \cup E_2)$. Let $G := G_1 \cup G_2$. We say that the set $\mathcal{S}_G$ is *decomposable into the sets $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$* if

$$\operatorname{conv} \mathcal{S}_G = \operatorname{conv} \bar{\mathcal{S}}_{G_1} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_2},$$

where $\bar{\mathcal{S}}_{G_1}$ (resp. $\bar{\mathcal{S}}_{G_2}$) is the set of all points in the space of $\mathcal{S}_G$ whose projection in the space defined by G_1 (resp. G_2) is $\mathcal{S}_{G_1}$ (resp. $\mathcal{S}_{G_2}$).

Other known decomposition results for multilinear sets are Theorem 1 in [15], Theorem 5 in [14], Theorem 1 in [16], and Theorem 4 in [10]. In all prior decomposition results, the hypergraphs G_1 and G_2 are assumed to be section hypergraphs of G. Recall that G_1 is a *section hypergraph* of $G = (V, E)$ if $G_1 = (V_1, E_1)$, where $V_1 \subset V$ and $E_1 = \{e \in E : e \subseteq V_1\}$. This means that G_1 and G_2 inherit all edges of G contained in their respective node sets. On the contrary, in our new decomposition result, G_1 is generally not a section hypergraph of G, and this key difference allows G_1 to have a very simple structure that will be exploited in Section 3.

In the remainder of the paper, for notational simplicity, we sometimes write a node variable z_v as $z_{\{v\}}$. This can happen, for example, when we have an edge e of cardinality two, $e = \{u, v\}$, and we write the variable corresponding to u as $z_{e \setminus \{v\}} = z_{\{u\}}$. We now present our decomposition result.

Theorem 4. *Let $G = (V, E)$ be a hypergraph, let v be a nest point of G, let $e_1 \subset e_2 \subset \cdots \subset e_k$ be the edges of G containing v, and let $E_v := \{e_1, \ldots, e_k\}$. For each $i \in [k] := \{1, \cdots, k\}$, let $p_i := e_i \setminus \{v\}$*

and define $P_v := \{p \in \{p_1, \dots, p_k\} : |p| \geq 2\}$. *Assume that* $P_v \subseteq E$. *Let* $G_1 := (e_k, E_v \cup P_v)$ *and let* $G_2 := G - v$. *Then the set* $\mathcal{S}_G$ *is decomposable into* $\mathcal{S}_{G_1}$ *and* $\mathcal{S}_{G_2}$.

Proof. We assume $k \geq 1$, as otherwise the result is obvious. We now explain how we write, in the rest of the proof, a vector z in the space defined by G by partitioning its components in a number of subvectors. The vector $z_\cap$ contains the components of z corresponding to nodes and edges in $G_1 \cap G_2$, i.e., nodes in $e_k \setminus \{v\}$ and edges in P_v. The vector z_1 contains the components of z corresponding to nodes and edges in G_1 but not in G_2, i.e., node v and edges in E_v. Finally, the vector z_2 contains the components of z corresponding to nodes and edges in G_2 but not in G_1. Using these definitions, we can now write, up to reordering variables, $z = (z_1, z_\cap, z_2)$. Similarly, we can write a vector z in the space defined by G_1 as $(z_1, z_\cap)$, and a vector z in the space defined by G_2 as $z = (z_\cap, z_2)$.

We now proceed with the proof of the theorem. To this end, we show the two inclusions $\operatorname{conv} \mathcal{S}_G \subseteq \operatorname{conv} \bar{\mathcal{S}}_{G_1} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_2}$ and $\operatorname{conv} \mathcal{S}_G \supseteq \operatorname{conv} \bar{\mathcal{S}}_{G_1} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_2}$. The first inclusion clearly holds, since $\mathcal{S}_G \subseteq \bar{\mathcal{S}}_{G_1} \cap \bar{\mathcal{S}}_{G_2}$. Thus, it suffices to show the inclusion $\operatorname{conv} \mathcal{S}_G \supseteq \operatorname{conv} \bar{\mathcal{S}}_{G_1} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_2}$. Let $\tilde{z} \in \operatorname{conv} \bar{\mathcal{S}}_{G_1} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_2}$. We will show that $\tilde{z} \in \operatorname{conv} \mathcal{S}_G$.

To prove $\tilde{z} \in \operatorname{conv} \mathcal{S}_G$, we will write $\tilde{z}$ explicitly as a convex combinations of vectors in $\mathcal{S}_G$. In the next claim, we show how a vector in $\mathcal{S}_{G_1}$ and a vector in $\mathcal{S}_{G_2}$ can be combined to obtain a vector in $\mathcal{S}_G$.

Claim 1. *Let* $(z_1, z_\cap) \in \mathcal{S}_{G_1}$ *and* $(z'_\cap, z'_2) \in \mathcal{S}_{G_2}$ *such that* $z_{p_i} = z'_{p_i}$ *for every* $i \in [k]$. *Then,* $(z_1, z'_\cap, z'_2) \in \mathcal{S}_G$.

Proof of claim. It suffices to show that $(z_1, z'_\cap) \in \mathcal{S}_{G_1}$. The edges of G_1 whose components are in $z'_\cap$ are the edges in P_v. Each edge in P_v contains only nodes with components in $z'_\cap$, thus we have $z'_{p_i} = \prod_{u \in p_i} z'_u$, for each $p_i \in P_v$. The edges of G_1 whose components are in z_1 are the edges in E_v, thus we only need to show $z_{e_i} = z_v \prod_{u \in p_i} z'_u$, for each $i \in [k]$. This equality holds since

$$z_{e_i} = z_v z_{p_i} = z_v z'_{p_i} = z_v \prod_{u \in p_i} z'_u.$$

◇

In the remainder of the proof, we show how to write explicitly $\tilde{z}$ as a convex combination of the vectors in $\mathcal{S}_G$ obtained in Claim 1.

By assumption, the vector $(\tilde{z}_1, \tilde{z}_\cap)$ is in $\operatorname{conv} \mathcal{S}_{G_1}$. Thus, it can be written as a convex combination of points in $\mathcal{S}_{G_1}$; i.e., there exists $\mu \geq 0$ such that

$$\begin{aligned} \sum_{(z_1, z_\cap) \in \mathcal{S}_{G_1}} \mu_{(z_1, z_\cap)} &= 1 \\ \sum_{(z_1, z_\cap) \in \mathcal{S}_{G_1}} \mu_{(z_1, z_\cap)} (z_1, z_\cap) &= (\tilde{z}_1, \tilde{z}_\cap). \end{aligned} \tag{1}$$

Similarly, the vector $(\tilde{z}_\cap, \tilde{z}_2)$ is in $\operatorname{conv} \mathcal{S}_{G_2}$ and it can be written as a convex combination of points in $\mathcal{S}_{G_2}$; i.e., there exists $\nu \geq 0$ such that

$$\begin{aligned} \sum_{(z_\cap, z_2) \in \mathcal{S}_{G_2}} \nu_{(z_\cap, z_2)} &= 1 \\ \sum_{(z_\cap, z_2) \in \mathcal{S}_{G_2}} \nu_{(z_\cap, z_2)} (z_\cap, z_2) &= (\tilde{z}_\cap, \tilde{z}_2). \end{aligned} \tag{2}$$

For ease of notation, we define, for $i \in [k+1]$,

$$m(i) := \begin{cases} 1 - \tilde{z}_{p_1} & \text{if } i = 1 \\ \tilde{z}_{p_{i-1}} - \tilde{z}_{p_i} & \text{if } i \in \{2, \dots, k\} \\ \tilde{z}_{p_k} & \text{if } i = k+1. \end{cases}$$

In the remainder of the proof, given binary $z_{p_1}, \dots, z_{p_k}$, we will consider the number $\min\{j \in [k+1] : z_{p_j} = 0\} \in \{1, \dots, k+1\}$, with the understanding that this number equals $k+1$ when $z_{p_1} = \cdots = z_{p_k} = 1$. For every $(z_1, z_\cap) \in \mathcal{S}_{G_1}$ and $(z'_\cap, z'_2) \in \mathcal{S}_{G_2}$ such that $z_{p_i} = z'_{p_i}$ for every $i \in [k]$, we define

$$\lambda_{(z_1, z'_\cap, z'_2)} := \frac{\mu_{(z_1, z_\cap)} \nu_{(z'_\cap, z'_2)}}{m(i)},$$

where $i := \min\{j \in [k+1] : z_{p_j} = 0\} = \min\{j \in [k+1] : z'_{p_j} = 0\}$. In the next claims we show that the vector λ that we just defined serves as the vector of multipliers to write $\tilde{z}$ as a convex combination of the vectors in $\mathcal{S}_G$ obtained in Claim 1. We start with a technical claim.

Claim 2. *For $i \in [k+1]$, we have*

$$\sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ \min\{j \in [k+1] : z_{p_j} = 0\} = i}} \mu_{(z_1, z_\cap)} = \sum_{\substack{(z_\cap, z_2) \in \mathcal{S}_{G_2} \\ \min\{j \in [k+1] : z_{p_j} = 0\} = i}} \nu_{(z_\cap, z_2)} = m(i)$$

Proof of claim. By considering the component of (1) corresponding to p_i, for $i \in [k]$, we obtain

$$\sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_i} = 1}} \mu_{(z_1, z_\cap)} = \tilde{z}_{p_i}.$$

We first consider the case $i = k+1$. We have

$$\sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ \min\{j \in [k+1] : z_{p_j} = 0\} = k+1}} \mu_{(z_1, z_\cap)} = \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_k} = 1}} \mu_{(z_1, z_\cap)} = \tilde{z}_{p_k}.$$

Next, we consider the case $i = 1$. We have

$$\sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ \min\{j \in [k+1] : z_{p_j} = 0\} = 1}} \mu_{(z_1, z_\cap)} = \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_1} = 0}} \mu_{(z_1, z_\cap)} = 1 - \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_1} = 1}} \mu_{(z_1, z_\cap)} = 1 - \tilde{z}_{p_1}.$$

Next, we consider the case $i \in [k]$. We have

$$\begin{aligned} \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ \min\{j \in [k+1] : z_{p_j} = 0\} = i}} \mu_{(z_1, z_\cap)} &= \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_{i-1}} = 1 \\ z_{p_i} = 0}} \mu_{(z_1, z_\cap)} \\ &= \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_{i-1}} = 1}} \mu_{(z_1, z_\cap)} - \sum_{\substack{(z_1, z_\cap) \in \mathcal{S}_{G_1} \\ z_{p_i} = 1}} \mu_{(z_1, z_\cap)} \\ &= \tilde{z}_{p_{i-1}} - \tilde{z}_{p_i}. \end{aligned}$$

The statement for ν follows symmetrically, starting with (2) rather than (1). ◇

In the next claim, we show that the multipliers λ are nonnegative and sum to one.

Claim 3. *We have* $\lambda \geq 0$ *and*

$$\sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\z_{p_i}=z'_{p_i}\forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)} = 1$$

Proof of claim. It follows from Claim 2 that $m(i) \geq 0$ for all $i \in [k+1]$. Thus, using the definition of λ, we obtain $\lambda \geq 0$. Using Claim 2, we obtain

$$\begin{aligned}
\sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\z_{p_i}=z'_{p_i}\forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)} &= \sum_{i\in[k+1]} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\\min\{j\in[k+1]:z_{p_j}=0\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\\min\{j\in[k+1]:z'_{p_j}=0\}=i}} \lambda_{(z_1,z'_\cap,z'_2)} \\
&= \sum_{i\in[k+1]} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\\min\{j\in[k+1]:z_{p_j}=0\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\\min\{j\in[k+1]:z'_{p_j}=0\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)} \\
&= \sum_{i\in[k+1]} \frac{1}{m(i)} \left(\sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\\min\{j\in[k+1]:z_{p_j}=0\}=i}} \mu_{(z_1,z_\cap)} \right) \left(\sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\\min\{j\in[k+1]:z'_{p_j}=0\}=i}} \nu_{(z'_\cap,z'_2)} \right) \\
&= \sum_{i\in[k+1]} \frac{(m(i))^2}{m(i)} = \sum_{i\in[k+1]} m(i) = 1.
\end{aligned}$$

◇

Our last claim, which concludes the proof of the theorem, shows that the multipliers λ yield $\tilde{z}$ as a convex combination of the vectors in $\mathcal{S}_G$ obtained in Claim 1.

Claim 4. *We have*

$$(\tilde{z}_1, \tilde{z}_\cap, \tilde{z}_2) = \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\z_{p_i}=z'_{p_i}\forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)}(z_1, z'_\cap, z'_2), \tag{3}$$

Proof of claim. Using the definition of λ, we rewrite (3) in the form

$$\begin{aligned}
(\tilde{z}_1, \tilde{z}_\cap, \tilde{z}_2) &= \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\z_{p_i}=z'_{p_i}\forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)}(z_1, z'_\cap, z'_2) \\
&= \sum_{i\in[k+1]} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\\min\{j\in[k+1]:z_{p_j}=0\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\\min\{j\in[k+1]:z'_{p_j}=0\}=i}} \lambda_{(z_1,z'_\cap,z'_2)}(z_1, z'_\cap, z'_2) \\
&= \sum_{i\in[k+1]} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\\min\{j\in[k+1]:z_{p_j}=0\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\\min\{j\in[k+1]:z'_{p_j}=0\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)}(z_1, z'_\cap, z'_2).
\end{aligned}$$

We now verify the obtained inequality, first for components $\tilde{z}_1$, and then for components $\tilde{z}_\cap, \tilde{z}_2$. We start with components $\tilde{z}_1$. Using Claim 2, we obtain

$$\begin{aligned}
\tilde{z}_1 &= \sum_{i\in[k+1]} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\ \min\{j\in[k+1]:z_{p_j}=0\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\ \min\{j\in[k+1]:z'_{p_j}=0\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)} z_1 \\
&= \sum_{i\in[k+1]} \frac{1}{m(i)} \left(\sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\ \min\{j\in[k+1]:z_{p_j}=0\}=i}} \mu_{(z_1,z_\cap)} z_1 \right) \left(\sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\ \min\{j\in[k+1]:z'_{p_j}=0\}=i}} \nu_{(z'_\cap,z'_2)} \right) \\
&= \sum_{i\in[k+1]} \frac{m(i)}{m(i)} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\ \min\{j\in[k+1]:z_{p_j}=0\}=i}} \mu_{(z_1,z_\cap)} z_1 \\
&= \sum_{(z_1,z_\cap)\in\mathcal{S}_{G_1}} \mu_{(z_1,z_\cap)} z_1,
\end{aligned}$$

and the resulting equation is implied by (1).

Next, we consider components $\tilde{z}_\cap, \tilde{z}_2$. Using Claim 2, we obtain

$$\begin{aligned}
(\tilde{z}_\cap, \tilde{z}_2) &= \sum_{i\in[k+1]} \sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\ \min\{j\in[k+1]:z_{p_j}=0\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\ \min\{j\in[k+1]:z'_{p_j}=0\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)} (z'_\cap, z'_2) \\
&= \sum_{i\in[k+1]} \frac{1}{m(i)} \left(\sum_{\substack{(z_1,z_\cap)\in\mathcal{S}_{G_1}\\ \min\{j\in[k+1]:z_{p_j}=0\}=i}} \mu_{(z_1,z_\cap)} \right) \left(\sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\ \min\{j\in[k+1]:z'_{p_j}=0\}=i}} \nu_{(z'_\cap,z'_2)} (z'_\cap, z'_2) \right) \\
&= \sum_{i\in[k+1]} \frac{m(i)}{m(i)} \sum_{\substack{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}\\ \min\{j\in[k+1]:z'_{p_j}=0\}=i}} \nu_{(z'_\cap,z'_2)} (z'_\cap, z'_2) \\
&= \sum_{(z'_\cap,z'_2)\in\mathcal{S}_{G_2}} \nu_{(z'_\cap,z'_2)} (z'_\cap, z'_2),
\end{aligned}$$

and the resulting equation is (2). $\diamond$

$\square$

An example of hypergraphs G, G_1, G_2 satisfying the assumptions of Theorem 4 is given in Figure 1. Theorem 4 provides a decomposition scheme for multilinear sets whose hypergraphs contain nest points. Moreover, the hypergraph G_1 defined in the statement of the theorem has a very special structure. In the next section, we characterize the multilinear polytope of G_1 using a direct approach. This result together with the decomposition result of Theorem 4 enables us to obtain a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs.

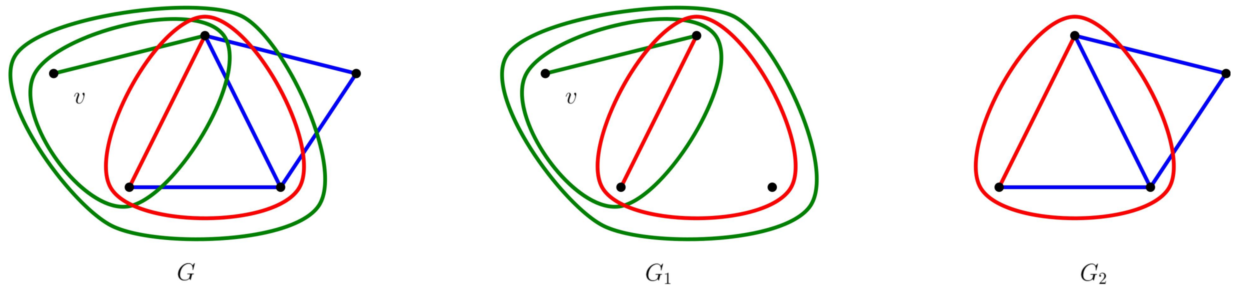


Figure 1: An instance of hypergraphs G, G_1, G_2 that satisfy the assumptions of Theorem 4: the multilinear set $\mathcal{S}_G$ is decomposable into sets $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$.

3 The multilinear polytope of pointed hypergraphs

In this section, we characterize the multilinear polytope for a special type of hypergraphs that serve as the building block for our proposed extended formulations. We call a hypergraph $G = (V, E)$ *pointed* at v, if

- the edges in E containing v are $e_1 \subset e_2 \subset \cdots \subset e_k$,
- $V = e_k$,
- $E = E_v \cup P_v$, where $E_v := \{e_1, \ldots, e_k\}$ and $P_v := \{e_i \setminus \{v\} : i \in [k],\ |e_i \setminus \{v\}| \geq 2\}$.

It then follows that the hypergraph G_1 defined in the statement of Theorem 4 is pointed at v. The next theorem provides an explicit description for the multilinear polytope of pointed hypergraphs.

Theorem 5. *Consider a hypergraph $G = (V, E)$ pointed at v. Define $p_i = e_i \setminus \{v\}$ for all $i \in [k]$. Then MP_G is defined by the following inequalities:*

$$
\begin{aligned}
z_u &\leq 1 && \forall u \in V \\
z_{e_k} &\geq 0 && \\
z_{e_k} &\leq z_{p_k} && \\
z_{e_{i+1}} &\leq z_{e_i} && \forall i \in [k-1] \\
-z_{p_i} + z_{p_{i+1}} + z_{e_i} - z_{e_{i+1}} &\leq 0 && \forall i \in [k-1] \\
z_{p_{i+1}} &\leq z_u && \forall u \in e_{i+1} \setminus e_i,\ \forall i \in [k-1] \\
\sum_{u \in e_{i+1} \setminus e_i} z_u + z_{p_i} - z_{p_{i+1}} &\leq |e_{i+1} \setminus e_i| && \forall i \in [k-1] \\
z_{e_1} &\leq z_v && \\
z_v + z_{p_1} - z_{e_1} &\leq 1 && \\
z_{p_1} &\leq z_u && \forall u \in p_1 \\
\sum_{u \in p_1} z_u - z_{p_1} &\leq |p_1| - 1. &&
\end{aligned}
\tag{4}
$$

Proof. Denote by G_0 (resp. G_1) the hypergraph corresponding to the face of MP_G with $z_v = 0$ (resp. $z_v = 1$). We then have:

$$\mathrm{MP}_G = \mathrm{conv}(\mathrm{MP}_{G_0} \cup \mathrm{MP}_{G_1}).$$

It can be checked that both G_0 and G_1 are γ-acyclic hypergraphs and hence their multilinear polytopes coincide with their flower relaxations (see Theorem 14 in [14]). Denote by $\bar{z}$ the vector consisting of z_u for all $u \in V \setminus \{v\}$ and z_e for all $e \in P_v$. It then follows that MP_{G_0} and MP_{G_1} are given by:

$$\mathrm{MP}_{G_0} = \Big\{ z \in \mathbb{R}^{|V|+|E|} : z_v = 0,\ z_{e_i} = 0, \forall i \in [k],\ \bar{z} \in \mathcal{Q} \Big\}$$
$$\mathrm{MP}_{G_1} = \Big\{ z \in \mathbb{R}^{|V|+|E|} : z_v = 1,\ z_{e_i} = z_{p_i},\ \forall i \in [k],\ \bar{z} \in \mathcal{Q} \Big\},$$

where

$$\begin{aligned}\mathcal{Q} = \Big\{ & z \in \mathbb{R}^{|V|+|P_v|-1} :\ z_u \leq 1,\ \forall u \in V \setminus \{v\},\ z_{p_k} \geq 0,\ z_{p_{i+1}} \leq z_{p_i},\ \forall i \in [k-1], \\ & z_{p_{i+1}} \leq z_u,\ \forall u \in e_{i+1} \setminus e_i,\ \forall i \in [k-1],\ \sum_{u \in e_{i+1} \setminus e_i} z_u + z_{p_i} - z_{p_{i+1}} \leq |e_{i+1} \setminus e_i|,\ \forall i \in [k-1] \\ & z_{p_1} \leq z_u,\ \forall u \in p_1, \sum_{u \in p_1} z_u - z_{p_1} \leq |p_1| - 1 \Big\},\end{aligned}$$

where the description of $\mathcal{Q}$ follows from Theorem 14 in [14]. Using Balas' formulation for the union of polytopes [1], it follows that the polytope MP_G is the projection onto the space of the z variables of the polyhedron defined by the following system (5)–(7):

$$\begin{aligned} \lambda_0 + \lambda_1 = 1,\ \lambda_0 \geq 0,\ & \lambda_1 \geq 0 \\ z_u = z^0_u + z^1_u \qquad & \forall u \in V \\ z_{p_i} = z^0_{p_i} + z^1_{p_i} \qquad & \forall i \in [k] \\ z_{e_i} = z^0_{e_i} + z^1_{e_i} \qquad & \forall i \in [k] \end{aligned} \tag{5}$$

$$\begin{aligned} z^0_v &= 0 \\ z^0_{e_i} &= 0 && \forall i \in [k] \\ z^0_u &\leq \lambda_0 && \forall u \in V \setminus \{v\} \\ z^0_{p_k} &\geq 0 \\ z^0_{p_{i+1}} &\leq z^0_{p_i} && \forall i \in [k-1] \\ z^0_{p_{i+1}} &\leq z^0_u && \forall u \in e_{i+1} \setminus e_i,\ \forall i \in [k-1] \\ \sum_{u \in e_{i+1} \setminus e_i} z^0_u + z^0_{p_i} - z^0_{p_{i+1}} &\leq |e_{i+1} \setminus e_i| \lambda_0 && \forall i \in [k-1] \\ z^0_{p_1} &\leq z^0_u && \forall u \in p_1 \\ \sum_{u \in p_1} z^0_u - z^0_{p_1} &\leq (|p_1| - 1)\lambda_0 \end{aligned} \tag{6}$$

$$
\begin{aligned}
z^1_v &= \lambda_1 && \\
z^1_{e_i} &= z^1_{p_i} && \forall i \in [k] \\
z^1_u &\le \lambda_1 && \forall u \in V \setminus \{v\} \\
z^1_{p_k} &\ge 0 && \\
z^1_{p_{i+1}} &\le z^1_{p_i} && \forall i \in [k-1] \\
z^1_{p_{i+1}} &\le z^1_u && \forall u \in e_{i+1} \setminus e_i,\ \forall i \in [k-1] \\
\sum_{u \in e_{i+1} \setminus e_i} z^1_u + z^1_{p_i} - z^1_{p_{i+1}} &\le |e_{i+1} \setminus e_i| \lambda_1 && \forall i \in [k-1] \\
z^1_{p_1} &\le z^1_u && \forall u \in p_1 \\
\sum_{u \in p_1} z^1_u - z^1_{p_1} &\le (|p_1| - 1)\lambda_1. &&
\end{aligned}
\tag{7}
$$

In the remainder of this proof, we project out $z^0, z^1, \lambda_0, \lambda_1$ from system (5)–(7) and obtain the description of MP_G in the original space. Using $z_v = z^0_v + z^1_v$, $z^0_v = 0$, and $z^1_v = \lambda_1$, we deduce that $\lambda_0 = 1 - z_v$ and $\lambda_1 = z_v$. Moreover, from $z_{e_i} = z^0_{e_i} + z^1_{e_i}$, $z^0_{e_i} = 0$, and $z^1_{e_i} = z^1_{p_i}$ for all $i \in [k]$, it follows that $z^1_{e_i} = z^1_{p_i} = z_{e_i}$ for all $i \in [k]$, which together with $z_{p_i} = z^0_{p_i} + z^1_{p_i}$ implies that $z^0_{p_i} = z_{p_i} - z_{e_i}$ for all $i \in [k]$. Finally using $z_u = z^0_u + z^1_u$ to project out z^0_u, for all $u \in V \setminus \{v\}$, the projection of system (5)–(7) onto the space $z, z^1_u, u \in V \setminus \{v\}$ is given by:

$$0 \le z_v \le 1 \tag{8}$$

$$z_{e_k} \ge 0,\ z_{e_k} \le z_{p_k} \tag{9}$$

$$z_{e_{i+1}} \le z_{e_i} \qquad \forall i \in [k-1] \tag{10}$$

$$-z_{p_i} + z_{p_{i+1}} + z_{e_i} - z_{e_{i+1}} \le 0 \qquad \forall i \in [k-1] \tag{11}$$

and by

$$
\begin{aligned}
z_u - z^1_u &\le 1 - z_v && \forall u \in V \setminus \{v\} \\
z_{p_{i+1}} - z_{e_{i+1}} &\le z_u - z^1_u && \forall u \in e_{i+1} \setminus e_i,\ \forall i \in [k-1] \\
\sum_{u \in e_{i+1} \setminus e_i} (z_u - z^1_u) + z_{p_i} - z_{e_i} - z_{p_{i+1}} + z_{e_{i+1}} &\le |e_{i+1} \setminus e_i|(1 - z_v) && \forall i \in [k-1] \\
z_{p_1} - z_{e_1} &\le z_u - z^1_u && \forall u \in p_1 \\
\sum_{u \in p_1} (z_u - z^1_u) - z_{p_1} + z_{e_1} &\le (|p_1| - 1)(1 - z_v) && \\
z^1_u &\le z_v && \forall u \in V \setminus \{v\} \\
z_{e_{i+1}} &\le z^1_u && \forall u \in e_{i+1} \setminus e_i,\ \forall i \in [k-1] \\
\sum_{u \in e_{i+1} \setminus e_i} z^1_u + z_{e_i} - z_{e_{i+1}} &\le |e_{i+1} \setminus e_i| z_v && \forall i \in [k-1] \\
z_{e_1} &\le z^1_u && \forall u \in p_1 \\
\sum_{u \in p_1} z^1_u - z_{e_1} &\le (|p_1| - 1) z_v. &&
\end{aligned}
\tag{12}
$$

First consider inequalities (8)–(11); besides the redundant inequality $z_v \ge 0$, all remaining inequalities are present in system (4). Hence, to complete the proof, it suffices to project out z^1_u, $u \in V \setminus \{v\}$ from system (12).

We start by projecting out variables z^1_u, $u \in p_1$ from system (12).

Claim 5. *Consider all inequalities of system* (12) *containing variables* z^1_u, $u \in p_1$*:*

$$z^1_u \le z_v \qquad \forall u \in p_1 \tag{13}$$

$$z_{p_1} - z_{e_1} \le z_u - z^1_u \qquad \forall u \in p_1 \tag{14}$$

$$\sum_{u \in p_1} z^1_u - z_{e_1} \le (|p_1| - 1) z_v \tag{15}$$

$$z_u - z^1_u \le 1 - z_v \qquad \forall u \in p_1 \tag{16}$$

$$z_{e_1} \le z^1_u \qquad \forall u \in p_1 \tag{17}$$

$$\sum_{u \in p_1} (z_u - z^1_u) - z_{p_1} + z_{e_1} \le (|p_1| - 1)(1 - z_v). \tag{18}$$

Then by projecting out z^1_u, $u \in p_1$ *from the above system, we obtain*

$$z_u \le 1 \qquad \forall u \in p_1 \tag{19}$$

$$z_{e_1} \le z_v \tag{20}$$

$$z_v + z_{p_1} - z_{e_1} \le 1 \tag{21}$$

$$z_{p_1} \le z_u \qquad \forall u \in p_1 \tag{22}$$

$$\sum_{u \in p_1} z_u - z_{p_1} \le |p_1| - 1. \tag{23}$$

Proof of claim. Using Fourier–Motzkin elimination, we project out z^1_u, $u \in p_1$ from inequalities (13)–(18). To this end we first consider the following simple cases:

1. Projecting out z^1_u, $u \in p_1$ from inequalities (13) and (16), we obtain inequalities (19).
2. Projecting out z^1_u, $u \in p_1$ from inequalities (13) and (17), we obtain inequality (20).
3. Projecting out z^1_u, $u \in p_1$ from inequalities (14) and (16), we obtain inequality (21).
4. Projecting out z^1_u, $u \in p_1$ from inequalities (14) and (17), we obtain inequality (22).

To complete the proof, it suffices to project out z^1_u, $u \in p_1$ from inequalities (15) and (18). Consider a variable $z^1_{\bar{u}}$ for some $\bar{u} \in p_1$. Projecting out this variable from inequalities (15) and (18), we obtain inequality (23). Hence, to project out z^1_u, $u \in p_1$ from inequality (15) (resp. inequality (18)), it suffices to consider inequalities (16) and (17) (resp. inequalities (13) and (14)).

First consider inequality (15); let $p_1 = p^1_1 \cup p^2_1$ such that $p^1_1 \cap p^2_1 = \emptyset$. Projecting out z^1_u, $u \in p^1_1$ from inequalities (15) and (16), and projecting out z^1_u, $u \in p^2_1$ from inequalities (15) and (17), we obtain:

$$\sum_{u \in p^1_1} z_u + (|p^2_1| - 1) z_{e_1} \le (|p^2_1| - 1) z_v + |p^1_1|. \tag{24}$$

First let $|p^2_1| = 0$; in this case inequality (24) simplifies to

$$\sum_{u \in e_1} z_u - z_{e_1} \le |e_1| - 1,$$

which is a redundant inequality as it is implied by inequalities (21) and (23). Now let $|p^2_1| \ge 1$. In this case, inequality (24) is implied by inequalities (19) and (20) and hence is redundant.

Finally, consider inequality (18); let $p_1 = p_1^1 \cup p_1^2$ such that $p_1^1 \cap p_1^2 = \emptyset$. Projecting out z_u^1, $u \in p_1^1$ from inequalities (13) and (18), and projecting out z_u^1, $u \in p_1^2$ from inequalities (14) and (18), we obtain:

$$\sum_{u \in p_1^1} z_u + (|p_1^2| - 1)(z_v + z_{p_1} - z_{e_1}) \leq (|p_1| - 1). \tag{25}$$

First let $|p_1^2| = 0$; in this case inequality (25) simplifies to

$$\sum_{u \in p_1} z_u - z_v - z_{p_1} + z_{e_1} \leq |p_1| - 1,$$

which is a redundant inequality as it is implied by inequalities (20) and (23). Now let $|p_1^2| \geq 1$. In this case, inequality (25) is redundant as it is implied by inequalities (19) and (21), and this completes the proof. $\diamond$

Next, we project out variables z_u^1, $u \in e_{i+1} \setminus e_i$, $i \in [k-1]$ from system (12).

Claim 6. *Let $i \in [k-1]$ and consider all inequalities of system (12) containing variables z_u^1, $u \in e_{i+1} \setminus e_i$:*

$$z_u^1 \leq z_v \qquad \forall u \in e_{i+1} \setminus e_i \tag{26}$$
$$z_{p_{i+1}} - z_{e_{i+1}} \leq z_u - z_u^1 \qquad \forall u \in e_{i+1} \setminus e_i \tag{27}$$
$$\sum_{u \in e_{i+1} \setminus e_i} z_u^1 + z_{e_i} - z_{e_{i+1}} \leq |e_{i+1} \setminus e_i| z_v \tag{28}$$
$$z_u - z_u^1 \leq 1 - z_v \qquad \forall u \in e_{i+1} \setminus e_i \tag{29}$$
$$z_{e_{i+1}} \leq z_u^1 \qquad \forall u \in e_{i+1} \setminus e_i \tag{30}$$
$$\sum_{u \in e_{i+1} \setminus e_i} (z_u - z_u^1) + z_{p_i} - z_{e_i} - z_{p_{i+1}} + z_{e_{i+1}} \leq (|e_{i+1} \setminus e_i|)(1 - z_v). \tag{31}$$

Then by projecting out z_u^1, $u \in e_{i+1} \setminus e_i$ from the above system, we obtain a system of inequalities that is implied by inequalities (10)-(11), *inequalities* (19)-(23) *and the following inequalities:*

$$z_u \leq 1 \qquad \forall u \in e_{i+1} \setminus e_i \tag{32}$$
$$z_{p_{i+1}} \leq z_u \qquad \forall u \in e_{i+1} \setminus e_i \tag{33}$$
$$\sum_{u \in e_{i+1} \setminus e_i} z_u + z_{p_i} - z_{p_{i+1}} \leq |e_{i+1} \setminus e_i|. \tag{34}$$

Proof of claim. Using Fourier–Motzkin elimination, we project out z_u^1, $u \in e_{i+1} \setminus e_i$ from inequalities (26)–(31). To this end, it suffices to consider the following cases:

1. Projecting out z_u^1, $u \in e_{i+1} \setminus e_i$ from inequalities (26) and (29), we obtain inequalities (32).
2. Projecting out z_u^1, $u \in e_{i+1} \setminus e_i$ from inequalities (26) and (30), we obtain $z_{e_{i+1}} \leq z_v$, which is a redundant inequality as it is implied by inequalities (10) and (20).
3. Projecting out z_u^1, $u \in e_{i+1} \setminus e_i$ from inequalities (27) and (29), we obtain $z_v + z_{p_{i+1}} - z_{e_{i+1}} \leq 1$, whose redundancy follows from inequalities (11) and (21).
4. Projecting out z_u^1, $u \in e_{i+1} \setminus e_i$ from inequalities (27) and (30), we obtain inequalities (33).
5. Projecting out z_u^1, $u \in e_{i+1} \setminus e_i$ from inequalities (28) and (31), we obtain inequalities (34).

It remains to project out z^1_u, $u \in e_{i+1} \backslash e_i$ from inequality (28) (resp. (31)) together with inequalities (29) and (30) (resp. inequalities (26) and (27)). It can be checked that the resulting inequalities are redundant. We do not include the proof here as it follows from a similar line of arguments to those establishing redundancy of inequalities (24) and (25) in the proof of Claim 5. $\diamond$

Therefore, by inequalities (8)–(11), Claim 5, and Claim 6, we conclude that MP_G is defined by system (4). $\square$

It is interesting to note that, in spite of its simple structure, the constraint matrix of the multilinear polytope of a pointed hypergraph is not totally unimodular. The following example demonstrate this fact. For notational simplicity, in all examples, given a node v_i, we write z_i instead of z_{v_i}. Similarly, given an edge $\{v_i, v_j, v_k\}$, we write z_{ijk} instead of $z_{\{v_i,v_j,v_k\}}$.

Example 1. *Consider $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3, v_4\}, V\}$. It is simple to check that the G is a pointed hypergraph at v_1; now consider the following inequalities all of which are present in the description of MP_G:*

$$
\begin{aligned}
z_{234} &\leq z_3 \\
z_{234} &\leq z_4 \\
-z_2 + z_{12} + z_{234} - z_{1234} &\leq 0 \\
z_2 + z_3 + z_4 - z_{234} &\leq 2.
\end{aligned}
$$

It can be checked that all above inequalities are facet-defining for MP_G. Now consider the submatrix of these inequalities corresponding to variables z_2, z_3, z_4, z_{234}. It can be checked that the determinant of this submatrix equals -2, implying the constraint matrix of MP_G is not totally unimodular.

We should also remark that one cannot use the concept of balanced matrices to prove the integrality of system (4) (see Theorem 6.13 in [7]). In order to use this result, each inequality $ax \leq b$ defining the system should satisfy $b = 1 - n(a)$, where $n(a)$ denotes the number of elements in a equal to -1. The inequality $-z_{p_i} + z_{p_{i+1}} + z_{e_i} - z_{e_{i+1}} \leq 0$ does not satisfy this assumption as for this inequality we have $n(a) = 2$ and $b = 0$.

4 Extended formulations and hypergraphs with nest points

In this section, we represent the multilinear polytope of hypergraphs with nest points in terms of multilinear polytopes of simpler hypergraphs. As a result, we obtain a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs. To this end, we first introduce expanded hypergraphs, a class of hypergraphs that determine the extended space to which our proposed extended formulations belong.

4.1 Expanded hypergraphs

We say that a hypergraph $G = (V, E)$ is *expanded w.r.t.* $v_1, \ldots, v_s$, if $v_1, \ldots, v_s$ is a nest point sequence of G, and for every edge $e \in E$, the set E also contains the sets of cardinality at least two among $e \setminus \{v_1\}$, $e \setminus \{v_1, v_2\}$, $\ldots$, $e \setminus \{v_1, \ldots, v_s\}$. The following three lemmas establish some basic properties of expanded hypergraphs which we will use for our convex hull characterizations:

Lemma 1. *Let G be a hypergraph expanded w.r.t. $v_1, \ldots, v_s$, for $s \geq 1$. Then, $G - v_1$ is expanded w.r.t. $v_2, \ldots, v_s$.*

Proof. Let $G = (V, E)$ and let $G - v_1 = (V', E')$, where $V' := V \setminus \{v_1\}$, and $E' := \{e \setminus \{v_1\} : e \in E,\ |e \setminus \{v_1\}| \geq 2\}$. Since $v_1, \dots, v_s$ is a nest point sequence of G, then $v_2, \dots, v_s$ is a nest point sequence of $G - v_1$. Thus, we only need to show that, for every $e \in E'$, the set E' contains the sets of cardinality at least two among $e \setminus \{v_2\}$, $e \setminus \{v_2, v_3\}$, $\dots$, $e \setminus \{v_2, \dots, v_s\}$.

Let $e \in E'$. We show that we have $e \in E$. By definition of $G - v_1$, either $e \in E$, or $e \cup \{v_1\} \in E$. In the first case we are done; In the second case, since G is expanded, we have $(e \cup \{v_1\}) \setminus \{v_1\} \in E$, thus $e \in E$, and we are done. Since $e \in E$ and G is expanded, the set E also contains the sets of cardinality at least two among $e \setminus \{v_1\}$, $e \setminus \{v_1, v_2\}$, $\dots$, $e \setminus \{v_1, \dots, v_s\}$. Since $v_1 \notin e$, these are the sets of cardinality at least two among $e \setminus \{v_2\}$, $e \setminus \{v_2, v_3\}$, $\dots$, $e \setminus \{v_2, \dots, v_s\}$. By definition of $G - v_1$, the set E' also contains the sets of cardinality at least two among $e \setminus \{v_2\}$, $e \setminus \{v_2, v_3\}$, $\dots$, $e \setminus \{v_2, \dots, v_s\}$. Hence, $G - v_1$ is expanded w.r.t. $v_2, \dots, v_s$. □

Let $G = (V, E)$ be a hypergraph and let $v_1, \dots, v_s$ be a nest point sequence of G. The *expansion of G w.r.t. $v_1, \dots, v_s$* is the hypergraph $G' = (V, E')$, where E' is obtained from E by adding, for each $e \in E$, the sets of cardinality at least two among $e \setminus \{v_1\}$, $e \setminus \{v_1, v_2\}$, $\dots$, $e \setminus \{v_1, \dots, v_s\}$.

Lemma 2. *Let G be a hypergraph, let $v_1, \dots, v_s$ be a nest point sequence of G, and let G' be the expansion of G w.r.t. $v_1, \dots, v_s$. Then G' is expanded w.r.t. $v_1, \dots, v_s$.*

Proof. Let $G' = (V, E')$. Clearly, for every edge $e \in E'$, the set E' also contains the sets of cardinality at least two among $e \setminus \{v_1\}$, $e \setminus \{v_1, v_2\}$, $\dots$, $e \setminus \{v_1, \dots, v_s\}$. Thus, we only need to show that $v_1, \dots, v_s$ is a nest point sequence of G'. Note that, by construction of G', for every $i = 1, \dots, s$, the edges of $G - v_1 - \cdots - v_{i-1}$ containing v_i coincide with the edges of $G' - v_1 - \cdots - v_{i-1}$ containing v_i. Hence, for $i = 1, \dots, s$, the fact that v_i is a nest point of $G - v_1 - \cdots - v_{i-1}$ implies that v_i is a nest point of $G' - v_1 - \cdots - v_{i-1}$. Thus, $v_1, \dots, v_s$ is a nest point sequence of G'. □

Lemma 3. *Let $G = (V, E)$ be a hypergraph expanded w.r.t. $v_1, \dots, v_s$. Let $e \in E$ such that $e \cap \{v_1, \dots, v_s\} \neq \emptyset$, and let v_i be the first node in the sequence $v_1, \dots, v_s$ contained in e. Then, $e \setminus \{v_i\} \in E$, if $|e \setminus \{v_i\}| \geq 2$. Furthermore, if there exists at least one edge in E strictly contained in e and containing v_i, then there exists only one edge $f \in E$ of maximum cardinality, and $f \setminus \{v_i\} \in E$, if $|f| \geq 3$.*

Proof. If $|e \setminus \{v_i\}| \geq 2$, we have $e \setminus \{v_i\} \in E$ since G is expanded.

In the rest of the proof, we assume that the set of edges $\bar{E} := \{g \in E : g \subset e,\ v_i \in g\}$ is nonempty. We show that there exists one edge $f \in \bar{E}$ containing all the edges in $\bar{E}$.

Since G is expanded w.r.t. $v_1, \dots, v_s$, node v_i is a nest point of $G - v_1 - \cdots - v_{i-1}$. Hence, the edges of $G - v_1 - \cdots - v_{i-1}$ containing v_i are totally ordered. Since G is expanded, the edges of $G - v_1 - \cdots - v_{i-1}$ containing v_i coincide with the edges of G containing v_i and not containing $v_1, \dots, v_{i-1}$. Hence, these edges are totally ordered. Assume these are edges $e_1 \subset e_2 \subset \cdots \subset e_k$. Then, $e = e_j$, for $j \geq 2$, and we set $f := e_{j-1}$.

Furthermore, since f is contained in e and it contains v_i, v_i is the first node in the sequence $v_1, \dots, v_s$ such that $v_i \in f$, thus $f \setminus \{v_i\} \in E$, if $|f| \geq 3$, since G is expanded. □

4.2 Convex hull characterizations

In this section, we study the multilinear polytope of expanded hypergraphs. First we consider the general case in which the hypergraph G is expanded w.r.t. $v_1, \cdots, v_s$ for some $s \geq 1$ and characterize MP_G in terms of multilinear polytopes of simpler hypergraphs. Subsequently, we consider the important special case with $s = |V|$ and present a polynomial-size formulation for MP_G. This in turn enables us to obtain a polynomial-size extended formulation for the multilinear polytope of β-acyclic hyerpgraphs. Recall that $G' = (V', E')$ is called a *partial hypergraph* of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

Theorem 6. *Let $G = (V, E)$ be a hypergraph expanded w.r.t. $v_1, \ldots, v_s$ for some $s \geq 1$. For each $i \in [s]$, denote by $\tilde{G}_i$ the partial hypergraph of $G - v_1 - v_2 - \cdots - v_{i-1}$ pointed at v_i; that is, denoting by $E_{v_i} := \cup_{j \in [k]}\{e_j\}$ the set of edges of $G - v_1 - v_2 - \cdots - v_{i-1}$ containing v_i, and letting $e_1 \subset e_2 \subset \cdots \subset e_k$, we have $\tilde{G}_i = (\tilde{V}_i, \tilde{E}_i)$, where $\tilde{V}_i = e_k$ and $\tilde{E}_i = E_{v_i} \cup \{e \setminus \{v_i\} : |e \setminus \{v_i\}| \geq 2, e \in E_{v_i}\}$. Then, MP_G is given by a description of $MP_{G - v_1 - \cdots - v_s}$ together with a description of $MP_{\tilde{G}_i}$ for all $i \in [s]$, where $MP_{\tilde{G}_i}$ is characterized in Theorem 5.*

Proof. The proof is by induction on the number of nest points s of G. In the base case we have $s = 0$. In this case we do not have any pointed hypergraphs $\tilde{G}_i$ and we have $G - v_1 - \cdots - v_s = G$, hence the statement trivially holds.

We now show the inductive step. Node v_1 is a nest point of G; define $G_1 = \tilde{G}_1$, i.e., the partial hypergraph of G pointed at v_1, and $G_2 = G - v_1$. Then by Theorem 4, the set $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$. That is, MP_G is defined by inequalities defining $\mathrm{MP}_{\tilde{G}_1}$ together with those defining $\mathrm{MP}_{G - v_1}$. Since $\tilde{G}_1$ is a pointed hypergraph at v_1, its multilinear polytope $\mathrm{MP}_{\tilde{G}_1}$ is given by Theorem 5. From Lemma 1, it follows that $G - v_1$ is expanded w.r.t. $v_2, \ldots, v_s$, and it has one fewer nest point than G. Hence, by the induction hypothesis, the polytope $\mathrm{MP}_{G - v_1}$ is given by a description of $\mathrm{MP}_{G - v_1 - v_2 - \cdots - v_s}$ together with a description of $\mathrm{MP}_{\tilde{G}_i}$ for all $i \in \{2, \cdots, s\}$, and this completes the proof. $\square$

Theorem 7. *Let $G = (V, E)$ be a β-acyclic hypergraph expanded w.r.t. $v_1, \ldots, v_n$. For every $e \in E$, we denote by $v(e)$ the first node in the sequence $v_1, \ldots, v_n$ contained in e, and we define $p(e) := e \setminus \{v(e)\}$. Define $M := \{e \in E : \exists g \in E, g \subset e, v(e) \in g\}$. For every $e \in M$, let $f(e) \subset e$ be the edge of maximum cardinality with $v(e) \in f(e)$ (unique by Lemma 3), and let $f'(e) := f(e) \setminus \{v(e)\}$. Finally, denote by $\bar{E}$ the set of maximal edges of G; i.e., $\bar{E} = \{e \in E : \nexists g \in E,\ g \supset e\}$. Then, MP_G is defined by the following system of linear inequalities:*

$$
\begin{aligned}
0 \leq z_u &\leq 1 && \forall u \in V && (35)\\
z_e &\geq 0 && \forall e \in \bar{E} && (36)\\
z_e - z_{p(e)} &\leq 0 && \forall e \in E && (37)\\
z_e - z_{f(e)} &\leq 0 && \forall e \in M && (38)\\
-z_{f'(e)} + z_{p(e)} + z_{f(e)} - z_e &\leq 0 && \forall e \in M && (39)\\
z_{v(e)} + z_{p(e)} - z_e &\leq 1 && \forall e \in E \setminus M && (40)\\
z_e - z_{v(e)} &\leq 0 && \forall e \in E \setminus M. && (41)
\end{aligned}
$$

Proof. First note that the all variables that appear in system (35)–(41) are present in $\mathcal{S}_G$ due to Lemma 3. The proof is by induction on $|V|$. In the base case we have $V = \{u\}$ and $E = \emptyset$. Clearly, MP_G is then given by $0 \leq z_u \leq 1$.

We now show the inductive step. Node v_1 is a nest point of G, and for ease of notation we set $v := v_1$. Without loss of generality, assume that v is not an isolated node. Let $e_1 \subset e_2 \subset \cdots \subset e_k$ for some $k \geq 1$, be the edges of G containing v. For each $i \in [k]$, let $p_i := e_i \setminus \{v\}$, let $E_v := \{e_1, \ldots, e_k\}$, and let $P_v := \{p \in \{p_1, \ldots, p_k\} : |p| \geq 2\}$. Define $G_1 := (e_k, E_v \cup P_v)$ and $G_2 := G - v$. Since G is expanded w.r.t. $v_1, \cdots, v_n$, we have $G = G_1 \cup G_2$. Then from Theorem 4, it follows that the set $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_1}$ and $\mathcal{S}_{G_2}$. That is, MP_G is defined by inequalities defining MP_{G_1} together with those defining MP_{G_2}.

By Theorem 5, the polytope MP_{G_1} is given by:

$$
\begin{aligned}
0 \leq z_u &\leq 1 \qquad \forall u \in e_k && (42)\\
z_{e_k} &\geq 0 && (43)
\end{aligned}
$$

$$z_{e_i} - z_{p_i} \leq 0 \qquad \forall i \in [k] \tag{44}$$

$$z_{e_{i+1}} - z_{e_i} \leq 0 \qquad \forall i \in [k-1] \tag{45}$$

$$-z_{p_i} + z_{p_{i+1}} + z_{e_i} - z_{e_{i+1}} \leq 0 \qquad \forall i \in [k-1] \tag{46}$$

$$z_v + z_{p_1} - z_{e_1} \leq 1 \tag{47}$$

$$z_{e_1} - z_v \leq 0 \tag{48}$$

$$z_{p_{i+1}} \leq z_u \qquad \forall u \in e_{i+1} \setminus e_i, \; \forall i \in [k-1] \tag{49}$$

$$\sum_{u \in e_{i+1} \setminus e_i} z_u + z_{p_i} - z_{p_{i+1}} \leq |e_{i+1} \setminus e_i| \qquad \forall i \in [k-1] \tag{50}$$

$$z_{p_1} \leq z_u \qquad \forall u \in p_1 \tag{51}$$

$$\sum_{u \in p_1} z_u - z_{p_1} \leq |p_1| - 1. \tag{52}$$

We should remark that the valid inequalities $z_u \geq 0$ for all $u \in e_k$ and $z_{e_i} \leq z_{p_i}$, for all $i \in [k-1]$ are not present in the description of MP_{G_1} as given by Theorem 5, and hence are redundant. However, we includ them in the above system as they simplify the proof. From Lemma 1, it follows that $G - v_1$ is a β-acyclic hypergraph expanded w.r.t. $v_2, \ldots, v_n$, and it has one fewer node than G. Hence, by the induction hypothesis, the polytope MP_{G_2} is given by

$$0 \leq z_u \leq 1 \qquad \forall u \in V \setminus \{v\} \tag{53}$$

$$z_e \geq 0 \qquad \forall e \in \bar{E} \setminus \{e_k\} \tag{54}$$

$$z_e - z_{p(e)} \leq 0 \qquad \forall e \in E \setminus E_v \tag{55}$$

$$z_e - z_{f(e)} \leq 0 \qquad \forall e \in M \setminus \cup_{i=2}^k \{e_i\} \tag{56}$$

$$-z_{f'(e)} + z_{p(e)} + z_{f(e)} - z_e \leq 0 \qquad \forall e \in M \setminus \cup_{i=2}^k \{e_i\} \tag{57}$$

$$z_{v(e)} + z_{p(e)} - z_e \leq 1 \qquad \forall e \in E \setminus (M \cup \{e_1\}) \tag{58}$$

$$z_e - z_{v(e)} \leq 0 \qquad \forall e \in E \setminus (M \cup \{e_1\}). \tag{59}$$

Hence, to complete the proof it suffices to show that combining inequalities (42)–(52) and inequalities (53)–(59), we obtain inequalities (35)–(41). We start by making the following observations:

1. Inequalities (42) and (53) are equivalent to inequalities (35).
2. Inequalities (43) and (54) are equivalent to inequalities (36).
3. Inequalities (44) and (55) are equivalent to inequalities (37).
4. Noting that $f(e_{i+1}) = e_i$, for all $i \in [k-1]$, it follows that inequalities (45) and (56) are equivalent to inequalities (38).
5. Noting that $f'(e_{i+1}) = p_i$, for all $i \in [k-1]$, it follows that inequalities (46) and (57) are equivalent to inequalities (39).
6. Inequalities (47) and (58) are equivalent to inequalities (40).
7. inequalities (48) and (59) are equivalent to inequalities (41).

Hence it remains to show that inequalities (49)–(52) are implied by inequalities (35)–(41). First, let us consider inequalities (49); namely,

$$z_{p_{i+1}} \leq z_u \qquad u \in e_{i+1} \setminus e_i, \; i \in [k-1]. \tag{60}$$

Fix $i \in [k-1]$ and fix $u \in p_{i+1}$; note that since G is expanded w.r.t. v, we have $p_{i+1} \in E$. For notational simplicity let $q_0 := p_{i+1}$. Consider the two elements $p(q_0), v(q_0)$, if $q_0 \in E \setminus M$ (resp. $p(q_0), f(q_0)$, if $q_0 \in M$), as defined in system (53)- (59). Let q_1 be one of these two elements such that $u \in q_1$. Clearly the inequality $z_{q_0} \leq z_{q_1}$ is present in system (53)- (59). If $q_1 = u$, then we are done. Otherwise, as before we consider $p(q_1), v(q_1)$, if $q_1 \in E \setminus M$ (resp. resp. $p(q_1), f(q_1)$, if $q_1 \in M$), and let q_2 be one of these two elements such that $u \in q_2$. Again the inequality $z_{q_1} \leq z_{q_2}$ is present in system (53)- (59). We continue this recursion until $q_k = u$. It then follows that the collection of inequalities $z_{p_{i+1}} = z_{q_0} \leq z_{q_1}$, $z_{q_1} \leq z_{q_2}, \cdots, z_{q_{k-1}} \leq z_{q_k} = z_u$ imply inequality (60). The redundancy of inequalities (51) follows from a similar line of arguments.

Next, let us consider inequalities (50), which can be written as:

$$\sum_{u \in p_{i+1} \setminus p_i} z_u + z_{p_i} - z_{p_{i+1}} \leq |p_{i+1} \setminus p_i| \qquad i \in [k-1], \tag{61}$$

where as before for any $i \in [k-1]$, we assume $|p_{i+1}| \geq 2$ and hence by construction $p_{i+1} \in E$. In the following we show that these inequalities are implied by system (35)–(41). To this end, we prove a more general statement, i.e., for any $q_1 \subset q_2$ with $q_1 \in V \cup E$ and $q_2 \in E$, we show that the inequality

$$\sum_{u \in q_2 \setminus q_1} z_u + z_{q_1} - z_{q_2} \leq |q_2 \setminus q_1|, \tag{62}$$

is implied by the following inequalities of system (35)–(41):

$$-z_{f'(e)} + z_{p(e)} + z_{f(e)} - z_e \leq 0 \qquad \forall e \in M : e \subseteq q_2,\ e \not\subseteq q_1 \tag{63}$$

$$z_{v(e)} + z_{p(e)} - z_e \leq 1 \qquad \forall e \in E \setminus M : e \subseteq q_2,\ e \not\subseteq q_1. \tag{64}$$

Then setting $q_1 = p_i$ and $q_2 = p_{i+1}$ completes the proof. The proof is by induction on the number of nodes in q_2. In the base case we have $q_2 = \{u_1, u_2\}$ for which inequality (62) simplifies to $z_{u_1} + z_{u_2} - z_{q_2} \leq 1$, which is present among inequalities (64) since from $|q_2| = 2$, it follows that $q_2 \in E \setminus M$.

We now proceed with the inductive step. Let $|q_2| = k$ for some $k \geq 3$. Two cases arise:

(i) $q_1 \subseteq p(q_2)$: in this case the following inequality is present among (if $q_1 \in E \setminus M$) or is implied by (if $q_1 \in M$) inequalities (63) and (64):

$$z_{v(q_2)} + z_{p(q_2)} - z_{q_2} \leq 1. \tag{65}$$

Since by assumption $q_1 \subseteq p(q_2)$ and $|p(q_2)| = k-1$, by the induction hypothesis, the inequality

$$\sum_{u \in p(q_2) \setminus q_1} z_u + z_{q_1} - z_{p(q_2)} \leq |p(q_2) \setminus q_1|, \tag{66}$$

is implied by inequalities

$$-z_{f'(e)} + z_{p(e)} + z_{f(e)} - z_e \leq 0 \qquad \forall e \in M : e \subseteq p(q_2),\ e \not\subseteq q_1$$

$$z_{v(e)} + z_{p(e)} - z_e \leq 1 \qquad \forall e \in E \setminus M : e \subseteq p(q_2),\ e \not\subseteq q_1,$$

which are in turn present among inequalities (63) and (64), since $q_1 \subseteq p(q_2) \subseteq q_2$. Summing up inequalities (65) and (66) we obtain inequality (62).

(ii) $q_1 \not\subseteq p(q_2)$: in this case, we must have $q_2 \in M$. Hence the following inequality is present among inequalities (63):

$$-z_{f'(q_2)} + z_{f(q_2)} + z_{p(q_2)} - z_{q_2} \leq 0. \tag{67}$$

By Lemma 3, we have $q_1 \subseteq f(p_2)$. Then using a similar line of arguments to those in case (i) above, we conclude that the following are implied by inequalities (63) and (64):

$$
\begin{aligned}
\sum_{u \in p(q_2) \setminus f'(q_2)} z_u + z_{f'(q_2)} - z_{p(q_2)} \leq |p(q_2) \setminus f'(q_2)| \\
\sum_{u \in f(q_2) \setminus q_1} z_u + z_{q_1} - z_{f(q_2)} \leq |f(q_2) \setminus q_1|.
\end{aligned} \tag{68}
$$

Summing up inequalities (67) and (68) we obtain inequality (62) implying it is redundant.

The redundancy of inequalities (52) then immediately follows by setting $q_1 = v$ for some $v \in p_1$ and $q_2 = p_1$. □

Let $G = (V, E)$ be a β-acyclic hypergraph expanded with respect to $v_1, \cdots, v_n$, where $n = |V|$. By Theorem 7, MP_G is given by system (35)–(41). We should remark that, in spite of its simplicity, the constraint matrix of MP_G is *not* totally unimodular. The following example demonstrates this fact.

Example 2. *Consider the hypergraph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and*

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, V\}.$$

It is simple to check that G is β-acyclic and is expanded with respect to v_1, v_2, v_3, v_4. By Theorem 7, MP_G contains the following inequalities:

$$
\begin{aligned}
z_1 + z_2 - z_{12} &\leq 1 \\
z_2 + z_3 - z_{23} &\leq 1 \\
z_{123} - z_{12} &\leq 0 \\
-z_3 + z_{34} + z_{123} - z_{1234} &\leq 0.
\end{aligned}
$$

It can be checked that all above inequalities are facet-defining. The determinant of the submatrix corresponding to variables $z_2, z_3, z_{12}, z_{123}$ equals -2, implying the constraint matrix of MP_G is not totally unimodular.

Moreover, one cannot use the concept of balanced matrices to prove the integrality of system (35)–(41) (see Theorem 6.13 in [7]). In order to use this result, each inequality $ax \leq b$ defining the system should satisfy $b = 1 - n(a)$, where $n(a)$ denotes the number of elements in a equal to -1. Clearly, the inequality $-z_{f'(e)} + z_{p(e)} + z_{f(e)} - z_e \leq 0$ does not satisfy this assumption.

From Theorems 6 and 7, we directly obtain the following results on extended formulations of multilinear polytopes:

Theorem 3. *Let $G = (V, E)$ be a hypergraph of rank r, and let $v_1, \ldots, v_s$ be a nest point sequence of G. Then an extended formulation of MP_G is given by a description of $MP_{G-v_1-\cdots-v_s}$, together with a system of at most $|V| + 2|E| + 4rs$ linear inequalities, including at most $(r-2)s$ extended variables. The system is characterized in Theorem 6.*

Proof. Let $G' = (V, E')$ be the expansion of G w.r.t. $v_1, \ldots, v_s$. From Lemma 2, it follows that G' is expanded w.r.t. $v_1, \ldots, v_s$. We then apply Theorem 6 to G', and observe that $G' - v_1 - \cdots - v_s = G - v_1 - \cdots - v_s$. The total number of inequalities associated with multilinear polytopes of pointed partial hypergraphs of G' at v_i, $i \in [s]$ is upper bounded by $|V| + 2|\bar{E}| + 4rs$, where $\bar{E}$ denotes the set of maximal edges of G'. To see this, consider system (4) defining the convex hull of a pointed hypergraph. First, note that we have a total number of $|V|$ inequalities of the form $z_u \leq 1$. The total number of nonredundant inequalities of the form $z_{e_k} \geq 0$ and $z_{e_k} \leq z_{p_k}$ is $2|\bar{E}|$. Moreover, the total number of inequalities of the form $z_{e_{i+1}} \leq z_{e_i}$, $i \in [k-1]$ and $z_{e_1} \leq z_v$ is upper bounded by rs. Similarly the total number of inequalities of the form $-z_{p_i} + z_{p_{i+1}} + z_{e_i} - z_{e_{i+1}} \leq 0$, $i \in [k-1]$ and $z_v + z_{p_1} - z_{e_1} \leq 1$ is upper bounded by rs. Also, the total number of inequalities of the form $z_{p_{i+1}} \leq z_u$, $u \in e_{i+1} \setminus e_i$, $i \in [k-1]$, $z_{p_1} \leq z_u$, $u \in p_1$ is upper bounded by rs. Finally, the total number of inequalities of the form $\sum_{u \in e_{i+1} \setminus e_i} z_u + z_{p_i} - z_{p_{i+1}} \leq |e_{i+1} \setminus e_i|$, $i \in [k-1]$ and $\sum_{u \in p_1} z_u - z_{p_1} \leq |p_1| - 1$ is upper bounded by rs. An upper bound on the number of corresponding linear inequalities can then be obtained using $|\bar{E}| \leq |E|$. Finally, notice that in a rank r hyperpgraph each nest point is present in at most $r-1$ edges implying that the number of extended variables does not exceed $(r-2)s$.

□

We are now ready to prove the main result of this paper, Theorem 1, which we recall below.

Theorem 1. *Let $G = (V, E)$ be a β-acyclic hypergraph of rank r. Then there exists an polynomial-size extended formulation of MP_G comprising of at most $(3r-4)|V| + 4|E|$ inequalities, with at most $(r-2)|V|$ extended variables. The system is explicitly given in Theorem 7.*

Proof. Since G is β-acyclic, by Theorem 2, it has a nest point sequence of length $|V|$, say $v_1, \ldots, v_n$. Let $G' = (V, E')$ be the expansion of G w.r.t. $v_1, \ldots, v_n$. From Lemma 2, G' is expanded w.r.t. $v_1, \ldots, v_n$. We then apply Theorem 7 to G'. System (35)–(41) consists of $2|V| + 3|E'| + |\bar{E}|$ inequalities, where $\bar{E}$ denotes the set of maximal edges of G'. The result then follows using the fact that $|E'| \leq (r-2)|V| + |E|$ and $\bar{E} \leq E$. □

We remark that Theorem 3 allows us to obtain an extended formulation for the multilinear polytope of certain hypergraphs that are not β-acyclic. This happens precisely when a description of $\mathrm{MP}_{G-v_1-\cdots-v_s}$ is available. The following example demonstrates this fact.

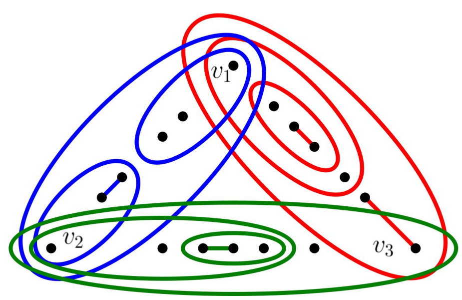


Figure 2: Illustration of the hypergraph considered in Example 3.

Example 3. *Consider the hypergraph G depicted in Figure 2. A nest point sequence of G is given by all nodes of G (in any order), except for v_1, v_2, v_3. The hypergraph $G' = (V', E')$ obtained from G by removing all nodes except for v_1, v_2, v_3 is a "triangle", i.e. $V' = \{v_1, v_2, v_3\}$ and $E' = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$. It is well known that $MP_{G'}$ is obtained by adding triangle inequalities to the standard linearization $MP^{\mathrm{LP}}_{G'}$ [25]. Theorem 3 then gives an extended formulation of MP_G.*

5 The original space

In Section 4, we presented a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs. It is often desirable to obtain an explicit description for the multilinear polytope in the original space. To this end, one can employ Fourier-Motzkin elimination to project out the extended variables from system (35)–(41). In [14], the authors show that in the original space, the multilinear polytope of a γ-acyclic hypergraph $G = (V, E)$ contains exponentially many facet-defining inequalities (as a function of $|V|, |E|$), in general. As β-acyclicity subsumes γ-acyclicity, this result implies that the multilinear polytope of a β-acyclic hypergraph contains exponentially many facet-defining inequalities, in general.

From a computational perspective, sparsity is key to the effectiveness of cutting planes in a branch-and-cut framework. Indeed, all existing families of cutting planes for multilinear sets, such as flower inequalities [14] and running intersection inequalities [16] are sparse. Namely, for a rank r hypergraph, flower inequalities contain at most $\frac{r}{2}$ nonzero coefficients, and running intersection inequalities contain at most $2(r-1)$ nonzero coefficients. When added to the standard linearization, flower inequalities characterize the multilinear polytope of γ-acyclic hypergraphs [14], and running intersection inequalities characterize the multilinear polytope of kite-free β-acyclic hypergraphs [16]. However, as we detail in the following, the multilinear polytope of a β-acyclic hypergraph $G = (V, E)$ may contain very dense facets, in general. That is, inequalities containing as many as $\theta(|E|)$ nonzero coefficients. This is significant, as almost for all multilinear sets appearing in nonconvex problems, we have $r \ll |E|$.

5.1 The multilinear polytope of beta-acyclic hypergraphs with dense facets

In the following, we present a family of β-acyclic hypergraphs $G = (V, E)$ whose multilinear polytope contains facet-defining inequalities with $|E|$ non-zero coefficients.

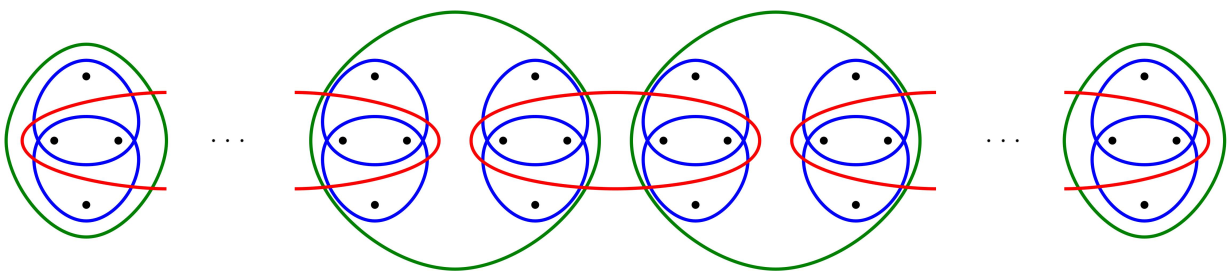


Figure 3: Illustration of the family of hypergraphs considered in Proposition 1.

Proposition 1. *Let $n \geq 2$ and consider the β-acyclic hypergraph $G = (V, E)$ with*

$$V = \bigcup_{i \in [n]} V^i, \qquad E = H \cup \bigcup_{i \in [n]} E^i,$$

where $V^1 = \{v^1_3, v^1_4, v^1_7, v^1_8\}$, $V^i = \{v^i_1, \cdots, v^i_8\}$ for all $i \in [n-1] \setminus \{1\}$, $V^n = \{v^n_1, v^n_2, v^n_5, v^n_6\}$,

$$\begin{aligned}
H &= \Big\{\{v^i_3, v^i_4, v^{i+1}_1, v^{i+1}_2\},\ i \in [n-1]\Big\} \\
E^1 &= \Big\{\{v^1_3, v^1_4, v^1_7\}, \{v^1_3, v^1_4, v^1_8\}, V^1\Big\} \\
E^i &= \Big\{\{v^i_1, v^i_2, v^i_5\}, \{v^i_1, v^i_2, v^i_6\}, \{v^i_3, v^i_4, v^i_7\}, \{v^i_3, v^i_4, v^i_8\}, V^i\Big\} \qquad \forall i \in [n-1] \setminus \{1\} \\
E^n &= \Big\{\{v^n_1, v^n_2, v^n_5\}, \{v^n_1, v^n_2, v^n_6\}, V^n\Big\}.
\end{aligned}$$

See Figure 3. Then the following inequality containing $|E|$ nonzero coefficients defines a facet of MP_G:

$$-\sum_{i \in [n]} z_{V^i} - \sum_{e \in H} z_e + \sum_{i \in [n]} \sum_{e \in E^i \setminus \{V^i\}} z_e \leq 2n - 3. \tag{69}$$

Proof. For notational simplicity in the following, we define $h^i := \{v_3^i, v_4^i, v_1^{i+1}, v_2^{i+1}\}$, for all $i \in [n-1]$, $e_1^i := \{v_1^i, v_2^i, v_5^i\}$, and $e_2^i := \{v_1^i, v_2^i, v_6^i\}$ for all $i \in [n] \setminus \{1\}$. Moreover, we define $e_3^i := \{v_3^i, v_4^i, v_7^i\}$, and $e_4^i := \{v_3^i, v_4^i, v_8^i\}$ for all $i \in [n-1]$.

We start by proving the validity of inequality (69) for MP_G. First, we construct the hypergraph $G' = (V, E \cup E')$, where

$$E' := \Big\{ f^i := \{v_1^i, v_2^i\}, i \in [n] \setminus \{1\} \Big\} \cup \Big\{ g^i := \{v_3^i, v_4^i\}, i \in [n-1] \Big\}.$$

The following inequalities are all extended running intersection inequalities and hence are valid for $\mathrm{MP}_{G'}$ (see Section 5.2 for the definition of extended running intersection inequalities):

$$\begin{aligned}
&-z_{g^1} + z_{e_3^1} + z_{e_4^1} - z_{V^1} \leq 0\\
&-z_{f^n} + z_{e_1^n} + z_{e_2^n} - z_{V^n} \leq 0\\
&\quad z_{g^i} + z_{f^{i+1}} - z_{h^i} \leq 1, \quad \forall i \in [n-1]\\
&-z_{f^i} - z_{g^i} + z_{e_1^i} + z_{e_2^i} + z_{e_3^i} + z_{e_4^i} - z_{V^i} \leq 1, \quad \forall i \in [n-1] \setminus \{1\}.
\end{aligned}$$

Summing up the above inequalities we obtain inequality (69) implying its validity for $\mathrm{MP}_{G'}$. Since $\mathrm{MP}_{G'} \subset \mathrm{MP}_G$, we conclude that inequality (69) is valid for MP_G as well.

We now show that inequality (69) defines a facet of MP_G. Consider a nontrivial valid inequality $az \leq \alpha$ for MP_G that is satisfied tightly by any point in $\mathcal{S}_G$ satisfying inequality (69) tightly. In the following, we show that the two inequalities (69) and $az \leq \alpha$ coincide up to a positive scaling, which by full dimensionality of MP_G (see Proposition 1 in [13]) implies that inequality (69) is defines a facet of MP_G.

First consider a point $z^1 \in \mathcal{S}_G$ with $z^1_{v_3^i} = z^1_{v_4^i} = z^1_{v_7^i} = 1$ for all $i \in [n-1]$, $z^1_{v_8^i} = 1$ for all $i \in [n-1] \setminus \{1\}$, and for every other $v \in V$, we have $z^1_v = 0$. It can be checked that inequality (69) is satisfied tightly at this point. Now consider a second tight point $z^2 \in \mathcal{S}_G$ whose components are equal to z^1 except for one component $z^2_{v_1^j} = 1$ for some $j \in [n] \setminus \{1\}$. Substituting these two tight points in $az = \alpha$, yields $a_{v_1^j} = 0$. Using a similar line of arguments, we obtain:

$$a_v = 0 \qquad \forall v \in V. \tag{70}$$

Let us again consider the tight point $z^1 \in \mathcal{S}_G$ defined above. Construct another tight point $z^3 \in \mathcal{S}_G$ with $z^3_{v_3^i} = z^3_{v_4^i} = z^3_{v_7^i} = 1$ for all $i \in [n-1]$, $z^3_{v_8^i} = 1$ for all $i \in [n-1] \setminus \{1\}$, $z^3_{v_1^j} = z^3_{v_2^j} = z^3_{v_5^j} = 1$, for some $j \in [n] \setminus \{1\}$, and for every other $v \in V$, we have $z^3_v = 0$. Substituting z^1, z^3 in $az = \alpha$, yields $a_{e_1^j} + a_{h^{j-1}} = 0$. Using a similar line of arguments, we obtain:

$$a_{e_1^i} = a_{e_2^i} = a_{e_3^{i-1}} = a_{e_4^{i-1}} = -a_{h^{i-1}} \qquad \forall i \in [n] \setminus \{1\}. \tag{71}$$

Now consider a tight point $z^4 \in \mathcal{S}_G$ with $z^4_v = 1$ for all $v \in V$, and construct another tight point $z^5 \in \mathcal{S}_G$ with $z^5_v = 1$ for all $v \in V \setminus \{v_5^j\}$ and $z^5_{v_5^j} = 0$ for some $j \in [n] \setminus \{1\}$. Substituting z^4, z^5 in $az = \alpha$, yields $a_{e_1^j} + a_{V^j} = 0$. Using a similar line of arguments, we obtain:

$$\begin{aligned}
&a_{e_1^i} = a_{e_2^i} = a_{e_3^i} = a_{e_4^i} = -a_{V^i} \qquad \forall i \in [n-1] \setminus \{1\}\\
&a_{e_3^1} = a_{e_4^1} = -a_{V^1}\\
&a_{e_1^n} = a_{e_2^n} = -a_{V^n}.
\end{aligned} \tag{72}$$

Combining (71) and (72) and using the fact that z^4 defined above is a tight point of inequality (69), we obtain:

$$a_{e_1^i} = a_{e_2^i} = a_{e_3^j} = a_{e_4^j} = -a_{h^j} = -a_{V^k} = \frac{\alpha}{2n-3} \qquad \forall i \in [n] \setminus \{1\}, j \in [n-1], k \in [n]. \tag{73}$$

Since $az \leq \alpha$ is nontrivial and valid for $\mathcal{S}_G$, we have $\alpha > 0$. Hence, by (73), we conclude that inequality (69) coincides with $az \leq \alpha$ up to a positive scaling implying that it defines a facet of MP_G. $\square$

Notice that the hypergraph G in Proposition 1 has a fixed rank $r = 8$, while $|E| = 6n - 5$ for all $n \geq 2$.

On the positive side, as a corollary to our main results, we obtain an interesting property of the coefficients in facet-defining inequalities for the multilinear poytope of β-acyclic hypergraphs.

Corollary 1. *Let $G = (V, E)$ be a β-acyclic hypergraph and let $az \leq b$ be a facet-defining inequality of MP_G different from $z_p \geq 0$, for $p \in V \cup E$. Then, $\sum_{p \in V \cup E} a_p = b$.*

Proof. Since G is β-acyclic, by Theorem 2, it has a nest point sequence of length $|V|$, say $v_1, \ldots, v_n$. Let $G' = (V, E')$ be the expansion of G w.r.t. $v_1, \ldots, v_n$. From Lemma 2, G' is expanded w.r.t. $v_1, \ldots, v_n$. We then apply Theorem 7 to G'. Denote by $\bar{E}$ the set of maximal edges of G', and note that $\bar{E} \subseteq E$. System (35)–(41) contains nonnegativity constraints for the edges in $\bar{E}$, and every other inequality $cx \leq d$ satisfies $\sum_{p \in V \cup E'} c_p = d$. The extended variables correspond to the edges in $E' \setminus E$, and none of them are in $\bar{E}$. The inequality $az \leq b$ is then obtained from System (35)–(41), by projecting out all the variables in $E' \setminus E$ via Fourier-Motzkin elimination. The projection consists of nonnegativity constraints on the edges in $\bar{E}$, and of inequalities that are sums of constraints of the form $cx \leq d$ with $\sum_{p \in V \cup E'} c_p = d$. Since $az \leq b$ is not a nonnegativity constraint, we have $\sum_{p \in V \cup E} a_p = b$. $\square$

We remark that, using Proposition 6 in [13], Corollary 1 also holds for facet-defining inequalities of MP_G, for general a hypergraph G, provided that their support hypergraphs are β-acyclic.

5.2 Extended running intersection inequalities

Let us consider again the description for the multilinear polytope of an expansion of a β-acyclic hypergraph G given by inequalities (35)–(41). Inequalities (35)–(38) and inequalities (40)–(41) are either flower inequalities or are present in the standard linearization. Now consider inequalities (39): these inequalities are running intersection inequalities, if $f'(e)$ is a node of G, but are not implied by any previously known inequalities for multilinear sets, if $f'(e)$ is an edge of G. Motivated by this observation, we next introduce a new class of cutting planes for multilinear sets that serve as a generalization of running intersection inequalities, introduced in [16].

In order to define the new inequalities, we first introduce the notion of running intersection property [2]. A set F of subsets of a finite set V has the *running intersection property* if there exists an ordering $p_1, p_2, \ldots, p_m$ of the sets in F such that

$$\text{for each } k = 2, \ldots, m, \text{ there exists } j < k \text{ such that } p_k \cap \Big(\bigcup_{i<k} p_i \Big) \subseteq p_j. \tag{74}$$

Henceforth, we refer to an ordering $p_1, p_2, \ldots, p_m$ satisfying (74) as a *running intersection ordering* of F. Each running intersection ordering $p_1, p_2, \ldots, p_m$ of F induces a collection of sets

$$N(p_1) := \emptyset, \qquad N(p_k) := p_k \cap \Big(\bigcup_{i<k} p_i \Big) \text{ for } k = 2, \ldots, m. \tag{75}$$

Definition 1. *Consider a hypergraph $G = (V, E)$. Let $e_0 \in E$ and let e_k, $k \in K$, be a collection of edges in E with $e_0 \cap e_k \neq \emptyset$ for all $k \in K$, such that the set $\tilde{E} := \{e_0 \cap e_k : k \in K\}$ has the running intersection property. Consider a running intersection ordering of $\tilde{E}$ with the corresponding sets $N(e_0 \cap e_k)$, for all $k \in K$, as defined in (75). For each $k \in K$, let $w_k \subseteq N(e_0 \cap e_k)$ such that*

$w_k \in \{\emptyset\} \cup V \cup E$. *We define an* extended running intersection inequality *centered at* e_0 *with neighbors* e_k*,* $k \in K$ *as:*

$$-\sum_{k\in K} z_{w_k} + \sum_{v\in e_0 \setminus \bigcup_{k\in K} e_k} z_v + \sum_{k\in K} z_{e_k} - z_{e_0} \leq \omega - 1, \tag{76}$$

where we define $z_\emptyset = 0$*, and*

$$\omega = \Big|e_0 \setminus \bigcup_{k\in K} e_k\Big| + \Big|\Big\{k \in K : N(e_0 \cap e_k) = \emptyset\Big\}\Big|.$$

We do not include the proof of validity for extended running intersection inequalities, as the proof mirrors the proof of validity for running intersection inequalities (see Proposition 1 in [16]). In [16], the authors prove that the system of all running intersection inequalities centered at e_0 with neighbors e_k, $k \in K$, is independent of the running intersection ordering (see Proposition 2 in [16]). The same statement holds for extended running intersection inequalities.

Remark 1. *In the special case where the sets* w_k *for all* $k \in K$ *with* $N(e_0 \cap e_k) \neq \emptyset$ *are nodes of* G*, extended running intersection inequalities simplify to running intersection inequalities introduced in [16]. In an even more restrictive setting where* $w_k = \emptyset$ *for all* $k \in K$*, extended running intersection inequalities simplify to flower inequalities introduced in [14].*

We now define the *extended running intersection relaxation* of the multilinear set $\mathcal{S}_G$, denoted by $\mathrm{MP}^{\mathrm{ERI}}_G$, as the polytope obtained by adding to the standard linearization, all possible extended running intersection inequalities of $\mathcal{S}_G$. For a general hypergraph G, many of the extended running intersection inequalities are redundant for $\mathrm{MP}^{\mathrm{ERI}}_G$. The following proposition provides sufficient conditions to identify such redundant inequalities.

Proposition 2. *Consider an extended running intersection inequality centered at* e_0 *with neighbors* e_k*,* $k \in K$*, as defined by* (76)*. If this inequality defines a facet of* MP^{ERI}_G*, then it satisfies the following conditions:*

(i) for any $k \neq k' \in K$*, we have* $e_0 \cap e_k \not\subseteq e_0 \cap e_{k'}$*,*

(ii) for each $k \in K$*, we have* $|e_0 \cap e_k| \geq 2$*,*

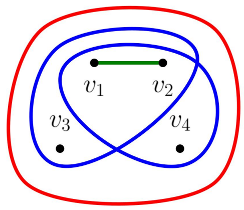
(iii) for any $k \neq k' \in K$*, with* $w_k, w_{k'} \in N(e_0 \cap e_k) \cap N(e_0 \cap e_{k'})$*, we have* $w_k = w_{k'}$*.*

(iv) for each $k \in K$*, we have* $w_k \not\subset p \subseteq N(e_0 \cap e_k)$ *for any* $p \in E$*.*

Proof. The proof of redundancy of an extended running intersection inequality not satisfying one of the conditions (i)–(iii) follows from the proof of Proposition 3 in [16] regarding the redundancy of running intersection inequalities. Hence it suffices to show that if an extended running intersection inequality does not satisfy condition (iv), then it is implied by other inequalities in $\mathrm{MP}^{\mathrm{ERI}}_G$.

Consider an extended running intersection inequality centered at e_0 with neighbors e_k, $k \in K$ such that for some $\bar{k} \in K$ we have $w_{\bar{k}} \subset p \subseteq N(e_0 \cap e_{\bar{k}})$ for some $p \in E$. Then consider another extended running intersection inequality that is identical to the first one except for w_k replaced by p. Moreover, consider the inequality $z_p \leq z_{w_{\bar{k}}}$ present in the standard linearization and hence present in $\mathrm{MP}^{\mathrm{ERI}}_G$. Summing up the latter two inequalities, we obtain the first extended running intersection inequality, and this completes the proof. ☐

Condition (iv) of Proposition 2 identifies conditions under which running intersection inequalities are implied by extended running intersection inequalities. The following example demonstrates this fact.



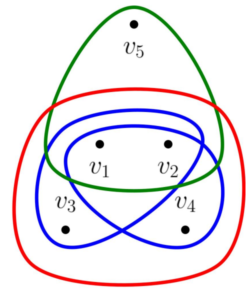


Figure 4: Illustration of the hypergraphs considered in Examples 4 and 5.

Example 4. *Consider the hypergraph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and*

$$E = \Big\{\{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, V\Big\}.$$

See Figure 4 (a). The running intersection inequalities centered at V with neighbors $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$ are given by:

$$\begin{aligned} &- z_1 + z_{123} + z_{124} - z_{1234} \leq 0 \\ &- z_2 + z_{123} + z_{124} - z_{1234} \leq 0. \end{aligned} \tag{77}$$

Moreover, the additional extended running intersection inequality centered at V with neighbors $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$ is given by:

$$-z_{12} + z_{123} + z_{124} - z_{1234} \leq 0. \tag{78}$$

Since $z_{12} \leq z_1$ and $z_{12} \leq z_2$, inequality (78) implies inequalities (77). It can be checked that inequality (78) defines a facet of MP_G. In fact by adding inequality (78) together with flower inequalities to MP_G^{LP}, we obtain MP_G.

Now suppose that G is β-acyclic. Notice that extended running intersection inequalities are sparse; that is, for a rank r hypergraph, extended running intersection inequalities contain at most $2(r-1)$ nonzero coefficients, implying by Proposition 1 that $\mathrm{MP}_G^{\mathrm{ERI}}$ does not coincide with the multilinear polytope of β-acyclic hypergraphs. We leave as an open question the problem of characterizing the class of hypergraphs G for which we have $\mathrm{MP}_G = \mathrm{MP}_G^{\mathrm{ERI}}$. The following example provides perhaps the simplest β-acyclic hypergraph G for which we have $\mathrm{MP}_G \subset \mathrm{MP}_G^{\mathrm{ERI}}$.

Example 5. *Consider the β-acyclic hypergraph $G = (V, E)$ with $V = \{v_1, \cdots, v_5\}$ and*

$$E = \Big\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_3, v_4\}\Big\}.$$

See Figure 4 (b). It can be checked that the following inequality defines a facet of MP_G:

$$z_5 - z_{125} + z_{123} + z_{124} - z_{1234} \leq 1. \tag{79}$$

However, the above inequality is not an extended running intersection inequality since $\{v_1, v_2, v_5\} \notin \{v_1, v_2, v_3\} \cap \{v_1, v_2, v_4\}$. In fact, inequality (79) can be obtained as follows. Let $G' = (V, E')$ denote the expansion of G with respect to the nested sequence $\{v_5, v_4, v_3, v_2, v_1\}$. Then we have $E' = E \cup \{\{v_1, v_2\}\}$. By Theorem 7, the following inequalities are implied by $MP_{G'}$:

$$-z_{12} + z_{123} + z_{124} - z_{1234} \leq 0$$

$$z_5 + z_{12} - z_{125} \leq 1.$$

Projecting out z_{12} *from above inequalities, we obtain inequality* (79)*. Indeed employing this technique in a recursive manner, one can obtain dense facet-defining inequalities for the multilinear polytope of* β*-acyclic hypergraphs.*

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