Transportation and Inventory Planning in Serial Supply Chain with Heterogeneous Capacitated Vehicles

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We study serial supply chain problems where a product is transported from a supplier to a warehouse (inbound transportation), and then from the warehouse (outbound transportation) to a retailer such that demand for a given planning horizon is satisfied. We consider heterogeneous vehicles of varying capacities for the transportation in each time period, and the objective is to plan inbound and outbound transportation along with inventory in each time period such that the overall inventory and transportation costs are minimized. These problems belong to the class of two-echelon lot-sizing problems (2-ELS) with warehouse and retailer as first- and second-echelons, respectively. We address an open question raised in van Hoesel, Romeijn, Morales, Wagelmans [Management Science 51(11):1706-1719, 2005]: Does there exist a polynomial time algorithm for 2-ELS with a single capacitated vehicle for each of the inbound and outbound transportation? Specifically, we introduce polynomial time algorithms for this problem and its three generalizations with multiple capacitated vehicles for inbound and/or outbound transportation, thereby generalizing the results of Kaminsky and Simchi-Levi [IIE Transactions 35(11):1065-1075, 2003] and Sargut and Romeijn [IIE Transactions 39(11):1031-1043, 2007] for 2-ELS with a single capacitated vehicle for inbound transportation and uncapacitated outbound transportation.

Key words: two-echelon lot-sizing, serial supply chain, inbound and outbound transportation, multi-mode, multi-module, outsourcing, dynamic programming, polynomial algorithm

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1. Introduction

In today's economy and commerce, globalization of exchanges has forced researchers to reiterate the importance of serial supply chain systems. Products in such systems are manufactured at distant facilities, and then stored at intermediate warehouses or third-party logistics before finally making it to the consumer/retailer. According to a global survey conducted by Geodis (2017), optimizing inventory costs, reducing transportation and warehouse costs, and improving product availability are among the most important objectives in today's supply chain. Therefore, it is imperative to plan and acquire resources with limited capacities, and utilize them in a meaningful way by efficiently planning transportation and inventory over a time horizon. It is well established that integrating the supply chain decisions (production, transportation, and inventory) and looking at the problem holistically yields lower costs than treating each entity of the system separately.

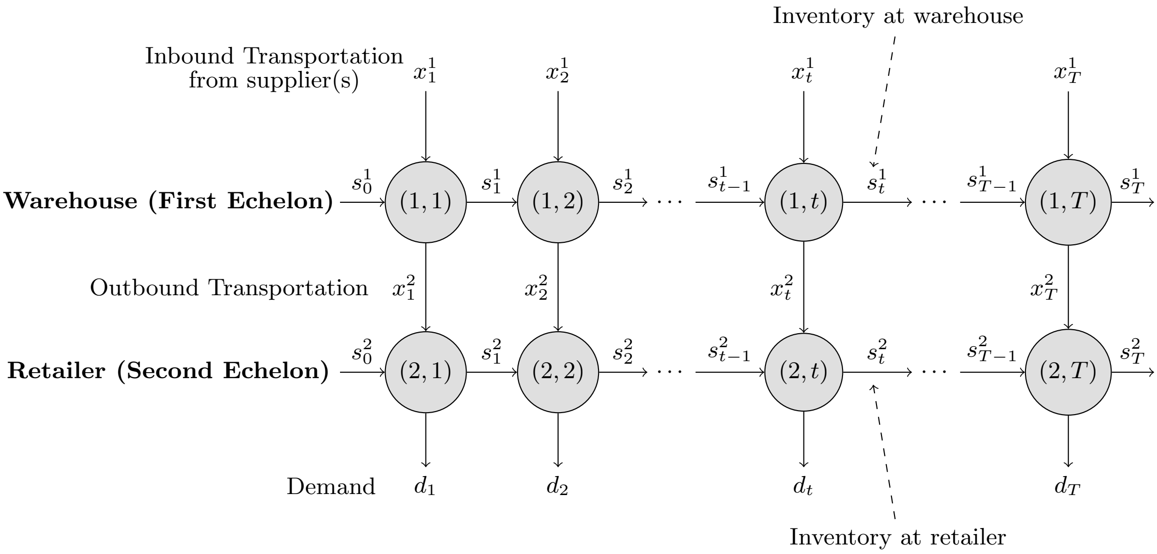


Figure 1 Network Flow Representation of the Two-Echelon Lot-Sizing Problem

We consider an integrated serial supply chain defined over a planning horizon that comprises of transportation of products from supplier(s) or manufacturing facilities to an intermediate warehouse (inbound transportation) and from the warehouse to a retailer (outbound transportation) using heterogeneous vehicles of varying capacities in each time period. Such problems are referred to as the two-echelon lot-sizing (2-ELS) problems where the first- and second-echelon represents the warehouse and the retailer, respectively. In Figure 1, we provide a network flow representation of the 2-ELS problem where in each echelon $e \in \{1,2\}$, there are T nodes, each corresponding to a time period $t \in \mathcal{T} := \{1,2,\ldots,T\}$ in the planning horizon. Each node in this figure is labelled as (e,t) where $e \in \{1,2\}$ denotes its echelon and $t \in \mathcal{T}$ denotes its time period. Variable x_t^1 denotes the amount transported from the supplier to the warehouse during period t and x_t^2 denotes the amount transported from the warehouse to the retailer during period $t \in \mathcal{T}$. Also, s_t^1 and s_t^2 denote the inventory held by the warehouse and the retailer at the end of period $t \in \mathcal{T}$, respectively.

Many researchers have studied 2-ELS and its variants (Lee et al. 2003, Kaminsky and Simchi-Levi 2003, van Hoesel et al. 2005, Hwang et al. 2016); refer to Section 2 for detailed literature review. However, there is no study in the literature that allows capacitated vehicle(s) for outbound transportation, and heterogeneous capacitated vehicles for inbound and outbound transportation (Zhao and Zhang 2020). In this paper, we study 2-ELS problems where single or multiple heterogeneous capacitated vehicles are available for both inbound and outbound transportation. Note that the capacities of vehicles available for inbound transportation can be different from capacities of vehicles available for outbound transportation. The motivation behind studying these problems stems from a plethora of real-world problems impacting semiconductor manufacturing, convoy resupply planning, and climate change that we will discuss in Section 1.2. Additionally, the 2-ELS problems also arise in integrated manufacturing and transportation systems where the first-echelon represents a manufacturing facility with heterogeneous machines of varying capacities. In other words, production decisions in such systems are equivalent to the inbound transportation decisions. Throughout the rest of the paper, for clear distinction between the echelons and to avoid confusion between inbound and outbound transportation, we refer to the first and second echelons as production and retailer echelons, respectively. Moreover, we refer to capacitated vehicles and machines as capacitated modules.

1.1. Problem Definition and Contributions

Given a planning horizon $\mathcal{T}$ with demand d_t for $t \in \mathcal{T}$ in the retailer (second) echelon, we formulate the 2-ELS problem with two capacitated modules in each of the echelons, i.e., C_p^1 and C_p^2 in first-echelon and C_r^1 and C_r^2 in second-echelon, as follows:

$$\text{Minimize} \sum_{t=1}^{T}\sum_{e=1}^{2}\left(h_t^e(s_t^e) + \sum_{i=1}^{2}\left(q_t^{e,i}y_t^{e,i} + p_t^{e,i}(x_t^{e,i})\right)\right) \tag{$\mathcal{P}$}$$

$$\text{s.t. } s_{t-1}^1 + x_t^1 = s_t^1 + x_t^2,\ t \in \mathcal{T}, \tag{1a}$$

$$s_{t-1}^2 + x_t^2 = d_t + s_t^2,\ t \in \mathcal{T}, \tag{1b}$$

$$x_t^{1,i} \le C_p^i y_t^{1,i},\quad i \in \{1,2\},\ t \in \mathcal{T}, \tag{1c}$$

$$x_t^{2,i} \le C_r^i y_t^{2,i},\quad i \in \{1,2\},\ t \in \mathcal{T}, \tag{1d}$$

$$x_t^1 = \sum_{i=1}^{2} x_t^{1,i} \le C_p^1 y_t^{1,1} + C_p^2 y_t^{1,2},\ t \in \mathcal{T}, \tag{1e}$$

$$x_t^2 = \sum_{i=1}^{2} x_t^{2,i} \le C_r^1 y_t^{2,1} + C_r^2 y_t^{2,2},\ t \in \mathcal{T}, \tag{1f}$$

$$y_t^{e,i} \in \{0,1\},\ x_t^{e,i}, x_t^e, s_t^e \ge 0,\ t \in \mathcal{T},\quad e \in \{1,2\},\quad i \in \{1,2\}, \tag{1g}$$

where $x_t^{1,i}$ denotes the amount produced in period t using module $i \in \{1,2\}$ in the first-echelon, and and $x_t^{2,i}$ denotes amount transported in period $t \in \mathcal{T}$ from first-echelon to second-echelon using module $i \in \{1,2\}$. Variables s_t^1 and s_t^2 denote the amount of inventory held at the end of period t in production and retailer echelons, respectively. Moreover, $y_t^{e,i}$ for $e \in \{1,2\}$, $i \in \{1,2\}$ and $t \in \mathcal{T}$ is a binary variable which is equal to 1 if module i is used in echelon e in period t, and 0 otherwise. The production and holding cost functions $p_t^{e,i}(\cdot)$ and $h_t^e(\cdot)$ are assumed to be concave, and in addition, a setup cost of $q_t^{e,i}$ is incurred whenever $y_t^{e,i}$ is equal to 1. We also assume that the initial and final inventories in each of the echelons are zero, i.e., $s_0^e = 0$ and $s_T^e = 0$ for $e \in \{1,2\}$. Note that this is a reasonable assumption that has been considered widely in the literature (van Hoesel et al. 2005, Florian and Klein 1971, Zangwill 1969), but it cannot be stated without the loss of generality.

Constraints (1a) and (1b) are the classical inventory or flow balance constraints to ensure that the incoming flow at each node is equal to the outgoing flow. Constraints (1c) and (1d) are the capacity constraints that ensure the amount produced or transported using each module to be less than or equal to the capacity of the module, if it is used (i.e. if $y_t^{e,i} = 1$). It should also be noted that the capacities of modules in both the echelons are assumed to be time invariant. If the capacities are not stationary across the planning horizon, the problem is NP-hard even for the single echelon problem (Florian et al. 1980).

Contributions of this paper. Kaminsky and Simchi-Levi (2003), van Hoesel et al. (2005) and Hwang et al. (2013, 2016) studied a special case of Problem $\mathcal{P}$ with a single uncapacitated transportation module, i.e., $C_r^1 \leftarrow \infty$ and $C_r^2 = 0$, and single capacitated production module, i.e., $C_p^2 = 0$. We denote this problem by 2-ELS-PCTU where 'PC' denotes single capacitated production module and 'TU' denotes single uncapacitated transportation module. It is well known that addition of the capacity constraints to an uncapacitated lot-sizing problem makes the problem more difficult to solve. This is mainly because the classical Zero Ordering Inventory property does not hold since we can produce/transport limited units in each time period. We present dynamic programming based polynomial time exact algorithms for the following four generalization of 2-ELS-PCTU:

(i) 2-ELS with single capacitated production and transportation modules, i.e., $\mathcal{P}$ with $C_p^2 = C_r^2 = 0$, denoted by 2-ELS-PCTC.

(ii) 2-ELS with single capacitated production module and two capacitated transportation modules, i.e., $\mathcal{P}$ with $C_p^2 = 0$, denoted by 2-ELS-PC2TC.

(iii) 2-ELS with two capacitated production modules and single capacitated transportation module, i.e., $\mathcal{P}$ with $C_r^2 = 0$, denoted by 2-ELS-2PCTC. Note that Sargut and Romeijn (2007) studied its special case with $C_p^1 \leftarrow \infty$ and $C_r^1 \leftarrow \infty$, i.e., transportation module and one of the two production modules are uncapacitated.

(iv) 2-ELS with two (one capacitated and another uncapacitated) production and transportation modules, i.e., $\mathcal{P}$ with $C_p^2 \leftarrow \infty$ and $C_r^2 \leftarrow \infty$. Since an uncapacitated (production or transportation) module also implies an additional option of uncapacitated subcontracting/outsourcing, we denote this problem by 2-ELS-PCTC-O.

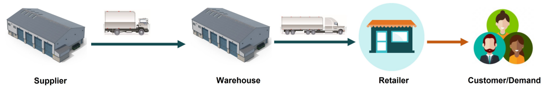
As per our knowledge, these four problems have not been studied in the literature, and the existence of polynomial time algorithm, even for 2-ELS-PCTC has been an open question (van Hoesel et al. 2005, Zhao and Zhang 2020). We summarize the contributions of this paper in Table 1 and pictorially represent the aforementioned four problems in Figure 2.

Significance of Results in Table 1. Apart from addressing an open question, the polynomial time algorithms provided in this paper also build stepping stones for various serial supply chain and lot-sizing problems with multiple heterogeneous capacitated vehicles in multiple echelons. Additionally, the dynamic programming algorithms can be used to obtain: (a) effective reformulations of these problems using variable redefinition (Eppen and Martin 1987), (b) feasible sub-optimal solutions and an efficient heuristics, and (c) cutting planes by filtering out several infeasible/suboptimal solutions (Hartman et al. 2010).

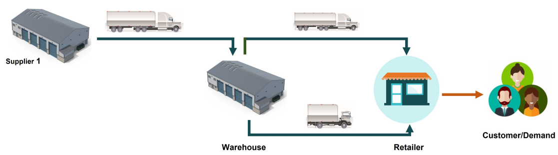
C_p^1	C_p^2	C_r^1	C_r^2	Contributor	Complexity
∞	0	∞	0	Zangwill (1969)	$O(T^4)$
				Love (1972)	$O(T^3)$
				Melo and Wolsey (2010)	$O(T^2 \log T)$
Arbitrary	0	∞	0	Kaminsky and Simchi-Levi (2003)	$O(T^8)$
				van Hoesel et al. (2005)	$O(T^7)$
				Hwang et al. (2016)	$O(T^6)$
Arbitrary	Arbitaray	∞	0	Sargut and Romeijn (2007)	$O(T^{10})$**
Arbitrary	∞	∞	0	Sargut and Romeijn (2007)	$O(T^8)$**
Arbitrary	Arbitrary	∞	0	Kulkarni and Bansal (2022b)	$O(T^{12})$
Arbitrary	0	Arbitrary	0	This article	$O(T^{10})$
Arbitrary	Arbitrary	Arbitrary	0	This article	$O(T^{16})$
Arbitrary	0	Arbitrary	Arbitrary	This article	$O(T^{13})$
Arbitrary	∞	Arbitrary	∞	This article	$O(T^{13})$

Table 1 Polynomial Algorithms for Two-Echelon Lot-Sizing Problem $\mathcal{P}$

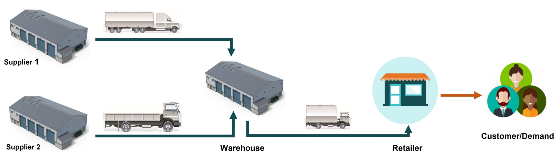
** Under assumption that setup costs are zero, i.e., $q_t^{1,i} = 0$ for all (i,t)



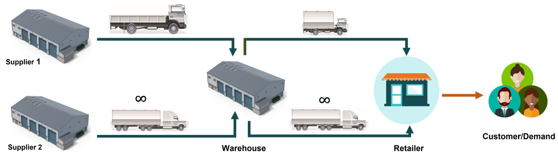
(a) 2-ELS-PCTC



(b) 2-ELS-PC2TC



(c) 2-ELS-2PCTC



(d) 2-ELS-PCTC-O

Figure 2 Pictorial single-period representation of four two-echelon lot-sizing problems studied in this paper

1.2. Other Practical Applications and Key Features of Problem $\mathcal{P}$

Other Practical Applications. In addition to applications in supply chain, logistics, and integrated manufacturing and transportation systems (discussed earlier), the two-echelon lot-sizing problems also arise in *assembly line systems* (Pochet and Wolsey 2006). The two echelons, namely sub-assembly and final-assembly, are common especially in semiconductor manufacturing industry (Sarin et al. 2011). Given the demand of the final assembled product over a planning horizon, the goal is to schedule the production on each of the two stages such that the overall sum of production costs, machines setup costs, and holding costs of the sub-assembled and final assembled items is minimized. It must be noted that production at either of the stages is more likely to be capacitated, since machines tend to have an operating capacity. Another area where 2-ELS problems arise is in *convoy resupply planning* where resources are transported from the suppliers to a centralized depot, and finally from the depot to the army base.

Key Features. One of the key features of the problems studied in this paper is that they allow presence of multiple capacitated modules in both the echelons. The notion of having multiple resources is motivated from the rise in global greenhouse gas emissions and stricter emission regulations due to which companies are using two types of resources: conventional resources and green resources, and each of these resources have their own costs and capacities (Hong et al. 2016). Another key feature of our models is consideration of fixed cost $q_t^{e,i}$ to represent the machine setup costs or the fixed ordering costs, along with variable concave cost functions $p_t^{e,i}(.)$ to address economies of scale.

Organization of the paper. The rest of the paper is organized as follows. In Section 2, we review existing studies on lot-sizing problems that are closely related to problem $\mathcal{P}$. In Section 3, we present a dynamic programming algorithm to solve 2-ELS-PCTC to optimality. We then present polynomial time algorithms for 2-ELS-2PCTC and 2-ELS-PC2TC in Section 4, and for 2-ELS-PCTC-O in Section 5. Lastly, we provide concluding remarks and potential directions for future research in Section 6.

2. Literature Review

In this section, we review algorithmic results in the literature for problems that are either special cases or closely related to problem $\mathcal{P}$ by classifying them into two categories: single-echelon lot-sizing problems and multi-echelon lot-sizing problems. Note that 2-ELS-PCTC reduces to a single-echelon problem when the inventory held at the end of each period t in the second-echelon is equal to zero, i.e., $s_t^2 = 0$ for $t \in \mathcal{T}$, and $p_t^{2,i} = q_t^{2,i} = 0$ for $t \in \mathcal{T}$ and $i \in \{1, 2\}$. For such cases, we denote the problems using the prefix 1-ELS- instead of 2-ELS-. Furthermore, since there are no second echelon variables in the single-echelon problems, we drop the suffix 'TC' or 'TU' from the problem notations.

Single-Echelon Lot-Sizing Problems. Single-echelon lot-sizing problem with single capacitated module (denoted by 1-ELS-PC) is NP-hard in general when the capacities are time-varying across the planning horizon. In a seminal paper, Wagner and Whitin (1958) presented an $O(T^2)$ algorithm to solve uncapacitated lot-sizing problem i.e. 1-ELS-PC with $C_p^1 = \infty$, that was later improved to $O(T \log T)$ by Aggarwal and Park (1993). It is well-known that adding capacities increases the complexity of the problem. When the capacity is stationary, Florian and Klein (1971) provided an exact dynamic programming algorithm that solves 1-ELS-PC in $O(T^4)$ time. van Hoesel and Wagelmans (1996) provided an improved $O(T^3)$ algorithm to solve the foregoing problem when the holding costs are assumed to be linear. Atamtürk and Hochbaum (2001) studied the lot-sizing problem with single constant capacity and an additional option of uncapacitated subcontracting, i.e., 1-ELS-2PC with $C_p^2 = \infty$. Kulkarni and Bansal (2022a,b) studied lot-sizing problems with multiple ($n \in \mathbb{Z}_+$) capacitated modules available for utilization in each time period, with additional features such as subcontracting and backlogging, and for a fixed n, they provided polynomial time exact algorithms for each of these variants.

Multi-Echelon Lot-Sizing Problems. Over the years, researchers have been studying lot-sizing problems with $L \geq 2$ echelons. We use the prefix 'L-ELS-' instead of '2-ELS-' to denote this general class of problems. Since none of the problems studied in the literature consider finite capacity module(s) in echelons 2 to L, we denote these classes of problems using the suffix 'TU' (Transportation Uncapacitated) at the end, instead of 'TC'. Also, for problems where the production module is uncapacitated, i.e., $C_p^1 = \infty$, we denote them using the suffix 'PU' (Production Uncapacitated).

Production Uncapacitated and Transportation Uncapacitated. For L-ELS-PUTU, Zangwill (1969) presented an $O(LT^4)$ dynamic programming algorithm, and van Hoesel et al. (2005) showed that for $L = 2$, it takes $O(T^3)$ time. Thereafter, Melo and Wolsey (2010) introduced an improved dynamic programming algorithm for the 2-ELS-PUTU that runs in $O(T^2 \log T)$ time. Lee et al. (2003) presented an $O(T^6)$ algorithm for a variant of 2-ELS-PUTU where the transportation cost function is a step-wise function of the transportation capacity module, the production cost function is constant, and back-ordering is allowed in the second-echelon. To solve L-ELS-PUTU with demands in each echelon, Zhang et al. (2012) and Zhao and Zhang (2020) developed exact algorithms that take $O(T^4)$ time for $L = 2$ and $O(T^{3L+1})$ time for $L \geq 2$, respectively.

Production Capacitated and Transportation Uncapacitated. Kaminsky and Simchi-Levi (2003) studied a generalization of 3-ELS-PCTU where the first- and third-echelon are capacitated and second-echelon is uncapacitated with fixed (or concave) production/transportation costs. They showed that under several assumptions for the cost structures, this three-echelon problem can be reduced to a special case of 2-ELS-PCTU and they proposed an $O(T^8)$ time polynomial algorithm to solve the latter. van Hoesel et al. (2005) analyzed the structure of the optimal solution of 2-ELS-PCTU, and presented an $O(T^7)$ time algorithm to solve it. Later, Hwang et al. (2016) provided an improved $O(T^6)$ running time algorithm for 2-ELS-PCTU. Sargut and Romeijn (2007) studied the following two variants of 2-ELS-PCTU: (a) 2-ELS-2PCTU with $q_t^{1,1} = q_t^{1,2} = 0$ for $t \in \mathcal{T}$, i.e., two capacitated modules in the first-echelon with no setup costs, and (b) 2-ELS-2PCTU with $q_t^{1,1} = q_t^{1,2} = 0$ and $C_p^2 = \infty$, i.e., one capacitated module and one uncapacitated module (outsourcing) available for production in the first echelon. They developed $O(T^{10})$ and $O(T^8)$ time algorithms for them, respectively. Kulkarni and Bansal (2022b) also studied a generalization of 2-ELS-PCTU with n capacitated production modules, and developed an exact fixed parameter tractable algorithm that runs in $O(T^{4n+4})$ time. For $n = 2$, it takes $O(T^{12})$ time and is a special case of 2-ELS-2PCTC.

van Hoesel et al. (2005) extended the results for 2-ELS-PCTU to L-ELS-PCTU by presenting an $O(LT^{2L+3})$ time algorithm to solve it. Hwang et al. (2013) presented a significantly improved algorithm for L-ELS-PCTU with general concave costs that runs in $O(LT^8)$ time. Recently, Zhao and Zhang (2020) presented an $O(T^{2L^2+2})$ polynomial time algorithm to solve L-ELS-PCTU with demands present in intermediate echelons. If L is a part of the input, then the L-ELS-PCTU problem with intermediate demands is NP-hard (Zhao and Zhang 2020).

Production Capacitated and Transportation Capacitated. Existence of a polynomial time algorithm for 2-ELS-PCTC has been an open question (van Hoesel et al. 2005, Zhao and Zhang 2020) that we address in this paper.

3. Exact Polynomial Time Algorithm for 2-ELS-PCTC

In this section, we present an exact polynomial time dynamic programming algorithm to solve 2-ELS-PCTC. Since there is only one capacitated module in each echelon, we drop ",1" from the superscripts of $x_t^{1,1}$, $x_t^{2,1}$, $p_t^{e,1}$, $q_t^{e,1}$, and $y_t^{e,1}$ to get x_t^1, x_t^2, p_t^e, q_t^e, and y_t^e, respectively. Likewise, we also drop 1 from the superscripts of C_p^1 and C_r^1 to obtain C_p and C_r, respectively. We denote the cumulative sum of demands from periods t_1 through t_2 by $d_{t_1,t_2} = \sum_{j=t_1}^{t_2} d_j$.

3.1. Definitions and Characteristics of Extreme Points

First, we introduce some definitions that are needed to present the algorithm. We also analyze the structure of the optimal solutions of 2-ELS-PCTC. Since the objective function is concave, an optimal solution of the problem lies at an extreme point of the feasible region. Hence, we study the properties of the extreme points of the feasible region of 2-ELS-PCTC.

Definition 1. (*Subplan* $[a_1, a_2, b_1, b_2]$). *A collection of nodes* $(1, a_1), \ldots, (1, b_1)$, $(2, a_2), \ldots, (2, b_2)$, *where* $1 \le a_1 \le a_2 \le b_1 \le b_2 \le T$, *is a subplan if* $s^1_{a_1-1} = 0$, $s^2_{a_2-1} = 0$, $s^1_{b_1} = 0$, $s^2_{b_2} = 0$, $s^1_t > 0$ *for* $t \in \{a_1, \ldots, a_2 - 1\}$, $s^2_t > 0$ *for* $t \in \{b_1, \ldots, b_2 - 1\}$, *and at most one among* s^1_t *and* s^2_t *is equal to zero for* $t \in \{a_2, \ldots, b_1 - 1\}$. *Subplans* $[a_1, a_2, b_1, b_2]$ *and* $[a'_1, a'_2, b'_1, b'_2]$ *are referred to as consecutive subplans if* $a'_1 = b_1 + 1$ *and* $a'_2 = b_2 + 1$.

Definition 2. (*Block* $[k, l, type]$). *Given a subplan* $[a_1, a_2, b_1, b_2]$, *a collection of nodes* $(1, k), \ldots, (1, l)$ *and* $(2, k), \ldots, (2, l)$ *where* $a_2 \le k \le l \le b_1$, *such that* $s^e_t > 0$ *for all* $e \in \{1, 2\}$ *and* $t \in \{k, \ldots, l - 1\}$, *is referred to as a block associated to the subplan if the following conditions are satisfied depending on its "type":*

(a) $s^1_{k-1} > 0$, $s^1_l \ge 0$, $s^2_{k-1} = 0$, *and* $s^2_l = 0$ *if* $type = A_{11}$

(b) $s^1_{k-1} \ge 0$, $s^1_l = 0$, $s^2_{k-1} = 0$, *and* $s^2_l > 0$ *if* $type = A_{12}$

(c) $s^1_{k-1} = 0$, $s^1_l > 0$, $s^2_{k-1} \ge 0$, *and* $s^2_l = 0$ *if* $type = A_{21}$

(d) $s^1_{k-1} = 0$, $s^1_l = 0$, $s^2_{k-1} > 0$, *and* $s^2_l \ge 0$ *if* $type = A_{22}$

In Figure 3, we present an example for each of the four types of blocks for $k = 2$ and $l = 5$. Note that block type A_{ij}, $i, j \in \{1, 2\}$, corresponds to the inventory at the beginning of period k in echelon i, i.e., s^i_{k-1}, and inventory at the end of period of l in echelon j, i.e., s^j_l. For example, in block $[2, 5, A_{11}]$, inventory enters period 2 and leaves period 5 of the first-echelon.

Definition 3. *Fractional Production Period (FPP). A period* t *is a fractional production period in a subplan* $[a_1, a_2, b_1, b_2]$, *if* $0 < x^1_t < C_p$ *for some* $t \in \{a_1, \ldots, b_1\}$.

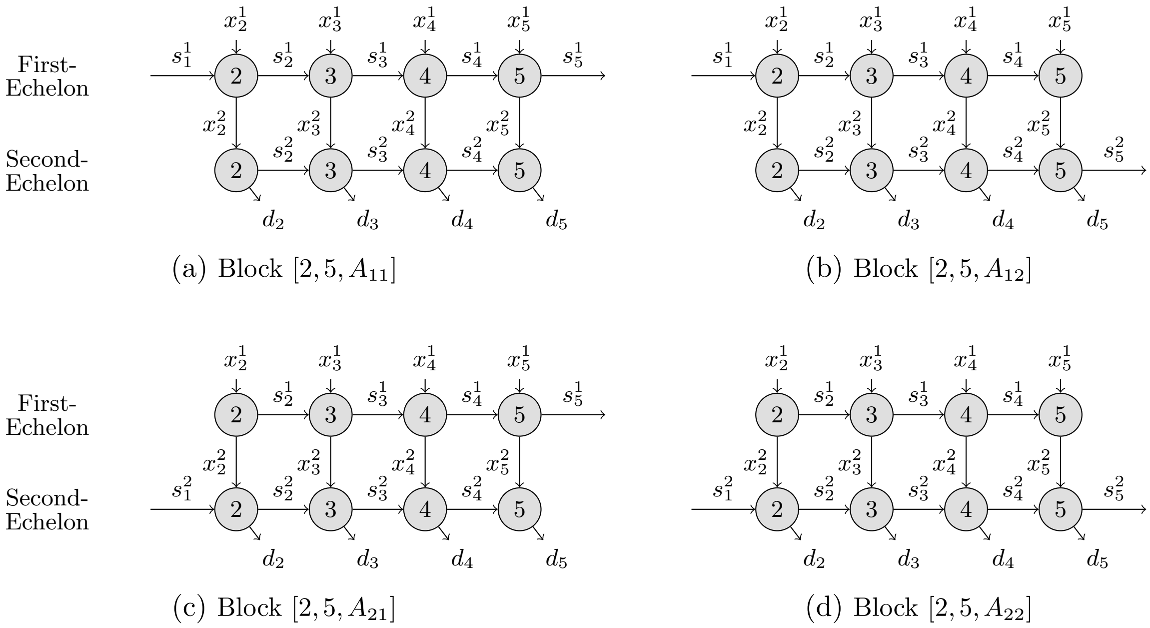


Figure 3 Four different types of blocks for $k=2$ and $l=5$

Definition 4. *Fractional Transportation Period (FTP). A period t is a fractional transportation period in a subplan $[a_1,a_2,b_1,b_2]$, if $0<x_t^2<C_r$ for some $t\in\{a_2,\dots,b_2\}$.*

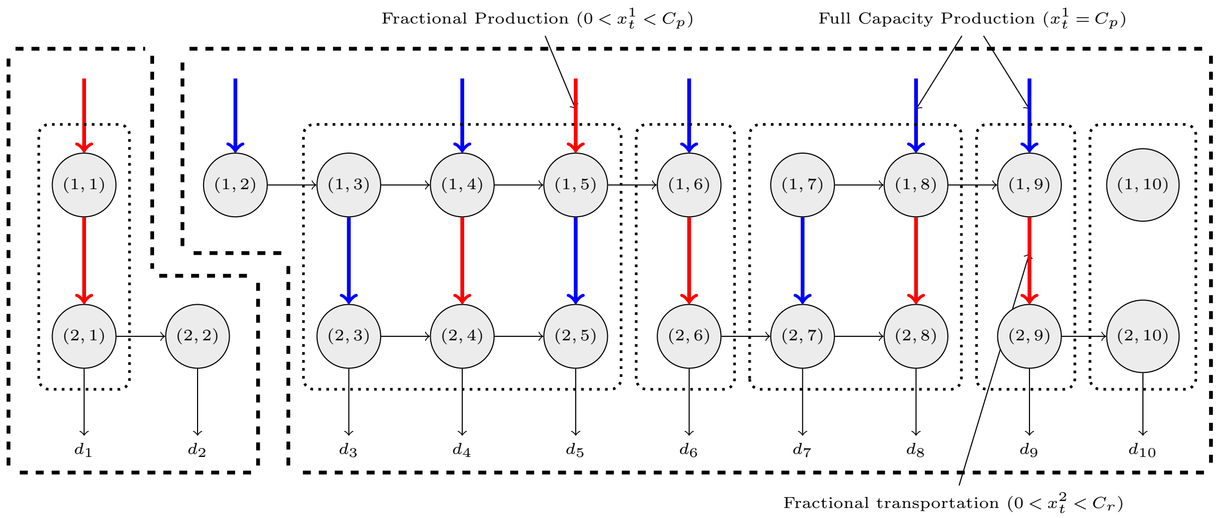


Figure 4 Extreme Point Solution of 10 period two echelon problem

Properties of Extreme Point Solution: In Figure 4, we provide an example of an extreme point solution of 2-ELS-PCTC with 10 time periods in the planning horizon. This extreme point has two subplans: $[1,1,1,2]$ and $[2,3,10,10]$ (outlined using dashed lines). The red arcs represent fractional production or fractional transportation and the blue arcs represent the full capacity production $(x_t^1=C_p)$ or transportation $(x_t^2=C_r)$. It can be observed that there is at most one fractional production period in each of the two subplans. Also, note that within subplan $[1,1,1,2]$ there is one block $[1,1,A_{11}]$ (outlined using dotted lines) and, within subplan $[2,3,10,10]$, there are five blocks: $[3,5,A_{11}]$, $[6,6,A_{12}]$, $[7,8,A_{21}]$, $[9,9,A_{12}]$, and $[10,10,A_{21}]$. Each block has at most one fractional transportation period.

Definition 5 (Consecutive Blocks)**.** *Blocks* $[k', l', type']$ *and* $[k'', l'', type'']$ *are consecutive if* $k'' = l' + 1$ *and either* $type' \in \{A_{i1}\}_{i\in\{1,2\}}$ *and* $type'' \in \{A_{1j}\}_{j\in\{1,2\}}$*, or* $type' \in \{A_{i2}\}_{i\in\{1,2\}}$ *and* $type'' \in \{A_{2j}\}_{j\in\{1,2\}}$*. For example, blocks* $[3, 5, A_{11}]$ *and* $[6, 6, A_{12}]$ *in Figure 4 are consecutive blocks. Actually, all five blocks in the subplan* $[2, 3, 10, 10]$ *are consecutive and they span periods* $a_2 = 3$ *through* $b_1 = 10$*.*

Free Arc Property. A free arc in a network flow is defined as any arc that carries a non-negative flow which is not at the full capacity. A free arc diagram is a network with only free arcs as undirected edges. Ahuja et al. (1988) studied the minimum cost network flow problem with concave costs and proved that the free arc network flow diagram associated with each of its extreme points does not contain any cycle. This result is widely referred to as the *free arc property*, and we utilize it to derive the following theorem.

Theorem 1. *There exists an optimal solution of 2-ELS-PCTC that consists of a series of consecutive subplans spanning over the entire planning horizon such that:*

(a) Each subplan has at most one fractional production period, and

(b) Each subplan $[a_1, a_2, b_1, b_2]$ *consists of a series of consecutive blocks spanning periods* a_2 *through* b_1 *such that each block has at most one fractional transportation period.*

Proof. The 2-ELS-PCTC can also be defined as a minimum cost network flow problem by constructing a source node connected to each node of the first-echelon with x_t^1 as outgoing flow from the source and a sink node connected to each node of the second-echelon with d_t as incoming flow to the sink. Therefore, each of its extreme point satisfies the free arc property. Now, assume that an extreme point solution has two fractional production periods within a subplan. Then, the free arc diagram associated with this extreme point will have a cycle. Similarly, if an extreme point solution has two fractional transportation periods within a block, the free arc diagram will not be acyclic. □

Remark 1. *Observe that according to this theorem, an optimal solution can have a subplan with more than one fractional transportation periods. As shown in Figure 4, even though Subplan* $[2, 3, 10, 10]$ *has 4 FTPs, its free arc diagram is acyclic.*

Remark 2. *It should be noted that this procedure to construct the graph for each subplan generalizes the procedure developed by van Hoesel et al. (2005) for 2-ELS-PCTU where the concept of blocks is not needed. In case of 2-ELS-PCTU with the uncapacitated retailer echelon, a subplan associated with any optimal solution has only single-period blocks* $[k, k, A_{ij}]$ *for* $k \in \{a_2, \ldots, b_1\}$*. On the contrary, we showed that in 2-ELS-PCTC, there may exist multi-period blocks within any extreme point solution.*

Definition 6. *Cumulative production quantity in period t of Subplan $[a_1, a_2, b_1, b_2]$ is the total amount produced in periods $a_1, \dots, t$. We denote it by ξ_t. According to Theorem 1, we explore solutions where in each period $t \in \{a_1, \dots, b_1\}$, production amount x_t^1 is either 0, C_p, or $f_p = d_{a_2,b_2} - C_p\lfloor \frac{d_{a_2,b_2}}{C_p} \rfloor < C_p$ (fractional production quantity). Therefore, ξ_t belongs to a finite set $\Xi_t = \Xi_t^1 \cup \Xi_t^2$ where*

$$\Xi_t^1 := \{0, C_p, 2C_p, \dots, (t - a_1 + 1)C_p\} \text{ , and } \Xi_t^2 := \{f^p, C_p + f^p, \dots, (t - a_1)C_p + f^p\}.$$

Note that for $t \in \{b_1, \dots, b_2\}$, $\Xi_t = \{d_{a_2,b_2}\}$ since the overall demand of the subplan must be satisfied. In the ensuing sections, we compute optimal cost for a given subplan and utilize it for computing optimal solution for 2-ELS-PCTC.

3.2. Computing optimal costs for a given subplan $[a_1, a_2, b_1, b_2]$

To compute the minimum cost for a given subplan $\phi = [a_1, a_2, b_1, b_2]$, we construct a directed acyclic graph $\mathcal{N}^\phi = (V, E)$ where $V := \{(k, e, \xi_k) : k \in \{a_2, \dots, b_1\},\ e \in \{1, 2\},\ \xi_k \in \Xi_k\} \bigcup \{(k, 1, \xi_k) : k \in \{a_1, \dots, a_2 - 1\},\ \xi_k \in \Xi_k\} \bigcup \{(k, 2, \xi_k) : k \in \{b_1 + 1, \dots, b_2\},\ \xi_k \in \Xi_k\}$ is the collection of nodes in the graph. Additionally, there is a source node labeled as $(a_1 - 1, 1, 0)$ and a destination node labeled as $(b_2, 2, d_{a_2,b_2})$. The overall optimal cost for each subplan is computed by finding the shortest path from the source node to the destination node. We now discuss the procedure to construct set of arcs E in the graph, and to assign corresponding arc weights. Note that $\mathcal{N}^\phi$ has $O(T^2)$ nodes and $O(T^4)$ arcs as Ξ_k has $O(T)$ elements. In Figure 5, we showcase construction of the nodes, the arcs, and their arc weights for $k \in \{a_1, \dots, a_2\}$ and for $k \in \{b_1, \dots, b_2\}$.

If $k \in \{a_1, \dots, a_2 - 1\}$, we make first-echelon decisions only because there is no block that exists between periods a_1 and a_2. Therefore, we create forward arc from node $(k-1, 1, \xi_{k-1})$ to $(k, 1, \xi_k)$ for all $k \in \{a_1, \dots, a_2 - 1\}$, $\xi_{k-1} \in \Xi_{k-1}$, and $\xi_k \in \Xi_k$ such that $\xi_{k-1} - \xi_k \in \{0, C_p, f^p\}$. The associated weight, denoted by $\Delta_{k,k}^1(\xi_{k-1}, \xi_k)$, of this arc is the sum of costs incurred to produce $\xi_k - \xi_{k-1}$ and hold ξ_k during period k, i.e.,

$$\Delta_{k,k}^1(\xi_{k-1}, \xi_k) = \begin{cases} h_k^1(\xi_k), & \text{if } \xi_k = \xi_{k-1} \\ p_k^1(\xi_k - \xi_{k-1}) + q_k^1 + h_k^1(\xi_k), & \text{otherwise.} \end{cases}$$

If $k \in \{b_1 + 1, \dots, b_2\}$, we make no decisions as the first echelon nodes during these time periods are not part of the subplan. Since the total amount that should be produced and transported prior to period $b_1 + 1$ is d_{a_2,b_2}, only holding costs are incurred at the end of every period. For $k \in \{b_1 + 1, \dots, b_2\}$, we create forward arcs from nodes $(k-1, 2, \xi_{k-1})$ to $(k, 2, \xi_k)$ and assign the edge weights $\Delta_{k,k}^2(\xi_{k-1}, \xi_k) = h_k^2(\xi_k - d_{a_2,k})$ where $\xi_k = \xi_{k-1} = d_{a_2,b_2}$.

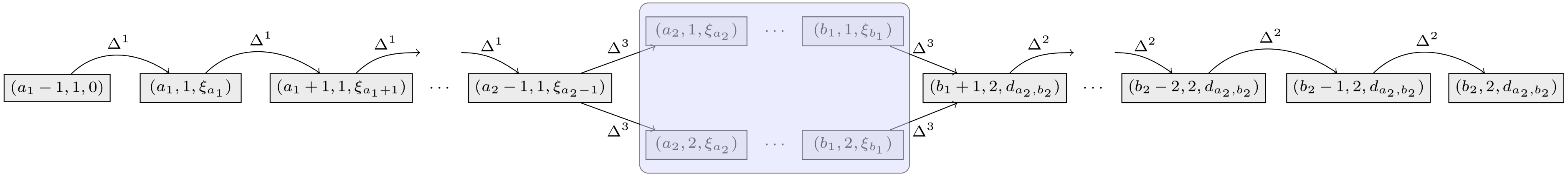


Figure 5 **Construction of directed acyclic graph $\mathcal{N}^\phi$ for a given subplan** (a_1, a_2, b_1, b_2)

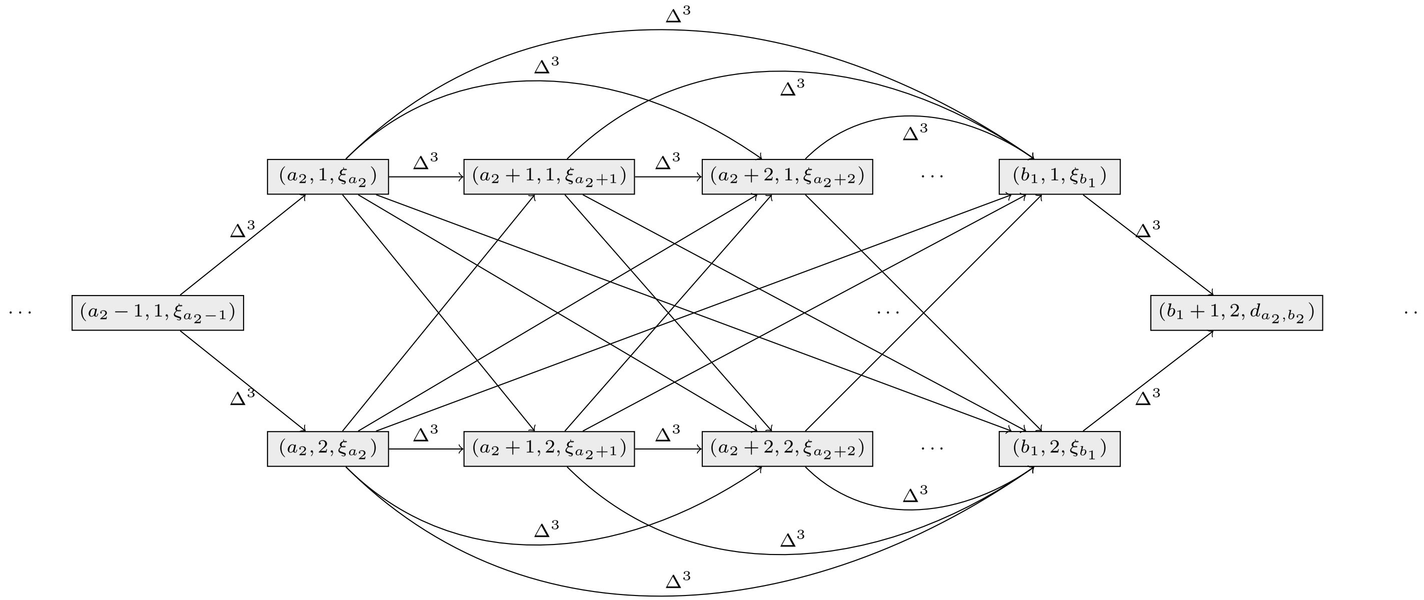


Figure 6 **Construction of subgraph for $t \in \{a_2-1, \ldots, b_1+1\}$ in a subplan** (a_1, a_2, b_1, b_2)

If $k \in \{a_2, \dots, b_1\}$, we create forward arcs from node $(k-1, i, \xi_{k-1})$ to (l, j, ξ_l) for all $k \le l \le b_1$, $i, j \in \{1, 2\}$, $\xi_{k-1} \in \Xi_{k-1}$, and $\xi_l \in \Xi_l$ such that $0 \le \xi_l - \xi_{k-1} \le (l-k+1)C_p$. We do not create an arc between two nodes if $\xi_{k-1} \in \Xi^2_{k-1}$ and $\xi_l \in \Xi^1_l$ since this implies that fractional production has occurred up to period $k-1$, but not upto period l, which is impossible. The weight for an arc from $(k-1, i, \xi_{k-1})$ to (l, j, ξ_l), denoted by $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, A_{ij})$, is the minimum cost of satisfying the demands of block $[k, l, A_{ij}]$, given that ξ_{k-1} units have been produced up to period $k-1$, and ξ_l units have been produced up to period l. We demonstrate the construction of the foregoing network for $k \in \{a_2, \dots, b_1\}$ in Figure 6.

Notice that, irrespective of the block type, the amount produced from periods k upto l is equal to $\xi_l - \xi_{k-1}$. In contrast, the amount transported from periods k through l (denoted by $\rho^{ij}_{k,l}$) depends on the type of block, i.e.

$$\rho^{ij}_{k,l} = \begin{cases} d_{k,l}, & \text{if } A_{ij} = A_{11} \\ \xi_l - d_{a_2,k-1}, & \text{if } A_{ij} = A_{12} \\ d_{1,l} - \xi_{k-1}, & \text{if } A_{ij} = A_{21} \\ \xi_l - \xi_{k-1}, & \text{if } A_{ij} = A_{22} \end{cases} \tag{2}$$

From Theorem 1, we know that there exists at most one FTP within any block $[k, l, A_{ij}]$. As a result, there will be $\mu^{max}_{ij} = \min\{l-k+1, \lfloor \frac{\rho^{ij}_{k,l}}{C_r} \rfloor\}$ periods where transportation occurs at full capacity and there will be at most one period with a fractional transportation quantity of $f^r = \rho^{ij}_{k,l} - \lfloor \frac{\rho^{ij}_{k,l}}{C_r} \rfloor C_r$ units. In a nutshell, there will be either (a) $l-k+1$ periods where we transport either 0 or C_r units, or (b) $l-k$ periods where we produce either 0 or C_p units and exactly one period where we transport f^r units. We now compute $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, A_{ij})$ using the following two steps:

(A) Compute the minimum cost of producing $\xi_{k-1} - \xi_l$ units and transporting $\rho^{ij}_{k,l}$ units such that there is no FTP within block $[k, l, type]$ (Section 3.2.1). We denote this cost function by $G^{\xi_{k-1},\xi_l}_{k,l,type}(l, \xi_l, \mu^{max}_{ij}, 0)$.

(B) Compute the minimum cost of producing $\xi_{k-1} - \xi_l$ units and transporting $\rho^{ij}_{k,l}$ units such that there is one FTP within block $[k, l, type]$ (Section 3.2.2). We denote this cost function by $G^{\xi_{k-1},\xi_l}_{k,l,type}(l, \xi_l, \mu^{max}_{ij}, 1)$.

Specifically, $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type) = \min\left\{G^{\xi_{k-1},\xi_l}_{k,l,type}(l, \xi_l, \mu^{max}_{ij}, 0), G^{\xi_{k-1},\xi_l}_{k,l,type}(l, \xi_l, \mu^{max}_{ij}, 1)\right\}$.

3.2.1. Minimum cost for a given block $[k, l, type]$ **without FTP.** Using the following recursive equations, we define a cost function $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ to compute the minimum cost of producing $\xi_t - \xi_{k-1}$ units and transporting μC_r units from first-echelon to second-echelon during periods k to t, for $t \in \{k, \dots, l\}$, $\xi_t \in \Xi_t$, and $\mu \in \{0, \dots, \mu^{max}_{ij}\}$:

$$G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,0) = \min \begin{cases} G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t,\mu,0) + \mathcal{H}_t, & \text{(3a)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t,\mu-1,0) + p^2_t(C_r) + q^2_t + \mathcal{H}_t, & \text{(3b)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t - C_p,\mu,0) + p^1_t(C_p) + q^1_t + \mathcal{H}_t, & \text{(3c)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t - C_p,\mu-1,0) + p^1_t(C_p) + q^1_t + p^2_t(C_r) + q^2_t + \mathcal{H}_t, & \text{(3d)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t - f^p,\mu,0) + p^1_t(f^p) + q^1_t + \mathcal{H}_t, & \text{(3e)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t - f^p,\mu-j,0) + p^1_t(f^p) + q^1_t + p^2_t(C_r) + q^2_t + \mathcal{H}_t. & \text{(3f)} \end{cases}$$

Here, $\mathcal{H}_t = h^1_t(s^1_t) + h^2_t(s^2_t)$ is the overall holding cost incurred in period t where the values of s^1_t and s^2_t are dependent on the type of block, and are provided in Table 2.

Block Type	s^1_t	s^2_t
A_{11}	$\xi_t - d_{a_2,k-1} - \mu C_r$	$\xi_{k-1} + \mu C_r - d_{a_1,t}$
A_{12}	$\xi_t - d_{a_2,k-1} - \mu C_r$	$\xi_{k-1} + \mu C_r - d_{a_1,t}$
A_{21}	$\xi_t - \xi_{k-1} - \mu C_r$	$\mu C_r - d_{k,t}$
A_{22}	$\xi_t - \xi_{k-1} - \mu C_r$	$\mu C_r - d_{k,t}$

Table 2 Inventory in both echelons at the end of period t depending on block type

The aforementioned recursive equations are derived as follows. In each period $t \in \{k,\dots,l\}$, there are three possible amounts that can be produced which are 0, C_p, or f^p. Also, since fractional transportation has not occurred up to period t, the possible transportation amounts in period t are 0 and C_r. This leads to six possibilities in each time period t of the block $[k,l,type]$ that determine the value of the cost function $G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,0)$. Equations (3a)- (3b), (3c)-(3d), and (3e)- (3f) are corresponding to production of 0, C_p, and f^p in period t, respectively. Additionally, equations (3a, 3c, 3e) and (3b, 3d, 3f) are for 0 and C_r units of transportation during period t, respectively.

Also, if (a) $\xi_t \in \Xi^1_t$, or (b) $\xi_{k-1} \in \Xi^2_{k-1}$ and $\xi_l \in \Xi^2_l$, then we set expressions (3e) and (3f) to infinity since there is no FPP between period k and t. We also set the value of $G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,0)$ to infinity for the following infeasible cases: (a) $\xi_t < \xi_{k-1}$ or $\xi_t > \xi_l$ or $\xi_t \notin \Xi_t$, (b) $\mu > t-k+1$, (c) $\mu C_r > \xi_t - d_{1,k-1}$, (d) $\mu C_r < d_{k,t}$, (e) $\xi_{k-1} \in \Xi^1_{k-1}$, $\xi_l \in \Xi^1_l$, and $\xi_t \in \Xi^2_t$, and (f) $\xi_{k-1} \in \Xi^2_{k-1}$, $\xi_l \in \Xi^2_l$, and $\xi_t \in \Xi^1_t$.

3.2.2. Minimum cost for a given block $[k,l,type]$ with one FTP. For $t \in \{k,\dots,l\}$, $\xi_t \in \Xi_t$, and $\mu \in \{0,\dots,\min(l-k+1,\mu^{max})\}$, we also derive the following recursion equations to compute $G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,1)$ which is the minimum cost of producing $\xi_t - \xi_{k-1}$ units, and transporting $\mu C_r + f^r$ units from period k through t.

$$G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,1) = \min_{\alpha_t\in\{0,C_p,f^p\}} \begin{cases} G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t-\alpha_t,\mu,1) + \mathcal{Q}^1_t(\alpha_t) + \mathcal{H}^r_t, & \text{(4a)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t-\alpha_t,\mu-1,1) + \mathcal{Q}^1_t(\alpha_t) + p^2_t(C_r) + q^2_t + \mathcal{H}^r_t, & \text{(4b)} \\ G^{\xi_{k-1},\xi_l}_{k,l,type}(t-1,\xi_t-\alpha_t,\mu,0) + \mathcal{Q}^1_t(\alpha_t) + p^2_t(f^r) + q^2_t + \mathcal{H}^r_t, & \text{(4c)} \end{cases}$$

where $\mathcal{Q}^1_t(\alpha_t)$ is the cost of producing α_t during period t and it is defined as follows:

$$\mathcal{Q}^1_t(\alpha_t) = \begin{cases} 0, & \text{if } \alpha_t = 0, \\ p^1_t(C_p) + q^1_t, & \text{if } \alpha_t = C_p, \\ p^1_t(f^p) + q^1_t, & \text{if } \alpha_t = f^p. \end{cases} \tag{5}$$

Note that in each period $t \in \{k,\dots,l\}$, there are three options available for production, $x^1_t \in \{0, C_p, f^p\}$, and three options available for transportation, $x^2_t \in \{0, C_r, f^r\}$, thereby resulting in nine total possible options. Equation (4a), (4b), and (4c) represent the cases where α_t items are produced in period t which incurs a cost of $\mathcal{Q}^1_t(\alpha_t)$, and the amount transported during period t is 0, C_r, and f^r, respectively. Again, $\mathcal{H}^r_t = h^1_t(s^1_t) + h^2_t(s^2_t)$ denotes the holding cost incurred during period t where s^1_t and s^2_t are obtained using Table 2 by replacing μC_r with $\mu C_r + f^r$.

We set the value of $G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,1)$ to infinity for the following infeasible cases: (a) $\xi_t < \xi_{k-1}$ or $\xi_t > \xi_l$ or $\xi_t \notin \Xi_t$, (b) $\mu > t-k$, (c) $\mu C_r + f^r > \xi_t - d_{1,k-1}$, (d) $\mu C_r + f^r < d_{k,t}$, (e) $\xi_{k-1} \in \Xi^1_{k-1}$, $\xi_l \in \Xi^1_l$, and $\xi_t \in \Xi^2_t$, (f) $\xi_{k-1} \in \Xi^2_{k-1}$, $\xi_l \in \Xi^2_l$, and $\xi_t \in \Xi^1_t$. Moreover, $\alpha_t \in \{0, C_p\}$ instead of $\alpha_t \in \{0, C_p, f^p\}$ if either (i) $\xi_{k-1} \in \Xi^1_{k-1}$ and $\xi_l \in \Xi^1_l$, or (ii) $\xi_{k-1} \in \Xi^2_{k-1}$ and $\xi_l \in \Xi^2_l$.

3.3. Computing Optimal Solution and Cost for 2-ELS-PCTC

We compute the minimum costs for all the possible subplans within the planning horizon using results presented in the previous section. Thereafter, a network is constructed with a set of nodes $\mathcal{V} := \{(a_1,a_2) : a_1, a_2 \in \mathcal{T}\}$ such that an arc between the nodes (a_1,a_2) and $(b_1,b_2) \in \mathcal{V}$ represents a Subplan $[a_1,a_2,b_1,b_2]$ with optimal cost of the subplan as its weight. Finally, we find the best sequence of subplans using the shortest path algorithm, as follows. We define a function $OPT(a_1,a_2)$ that computes the overall optimal cost of planning the production schedules from periods $a_1,\dots,T$ and transportation schedules from periods $a_2,\dots,T$ where $a_1 \le a_2$, such that $s^1_{a_1-1} = 0$ and $s^2_{a_2-1} = 0$. We then apply the following backward recursion to compute $OPT(a_1,a_2)$ for all $a_1,a_2 \in \mathcal{T}$:

$$OPT(a_1,a_2) = \min_{\substack{b_1,b_2\in\mathcal{T} \\ a_2\le b_1\le b_2}} \left\{ OPT(b_1+1,b_2+1) + \psi([a_1,a_2,b_1,b_2]) \right\},$$

where $OPT(T+1,T+1) = 0$. The overall optimal solution of 2-ELS-PCTC can be obtained by computing the value of $OPT(1,1)$ which essentially provides the optimal cost of scheduling production and transportation from periods 1 to T.

Theorem 2. *Problem 2-ELS-PCTC is solved to optimality using the above algorithm in* $O(T^{10})$ *time.*

Proof. Refer to Appendix A

4. Exact algorithm for generalizations of 2-ELS-PCTC

In this section, we present polynomial time exact algorithms for two generalizations of 2-ELS-PCTC: (*i*) 2-ELS-PC2TC (Section 4.1), and (*ii*) 2-ELS-2PCTC (Section 4.2). The key difference between the algorithms for 2-ELS-PCTC, 2-ELS-PC2TC, and 2-ELS-2PCTC lies in computing the optimal cost of each subplan $[a_1, a_2, b_1, b_2]$.

4.1. Algorithm for 2-ELS-PC2TC

In 2-ELS-PC2TC, during each period of the planning horizon, we have one machine of capacity C^p available for production and two vehicles of capacities C_r^1 and C_r^2 available for transportation. Without loss of generality, we assume that $C_r^1 \le C_r^2$. We redefine FTP as a period where either (a) $0 < x_t^{2,1} < C_r^1$ and $x_t^{2,2} \in \{0, C_r^2\}$, or (b) $x_t^{2,1} \in \{0, C_r^1\}$ and $0 < x_t^{2,2} < C_r^2$. Similar to the results of Theorem 1, we observe that the subplans and blocks associated with each extreme point have the following properties: (a) Each subplan has at most one FPP, and (b) Each block has at most one FTP. The proof of this claim follows directly from the proof of Theorem 1 with a minor modification in construction of network flow diagram. Since there are two vehicles available for transportation, we construct two arcs with flow capacities of C_r^1 and C_r^2 from node $(1, t)$ to node $(2, t)$ for all $t \in \mathcal{T}$.

Again, the overall optimal cost for a given subplan $\phi = [a_1, a_2, b_1, b_2]$ is computed by constructing a network $\mathcal{N}^\phi$ using the steps discussed in Section 3.2, and finding the shortest path from the source node to the sink node. The computation of arc weights $\Delta^1_{k,k}(\xi_{k-1}, \xi_k)$ and $\Delta^2_{k,k}(\xi_{k-1}, \xi_k)$ is same as presented in Section 3.2. For the remainder of this section, we present an algorithm to compute $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$, which is different for this problem.

4.1.1. DP for Computing $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$. We observe that depending on the number of times vehicles of capacity C_r^1 and C_r^2 operate at full capacities within a block $[k, l, type]$, there can be multiple fractional transportation levels. This is unlike what we saw in case of 2-ELS-PCTC where there is only one possible fractional transportation level equal to $\rho^{ij}_{k,l} - C_r \lfloor \frac{\rho^{ij}_{k,l}}{C_r} \rfloor$. Therefore, we introduce the following steps to compute $\Delta^3_{k,l}(\cdot)$ within our approach for solving 2-ELS-PC2TC:

1. Computing minimum cost for block $[k, l, type]$ without any FTP,
2. Computing all possible fractional transportation levels,
3. Computing minimum cost for block $[k, l, type]$ with one FTP,
4. Computing the overall minimum cost $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$.

Let e_j be a two-dimensional unit vector where j^{th} element is equal to one. We also define a vector $\mu := (\mu_1, \mu_2)$ where μ_1 and μ_2 denote the number of times vehicles of capacities C_r^1 and C_r^2, respectively, used to transport items at full capacity. Let $\mu_{ij}^{max,1} = \min\{l - k + 1, \lfloor \frac{\rho^{ij}_{k,l}}{C_r^1} \rfloor\}$, and $\mu_{ij}^{max,2} = \min\{l - k + 1, \lfloor \frac{\rho^{ij}_{k,l}}{C_r^2} \rfloor\}$.

Step 1. Computing minimum cost for a given block $[k, l, type]$ **without any FTP:** We define $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ as a recursive function that computes minimum cost of producing $\xi_t - \xi_{k-1}$ units and transporting $\mu_1 C^1_r + \mu_2 C^2_r$ units from periods k up to t. In any given time period $t \in \{k, \dots, l\}$, we can choose to either produce nothing, or produce C^p units, or produce f^p units. In addition, since we are currently only computing minimum costs without fractional transportation, we can transport either zero, C^1_r, C^2_r, or $C^1_r + C^2_r$ units (equivalent to $\sum_{j \in \mathcal{S}^r} C^j_r$ for $\mathcal{S}^r \subseteq \{1, 2\}$) in a given time period t. Based on the foregoing observations, we compute $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ for all possible values of t, ξ_t, and μ:

$$G^{\xi_{k-1},\xi_l}_{k,l,type}(t,\xi_t,\mu,0) = \min_{\substack{\alpha_t \in \{0, C^p, f^p\} \\ \mathcal{S}^r \subseteq \{1,2\}}} \Bigg\{ G^{\xi_{k-1},\xi_l}_{k,l,type}\Big(t-1, \xi_t - \alpha_t, \mu - \sum_{j \in \mathcal{S}^r} e_j, 0\Big) + \mathcal{Q}^1_t(\alpha_t) + \sum_{j \in \mathcal{S}^r} (p^{2,j}_t(C^r_j) + q^{2,j}_t) + \mathcal{H}_t \Bigg\} \tag{6}$$

where $\mathcal{Q}^1_t(\alpha_t)$ is the cost of producing α_t units and $\mathcal{H}_t$ is the total holding cost at the end of period t. Refer to equation (5) and Table 2 in Section 3.2.1 for definitions of $\mathcal{Q}^1_t(\alpha_t)$ and $\mathcal{H}_t$, respectively.

Step 2. Computing possible fractional transportation levels: Depending on the block type and cumulative production quantities ξ_{k-1} and ξ_l, we now compute all the possible fractional transportation levels within a block $[k, l, type]$. We define a set $\Pi := \{\pi \in \mathbb{Z}^2_+ : 0 < \rho^{ij}_{k,l} - \pi_1 C^1_r - \pi_2 C^2_r < C^2_r$ and $\rho^{ij}_{k,l} - \pi_1 C^1_r - \pi_2 C^2_r \neq C^1_r\}$, where $\rho^{ij}_{k,l}$ is the total amount transported within block $[k, l, type]$ for a given ξ_{k-1} and ξ_l that can be obtained using equation (3.2). Finally, we define the set of all possible fractional transportation levels $F := \{f^r : f^r = \rho^{ij}_{k,l} - \pi_1 C^1_r - \pi_2 C^2_r$ for $\pi \in \Pi\}$.

Step 3. Computing minimum cost for block $[k, l, type]$ **with one FTP:** For each $f^r \in F$ and $t \in \{k, \dots, l\}$, we compute the minimum cost to plan production and transportation such that there is at exactly one period from k up to t where fractional transportation of f^r units takes place for one of the two vehicles. We define a cost function $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ that computes minimum cost of producing $\xi_t - \xi_{k-1}$ units and transporting $\mu_1 C^1_r + \mu_2 C^2_r + f^r$ units from periods k up to t. Again, we have three possible choices for production, i.e., $\alpha_t \in \{0, C^p, f^p\}$. Also, in each period, we have a choice to transport f^r units using one of the available vehicles and utilize the other vehicle, if needed, at full capacity. Below we provide the recursive equation to compute the value of the cost function $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ for all $t \in \{k, \dots, l\}$, $f^r \in F$, $\xi_t \in \Xi_t$, $\mu_1 \in \{0, \dots, \mu^{max,1}_{ij}\}$, and $\mu_2 \in \{0, \dots, \mu^{max,2}_{ij}\}$:

$$\min_{\substack{\alpha_t \in \{0, C^p, f^p\} \\ \mathcal{S}^r \subseteq \{1,2\}}} \begin{cases} G^{\xi_{k-1},\xi_l}_{k,l,type}\Big(t-1, \xi_t - \alpha_t, \mu - \sum_{j \in \mathcal{S}^r} e_j, 1\Big) + \mathcal{Q}^1_t(\alpha_t) + \sum_{j \in \mathcal{S}^r} (p^{2,j}_t(C^r_j) + q^{2,j}_t) + \mathcal{H}_t, \\ G^{\xi_{k-1},\xi_l}_{k,l,type}\Big(t-1, \xi_t - \alpha_t, \mu - \sum_{j \in \mathcal{S}^r} e_j, 0\Big) + \mathcal{Q}^1_t(\alpha_t) + \mathcal{Q}^2_t(f^r) + \sum_{j \in \mathcal{S}^r} (p^{2,j}_t(C^r_j) + q^{2,j}_t) + \mathcal{H}_t \end{cases}$$

where $\mathcal{Q}_t^1(\alpha_t)$ is the cost of producing α_t units in period t and is obtained using equation (5). Moreover, $\mathcal{Q}_t^2(f^r)$ is the cost of transporting f^r units and is defined as follows.

$$\mathcal{Q}_t^2(f^r) = \begin{cases} \min\{p_t^{2,1}(f^r) + q_t^{2,1}, p_t^{2,2}(f^r) + q_t^{2,2}\}, & \text{if } f^r < C_r^1, \text{ and } \mathcal{S}^r = \emptyset \\ p_t^{2,1}(f^r) + q_t^{2,1}, & \text{if } f^r < C_r^1, \text{ and } \mathcal{S}^r = \{2\} \\ p_t^{2,2}(f^r) + q_t^{2,2}, & \text{if } f^r < C_r^1, \text{ and } \mathcal{S}^r = \{1\} \\ p_t^{2,2}(f^r) + q_t^{2,2}, & \text{if } C_r^1 < f^r < C_r^2, \text{ and } \{2\} \notin \mathcal{S}^r \\ \infty, & \text{otherwise.} \end{cases} \tag{8}$$

Remark 3. *The conditions on* (t, ξ_t, μ) *for which* $G_{k,l,type}^{\xi_{k-1},\xi_l}(t, \xi_t, \mu, 0)$ *is infeasible are similar to those discussed in Section 3.2.1. Likewise, the infeasible cases for* $G_{k,l,type}^{\xi_{k-1},\xi_l}(t, \xi_t, \mu, f^r)$ *are same as the ones presented in Section 3.2.2.*

Step 4. Overall Optimal Cost: Finally, the overall optimal cost $\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type)$ is computed using the following expression:

$$\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type) = \min \begin{cases} \min\limits_{\mu \in \{0,\dots,l-k+1\}^2} G_{k,l,type}^{\xi_{k-1},\xi_l}(l, \xi_l, \mu, 0), \\ \min\limits_{\substack{f^r \in F \\ \mu \in \Pi}} G_{k,l,type}^{\xi_{k-1},\xi_l}(l, \xi_l, \mu, f^r) \end{cases}$$

Theorem 3. *Problem 2-ELS-PC2TC is solved to optimality using the foregoing algorithm in* $O(T^{13})$ *time.*

Proof. Refer to Appendix B.

4.2. Algorithm for 2-ELS-2PCTC

Recall that in 2-ELS-2PCTC, two machines of capacities C_p^1 and C_p^2 are available for production where $C_p^1 \leq C_p^2$ without loss of generality, and one vehicle of capacity C_r is available for transportation in each time period t of the planning horizon. Therefore, we redefine a fractional production period (FPP) as a period t where either (a) $x^{1,1} \in \{0, C_p^1\}$ and $0 < x_t^{1,2} < C_p^2$, or (b) $0 < x_t^{1,1} < C_p^1$ and $x^{1,2} \in \{0, C_p^2\}$. We can represent the problem as a minimum cost network flow problem where two arcs of capacities C_p^1 and C_p^2 are available to carry flow from the source node to each node in the first echelon. We observe that the free-arc network for any extreme point solution of 2-ELS-2PCTC is always acyclic. As a consequence, the properties of the optimal solution discussed in Theorem 1 hold even for 2-ELS-2PCTC. We use this property to decompose the planning horizon into smaller subplans, and then find the best sequence of subplans using the shortest path algorithm.

The key aspect differentiating the algorithm for 2-ELS-2PCTC from the algorithm for 2-ELS-PCTC is the set of feasible cumulative production schedules for each subplan. Unlike in the case of 2-ELS-PCTC where each subplan had only one possible fractional production level of $f^p = d_{a_2,b_2} - C^p \lfloor \frac{d_{a_2,b_2}}{C^p} \rfloor$, there are multiple fractional production levels that can occur during any FPP within a subplan. In order to find all the possible fractional production levels, we first define a set $\Gamma := \left\{(\tau_1, \tau_2) \in \mathbb{Z}_+^2 : 0 < d_{a_2,b_2} - \sum_{i=1}^{2} \tau_i C_p^i < C_p^2 \text{ and } d_{a_2,b_2} - \sum_{i=1}^{2} \tau_i C_p^i \neq C_p^1\right\}$. Notice that Γ is a set of (τ_1, τ_2) vectors where τ_1 and τ_2 are integer multiples of capacities C_p^1 and C_p^2. These vectors represent the number of times production has occurred using modules of capacities C_p^1 and C_p^2, at full capacity, from periods a_1 through b_1. We also define a set

$$F^p := \left\{f^{p,v} : f^{p,v} = d_{a_2,b_2} - \tau_1^v C_p^1 - \tau_2^v C_p^2 \text{ for all } (\tau_1^v, \tau_2^v) \in \Gamma\right\}$$

whose elements are fractional production levels corresponding to each $(\tau_1^v, \tau_2^v) \in \Gamma$. We let $\tau_1^{max} = \lfloor \frac{d_{b_1,b_2}}{C_p^1} \rfloor$, and $\tau_2^{max} = \lfloor \frac{d_{b_1,b_2}}{C_p^2} \rfloor$. We now redefine the set of feasible cumulative production quantities during each period t. For each $f^{p,v} \in F^p$ and the corresponding $(\tau_1^v, \tau_2^v) \in \Gamma$, we denote a set of feasible cumulative production quantities with at most one fractional period by

$$\Xi_t^v := \left\{\gamma_1 C_p^1 + \gamma_2 C_p^2 + \delta f^{p,v} : \ \gamma_1 \in \{0, \ldots, \tau_1^v\}, \ \gamma_2 \in \{0, \ldots, \tau_2^v\}, \text{ and } \delta \in \{0, 1\}\right\}.$$

Also, the overall set of feasible production quantities Ξ_t is equal to $\bigcup_{f^{p,v} \in F^p} \Xi_t^v$.

In order to find an optimal solution for a given subplan, we construct a graph using the procedure discussed in Section 3.2. One of main modifications is that the nodes are now labeled $(k, e, \xi_k, f^{p,v})$, for $k \in \{a_1, \ldots, b_2\}$, $e \in \{1, 2\}$, $f^{p,v} \in F^p$, and $\xi_k \in \Xi_k^v$. Moreover, forward arcs are only constructed between two nodes $(k-1, e, \xi_{k-1}, f^{p,v_1})$ and (l, e, ξ_k, f^{p,v_2}) if $v_1 = v_2$. All other conditions and the procedure for creating arcs between any two nodes and assigning them weights $\Delta_{k,k}^1(\xi_{k-1}, \xi_k, f^{p,v})$, $\Delta_{k,k}^2(\xi_{k-1}, \xi_k, f^{p,v})$, or $\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type, f^{p,v})$ is same as mentioned in Section 3.2. Again, computing $\Delta_{k,k}^1(\xi_{k-1}, \xi_k, f^{p,v})$ and $\Delta_{k,k}^2(\xi_{k-1}, \xi_k, f^{p,v})$ is straightforward and henceforth, we present modifications in algorithm to compute $\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type, f^{p,v})$.

4.2.1. Computing $\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type, f^{p,v})$ for a given set of input values. We now compute the optimal costs for each block $[k, l, type]$ with a given $\xi_{k-1} \in \Xi_{k-1}^v$, $\xi_l \in \Xi_l^v$, and $f^{p,v} \in F^p$. Similar to the functions defined in Sections 3.2.1 and 3.2.2 for computing optimal costs, for a given $f^{p,v} \in F^p$, we define $G_{k,l,type,v}^{\xi_{k-1},\xi_l}(t, \xi_t, \mu, 0)$ as the minimum cost of producing $\xi_t - \xi_{k-1}$ units and transporting μC_r units from periods k upto t. Likewise, we define $G_{k,l,type,v}^{\xi_{k-1},\xi_l}(t, \xi_t, \mu, 1)$ as the cost function that computes the minimum cost of producing $\xi_t - \xi_{k-1}$ units and transporting μC_r units from from periods k upto t. Both of these cost functions are computed using equations (3) and (4) where amount produced in each time period $\alpha_t \in \{0, C_p^1, C_p^2, C_p^1 + C_p^2, C_p^2 + f^{p,v}, C_p^1 + f^{p,v}\}$ if $f^{p,v} < C_p^1$, and $\alpha_t \in \{0, C_p^1, C_p^2, C_p^1 + C_p^2, C_p^1 + f^{p,v}\}$, otherwise.

Theorem 4. *Problem 2-ELS-2PCTC is solved to optimality using the above algorithm in* $O(T^{16})$ *time.*

Proof. Refer to Appendix C.

5. Problem 2-ELS-PCTC with Outsourcing in Both Echelons

In this section, we present an exact algorithm for solving an extension of 2-ELS-PCTC where outsourcing is allowed in each time period of the planning horizon for both of the echelons. As typically done in the lot-sizing literature (Atamtürk and Hochbaum 2001, Sargut and Romeijn 2007), it is assumed that the outsourcing is uncapacitated in both echelons. This also implies that in each time period and in each echelon, we have two available modules/machines/vehicles (one capacitated and the other uncapacitated) for production and transportation.

Recall the network flow representation of the two-echelon lot-sizing problem. Since outsourcing is uncapacitated, any non-negative flow in a given period through the outsourcing arc is considered as a free-arc. Bearing this in mind, we redefine fractional production period (FPP) as a period t in which either $0 < x_t^{1,1} < C_p^1$ or $x_t^{1,2} > 0$. Likewise, a period t is a fractional transportation period (FTP) when either $0 < x_t^{2,1} < C_r^1$ or $x_t^{2,2} > 0$. Similar to Theorem 1, we can easily prove that there exists an optimal solution of 2-ELS-PCTC-O that comprises of a series of consecutive subplans such that each subplan has at most one FPP. Moreover, within each subplan of the optimal solution, there exist a series of blocks such that each block has at most one FTP. This characteristic of the optimal solution again enables us to decompose the overall planning horizon into smaller subplans and then computing the optimal costs for each of the subplans. As done in case of 2-ELS-PCTC, 2-ELS-PC2TC, and 2-ELS-2PCTC, the overall optimal cost is computed by finding the best sequence of subplans using a shortest path algorithm. Henceforth, we focus solely on computing the optimal costs of a given subplan $[a_1, a_2, b_1, b_2]$.

5.1. Computing optimal cost of a given subplan

Similar to the 2-ELS-2PCTC, we observe that within each subplan, there can be several fractional production quantities depending on the number of times full capacitated production have occurred within the entire subplan. However, only one out of these fractional quantities is produced during at most one time period $t \in \{a_1, \ldots, b_1\}$. We redefine the sets $\Gamma := \{\tau \in \mathbb{Z}_+ : d_{b_1,b_2} - \tau C_p^1 > 0\}$ and $F^p := \{f^{p,v} : f^{p,v} = d_{b_1,b_2} - \tau^v C_p^1 \text{ for } \tau^v \in \Gamma\}$. Again, for each $f^{p,v} \in F^p$, we define the set of cumulative production quantities during period t as follows:

$$\Xi_t^v := \left\{ \gamma_1 C^p + \delta f^{p,v} : \ \gamma_1 \in \{0, \ldots, \tau_1^v\}, \text{ and } \delta \in \{0,1\} \right\}.$$

Also, $\Xi_t = \bigcup_{f^{p,v} \in F^p} \Xi_t^v$. Using these sets, we again create a network with source node and destination node labeled as $(a_1 - 1, 1, 0)$ and $(b_2, 2, d_{a_2,b_2})$, respectively. For each $f^{p,v} \in F^p$, we also have nodes labeled as $(t, e, \xi_t, f^{p,v})$ where $t \in \{a_1, \ldots, b_2\}$, $e \in \{1, 2\}$, and $\xi_t \in \Xi_t^v$. Again, we create forward arcs only between nodes with same $f^{p,v}$, and use the procedure discussed in Section 3.2 to create the arcs between the nodes of this network. Weights $\Delta_{k,k}^1(\xi_{k-1}, \xi_k, f^{p,v})$ and $\Delta_{k,k}^2(\xi_{k-1}, \xi_k, f^{p,v})$ are also computed and assigned in the same manner. The major difference in this algorithm lies in the computation of edge weights $\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type, f^{p,v})$ which denotes the minimum cost of producing $\xi_l - \xi_{k-1}$ units and transporting $\rho_{k,l}^{ij}$ units such that the demands of period k to l are satisfied. We now present the recursive equations that computes $\Delta_{k,l}^3(\xi_{k-1}, \xi_l, type, f^{p,v})$ in polynomial time for a given set of input values and block $[k, l, type]$.

5.2. Computing $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type, f^{p,v})$ for 2-ELS-PCTC-O

Similar to the algorithm of 2-ELS-PC2TC, we are only interested in solutions where there is at most one FTP within a block $[k,l,type]$. Therefore, we compute $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type, f^{p,v})$ using the following steps:

Step 1. Computing all possible fractional transportation levels: Let $\Pi := \{\pi \in \mathbb{Z}_+ : \rho^{ij}_{k,l} - \pi C^1_r > 0\}$ where $\rho^{ij}_{k,l}$ is the total amount transported from the first echelon to the second echelon within block $[k,l,type]$ for a given ξ_{k-1} and ξ_l. We also redefine the set of fractional transportation values $F^r := \{f^r : f^r = \rho^{ij}_{k,l} - \pi C^1_r \text{ for } \pi \in \Pi\}$.

Step 2. Computing minimum cost for block $[k,l,type]$ without any FTP: We define $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ as the cost of producing $\xi_t - \xi_{k-1}$ units and transporting μC^1_r units from period k upto t for $t \in \{k, \ldots, l\}$. In time period t, we can produce $\alpha_t \in \{0, C^1_p, f^{p,v}, C^1_p + f^{p,v}\}$ units in the first echelon at cost $Q^1_t(\alpha_t)$, which is defined by:

$$Q^1_t(\alpha_t) = \begin{cases} 0, & \text{if } \alpha_t = 0 \\ p^{1,1}_t(C^1_p) + q^{1,1}_t, & \text{if } \alpha_t = C^1_p \\ p^{1,2}_t(f^{p,v}) + q^{1,2}_t, & \text{if } \alpha_t = f^{p,v}, \text{ and } f^{p,v} > C^1_p \\ \min\{p^{1,1}_t(f^{p,v}) + q^{1,1}_t, p^{1,2}_t(f^{p,v}) + q^{1,2}_t\}, & \text{if } \alpha_t = f^{p,v} \text{ and } f^{p,v} < C^1_p \\ p^{1,1}_t(C^1_p) + q^{1,1}_t + p^{1,2}_t(f^{p,v}) + q^{1,2}_t, & \text{if } \alpha_t = C^1_p + f^{p,v}. \end{cases}$$

It should also be noted that when $\xi_{k-1} \in \Xi^v_{k-1}$ such that $\delta = 0$ and $\xi_l \in \Xi^v_l$ such that $\delta = 0$, no fractional production takes place within the block $[k,l,type]$. Same reasoning applies to the case when $\xi_{k-1} \in \Xi^v_{k-1}$ such that $\delta = 1$ and $\xi_l \in \Xi^v_l$ such that $\delta = 1$. In both these cases, $\alpha_t \in \{0, C^1_p\}$. Regarding second echelon decisions, since we are not considering any fractional transportation in this step, we can either transport 0 or C^1_r units from echelon 1 to echelon 2 during period t. Again, we let $\mu^{max}_{ij} = \min\{l - k + 1, \lfloor \frac{\rho^{ij}_{k,l}}{C_r} \rfloor\}$ which denotes the maximum number of times transportation can occur at full capacity. The conditions to filter out the infeasible values of ξ_t, t, and μ are same as in case of Step 2 of algorithm for 2-ELS-PC2TC (Section 4.1.1). Based on the foregoing discussion, we present the following recursive equation where $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ for a given feasible $t \in \{k, \ldots, l\}$, $\xi_t \in \Xi^v_t$, and $\mu \in \{0, \ldots, \mu^{max}_{ij}\}$ is equal to:

$$\min_{\substack{\alpha_t \in \{0, C^p, f^{p,v}, C^1_p + f^{p,v}\} \\ \mathcal{S}^r \subseteq \{1\}}} \left\{ G^{\xi_{k-1},\xi_l}_{k,l,type}\left(t-1, \xi_t - \alpha_t, \mu - \sum_{j \in \mathcal{S}^r} e_j, 0\right) + Q^1_t(\alpha_t) + \sum_{j \in \mathcal{S}^r} (p^{2,j}_t(C^j_r) + q^{2,j}_t) + \mathcal{H}_t \right\},$$

where $\mathcal{H}_t = h^1_t(s^1_t) + h^2_t(s^2_t)$ is the holding cost incurred during period t, and the values of s^1_t and s^2_t can be obtained from Table 2.

Step 3. Computing minimum cost for block $[k,l,type]$ with exactly one FTP: Let $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ be a function that computes the minimum cost of producing $\xi_t - \xi_{k-1}$ units and transporting $\mu C^1_r + f^r$ units from periods k upto t such that demands from periods k to t are satisfied. Again, in each time period, we can either produce 0, C^1_p, or f^p units for some $f^p \in F^p$. However, in the second echelon, in each period t, in addition to the choices of transporting 0 or

C_r^1 units, we can also transport f^r or $C_r^1 + f^r$ units. We present the recursive equation below to compute $G^{\xi_{k-1},\xi_l}_{k,l,type,f^r}(t, \xi_t, \mu, f^r)$:

$$\min_{\substack{\alpha_t \in \{0, C^p, f^{p,v}, C_p^1 + f^{p,v}\} \\ \mathcal{S}^r \subseteq \{1\}}} \begin{cases} G^{\xi_{k-1},\xi_l}_{k,l,type}\Big(t-1, \xi_t - \alpha_t, \mu - j, f^r\Big) + Q_t^1(\alpha_t) + \sum_{j \in \mathcal{S}^r} (p_t^{2,j}(C_r^j) + q_t^{2,j}) + \mathcal{H}_t^r, \\ G^{\xi_{k-1},\xi_l}_{k,l,type}\Big(t-1, \xi_t - \alpha_t, \mu - j, 0\Big) + Q_t^1(\alpha_t) + Q_t^2(f^r) + \sum_{j \in \mathcal{S}^r} (p_t^{2,j}(C_r^j) + q_t^{2,j}) + \mathcal{H}_t^r \end{cases}$$

where $\mathcal{H}_t^r$ is the holding cost incurred at the end of period t, and $Q_t^2(f^r)$ is the cost of transporting/outsourcing the fractional quantity f^r in period t. More specifically,

$$Q_t^2(f^r) = \begin{cases} \min\left\{p_t^{2,1}(f^r) + q_t^{2,1}, p_t^{2,2}(f^r) + q_t^{2,2}\right\}, & \text{if } \mathcal{S}^r = \emptyset \text{ and } f^r < C_r^1 \\ p_t^{2,2}(f^r) + q_t^{2,2}, & \text{otherwise.} \end{cases}$$

Again, values of t, ξ_t, and μ for which this function is infeasible are similar to the ones discussed in Section 3.2.2 for 2-ELS-PCTC.

Step 4: Overall Optimal Cost:

$$\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type, f^{p,v}) = \min_{\mu \in \{0, \ldots, \mu_{ij}^{max}\}} \left\{ G^{\xi_{k-1},\xi_l}_{k,l,type}(l, \xi_l, \mu, 0), \min_{f^r \in F^r} G^{\xi_{k-1},\xi_l}_{k,l,type}(l, \xi_l, \mu, f^r) \right\}. \quad (9)$$

Theorem 5. *Problem 2-ELS-PCTC-O can be solved using the above algorithm in $O(T^{13})$ time.*

Proof. Refer to Appendix D

6. Concluding Remarks

In this paper, we answered an open question related to existence of a polynomial time algorithm for two-echelon lot-sizing problem with constant capacitated production and transportation in first- and second-echelon, respectively (denoted by 2-ELS-PCTC). Specifically, we introduced an $O(T^{10})$ algorithm for 2-ELS-PCTC that generalizes the 2-ELS-PCTC with uncapacitated (or infinite capacity) transportation studied by Kaminsky and Simchi-Levi (2003), van Hoesel et al. (2005). We also presented polynomial time algorithms for the following two generalizations of 2-ELS-PCTC: (a) 2-ELS-PC2TC where two capacitated vehicles of varying capacities are available for transportation in each time period, and (b) 2-ELS-2PCTC where two capacitated machines of varying capacities are avaiable for production in each time period. Finally, we developed an $O(T^{13})$ time algorithm for 2-ELS-PCTC with an additional option of uncapacitated subcontracting for both production and transportation (denoted by 2-ELS-PCTC-O), which is a generalization of 2-ELS-PCTC-O with uncapacitated transportation studied by Sargut and Romeijn (2007). A potential future research direction is to utilize the proposed dynamic programming algorithms for obtaining extended formulations using variable redefinition, feasible sub-optimal solutions and an efficient heuristics, and cutting planes by filtering out several infeasible/suboptimal solutions. Also, the following question is still open: *For $L \geq 3$, does there exist a polynomial time algorithm for L-ELS-PCTC with stationary capacities in all echelons?*

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Appendices

Appendix A: Proof for Theorem 2

There are $O(T^3)$ subplans, and to find the optimal cost of each subplan, we create a directed acyclic graph with $O(T^4)$ arcs and find the shortest path from the source node to the sink node. The arc weights $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$ are obtained by computing $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu^{max}, 0)$ and $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_l, \mu^{max}, 1)$ for all $t \in \{k, \ldots, l\}$, $\xi_t \in \Xi_t$ such that $\xi_{k-1} \leq \xi_t \leq \xi_l$, and $\mu \in \{0, \ldots, \mu^{max}\}$. For a given (t, ξ_t, μ), both $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu^{max}, 0)$ and $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_l, \mu^{max}, 1)\}$ are computed in constant time. Since there are $O(T^3)$ number of (t, ξ_t, μ) vectors, the time taken to compute each arc weight is also $O(T^3)$. As a result, the overall running time of our algorithm for 2-ELS-PCTC is $O(T^3 \times T^4 \times T^3)$, i.e., $O(T^{10})$.

Appendix B: Proof for Theorem 3

The key difference between the algorithms for 2-ELS-PCTC and 2-ELS-PC2TC is the computation of arc weights $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$. Notice that the each of the foregoing values are computed by computing $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ and $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ for all $t \in \{k, \ldots, l\}$, $\xi_t \in \Xi_t$ such that $\xi_{k-1} \leq \xi_t \leq \xi_l$, $\mu_1 \in \{0, \ldots, \mu^{max}\}$ and $\mu_2 \in \{0, \ldots, \mu^{max}\}$, where $\mu^{max}_1 = \min\{l - k + 1, \lfloor \rho^{ij}_{k,l}/C^1_r \rfloor\}$ and $\mu^{max}_2 = \min\{l - k + 1, \lfloor \rho^{ij}_{k,l}/C^2_r \rfloor\}$. For a given set of (t, ξ_t, μ, f^r), functions $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$ and $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ are computed in constant time. Since there are $O(T^6)$ possible (t, ξ_t, μ, f^r) vectors, each arc weight $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$ can be computed in $O(T^6)$ time. Hence the overall running time of the algorithm for 2-ELS-PC2TC is $O(T^6 \times T^4 \times T^3)$, i.e., $O(T^{13})$.

Appendix C: Proof for Theorem 4

The key difference between the algorithms for 2-ELS-PCTC and 2-ELS-2PCTC is the number of nodes and arcs within each network $\mathcal{N}^\phi$ for a given subplan $\phi = [a_1, a_2, b_1, b_2]$. For each $t \in \{a_1, \ldots, b_2\}$, the cumulative production quantity belongs to the set Ξ_t, and the number of elements in Ξ_t is bounded from above by $O(T^4)$. This eventually leads to $O(T^5)$ nodes and $O(T^{10})$ arcs in the directed acyclic graph $\mathcal{N}^\phi$. Moreover, computing the arc weights $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$ dominates the computation of $\Delta^1_{k,l}(\xi_{k-1}, \xi_l, type)$ and $\Delta^2_{k,l}(\xi_{k-1}, \xi_l, type)$, and takes $O(T^3)$ time. Since we compute the optimal costs for $O(T^3)$ subplans, the overall running time of algorithm is $O(T^{10}) \times O(T^3) \times O(T^3)$ which is equal to $O(T^{16})$.

Appendix D: Proof for Theorem 5

We construct a network $\mathcal{N}^\phi$ for each of the $O(T^3)$ subplans. Since there are $O(T^2)$ possible values ξ_t for each $t \in \{a_1, \ldots, b_2\}$, there are $O(T^3)$ nodes and $O(T^6)$ arcs within the graph $\mathcal{N}^\phi$. The bottleneck in the construction of $\mathcal{N}^\phi$ is the step where arc weights $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$ are computed and in order to do this we need to compute values of $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$, and $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ for all feasible values of (t, ξ_t, μ, f^r) vectors. For a given (t, ξ_t, μ, f^r) vector, $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, 0)$, and $G^{\xi_{k-1},\xi_l}_{k,l,type}(t, \xi_t, \mu, f^r)$ are computed in constant time, and since there are $O(T^4)$ possible such vectors, each $\Delta^3_{k,l}(\xi_{k-1}, \xi_l, type)$ is computed in $O(T^4)$ time. Thus, the time complexity of the algorithm for 2-ELS-PCTC-O is $O(T^4) \times O(T^6) \times O(T^3)$ which is equal to $O(T^{13})$.