THE JORDAN ALGEBRAIC STRUCTURE OF THE ROTATED QUADRATIC CONE

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Abstract. In this paper, we look into the rotated quadratic cone and analyze its algebraic structure. We construct an algebra associated with this cone and show that this algebra is a Euclidean Jordan algebra (EJA) with a certain inner product. We also demonstrate some spectral and algebraic characteristics of this EJA. The rotated quadratic cone is then proven to be the cone of squares of the generated EJA. The obtained results can help optimization researchers improve specialized interior-point algorithms for rotated quadratic cone programming based on the generated EJA. Additionally, since it is known that the rotated quadratic cone is a special case of the power cone, another reason for this study may be to open the door to understanding the algebraic structure of the general power cone in the future.

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1. Introduction

The Jordan algebras were created to shed light on a wide range of mathematical problems. The revelation of their connection to symmetric cones was a startling discovery. According to this relationship, any symmetric cone can be represented as the cone of squares of some Euclidean Jordan algebra (EJA for short). The EJAs turned out to supply the instruments for dealing with symmetric cone optimization problems [1–10]. In a nutshell, it provides a fundamental mechanism for treating all symmetric cones at the same time.

The work of Nesterov and Todd in [11] led to the inspiration for the creation of the Jordan algebraic optimization technique in [12], where Güler [13] is credited with being the first to link Jordan algebras with optimization. In their work [11], Nesterov and Todd did not use a Jordan algebraic approach in their theoretical underpinning for the study of interior-point methods. Güler [13] noticed that the self-scaled cones' family [11] is equivalent to the set of symmetric cones for which a complete classification theory exists. This was a noteworthy attempt to extend Nesterov and Nemirovskii's theory of interior-point optimization methods (see [14])

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to optimization problems involving general classes of symmetric conicity constraints.

Symmetric cones include, but not limited to, the nonnegative orthant cone of $\mathbb{R}^n$, the second-order cone, and the cone of symmetric positive semidefinite matrices. These three types of cones have gotten a lot of attention in recent years due to their wide applicability with the addition of their efficiency algorithmically. Linear programming problems are known to be the most traditional class of optimization problems, and their conic extensions (especially second-order cone programming and semidefinite programming) have proven to be a very expressive and powerful way to introduce nonlinearities while keeping the great features of linear optimization.

The rotated quadratic cone is a special case of the power cone, which is one of the well-known convex cones. Let $\mathbb{R}_+$ be the set of positive real numbers and $\mathbb{R}^n$ be the space of all n-dimensional vectors. The power cone is defined as [15]

$$\mathcal{P}_+^{n|\alpha} \triangleq \left\{ \boldsymbol{x} = (x_1; x_2; \bar{\boldsymbol{x}}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-2} : x_1^{\alpha} x_2^{1-\alpha} \geq \|\bar{\boldsymbol{x}}\| \right\}, \tag{1}$$

where $0 < \alpha < 1$ is a fixed real number and $\|\cdot\|$ is the Euclidean norm. Figure 1.1 shows four special cases of the power cone. The most notable special case of (1) occurs when $\alpha = 0.5$, which is the case that gives rise to the rotated quadratic cone defined as

$$\mathcal{R}_+^n \triangleq \left\{ \boldsymbol{x} = (x_1; x_2; \bar{\boldsymbol{x}}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-2} : x_1 x_2 \geq \|\bar{\boldsymbol{x}}\|^2 \right\} = \mathcal{P}_+^{n|0.5}.$$

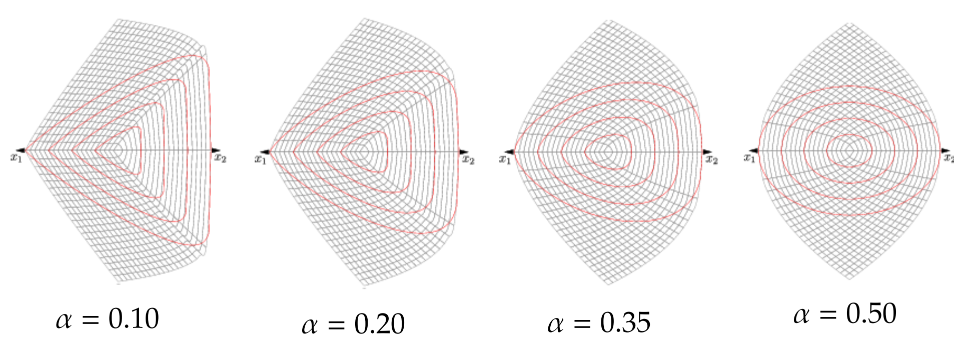

FIGURE 1.1. The picture on the right-hand side shows the boundary of the rotated quadratic cone (power cone boundary for α value of 0.5). The other three pictures show power cone boundaries for α values of 0.1, 0.2, and 0.35. These pictures are taken from [15].

Rotating the well-known second-order cone

$$\mathcal{E}_+^n \triangleq \left\{ \boldsymbol{x} \in \mathcal{R}^n : x_0 \geq \|\hat{\boldsymbol{x}}\|, x_0 \triangleq \sqrt{2}x_1, \hat{\boldsymbol{x}} \triangleq \left(\sqrt{2}x_2; \bar{\boldsymbol{x}} \right) \right\}$$

through an angle of 45° in the x_1, x_2-plane, yields the quadratic rotated cone $\mathcal{R}_+^n$. More specifically, it is known (and can be easily seen) that the constraint $x_1x_2 \geq \|\bar{x}\|^2$ is equivalent to the set of constraints

$$u = x_1 + x_2, \quad v = x_1 - x_2, \quad \boldsymbol{w} = (v; 2\bar{x}), \quad u \geq \|\boldsymbol{w}\|,$$

which are linear and second-order cone constraints. This means that the rotated quadratic cone is identical to the second-order cone under a linear transformation. Therefore, the rotated quadratic cone inherits symmetrical properties from the second-order cone. Among these properties, $\mathcal{R}_+^n$ is easily distinguishable as a regular cone since it is closed, convex, pointed, and solid cone. However, an algebra specialized for the quadratic rotated cone has not yet been built and it could do better practically in terms of efficiency.

Because second-order cone programs can be efficiently solved in polynomial time, researchers usually solve rotated quadratic cone programs by casting them as second-order cone programs using a linear transformation. To the best of our knowledge, there are no specialized algorithms for optimization problems over the rotated quadratic cone that make use its own unseen algebra. The purpose of this paper is to look into the rotated quadratic cone and analyze its algebraic structure. The obtained results can motivate algorithmists to develop efficient algorithms specialized for rotated quadratic cone programs (i.e. without casting them as second-order cone programs; see, for example, the work in [16], which is based on this principle). Additionally, since it is known that the rotated quadratic cone is a special case of the power cone, this work may also open the door to understanding the algebraic structure of the general power cone in the future.

As an algebraic paper, we particularly aim to construct an algebra associated with the rotated quadratic cone and to study its properties. We show that this cone is self-dual and homogeneous (hence, it is a symmetric cone) and that its corresponding algebra is a Jordan algebra. We also propose a spectral decomposition and establish some spectral and algebraic characteristics of this Jordan algebra. The generated algebra is also shown to be an EJA. Then we prove that the cone of squares of this EJA is the rotated quadratic cone itself. This can be seen as an alternative proof of the fact that a rotated quadratic cone is symmetric.

This paper is organized as follows. We end this section by introducing some notations that will be used in the sequel. Section 2 proves the self-duality of the cone and declares its homogeneity, affirming that it is a symmetric cone. In Section 3, we establish the algebraic structure of the cone and prove that its algebra is an EJA. Section 4 presents with proofs more algebraic and spectral properties and further characteristics of the operators associated with the cone. Section 5 contains some concluding remarks. Appendix A provides a proof of the power-associativity of the generated algebra, and Appendix B compares this algebra with that of the second-order cone.

1.1. **Notations.** Use "," to adjoin vectors and matrices in a row, and ";" to adjoin them in a column. As a result, the following is identical for the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = (\boldsymbol{u}^{\mathsf{T}}, \boldsymbol{v}^{\mathsf{T}})^{\mathsf{T}} = (\boldsymbol{u}; \boldsymbol{v}).$$

We denote by $\mathbb{R}^n_+$ the nonnegative orthant cone in $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ the set of all $n\times m$ real matrices, $\mathcal{S}^n$ the set of symmetric matrices in $\mathbb{R}^{n\times m}$, $\mathcal{S}^n_+$ the cone of positive semidefinite matrices in $\mathcal{S}^n$, $\mathcal{R}^n$ the nth-dimensional real Euclidean space $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-2}$ whose vectors are partitioned as $\boldsymbol{x} = (x_1; x_2; \bar{\boldsymbol{x}})$, and $\mathcal{R}^n_+$ the rotated quadratic cone in $\mathcal{R}^n$. That is,

$$\mathcal{R}^n_+ \triangleq \left\{\boldsymbol{x} \in \mathcal{R}^n : x_1x_2 \geq \|\bar{\boldsymbol{x}}\|^2,\ x_1, x_2 \geq 0\right\}.$$

The sets

$$\begin{aligned} \operatorname{int}\mathcal{R}^n_+ &\triangleq \{\boldsymbol{x} \in \mathcal{R}^n : x_1x_2 > \|\bar{\boldsymbol{x}}\|^2,\ x_1, x_2 \geq 0\};\\ \operatorname{bd}\mathcal{R}^n_+ &\triangleq \{\boldsymbol{x} \in \mathcal{R}^n : x_1x_2 = \|\bar{\boldsymbol{x}}\|^2,\ x_1, x_2 \geq 0\}, \end{aligned}$$

represent, respectively, the interior and boundary of the rotated quadratic cone.

We use $\mathbf{0}$ and O to denote the zero vector and zero matrix, respectively, with appropriate dimensions, and use I_n to denote the identity matrix in $\mathcal{S}^n$. We also define the matrices $H_n, R_n \in \mathcal{S}^n$ as

$$H_n \triangleq \begin{bmatrix} \frac{1}{2} & 0 & \mathbf{0}^{\mathsf{T}} \\ 0 & \frac{1}{2} & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{0} & I_{n-2} \end{bmatrix} \quad \text{and} \quad R_n \triangleq \begin{bmatrix} 0 & \frac{1}{2} & \mathbf{0}^{\mathsf{T}} \\ \frac{1}{2} & 0 & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{0} & -I_{n-2} \end{bmatrix}.$$

From now on, we will drop the index n from H_n and R_n unless the dimension of these two matrices does not follow from the context. Note that the cone $\mathcal{R}^n_+$ can be redefined as

$$\mathcal{R}^n_+ \triangleq \left\{\boldsymbol{x} \in \mathcal{R}^n : \boldsymbol{x}^{\mathsf{T}}R\boldsymbol{x} \geq 0,\ x_1, x_2 \geq 0\right\}.$$

Finally, throughout this paper, we will repeatedly make use of the inner product $\bullet : \mathcal{R}^n \times \mathcal{R}^n \longrightarrow \mathbb{R}$, which we define as

$$\boldsymbol{x} \bullet \boldsymbol{y} \triangleq \boldsymbol{x}^{\mathsf{T}}H\boldsymbol{y} = \frac{1}{2}(x_1y_1 + x_2y_2) + \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}},$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n$.

2. The symmetrical structure of the cone

In this section, we prove that the rotated quadratic cone is self-dual and homogeneous, which affirms that it is symmetric. Before that, in order to make the paper self-contained, we review some preliminary definitions before we prove the relevant results.

2.1. **Review of self-duality and homogeneity.** Like any cone (or, more generally, any set), the dual of the cone $\mathcal{R}^n_+$ is the set

$$\mathcal{R}^{n\star}_+ \triangleq \left\{ \boldsymbol{y} \in \mathcal{R}^n : \boldsymbol{x} \bullet \boldsymbol{y} \geq 0, \forall \boldsymbol{x} \in \mathcal{R}^n_+ \right\}.$$

Clearly, $\mathcal{R}^{n\star}_+$ is a cone. If the two cones $\mathcal{R}^n_+$ and $\mathcal{R}^{n\star}_+$ are equal, we say that $\mathcal{R}^n_+$ is a self-dual cone. That is, $\mathcal{R}^n_+$ is self-dual if it coincides with its dual cone.

The general linear group of degree n over $\mathbb{R}$ is denoted as $\mathrm{GL}_n(\mathbb{R})$ and defined to be the set of all nonsingular matrices of order n with entries from $\mathbb{R}$, together with the operation of ordinary matrix multiplication. The automorphism group of a cone $C \subset \mathbb{R}^n$ is set

$$\mathrm{AUT}(C) \triangleq \left\{ \varphi \in \mathrm{GL}_n(\mathbb{R}) : \varphi(C) = C \right\}.$$

As any regular cone, we say $\mathcal{R}^n_+$ is called homogeneous if for all $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{int} \mathcal{R}^n_+$, there exists an invertible linear map $\psi : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ such that the following two conditions hold: (i) $\psi(\boldsymbol{x}) = \boldsymbol{y}$; (ii) ψ is an automorphism of $\mathcal{R}^n_+$, i.e., $\psi(\mathcal{R}^n_+) = \mathcal{R}^n_+$. In other words, the subgroup $\mathrm{AUT}(\mathcal{R}^n_+)$ acts transitively into the interior of $\mathcal{R}^n_+$.

A regular cone is said to be symmetric if it is both homogeneous and self-dual. As we mentioned in the introduction, the rotated quadratic cone inherits symmetrical properties from the second-order cone. Therefore, it is not new to say that $\mathcal{R}^n_+$ is symmetric, but we think it is new to recover this with a proof.

2.2. **Known symmetrical results on the cone.** In this part, we strengthen an existing symmetrical property of the rotated quadratic cone with a new proof. More specifically, we prove the self-duality of the cone. We also declare its homogeneity.

Lemma 2.1. *$\mathcal{R}^n_+$ is self-dual.*

Proof. We need to show that $\mathcal{R}^n_+ = \mathcal{R}^{n\star}_+$. We begin by showing that $\mathcal{R}^n_+ \subseteq \mathcal{R}^{n\star}_+$ Let $\boldsymbol{x} \in \mathcal{R}^n_+$. Then for any $\boldsymbol{y} \in \mathcal{R}^n_+$, we have

$$x_1 x_2 y_1 y_2 \geq \|\bar{\boldsymbol{x}}\|^2 \, \|\bar{\boldsymbol{y}}\|^2, \tag{2}$$

which has to be true due to the rotated quadratic inequalities $x_1 x_2 \geq \|\bar{\boldsymbol{x}}\|^2$ and $y_1 y_2 \geq \|\bar{\boldsymbol{y}}\|^2$.

Using (2) and applying the arithmetic inequality for $x_1 y_1$ and $x_2 y_2$, we obtain

$$\|\bar{\boldsymbol{x}}\|^2 \, \|\bar{\boldsymbol{y}}\|^2 \leq x_1 x_2 y_1 y_2 \leq \frac{1}{4}(x_1 y_1 + x_2 y_2)^2.$$

Then, by taking the square root of both sides, we get

$$2\|\bar{\boldsymbol{x}}\| \, \|\bar{\boldsymbol{y}}\| \leq x_1 y_1 + x_2 y_2. \tag{3}$$

As a result, using (3) and applying the Cauchy–Schwartz inequality, we must have

$$\boldsymbol{x} \bullet \boldsymbol{y} = \frac{1}{2}(x_1 y_1 + x_2 y_2) + \bar{x}^{\mathsf{T}} \bar{y} \geq \|\bar{x}\| \, \|\bar{y}\| + \bar{x}^{\mathsf{T}} \bar{y} \geq \left|\bar{x}^{\mathsf{T}} \bar{y}\right| + \bar{x}^{\mathsf{T}} \bar{y} \geq 0. \tag{4}$$

This implies that $\boldsymbol{x} \in \mathcal{R}_+^{n\star}$, and hence $\mathcal{R}_+^n \subseteq \mathcal{R}_+^{n\star}$.

To show that $\mathcal{R}_+^{n\star} \subseteq \mathcal{R}_+^n$, let $\boldsymbol{y} \in \mathcal{R}_+^{n\star}$, and consider

$$\boldsymbol{x} \triangleq \left(y_2\|\bar{y}\|; y_1\|\bar{y}\|; -\sqrt{y_1 y_2}\,\bar{y}\right).$$

It can be easily checked that $\boldsymbol{x} \in \mathcal{R}_+^n$. Since $\boldsymbol{y} \in \mathcal{R}_+^{n\star}$ and $\boldsymbol{x} \in \mathcal{R}_+^n$, it follows that

$$0 \leq \boldsymbol{x} \bullet \boldsymbol{y} = y_1 y_2 \, \|\bar{y}\| - \sqrt{y_1 y_2} \, \|\bar{y}\|^2 = \sqrt{y_1 y_2} \, \|\bar{y}\| \, (\sqrt{y_1 y_2} - \|\bar{y}\|),$$

which yields $\sqrt{y_1 y_2} - \|\bar{y}\| \geq 0$. Thus $\boldsymbol{y} \in \mathcal{R}_+^n$, yielding $\mathcal{R}_+^{n\star} \subseteq \mathcal{R}_+^n$. The proof is complete. □

We give, without a proof, the following lemma. The result in Lemma 2.2 will be a corollary of a more general result on the symmetry of $\mathcal{R}_+^n$.

Lemma 2.2. *$\mathcal{R}_+^n$ is homogeneous.*

From Lemmas 2.1 and 2.2, we conclude that $\mathcal{R}_+^n$ is a symmetric cone. This conclusion, together with Lemma 2.2, will be given in the next section as a corollary (see Corollary 3.10) of a main theorem of the paper.

3. The algebraic structure of the cone

This is the main section of the paper and is intended to establish the algebraic structure of the rotated quadratic cone. Before that, in order to make the paper self-contained, we review some preliminary definitions before we state the new results.

3.1. **Review of Jordan algebras.** Let $\mathcal{J}$ be the nth dimensional real vector space over $\mathbb{R}$. A map $h : \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ is called bilinear if for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{J}$ and $\alpha, \beta \in \mathbb{R}$, the following two conditions hold:

(i) $h(\alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w}) = \alpha h(\boldsymbol{u}, \boldsymbol{w}) + \beta h(\boldsymbol{v}, \boldsymbol{w})$.
(ii) $h(\boldsymbol{w}, \alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha h(\boldsymbol{w}, \boldsymbol{u}) + \beta h(\boldsymbol{w}, \boldsymbol{v})$.

The vector space $\mathcal{J}$ is called an algebra over $\mathbb{R}$ if a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ exists. If $\boldsymbol{x} \in \mathcal{J}$, then for $n \geq 2$ we define $\boldsymbol{x}^n$ recursively as $\boldsymbol{x}^n \triangleq \boldsymbol{x} \circ \boldsymbol{x}^{n-1}$.

Definition 3.1. *A Jordan algebra is an algebra $(\mathcal{J}, \circ)$ in which the following two conditions hold:*

(i) $\boldsymbol{x} \circ \boldsymbol{y} = \boldsymbol{y} \circ \boldsymbol{x}$ *(commutativity).*
(ii) $(\boldsymbol{x} \circ \boldsymbol{y}) \circ \boldsymbol{x}^2 = \boldsymbol{x} \circ (\boldsymbol{y} \circ \boldsymbol{x}^2)$ *(Jordan's identity).*

A Jordan algebra $(\mathcal{J}, \circ)$ has an *identity element* if there exists a (necessarily unique) element $\boldsymbol{e} \in \mathcal{J}$ such that $\boldsymbol{x} \circ \boldsymbol{e} = \boldsymbol{e} \circ \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathcal{J}$. A Jordan algebra $(\mathcal{J}, \circ)$ is not necessarily associative, i.e., $\boldsymbol{x} \circ (\boldsymbol{y} \circ \boldsymbol{z}) = (\boldsymbol{x} \circ \boldsymbol{y}) \circ \boldsymbol{z}$ may not generally hold. However, it is power-associative, i.e., $\boldsymbol{x}^p \circ \boldsymbol{x}^q = \boldsymbol{x}^{p+q}$ for all integers $p, q \geq 0$.

Definition 3.2. *A Jordan algebra $(\mathcal{J}, \circ)$ is called Euclidean if there exists a symmetric, positive definite, associative quadratic form on $\mathcal{J}$. That is, $(\mathcal{J}, \circ)$ is called an EJA if there exists an inner product $\langle \cdot, \cdot \rangle : \mathcal{J} \times \mathcal{J} \longrightarrow \mathbb{R}$ such that for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{J}$, the following three conditions hold:*

(i) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ *(symmetry).*
(ii) $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \forall\, \boldsymbol{x} \neq 0$ *(positive definiteness).*
(iii) $\langle \boldsymbol{x}, \boldsymbol{y} \circ \boldsymbol{z} \rangle = \langle \boldsymbol{x} \circ \boldsymbol{y}, \boldsymbol{z} \rangle$ *(associativity).*

A set $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_r\}$ of nonzero elements, where

$$\boldsymbol{c}_i \circ \boldsymbol{c}_j = \begin{cases} \boldsymbol{c}_i, & \text{if } i = j, \\ \boldsymbol{0}, & \text{if } i \neq j, \end{cases} \quad \text{and} \quad \boldsymbol{c}_1 + \boldsymbol{c}_2 + \cdots + \boldsymbol{c}_r = \boldsymbol{e},$$

is called a complete system of orthogonal idempotents. A Jordan frame is a complete system of orthogonal primimive idempotents. By primitivity, we mean each idempotent is not a sum of two other idempotents. We have the following theorem [18].

Theorem 3.3. *Let $\mathcal{J}$ be an EJA. Then for each $\boldsymbol{x} \in \mathcal{J}$ there exist a Jordan frame $\{\boldsymbol{c}_1(\boldsymbol{x}), \boldsymbol{c}_2(\boldsymbol{x}), \ldots, \boldsymbol{c}_r(\boldsymbol{x})\}$ and real numbers $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x}), \ldots, \lambda_r(\boldsymbol{x})$ such that*

$$\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) + \cdots + \lambda_r(\boldsymbol{x})\boldsymbol{c}_r(\boldsymbol{x}).$$

In Theorem 3.3, we call the numbers "$\lambda_i(\boldsymbol{x})$" the eigenvalues of $\boldsymbol{x}$, the elements "$\boldsymbol{c}_i(\boldsymbol{x})$" the eigenvectors of $\boldsymbol{x}$, the factorization "$\sum_i \lambda_i(\boldsymbol{x})\boldsymbol{c}_i(\boldsymbol{x})$" the spectral decomposition of $\boldsymbol{x}$, and the positive integer "r" the rank of $\mathcal{J}$. The trace and determinant of $\boldsymbol{x}$ in $\mathcal{J}$ are respectively the numbers

$$\text{trace}(\boldsymbol{x}) \triangleq \sum_{i=1}^{r} \lambda_i(\boldsymbol{x}), \text{ and } \det(\boldsymbol{x}) \triangleq \prod_{i=1}^{r} \lambda_i(\boldsymbol{x}).$$

If f is any real-valued continuous function, then it is possible to define $f(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) \triangleq f(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x}) + \cdots + f(\lambda_r(\boldsymbol{x}))\boldsymbol{c}_r(\boldsymbol{x}).$$

For example, the square of $\boldsymbol{x}$, which is defined as $\boldsymbol{x}^2 \triangleq \boldsymbol{x} \circ \boldsymbol{x}$, can also be defined as $\boldsymbol{x}^2 \triangleq \sum_{i=1}^{r} \lambda_i^2(\boldsymbol{x})\boldsymbol{c}_i(\boldsymbol{x})$.

The three maps introduced in the following definition play an important role in the development of interior point methods for conic programming.

Definition 3.4. *Let $\boldsymbol{x}$ and $\boldsymbol{z}$ be elements in a Jordan algebra $\mathcal{J}$. Then:*

(i) The linear matrix of $\boldsymbol{x}$, $L(\boldsymbol{x}) : \mathcal{J} \longrightarrow \mathcal{J}$, is defined so that $L(\boldsymbol{x})\boldsymbol{y} \triangleq \boldsymbol{x} \circ \boldsymbol{y}$, for all $\boldsymbol{y} \in \mathcal{J}$.

(ii) *The quadratic operator of* $\boldsymbol{x}$ *and* $\boldsymbol{z}$, $Q_{\boldsymbol{x},\boldsymbol{z}} : \mathcal{J} \longrightarrow \mathcal{J}$, *is the matrix* $Q_{\boldsymbol{x},\boldsymbol{z}} \triangleq L(\boldsymbol{x})L(\boldsymbol{z}) + L(\boldsymbol{z})L(\boldsymbol{x}) - L(\boldsymbol{x} \circ \boldsymbol{z})$.

(iii) *The quadratic representation of* $\boldsymbol{x}$, $Q_{\boldsymbol{x}} : \mathcal{J} \longrightarrow \mathcal{J}$, *is the matrix* $Q_{\boldsymbol{x}} \triangleq 2L(\boldsymbol{x})^2 - L(\boldsymbol{x}^2) = Q_{\boldsymbol{x},\boldsymbol{x}}$.

The cone of squares of an $\mathcal{J}$ is the set $\mathcal{S}_{\mathcal{J}} \triangleq \{\boldsymbol{x}^2 : \boldsymbol{x} \in \mathcal{J}\}$. The Jordan algebraic characterization of symmetric cones is given in the following theorem [18].

Theorem 3.5. *A cone is symmetric iff it is the cone of squares of some EJA.*

In the preceding section, we have proved that the rotated quadratic cone $\mathcal{R}^n_+$ is symmetric. According to Theorem 3.5, $\mathcal{R}^n_+$ must be the cone of squares of some EJA. In the remaining part of this paper, our task is to explore such an EJA.

3.2. **New algebraic results on the cone.** In this part, our goal is to build an algebra on the space $\mathcal{R}^n$ which is intended to be an EJA and to have $\mathcal{R}^n_+$ as its cone of squares.

We start with the following observation:

$$x_1 x_2 \geq \|\bar{\boldsymbol{x}}\|^2 \iff (x_1 + x_2)^2 \geq (x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}},$$

which concludes that

$$\boldsymbol{x} \in \mathcal{R}^n_+ \iff \lambda_{1,2}(\boldsymbol{x}) \triangleq \frac{1}{2}\left(x_1 + x_2 \pm \sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}\right) \geq 0.$$

The values $\lambda_1(\boldsymbol{x})$ and $\lambda_2(\boldsymbol{x})$ defined above are set to be the eigenvalues of $\boldsymbol{x}$ in the space $\mathcal{R}^n$. The eigenvectors of $\boldsymbol{x}$ should be defined so that $\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})$. Simple calculations show that

$$\begin{aligned}
\boldsymbol{x} \;=\; & \overbrace{\left(\frac{x_1 + x_2}{2} + \frac{1}{2}\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}\right)}^{\lambda_1(\boldsymbol{x})} \overbrace{\begin{bmatrix} \frac{1}{2} + \frac{x_1 - x_2}{2\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \\ \frac{1}{2} - \frac{x_1 - x_2}{2\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \\ \frac{\bar{\boldsymbol{x}}}{\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \end{bmatrix}}^{\boldsymbol{c}_1(\boldsymbol{x})} \\
& + \underbrace{\left(\frac{x_1 + x_2}{2} - \frac{1}{2}\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}\right)}_{\lambda_2(\boldsymbol{x})} \underbrace{\begin{bmatrix} \frac{1}{2} - \frac{x_1 - x_2}{2\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \\ \frac{1}{2} + \frac{x_1 - x_2}{2\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \\ \frac{-\bar{\boldsymbol{x}}}{\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \end{bmatrix}}_{\boldsymbol{c}_2(\boldsymbol{x})}.
\end{aligned}$$

So, we define the eigenvectors of $\boldsymbol{x}$ in the space $\mathcal{R}^n$ as

$$\boldsymbol{c}_{1,2}(\boldsymbol{x}) \triangleq \frac{1}{2}\begin{bmatrix} 1 \pm \dfrac{x_1 - x_2}{\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \\ 1 \mp \dfrac{x_1 - x_2}{\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \\ \dfrac{\pm 2\bar{\boldsymbol{x}}}{\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}} \end{bmatrix}.$$

We define the trace and determinant of any vector $\boldsymbol{x} \in \mathcal{R}^n$, respectively, as

$$\begin{array}{rcl} \text{trace}(\boldsymbol{x}) & \triangleq & \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) = x_1 + x_2, \\ \det(\boldsymbol{x}) & \triangleq & \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}) = x_1 x_2 - \|\bar{\boldsymbol{x}}\|^2. \end{array}$$

We define the identity vector in the space $\mathcal{R}^n$ as $\boldsymbol{e} \triangleq \boldsymbol{c}_1(\boldsymbol{x}) + \boldsymbol{c}_2(\boldsymbol{x}) = (1; 1; \boldsymbol{0})$. Note that $\text{trace}(\boldsymbol{e}) = 2$ and $\det(\boldsymbol{e}) = 1$. In general, if f is any real-valued continuous function, we define $f(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) \triangleq f(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x}).$$

For example, the inverse of $\boldsymbol{x} \in \mathcal{R}^n$, with $\det(\boldsymbol{x}) \neq 0$, is defined as

$$\boldsymbol{x}^{-1} \triangleq \lambda_1^{-1}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{-1}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_2 \\ x_1 \\ -\bar{\boldsymbol{x}} \end{bmatrix} = \frac{1}{\det(\boldsymbol{x})} H^{-1} R \boldsymbol{x}, \tag{5}$$

and the square of $\boldsymbol{x} \in \mathcal{R}^n$ is defined as

$$\boldsymbol{x}^2 \triangleq \lambda_1^2(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = \begin{bmatrix} x_1^2 + \|\bar{\boldsymbol{x}}\|^2 \\ x_2^2 + \|\bar{\boldsymbol{x}}\|^2 \\ (x_1 + x_2)\bar{\boldsymbol{x}} \end{bmatrix}. \tag{6}$$

The linear matrix $\mathrm{L}(\boldsymbol{x})$ in the space $\mathcal{R}^n$ is defined so that $\mathrm{L}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2$. Simple calculations show that we must define $\mathrm{L}(\boldsymbol{x})$ as

$$\mathrm{L}(\boldsymbol{x}) \triangleq \begin{bmatrix} x_1 & 0 & \bar{\boldsymbol{x}}^{\mathsf{T}} \\ 0 & x_2 & \bar{\boldsymbol{x}}^{\mathsf{T}} \\ \frac{1}{2}\bar{\boldsymbol{x}} & \frac{1}{2}\bar{\boldsymbol{x}} & \frac{1}{2}(x_1 + x_2)I_{n-2} \end{bmatrix}. \tag{7}$$

Note that $\mathrm{L}(\boldsymbol{e}) = I_n$ and $\mathrm{L}(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}$. Note also that the matrix

$$H\mathrm{L}(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} x_1 & 0 & \bar{\boldsymbol{x}}^{\mathsf{T}} \\ 0 & x_2 & \bar{\boldsymbol{x}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}} & \bar{\boldsymbol{x}} & (x_1 + x_2)I_{n-2} \end{bmatrix} \in \mathcal{S}^n,$$

and that $\boldsymbol{x} \in \mathcal{R}^n_+$ if and only if $H\mathrm{L}(\boldsymbol{x}) \in \mathcal{S}^n_+$.

We define the binary operation $\circ : \mathcal{R}^n \times \mathcal{R}^n \longrightarrow \mathcal{R}^n$ as

$$\boldsymbol{x} \circ \boldsymbol{y} \triangleq \mathrm{L}(\boldsymbol{x})\boldsymbol{y} = \begin{bmatrix} x_1 y_1 + \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}} \\ x_2 y_2 + \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}} \\ \frac{1}{2}(y_1 + y_2)\bar{\boldsymbol{x}} + \frac{1}{2}(x_1 + x_2)\bar{\boldsymbol{y}} \end{bmatrix},$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n$. Note that

$$\boldsymbol{x} \circ \boldsymbol{e} = \boldsymbol{e} \circ \boldsymbol{x} = \boldsymbol{x},\ \boldsymbol{x} \circ \boldsymbol{x}^{-1} = \boldsymbol{x}^{-1} \circ \boldsymbol{x} = \boldsymbol{e},\ \boldsymbol{x} \circ \boldsymbol{x} = \boldsymbol{x}^2.$$

Note also that the inner product "$\bullet$" can be redefined as

$$\boldsymbol{x} \bullet \boldsymbol{y} \triangleq \frac{1}{2}\text{trace } (\boldsymbol{x} \circ y).$$

Let $\boldsymbol{x} \in (\mathcal{R}^n, \circ, \bullet)$. One can show that

$$\begin{array}{l} \boldsymbol{c}_1(\boldsymbol{x}) = H^{-1}R\boldsymbol{c}_2(\boldsymbol{x}),\ \boldsymbol{c}_2(\boldsymbol{x}) = H^{-1}R\boldsymbol{c}_1(\boldsymbol{x}), \\ \boldsymbol{c}_i(\boldsymbol{x}) \circ \boldsymbol{c}_i(\boldsymbol{x}) = \boldsymbol{c}_i(\boldsymbol{x}), \quad i = 1, 2, \\ \boldsymbol{c}_1(\boldsymbol{x}) \circ \boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{0},\ \boldsymbol{c}_1(\boldsymbol{x}) \bullet \boldsymbol{c}_2(\boldsymbol{x}) = 0, \\ \lambda_1(\boldsymbol{c}_1(\boldsymbol{x})) = \lambda_1(\boldsymbol{c}_2(\boldsymbol{x})) = 1, \\ \lambda_2(\boldsymbol{c}_1(\boldsymbol{x})) = \lambda_2(\boldsymbol{c}_2(\boldsymbol{x})) = 0. \end{array} \tag{8}$$

This concludes that the pair $\{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ is a Jordan frame. We have the following propositions concerning the pair $(\mathcal{R}^n, \circ)$ and the triple $(\mathcal{R}^n, \circ, \bullet)$.

Proposition 3.6. *The structure* $(\mathcal{R}^n, \circ)$ *is an algebra over* $\mathbb{R}$.

Proof. We show that the multiplication "$\circ$" is a bilinear map. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{R}^n$, then for all $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} (\alpha\boldsymbol{x} + \beta\boldsymbol{y}) \circ \boldsymbol{z} &= \begin{bmatrix} (\alpha x_1 + \beta y_1)z_1 + (\alpha\bar{\boldsymbol{x}}^\top + \beta\bar{\boldsymbol{y}}^\top)\bar{\boldsymbol{z}} \\ (\alpha x_2 + \beta y_2)z_1 + (\alpha\bar{\boldsymbol{x}}^\top + \beta\bar{\boldsymbol{y}}^\top)\bar{\boldsymbol{z}} \\ \frac{1}{2}(\alpha\bar{\boldsymbol{x}} + \beta\bar{\boldsymbol{y}})(z_1 + z_2) + \frac{1}{2}\bar{\boldsymbol{z}}(\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2) \end{bmatrix} \\ &= \alpha \begin{bmatrix} x_1 z_1 + \bar{\boldsymbol{x}}^\top \bar{\boldsymbol{z}} \\ x_2 z_2 + \bar{\boldsymbol{x}}^\top \bar{\boldsymbol{z}} \\ \frac{1}{2}\bar{\boldsymbol{x}}(z_1 + z_2) + \frac{1}{2}\bar{\boldsymbol{z}}(x_1 + x_2) \end{bmatrix} + \beta \begin{bmatrix} y_1 z_1 + \bar{\boldsymbol{y}}^\top \bar{\boldsymbol{z}} \\ y_2 z_2 + \bar{\boldsymbol{y}}^\top \bar{\boldsymbol{z}} \\ \frac{1}{2}\bar{\boldsymbol{y}}(z_1 + z_2) + \frac{1}{2}\bar{\boldsymbol{z}}(y_1 + y_2) \end{bmatrix} \\ &= \alpha(\boldsymbol{x} \circ \boldsymbol{z}) + \beta(\boldsymbol{y} \circ \boldsymbol{z}). \end{aligned}$$

We also have

$$\begin{aligned} \boldsymbol{z} \circ (\alpha\boldsymbol{x} + \beta\boldsymbol{y}) &= \begin{bmatrix} z_1(\alpha x_1 + \beta y_1) + \bar{\boldsymbol{z}}^\top(\alpha\bar{\boldsymbol{x}} + \beta\bar{\boldsymbol{y}}) \\ z_2(\alpha x_2 + \beta y_2) + \bar{\boldsymbol{z}}^\top(\alpha\bar{\boldsymbol{x}} + \beta\bar{\boldsymbol{y}}) \\ \frac{1}{2}\bar{\boldsymbol{z}}\,(\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2) + \frac{1}{2}(\alpha\bar{\boldsymbol{x}} + \beta\bar{\boldsymbol{y}})(z_1 + z_2) \end{bmatrix} \\ &= \alpha \begin{bmatrix} z_1 x_1 + \bar{\boldsymbol{z}}^\top \bar{\boldsymbol{x}} \\ z_2 x_2 + \bar{\boldsymbol{z}}^\top \bar{\boldsymbol{x}} \\ \frac{1}{2}\bar{\boldsymbol{z}}(x_1 + x_2) + \frac{1}{2}\bar{\boldsymbol{x}}(z_1 + z_2) \end{bmatrix} + \beta \begin{bmatrix} z_1 y_1 + \bar{\boldsymbol{z}}^\top \bar{\boldsymbol{y}} \\ z_2 y_2 + \bar{\boldsymbol{z}}^\top \bar{\boldsymbol{y}} \\ \frac{1}{2}\bar{\boldsymbol{z}}(y_1 + y_2) + \frac{1}{2}\bar{\boldsymbol{y}}(z_1 + z_2) \end{bmatrix} \\ &= \alpha(\boldsymbol{z} \circ \boldsymbol{x}) + \beta(\boldsymbol{z} \circ \boldsymbol{y}). \end{aligned}$$

Hence, the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \circ \boldsymbol{y}$ from $\mathcal{R}^n \times \mathcal{R}^n$ into $\mathcal{R}^n$ is bilinear. Therefore, $(\mathcal{R}^n, \circ)$ is an algebra over $\mathbb{R}$. □

Proposition 3.7. *The algebra* $(\mathcal{R}^n, \circ)$ *is a Jordan algebra.*

Proof. It is clear that the binary operation "$\circ$" is commutative. Now, we need to show that "$\circ$" satisfies Jordan's identity. That is, we show that $(\boldsymbol{x} \circ \boldsymbol{y}) \circ \boldsymbol{x}^2 = \boldsymbol{x} \circ (\boldsymbol{y} \circ \boldsymbol{x}^2)$ for all $\boldsymbol{x}, \boldsymbol{y} \in (\mathcal{R}^n, \circ)$. Due to the commutativity property, Jordan's identity for this algebra can be written as $\boldsymbol{x}^2 \circ (\boldsymbol{x} \circ \boldsymbol{y}) = \boldsymbol{x} \circ (\boldsymbol{x}^2 \circ \boldsymbol{y})$, or equivalently, $\mathrm{L}(\boldsymbol{x}^2)\mathrm{L}(\boldsymbol{x}) = \mathrm{L}(\boldsymbol{x})\mathrm{L}(\boldsymbol{x}^2)$ for all $\boldsymbol{x} \in (\mathcal{R}^n, \circ)$. This follows from the following calculation:

$$\mathrm{L}(\boldsymbol{x})\mathrm{L}\left(\boldsymbol{x}^2\right) = \begin{bmatrix} x_1 & 0 & \bar{\boldsymbol{x}}^\mathsf{T} \\ 0 & x_2 & \bar{\boldsymbol{x}}^\mathsf{T} \\ \frac{1}{2}\bar{\boldsymbol{x}} & \frac{1}{2}\bar{\boldsymbol{x}} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix} \begin{bmatrix} x_1^2 + \|\bar{x}\|^2 & 0 & (x_1+x_2)\bar{\boldsymbol{x}}^\mathsf{T} \\ 0 & x_2^2 + \|\bar{x}\|^2 & (x_1+x_2)\bar{\boldsymbol{x}}^\mathsf{T} \\ \frac{1}{2}(x_1+x_2)\bar{x} & \frac{1}{2}(x_1+x_2)\bar{x} & \frac{1}{2}(x_1^2+x_2^2+2\|\bar{x}\|^2)I_{n-2} \end{bmatrix}$$

$$= \begin{bmatrix} x_1^3 + \frac{1}{2}\|\bar{x}\|^2(3x_1+x_2) & \frac{1}{2}\|\bar{x}\|^2(x_1+x_2) & \frac{1}{2}(3x_1^2+x_2^2+2x_1x_2+2\|\bar{x}\|^2)\bar{\boldsymbol{x}}^\mathsf{T} \\ \frac{1}{2}\|\bar{x}\|^2(x_1+x_2) & x_2^3 + \frac{1}{2}\|\bar{x}\|^2(x_1+3x_2) & \frac{1}{2}(x_1^2+3x_2^2+2x_1x_2+\|\bar{x}\|^2)\bar{\boldsymbol{x}}^\mathsf{T} \\ \frac{1}{4}(3x_1^2+x_2^2+2x_1x_2+2\|\bar{x}\|^2)\bar{\boldsymbol{x}} & \frac{1}{4}(x_1^2+3x_2^2+2x_1x_2+2\|\bar{x}\|^2)\bar{\boldsymbol{x}} & \frac{1}{4}(x_1+x_2)\left(2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + (x_1^2+x_2^2+2\|\bar{x}\|^2)I_{n-2}\right) \end{bmatrix}$$

$$= \mathrm{L}\left(\boldsymbol{x}^2\right)\mathrm{L}(\boldsymbol{x}).$$

Thus, $(\mathcal{R}^n, \circ)$ is a Jordan algebra. The proof is complete. □

Proposition 3.8. $(\mathcal{R}^n, \circ, \bullet)$ *is an EJA.*

Proof. It is clear that the inner product "$\bullet$" is a symmetric and positive definite on $\mathcal{R}^n$. Therefore, to show that $(\mathcal{R}^n, \circ, \bullet)$ is an EJA, it suffices to show that "$\bullet$" is associative, i.e., $(\boldsymbol{x} \circ \boldsymbol{y}) \bullet \boldsymbol{z} = \boldsymbol{x} \bullet (\boldsymbol{y} \circ \boldsymbol{z})$, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{R}^n$. This follows from the following calculation:

$$\begin{aligned}
(\boldsymbol{x} \circ \boldsymbol{y}) \bullet \boldsymbol{z} &= \begin{bmatrix} x_1 y_1 + \bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} \\ x_2 y_2 + \bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} \\ \dfrac{(x_1+x_2)}{2}\bar{\boldsymbol{y}} + \dfrac{(y_1+y_2)}{2}\bar{\boldsymbol{x}} \end{bmatrix} \bullet \begin{bmatrix} z_1 \\ z_2 \\ \bar{\boldsymbol{z}} \end{bmatrix} \\
&= \frac{1}{2}\Big(x_1 y_1 z_1 + x_2 y_2 z_2 + (z_1+z_2)\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} + (x_1+x_2)\bar{\boldsymbol{y}}^\mathsf{T}\bar{\boldsymbol{z}} + (y_1+y_2)\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{z}}\Big) \\
&= \frac{1}{2}\left(x_1\left(y_1 z_1 + \bar{\boldsymbol{y}}^\mathsf{T}\bar{\boldsymbol{z}}\right) + x_2\left(y_2 z_2 + \bar{\boldsymbol{y}}^\mathsf{T}\bar{\boldsymbol{z}}\right) + 2\bar{x}^\mathsf{T}\left(\frac{z_1+z_2}{2}\bar{\boldsymbol{y}} + \frac{y_1+y_2}{2}\bar{\boldsymbol{z}}\right)\right) \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \bar{\boldsymbol{x}} \end{bmatrix} \bullet \begin{bmatrix} y_1 z_1 + \bar{\boldsymbol{y}}^\mathsf{T}\bar{\boldsymbol{z}} \\ y_2 z_2 + \bar{\boldsymbol{y}}^\mathsf{T}\bar{\boldsymbol{z}} \\ \dfrac{y_1+y_2}{2}\bar{\boldsymbol{z}} + \dfrac{z_1+z_2}{2}\bar{\boldsymbol{y}} \end{bmatrix} = \boldsymbol{x} \bullet (\boldsymbol{y} \circ \boldsymbol{z}).
\end{aligned}$$

The proof is complete. □

The quadratic representation of $\boldsymbol{x}$, $\mathrm{Q}_{\boldsymbol{x}} : \mathcal{R}^n \longrightarrow \mathcal{R}^n$, is defined to be the matrix:

$$\begin{aligned}
\mathrm{Q}_{\boldsymbol{x}} &\triangleq 2\mathrm{L}(\boldsymbol{x})^2 - L(\boldsymbol{x}^2) \\
&= \begin{bmatrix} x_1^2 & \|\bar{\boldsymbol{x}}\|^2 & 2x_1\bar{\boldsymbol{x}}^\mathsf{T} \\ \|\bar{\boldsymbol{x}}\|^2 & x_2^2 & 2x_2\bar{\boldsymbol{x}}^\mathsf{T} \\ x_1\bar{\boldsymbol{x}} & x_2\bar{\boldsymbol{x}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \det(\boldsymbol{x})I_{n-2} \end{bmatrix} = 2\boldsymbol{x}\boldsymbol{x}^\mathsf{T}H - \det(\boldsymbol{x})H^{-1}R.
\end{aligned} \tag{9}$$

More generally, the quadratic operator of $\boldsymbol{x}$ and $\boldsymbol{z}$, $\mathrm{Q}_{\boldsymbol{x},\boldsymbol{z}} : \mathcal{R}^n \longrightarrow \mathcal{R}^n$, is defined to be the matrix:

$$
\begin{aligned}
\mathrm{Q}_{\boldsymbol{x},\boldsymbol{z}} &\triangleq \mathrm{L}(\boldsymbol{x})L(\boldsymbol{z}) + L(\boldsymbol{z})\mathrm{L}(\boldsymbol{x}) - L(\boldsymbol{x} \circ \boldsymbol{z}) \\
&= \begin{bmatrix} x_1 z_1 & \bar{\boldsymbol{x}}^\top \bar{\boldsymbol{z}} & x_1 \bar{\boldsymbol{z}}^\top + z_1 \bar{\boldsymbol{x}}^\top \\ \bar{\boldsymbol{x}}^\top \bar{\boldsymbol{z}} & x_2 z_2 & x_2 \bar{\boldsymbol{z}}^\top + z_2 \bar{\boldsymbol{x}}^\top \\ \frac{z_1}{2}\bar{\boldsymbol{x}} + \frac{x_1}{2}\bar{\boldsymbol{z}} & \frac{z_2}{2}\bar{\boldsymbol{x}} + \frac{x_2}{2}\bar{\boldsymbol{z}} & \bar{\boldsymbol{x}}\bar{\boldsymbol{z}}^\top + \bar{\boldsymbol{z}}\bar{\boldsymbol{x}}^\top + \left(\frac{x_1 z_2 + x_2 z_1}{2} - \bar{\boldsymbol{x}}^\top \bar{\boldsymbol{z}}\right) I_{n-2} \end{bmatrix}.
\end{aligned} \tag{10}
$$

Note that $\mathrm{Q}_{\boldsymbol{e}} = I_n, \mathrm{Q}_{\boldsymbol{x}}\boldsymbol{e} = \boldsymbol{x}^2, \mathrm{Q}_{\boldsymbol{e},\boldsymbol{x}} = \mathrm{L}(\boldsymbol{x}), \mathrm{Q}_{\boldsymbol{x},\boldsymbol{x}} = \mathrm{Q}_{\boldsymbol{x}}$, and $\mathrm{Q}_{\boldsymbol{x},\boldsymbol{z}} = \mathrm{Q}_{\boldsymbol{z},\boldsymbol{x}}$. Recall that the matrix $L(\boldsymbol{x})$ is not symmetric, but $HL(\boldsymbol{x})$ is a symmetric matrix. Likewise, $Q_{\boldsymbol{x}}$ and $Q_{\boldsymbol{x},\boldsymbol{z}}$ are not symmetric matrices, but $HQ_{\boldsymbol{x}}$ and $HQ_{\boldsymbol{x},\boldsymbol{z}}$ are symmetric matrices.

One can see that, if $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in (\mathcal{R}^n, \circ)$, then $\boldsymbol{x} \circ (\boldsymbol{y} \circ \boldsymbol{z}) \neq (\boldsymbol{x} \circ \boldsymbol{y}) \circ \boldsymbol{z}$ in general, hence $(\mathcal{R}^n, \circ)$ is not associative. However, since it is a Jordan algebra, $(\mathcal{R}^n, \circ)$ must be power-associative, i.e., one has $\boldsymbol{x}^p \circ \boldsymbol{x}^q = \boldsymbol{x}^{p+q}$, for any $\boldsymbol{x} \in \mathcal{R}^n$ and integers $p, q \geq 1$.

As a main result of the paper, we present and prove the following theorem, which contains an important characterization of the cone under study.

Theorem 3.9. *The rotated quadratic cone $\mathcal{R}^n_+$ is the cone of squares of the EJA $(\mathcal{R}^n, \circ, \bullet)$. That is, $\mathcal{R}^n_+ = \mathcal{S}_{\mathcal{R}^n}$.*

Proof. The cone of squares of the EJA $(\mathcal{R}^n, \circ, \bullet)$ is the set:

$$
\mathcal{S}_{\mathcal{R}^n} \triangleq \left\{\boldsymbol{x}^2 : \boldsymbol{x} \in \mathcal{R}^n\right\} = \left\{ \begin{bmatrix} x_1^2 + \|\bar{\boldsymbol{x}}\|^2 \\ x_2^2 + \|\bar{\boldsymbol{x}}\|^2 \\ (x_1 + x_2)\bar{\boldsymbol{x}} \end{bmatrix} : \boldsymbol{x} \in \mathcal{R}^n \right\}.
$$

First, we show that $\mathcal{S}_{\mathcal{R}^n} \subseteq \mathcal{R}^n_+$. Let $\boldsymbol{x} \in \mathcal{S}_{\mathcal{R}^n}$, then $\exists \boldsymbol{y} \in \mathcal{R}^n$ such that

$$
\boldsymbol{x} = \begin{bmatrix} y_1^2 + \|\bar{\boldsymbol{y}}\|^2 \\ y_2^2 + \|\bar{\boldsymbol{y}}\|^2 \\ (y_1 + y_2)\bar{\boldsymbol{y}} \end{bmatrix}.
$$

The fact that $(y_1 y_2 - \|\bar{\boldsymbol{y}}\|^2)^2 \geq 0$ implies that

$$
y_1^2 y_2^2 + \|\bar{\boldsymbol{y}}\|^4 \geq 2 y_1 y_2 \|\bar{\boldsymbol{y}}\|^2. \tag{11}
$$

It follows that

$$
\begin{aligned}
x_1 x_2 &= (y_1^2 + \|\bar{\boldsymbol{y}}\|^2)(y_2^2 + \|\bar{\boldsymbol{y}}\|^2) \\
&= (y_1^2 + y_2^2)\|\bar{\boldsymbol{y}}\|^2 + y_1^2 y_2^2 + \|\bar{\boldsymbol{y}}\|^4 \\
&\geq (y_1^2 + y_2^2)\|\bar{\boldsymbol{y}}\|^2 + 2 y_1 y_2 \|\bar{\boldsymbol{y}}\|^2 \\
&= (y_1 + y_2)^2 \|\bar{\boldsymbol{y}}\|^2 \;=\; \|\bar{\boldsymbol{x}}\|^2,
\end{aligned}
$$

where the inequality follows by adding the term $(y_1^2 + y_2^2)\|\bar{\boldsymbol{y}}\|^2$ to both side of (11). Thus $\boldsymbol{x} \in \mathcal{R}^n_+$, and hence $\mathcal{S}_{\mathcal{R}^n} \subseteq \mathcal{R}^n_+$.

To prove that $\mathcal{R}^n_+ \subseteq \mathcal{S}_{\mathcal{R}^n}$, we need to show that for any $\boldsymbol{x} \in \mathcal{R}^n_+$, there exists $\boldsymbol{y} \in \mathcal{R}^n$ satisfying $\boldsymbol{x} = \boldsymbol{y}^2$. Taking $\boldsymbol{y} \triangleq \sqrt{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \sqrt{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x})$ gives the desired result. □

The following is an immediate corollary of Proposition 3.8 and Theorems 3.5 and 3.9.

Corollary 3.10. *The rotated quadratic cone $\mathcal{R}^n_+$ is symmetric. In particular, $\mathcal{R}^n_+$ is a homogeneous cone.*

The EJA associated with the rotated quadratic cone and that associated with the second-order cone are compared in Table B.1 of Appendix B.

4. Further properties of the algebra

This section presents further algebraic properties of $(\mathcal{R}^n, \circ)$. These properties are important in developing more specialized interior-point rotated quadratic cone algorithms.

4.1. **Spectral properties.** In this part, we state and prove properties of the spectral scalar functions trace$(\cdot)$ and det$(\cdot)$ in the algebra $(\mathcal{R}^n, \circ)$. We have the following theorems.

Theorem 4.1. *We assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n_+$ in the following items, except item (iii). We have*

(i) $\text{trace}(\boldsymbol{x} + \boldsymbol{y}) = \text{trace}(\boldsymbol{x}) + \text{trace}(\boldsymbol{y})$.
(ii) $\text{trace}(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) = \alpha\, \text{trace}(\boldsymbol{x}) + (1-\alpha)\text{trace}(\boldsymbol{y})$, *where* $\alpha \in \mathbb{R}$.
(iii) If $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{R}^n_+$, *then* $\text{trace}(\boldsymbol{x}) \geq \text{trace}(\boldsymbol{y})$.
(iv) $\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{y}) + \lambda_1(\boldsymbol{y})\lambda_2(\boldsymbol{x}) \leq \text{trace}(\boldsymbol{x} \circ \boldsymbol{y}) \leq \lambda_1(\boldsymbol{x})\lambda_1(\boldsymbol{y}) + \lambda_2(\boldsymbol{x})\lambda_2(\boldsymbol{y})$.

Proof. The first two items are trivial. For item (iii), assume that $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{R}^n_+$, then $x_1 - y_1 \geq 0$ and $x_2 - y_2 \geq 0$. It follows that $\text{trace}(\boldsymbol{x}) = x_1 + x_2 \geq y_1 + y_2 = \text{trace}(\boldsymbol{y})$. To verify item (iv), using Cauchy–Schwartz inequality, we have

$$
\begin{aligned}
&\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{y}) + \lambda_1(\boldsymbol{y})\lambda_2(\boldsymbol{x}) \\
&= \frac{1}{4}\left(x_1 + x_2 + \sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}\right)\left(y_1 + y_2 - \sqrt{(y_1 - y_2)^2 + 4\bar{\boldsymbol{y}}^{\mathsf{T}}\bar{\boldsymbol{y}}}\right) \\
&\quad + \frac{1}{4}\left(y_1 + y_2 + \sqrt{(y_1 - y_2)^2 + 4\bar{\boldsymbol{y}}^{\mathsf{T}}\bar{\boldsymbol{y}}}\right)\left(x_1 + x_2 - \sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}\right) \\
&= \frac{1}{4}\left(2(x_1 + x_2)(y_1 + y_2) - 2\sqrt{(x_1 - x_2)^2 + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}}\sqrt{(y_1 - y_2)^2 + 4\bar{\boldsymbol{y}}^{\mathsf{T}}\bar{\boldsymbol{y}}}\right) \\
&\leq \frac{1}{2}\left((x_1 + x_2)(y_1 + y_2) + \begin{bmatrix} x_1 - x_2 \\ 2\bar{\boldsymbol{x}} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} y_1 - y_2 \\ 2\bar{\boldsymbol{y}} \end{bmatrix}\right) \\
&= \frac{1}{2}\left((x_1 + x_2)(y_1 + y_2) + (x_1 - x_2)(y_1 - y_2) + 4\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\right) \\
&= x_1 y_1 + x_2 y_2 + 2\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}} \\
&= 2\,\boldsymbol{x} \bullet \boldsymbol{y} = \text{trace}\,(\boldsymbol{x} \circ \boldsymbol{y}).
\end{aligned}
$$

This shows the first inequality in item (iv). The second inequality in item (iv) can be also shown in the same way. □

Theorem 4.2. *For any $x, y \in \mathcal{R}^n_+$ and $\alpha \in (0,1)$, we have*

(i) $\det(Q_x y) = \det^2(x)\det(y)$.
(ii) $\det(x+y) \geq \det(x) + \det(y)$.
(iii) $\det(e+x) \geq 1 + \det(x)$.
(iv) $\det(\alpha x + (1-\alpha)y) \geq \alpha^2 \det(x) + (1-\alpha)^2 \det(y)$.
(v) $\det(x)\det(y) \geq \det(x \circ y)$.
(vi) $\det(e+x)\det(e+y) - \det(e+x+y) \geq \det(x)\det(y) + 2\, x \bullet y$.

Proof. We do the proof item by item.

(i) Let $z \triangleq Q_x y$, $\alpha \triangleq \bar{x}^\top \bar{y}$, and $\gamma \triangleq \det(x)$. We know that

$$Q_x y = \begin{bmatrix} x_1^2 y_1 + \|\bar{x}\|^2 y_2 + 2x_1 \bar{x}^\top \bar{y} \\ x_2^2 y_2 + \|\bar{x}\|^2 y_1 + 2x_2 \bar{x}^\top \bar{y} \\ (x_1 y_1 + x_2 y_2)\bar{x} + (2\bar{x}\bar{x}^\top + \det(x) I_{n-2})\bar{y} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \bar{z} \end{bmatrix}.$$

Then we have that

$$\begin{aligned} z_1 z_2 &= 4\alpha^2 x_1 x_2 + 2\alpha\Big(x_1 y_1 + x_2 y_2\Big)\Big(x_1 x_2 + \|\bar{x}\|^2\Big) \\ &\quad + \|\bar{x}\|^2\Big(x_1 y_1 + x_2 y_2\Big)^2 + y_1 y_2 \gamma^2. \end{aligned}$$

We also have that

$$\begin{aligned} \|\bar{z}\|^2 &= 4\alpha^2 \|\bar{x}\|^2 + \gamma^2 \|\bar{y}\|^2 + 4\alpha^2 \gamma + \|\bar{x}\|^2\Big(x_1 y_1 + x_2 y_2\Big)^2 \\ &\quad + 2\alpha\Big(x_1 y_1 + x_2 y_2\Big)\Big(x_1 x_2 + \|\bar{x}\|^2\Big). \end{aligned}$$

It follows that

$$\begin{aligned} \det(Q_x y) &= z_1 z_2 - \|\bar{z}\|^2 \\ &= 4\alpha^2\Big(x_1 x_2 - \|\bar{x}\|^2\Big) + \gamma^2\Big(y_1 y_2 - \|\bar{y}\|^2\Big) - 4\alpha^2 \gamma \\ &= \gamma^2 \det(y) \;=\; \det(x)^2 \det(y) \end{aligned}$$

as desired.

(ii) Note that

$$\begin{aligned} \det(x+y) &= \lambda_1(x+y)\lambda_2(x+y) \\ &= (x_1 + y_1)(x_2 + y_2) - \|\bar{x} + \bar{y}\|^2 \\ &= x_1 x_2 + x_1 y_2 + x_2 y_1 + y_1 y_2 - \|\bar{x}\|^2 - \|\bar{y}\|^2 - 2\bar{x}^\top \bar{y} \\ &= (x_1 x_2 - \|\bar{x}\|^2) + (y_1 y_2 - \|\bar{y}\|^2) + (x_1 y_2 + x_2 y_1 - 2\bar{x}^\top \bar{y}) \\ &= \det(x) + \det(y) + 2\,(x \bullet y^{-1})\det(y) \\ &\geq \det(x) + \det(y), \end{aligned}$$

where the inequality is followed from (4). The result has been obtained.

(iii) Note that

$$\begin{aligned} \det(e+x) &= (1 + x_1)(1 + x_2) - \|\bar{x}\|^2 \\ &= 1 + x_1 x_2 - \|\bar{x}\|^2 + x_1 + x_2 \\ &\geq 1 + x_1 x_2 - \|\bar{x}\|^2 \;=\; 1 + \det(x). \end{aligned}$$

The desired result has been obtained.

(iv) It is clear that, for all $\alpha \geq 0$, $\alpha \boldsymbol{x}$, $(1-\alpha)\boldsymbol{y} \in \mathcal{R}^n_+$ and $\det(\alpha \boldsymbol{x}) = \alpha^2 \det(\boldsymbol{x})$. Therefore, by using the inequality in item (ii), we have

$$\begin{array}{rcl} \det(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) & \geq & \det(\alpha \boldsymbol{x}) + \det((1-\alpha)\boldsymbol{y}) \\ & = & \alpha^2 \det(\boldsymbol{x}) + (1-\alpha)^2 \det(\boldsymbol{y}) \end{array}$$

as desired.

(v) Note that

$$\begin{array}{rl} & \det(\boldsymbol{x})\det(\boldsymbol{y}) - \det(\boldsymbol{x} \circ \boldsymbol{y}) \\ = & \left(x_1x_2 - \|\bar{\boldsymbol{x}}\|^2\right)\left(y_1y_2 - \|\bar{\boldsymbol{y}}\|^2\right) - \left(x_1y_1 + \bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}}\right)\left(x_2y_2 + \bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}}\right) \\ & + \left\| \dfrac{y_1+y_2}{2}\bar{\boldsymbol{x}} + \dfrac{x_1+x_2}{2}\bar{\boldsymbol{y}} \right\|^2 \\ = & \|\bar{\boldsymbol{x}}\|^2\|\bar{\boldsymbol{y}}\|^2 - x_1x_2\|\bar{\boldsymbol{y}}\|^2 - y_1y_2\|\bar{\boldsymbol{x}}\|^2 - (x_1y_1 + x_2y_2)\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} - \left(\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}}\right)^2 \\ & + \dfrac{(y_1+y_2)^2}{4}\|\bar{\boldsymbol{x}}\|^2 + \dfrac{(x_1+x_2)^2}{4}\|\bar{\boldsymbol{y}}\|^2 + \dfrac{(x_1+x_2)(y_1+y_2)}{2}\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} \\ \geq & \|\bar{\boldsymbol{x}}\|^2\|\bar{\boldsymbol{y}}\|^2 - x_1x_2\|\bar{\boldsymbol{y}}\|^2 - y_1y_2\|\bar{\boldsymbol{x}}\|^2 - (x_1y_1 + x_2y_2)\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} - \|\bar{\boldsymbol{x}}\|^2\|\bar{\boldsymbol{y}}\|^2 \\ & + \dfrac{(y_1+y_2)^2}{4}\|\bar{\boldsymbol{x}}\|^2 + \dfrac{(x_1+x_2)^2}{4}\|\bar{\boldsymbol{y}}\|^2 + \dfrac{(x_1+x_2)(y_1+y_2)}{2}\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} \\ = & \dfrac{(y_1-y_2)^2}{4}\|\bar{\boldsymbol{x}}\|^2 + \dfrac{(x_1-x_2)^2}{4}\|\bar{\boldsymbol{y}}\|^2 - \dfrac{(x_1-x_2)(y_1-y_2)}{2}\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} \\ = & \left\| \dfrac{y_1-y_2}{2}\bar{\boldsymbol{x}} - \dfrac{x_1-x_2}{2}\bar{\boldsymbol{y}} \right\|^2 \;\geq\; 0, \end{array}$$

where the inequality follows from the Cauchy–Schwartz inequality. Consequently, $\det(\boldsymbol{x})\det(\boldsymbol{y}) \geq \det(\boldsymbol{x} \circ \boldsymbol{y})$ as desired.

(vi) Note that

$$\begin{array}{rl} & \det(\boldsymbol{e}+\boldsymbol{x})\det(\boldsymbol{e}+\boldsymbol{y}) - \det(\boldsymbol{e}+\boldsymbol{x}+\boldsymbol{y}) \\ = & \Big((1+x_1)(1+x_2) - \|\bar{\boldsymbol{x}}\|^2\Big)\Big((1+y_1)(1+y_2) - \|\bar{\boldsymbol{y}}\|^2\Big) \\ & - (1+x_1+y_1)(1+x_2+y_2) + \|\bar{\boldsymbol{x}} + \bar{\boldsymbol{y}}\|^2 \\ = & (1+x_1)(1+x_2)(1+y_1)(1+y_2) - \|\bar{\boldsymbol{x}}\|^2(1+y_1)(1+y_2) \\ & - \|\bar{\boldsymbol{y}}\|^2(1+x_1)(1+x_2) + \|\bar{\boldsymbol{x}}\|^2\|\bar{\boldsymbol{y}}\|^2 + \|\bar{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{y}}\|^2 + 2\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}} \\ & - (1+x_1+y_1)(1+x_2+y_2) \\ = & x_1x_2(y_1+y_2) + y_1y_2(x_1+x_2) + \left(x_1y_1 + x_2y_2 + 2\bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{y}}\right) \\ & - \|\bar{\boldsymbol{x}}\|^2(y_1+y_2) - \|\bar{\boldsymbol{y}}\|^2(x_1+x_2) + x_1x_2y_1y_2 + \|\bar{\boldsymbol{x}}\|^2\|\bar{\boldsymbol{y}}\|^2 \\ & - y_1y_2\|\bar{\boldsymbol{x}}\|^2 - x_1x_2\|\bar{\boldsymbol{y}}\|^2 \\ \geq & \|\bar{\boldsymbol{x}}\|^2(y_1+y_2) + \|\bar{\boldsymbol{y}}\|^2(x_1+x_2) + 2(\boldsymbol{x} \bullet \boldsymbol{y}) + \det(\boldsymbol{x})\det(\boldsymbol{y}) \\ & - \|\bar{\boldsymbol{x}}\|^2(y_1+y_2) - \|\bar{\boldsymbol{y}}\|^2(x_1+x_2) \;=\; \det(\boldsymbol{x})\det(\boldsymbol{y}) + 2\,\boldsymbol{x} \bullet \boldsymbol{y} \;\geq\; 0, \end{array}$$

where the last inequality follows from the fact that $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n_+$. Consequently, $\det(\boldsymbol{e}+\boldsymbol{x})\det(\boldsymbol{e}+\boldsymbol{y}) \geq \det(\boldsymbol{e}+\boldsymbol{x}+\boldsymbol{y})$ as desired. □

4.2. **Properties of Q.** In this part, we list and prove properties of the quadratic operators Q. and $Q_{\cdot,\cdot}$ in the algebra $(\mathcal{R}^n, \circ)$. We have the following theorems.

Theorem 4.3. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n$, $\alpha \in \mathbb{R}$, and an integer t be given. Also, wherever it is necessary that $\boldsymbol{x}^{-1}$ and $\boldsymbol{y}^{-1}$ exist, we assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ are nonsingular. The following identities hold.*

(i) $Q_{\alpha\boldsymbol{x}} = \alpha^2 Q_{\boldsymbol{x}}$.
(ii) $Q_{\boldsymbol{x}^{-1}} = Q_{\boldsymbol{x}}^{-1}$. *More generally,* $Q_{\boldsymbol{x}^t} = Q_{\boldsymbol{x}}^t$.
(iii) $Q_{\boldsymbol{x}}\boldsymbol{x}^{-1} = \boldsymbol{x}$. *Similarly,* $Q_{\boldsymbol{x}^{-1}}\boldsymbol{x} = \boldsymbol{x}^{-1}$.
(iv) $(Q_{\boldsymbol{x}}\boldsymbol{y})^{-1} = Q_{\boldsymbol{x}^{-1}}\boldsymbol{y}^{-1}$.
(v) $Q_{Q_{\boldsymbol{x}}\boldsymbol{y}} = Q_{\boldsymbol{x}}Q_{\boldsymbol{y}}Q_{\boldsymbol{x}}$.
(vi) $Q_{\boldsymbol{x},\boldsymbol{y}} = \frac{1}{4}H^{-1}(Q_{\boldsymbol{x}+\boldsymbol{y}} - Q_{\boldsymbol{x}} - Q_{\boldsymbol{y}})$.
(vii) $Q_{\boldsymbol{x}}L(\boldsymbol{x}^{-1}) = L(\boldsymbol{x})$. *Hence,* $L(\boldsymbol{x})Q_{\boldsymbol{x}}^{-1} = L(\boldsymbol{x}^{-1}) = Q_{\boldsymbol{e},\boldsymbol{x}^{-1}}$.

Proof. It is trivial to prove the result in item (i). From (9), we find that

$$Q_{\boldsymbol{x}^{-1}} = \frac{1}{\det(\boldsymbol{x})^2}\begin{bmatrix} x_2^2 & \|\bar{\boldsymbol{x}}\|^2 & -2x_2\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \|\bar{\boldsymbol{x}}\|^2 & x_1^2 & -2x_1\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -x_2\bar{\boldsymbol{x}} & -x_1\bar{\boldsymbol{x}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_{n-2} \end{bmatrix}.$$

Based on this finding, the first part of item (ii) as well as item (iii) hold. Simple induction is used to demonstrate that the second part of item (ii) holds too. Since $(\mathcal{R}^n, \circ, \bullet)$ is an EJA, the results for both items (iv) and (v) can be found and taken from [18]. For item (vi), we have that

$$Q_{\boldsymbol{x}+\boldsymbol{y}} = \begin{bmatrix} (x_1+y_1)^2 & \|\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}\|^2 & 2(x_1+y_1)(\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}})^{\mathsf{T}} \\ \|\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}\|^2 & (x_2+y_2)^2 & 2(x_2+y_2)(\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}})^{\mathsf{T}} \\ (x_1+y_1)(\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}) & (x_2+y_2)(\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}) & 2(\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}})(\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}})^{\mathsf{T}} + \det(\boldsymbol{x}+\boldsymbol{y})I_{n-2} \end{bmatrix}$$

and that

$$Q_{\boldsymbol{x}} + Q_{\boldsymbol{y}} = \begin{bmatrix} x_1^2+y_1^2 & \|\bar{\boldsymbol{x}}\|^2+\|\bar{\boldsymbol{y}}\|^2 & 2(x_1\bar{\boldsymbol{x}}^{\mathsf{T}}+y_1\bar{\boldsymbol{y}}^{\mathsf{T}}) \\ \|\bar{\boldsymbol{x}}\|^2+\|\bar{\boldsymbol{y}}\|^2 & x_2^2+y_2^2 & 2(x_2\bar{\boldsymbol{x}}^{\mathsf{T}}+y_2\bar{\boldsymbol{y}}^{\mathsf{T}}) \\ (x_1\bar{\boldsymbol{x}}+y_1\bar{\boldsymbol{y}}) & (x_2\bar{\boldsymbol{x}}+y_2\bar{\boldsymbol{y}}) & 2(\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}+\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}^{\mathsf{T}}) + (\det(\boldsymbol{x})+\det(\boldsymbol{y}))I_{n-2} \end{bmatrix}.$$

The desired result in item (vi) is now easily obtained. Item (vii) may be verified using simple algebraic computation. The proof is complete. □

Theorem 4.4. *Let $x \in \mathcal{R}^n$. If it is necessary, let also the inverse x^{-1} exist. Then*

(i) $Q_{x^{-1},x}Q_x = L(x^2)$.
(ii) $2L(x)Q_{e,x} - Q_x = L(x^2)$.
(iii) $L(x^{-1})Q_x x^{-1} = e$.
(iv) $Q_{x,x^{-1}}e = Q_{e,x}x^{-1} = e$.
(v) $Q_{e,x}e = Q_{x,x^{-1}}x = x$.
(vi) $Q_{e,x}x = x^2$.
(vii) $Q_{x,x^{-1}}x^{-1} = x^{-1}$.

Proof. From (5)-(7), we find that

$$\mathrm{L}\left(x^{-1}\right) = \frac{1}{\det(x)}\begin{bmatrix} x_2 & 0 & -\bar{x}^{\mathsf{T}} \\ 0 & x_1 & -\bar{x}^{\mathsf{T}} \\ -\frac{1}{2}\bar{x} & -\frac{1}{2}\bar{x} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix},$$

and that

$$\mathrm{L}\left(x^{2}\right) = \begin{bmatrix} x_1^2 + \|\bar{x}\|^2 & 0 & (x_1+x_2)\bar{x}^{\mathsf{T}} \\ 0 & x_2^2 + \|\bar{x}\|^2 & (x_1+x_2)\bar{x}^{\mathsf{T}} \\ \frac{1}{2}(x_1+x_2)\bar{x} & \frac{1}{2}(x_1+x_2)\bar{x} & \frac{1}{2}(x_1^2+x_2^2+2\|\bar{x}\|^2)I_{n-2} \end{bmatrix}.$$

From (10), we also find that

$$\mathrm{Q}_{e,x} = \begin{bmatrix} x_1 & 0 & \bar{x}^{\mathsf{T}} \\ 0 & x_2 & \bar{x}^{\mathsf{T}} \\ \frac{1}{2}\bar{x} & \frac{1}{2}\bar{x} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix} = \mathrm{L}(x),$$

and that

$$\mathrm{Q}_{x,x^{-1}} = \frac{1}{\det(x)}\begin{bmatrix} x_1x_2 & -\|\bar{x}\|^2 & -(x_1-x_2)\bar{x}^{\mathsf{T}} \\ -\|\bar{x}\|^2 & x_1x_2 & (x_1-x_2)\bar{x}^{\mathsf{T}} \\ -\frac{1}{2}(x_1-x_2)\bar{x} & \frac{1}{2}(x_1-x_2)\bar{x} & -2\bar{x}\bar{x}^{\mathsf{T}} + \frac{1}{2}(x_1^2+x_2^2+2\|\bar{x}\|^2)I_{n-2} \end{bmatrix}.$$

Based on these findings, the results in all items can be obtained using some simple calculations. For instance, item (ii) follows by noting that

$$\begin{aligned}
\mathrm{L}\left(x^{2}\right) + \mathrm{Q}_x &= \begin{bmatrix} x_1^2 + \|\bar{x}\|^2 & 0 & (x_1+x_2)\bar{x}^{\mathsf{T}} \\ 0 & x_2^2 + \|\bar{x}\|^2 & (x_1+x_2)\bar{x}^{\mathsf{T}} \\ \frac{1}{2}(x_1+x_2)\bar{x} & \frac{1}{2}(x_1+x_2)\bar{x} & \frac{1}{2}(x_1^2+x_2^2+2\|\bar{x}\|^2)I_{n-2} \end{bmatrix} \\
&\quad + \begin{bmatrix} x_1^2 & \|\bar{x}\|^2 & 2x_1\bar{x}^{\mathsf{T}} \\ \|\bar{x}\|^2 & x_2^2 & 2x_2\bar{x}^{\mathsf{T}} \\ x_1\bar{x} & x_2\bar{x} & 2\bar{x}\bar{x}^{\mathsf{T}} + \det(x)I_{n-2} \end{bmatrix} \\
&= \begin{bmatrix} 2x_1^2 + \|\bar{x}\|^2 & \|\bar{x}\|^2 & (3x_1+x_2)\bar{x}^{\mathsf{T}} \\ \|\bar{x}\|^2 & 2x_2^2 + \|\bar{x}\|^2 & (x_1+3x_2)\bar{x}^{\mathsf{T}} \\ \frac{1}{2}(3x_1+x_2)\bar{x} & \frac{1}{2}(x_1+3x_2)\bar{x} & 2\bar{x}\bar{x}^{\mathsf{T}} + \frac{1}{2}(x_1+x_2)^2 I_{n-2} \end{bmatrix} \\
&= 2\begin{bmatrix} x_1^2 + \frac{1}{2}\|\bar{x}\|^2 & \frac{1}{2}\|\bar{x}\|^2 & \frac{1}{2}(3x_1+x_2)\bar{x}^{\mathsf{T}} \\ \frac{1}{2}\|\bar{x}\|^2 & x_2^2 + \frac{1}{2}\|\bar{x}\|^2 & \frac{1}{2}(x_1+3x_2)\bar{x}^{\mathsf{T}} \\ \frac{1}{4}(3x_1+x_2)\bar{x} & \frac{1}{4}(x_1+3x_2)\bar{x} & \bar{x}\bar{x}^{\mathsf{T}} + \frac{1}{2}(x_1+x_2)^4 I_{n-2} \end{bmatrix} \\
&= 2\begin{bmatrix} x_1 & 0 & \bar{x}^{\mathsf{T}} \\ 0 & x_2 & \bar{x}^{\mathsf{T}} \\ \frac{1}{2}\bar{x} & \frac{1}{2}\bar{x} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix}\begin{bmatrix} x_1 & 0 & \bar{x}^{\mathsf{T}} \\ 0 & x_2 & \bar{x}^{\mathsf{T}} \\ \frac{1}{2}\bar{x} & \frac{1}{2}\bar{x} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix} \\
&= 2\mathrm{L}(x)\mathrm{Q}_{e,x},
\end{aligned}$$

and item (iii) follows by noting that

$$\begin{aligned}
&\mathrm{L}\left(\boldsymbol{x}^{-1}\right)\mathrm{Q}_{\boldsymbol{x}}\boldsymbol{x}^{-1}\\
&= \frac{1}{\det(\boldsymbol{x})^2}\begin{bmatrix} x_2 & 0 & -\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 0 & x_1 & -\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{1}{2}\bar{\boldsymbol{x}} & -\frac{1}{2}\bar{\boldsymbol{x}} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix}\begin{bmatrix} x_1^2 & \|\bar{\boldsymbol{x}}\|^2 & 2x_1\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \|\bar{\boldsymbol{x}}\|^2 & x_2^2 & 2x_2\bar{\boldsymbol{x}}^{\mathsf{T}} \\ x_1\bar{\boldsymbol{x}} & x_2\bar{\boldsymbol{x}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_{n-2} \end{bmatrix}\begin{bmatrix} x_2 \\ x_1 \\ -\bar{\boldsymbol{x}} \end{bmatrix}\\
&= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_2 & 0 & -\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 0 & x_2 & -\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{1}{2}\bar{\boldsymbol{x}} & -\frac{1}{2}\bar{\boldsymbol{x}} & \frac{1}{2}(x_1+x_2)I_{n-2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \bar{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{e}.
\end{aligned}$$

The proof is complete. □

The following theorem is crucial to analyze specialized interior-point quadratic rotated cone algorithms. We will use Theorem 3.9 in the proof of this theorem.

Theorem 4.5. *Let $\boldsymbol{p} \in (\mathcal{R}^n, \circ)$ be nonsingular. Then we have*

(i) $\mathrm{Q}_{\boldsymbol{p}}(\mathcal{R}^n_+) = \mathcal{R}^n_+$. *(ii)* $\mathrm{Q}_{\boldsymbol{p}}(\operatorname{int}\mathcal{R}^n_+) = \operatorname{int}\mathcal{R}^n_+$.

Proof. First, we show that $\mathrm{Q}_{\boldsymbol{p}}(\mathcal{R}^n_+) \subseteq \mathcal{R}^n_+$. Let $\boldsymbol{y} \in \mathrm{Q}_{\boldsymbol{p}}(\mathcal{R}^n_+)$, then $\boldsymbol{y} = \mathrm{Q}_{\boldsymbol{p}}\boldsymbol{x}$ for some $\boldsymbol{x} \in \mathcal{R}^n_+$. Note that, from item (i) in Theorem 4.2, we have $\det(\boldsymbol{y}) \geq 0$. As a result, either $\boldsymbol{y} \in \mathcal{R}^n_+$ or $\boldsymbol{y} \in -\mathcal{R}^n_+$. To demonstrate that $\boldsymbol{y} \in \mathcal{R}^n_+$, it is enough to show that $\operatorname{trace}(\boldsymbol{y}) \geq 0$.

$$\begin{aligned}
\operatorname{trace}(\boldsymbol{y}) &= y_1 + y_2\\
&= \left(\mathrm{Q}_{\boldsymbol{p}}\boldsymbol{x}\right)_1 + \left(\mathrm{Q}_{\boldsymbol{p}}\boldsymbol{x}\right)_2\\
&= p_1^2x_1 + \|\bar{\boldsymbol{p}}\|^2x_2 + 2p_1\bar{\boldsymbol{p}}^{\mathsf{T}}\bar{\boldsymbol{x}} + p_2^2x_2 + \|\bar{\boldsymbol{p}}\|^2x_1 + 2p_2\bar{\boldsymbol{p}}^{\mathsf{T}}\bar{\boldsymbol{x}}\\
&= x_1\left(p_1^2 + \|\bar{\boldsymbol{p}}\|^2\right) + x_2\left(p_2^2 + \|\bar{\boldsymbol{p}}\|^2\right) + 2\left(p_1 + p_2\right)\bar{\boldsymbol{p}}^{\mathsf{T}}\bar{\boldsymbol{x}}\\
&= 2\,\boldsymbol{x} \bullet \boldsymbol{p}^2 \;\geq\; 0,
\end{aligned}$$

where the inequality follows from the self-duality of the cone $\mathcal{R}^n_+$ and due to the fact that $\boldsymbol{x} \in \mathcal{R}^n_+$, and that $\boldsymbol{p}^2 \in \mathcal{S}_{\mathcal{R}^n} = \mathcal{R}^n_+$ (from Theorem 3.9).

Now, we show that $\mathcal{R}^n_+ \subseteq \mathrm{Q}_{\boldsymbol{p}}(\mathcal{R}^n_+)$. The vector $\boldsymbol{p}^{-1}$ is nonsingular because of the nonsingularity $\boldsymbol{p}$. Therefore, $\mathrm{Q}_{\boldsymbol{p}^{-1}}\boldsymbol{x} \in \mathcal{R}^n_+$ for each $\boldsymbol{x} \in \mathcal{R}^n_+$. Since $\boldsymbol{x} = \mathrm{Q}_{\boldsymbol{p}}\mathrm{Q}_{\boldsymbol{p}^{-1}}\boldsymbol{x}$, it follows that $\mathcal{R}^n_+ \subseteq \mathrm{Q}_{\boldsymbol{p}}(\mathcal{R}^n_+)$. We have obtained the result in item (i). That in item (ii) can be obtained in the same way. The result is established. □

5. Concluding remarks

In this paper, we have studied the rotated quadratic cone from algebraic and analytical perspectives. We have built an algebra associated with this cone and have shown that this algebra is a Euclidean Jordan algebra with a certain inner product. We have also presented with proofs some spectral and algebraic properties of this algebra.

TABLE 5.1. The ranks and barriers of symmetric cones.

Symmetric cone	Constraint	Dimension	Rank	Optimization barrier
Nonnegative orthant cone $\mathbb{R}^n_+$	$\boldsymbol{x} \geq \boldsymbol{0}$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$-\sum_{i=1}^n \ln x_i$
Second-order cone $\mathcal{E}^n_+$	$x_0 \geq \|\|\hat{x}\|\|$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$-\ln\left(x_0^2 - \|\|\hat{x}\|\|^2\right)$
Rotated quadratic cone $\mathcal{R}^n_+$	$x_1x_2 \geq \|\|\bar{x}\|\|^2$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$-\ln\left(x_1x_2 - \|\|\bar{x}\|\|^2\right)$
Cone of positive semidefinite matrices $\mathcal{S}^n_+$	$X \succeq O$	$\mathcal{O}(n^2)$	$\mathcal{O}(n)$	$-\ln\det(X)$

Mainly, and most importantly, we have shown that the cone of squares of the generated algebra is the rotated quadratic cone itself. These results can help interested researchers improve specialized interior-point rotated cone algorithms based on the generated Jordan algebra. In addition, since the rotated quadratic cone is a special case of the power cone, another reason for this study may be to help open the door to understanding the algebraic structure of the general power cone in the future.

In optimization, particularly in the context of interior-point methods for conic programming, the barrier functions are useful in the analysis of Newton's method. If we follow the standard way of defining the logarithmic barriers in conic programming, then the logarithmic barrier function $b : \text{int}\,\mathcal{R}^n_+ \longrightarrow \mathbb{R}$ associated with the rotated quadratic cone can be defined as

$$b(\boldsymbol{x}) \triangleq -\ln\det(\boldsymbol{x}) = -\ln\left(x_1x_2 - \|\bar{\boldsymbol{x}}\|^2\right).$$

We mention that the function $b(\cdot)$ is strictly convex on int $\mathcal{R}^n_+$. This barrier function would be the counterpart of very well-known barriers in the interior-point theory of conic programming; see Table 5.1.

A future research work in this line will could start from the barrier function $b(\boldsymbol{x})$ defined above. In particular, a future research optimization paper will probably start from computing the gradient $\nabla_{\boldsymbol{x}} b(\boldsymbol{x})$ and the Hessian $\nabla^2_{\boldsymbol{xx}} b(\boldsymbol{x})$, which will be respectively given in terms of the inverse vector, $\boldsymbol{x}^{-1}$, and the quadratic representation matrix, $\mathrm{Q}_{\boldsymbol{x}^{-1}}$, that have been introduced in this paper in detail. More specifically, one can show that $\nabla_{\boldsymbol{x}} b(\boldsymbol{x}) = -2H\boldsymbol{x}^{-1}$ and $\nabla^2_{\boldsymbol{xx}} b(\boldsymbol{x}) = 2H\mathrm{Q}_{\boldsymbol{x}^{-1}}$. These derivatives will be used to show that $b(\boldsymbol{x})$ is a self-concordant barrier (see [14]) satisfying certain smoothness conditions and allowing efficient interior-point algorithms. Future algorithms based on this work are expected to become increasingly differentiated and systematically specialized over the rotated quadratic cone in years to come.

DECLARATIONS

Competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during this study.

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Appendix A. Proof of the power-associativity of the algebra $(\mathcal{R}^n, \circ)$

Let $\boldsymbol{x} \in (\mathcal{R}, \circ)$ and p be a nonnegative integer. Let us recursively define

$$\boldsymbol{x}^{(0)} \triangleq \boldsymbol{e}, \; \boldsymbol{x}^{(p)} \triangleq \boldsymbol{x} \circ \boldsymbol{x}^{(p-1)}.$$

We also define

$$\boldsymbol{x}^{p} \triangleq \lambda_1^p(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}).$$

To prove the power-associativity of the algebra $(\mathcal{R}^n, \circ)$, we first use induction to prove that $\boldsymbol{x}^{(p)} = \boldsymbol{x}^p$ for any p.

It is clear that the result is true for $p = 0$ as $\boldsymbol{x}^{(0)} = \boldsymbol{e} = \boldsymbol{c}_1(\boldsymbol{x}) + \boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{x}^0$. Let k be a nonnegative integer. Now, we assume that it is true that $\boldsymbol{x}^{(k)} = \boldsymbol{x}^k$, and need to prove that the result is true for $k+1$. Due to bilinearity of "$\circ$" and from (8), we have

$$\begin{array}{rcl}
\boldsymbol{x}^{(k+1)} & = & \boldsymbol{x} \circ \boldsymbol{x}^{(k)} \\
& = & \boldsymbol{x} \circ \boldsymbol{x}^{k} \\
& = & (\lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})) \circ \left(\lambda_1^k(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \\
& = & \lambda_1^{k+1}(\boldsymbol{x})\, \boldsymbol{c}_1^2(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\, \boldsymbol{c}_2^2(\boldsymbol{x}) + \lambda_1(\boldsymbol{x})\lambda_2^k(\boldsymbol{x})\, \boldsymbol{c}_1(\boldsymbol{x}) \circ \boldsymbol{c}_2(\boldsymbol{x}) \\
& & +\lambda_2(\boldsymbol{x})\lambda_1^k(\boldsymbol{x})\, \boldsymbol{c}_2(\boldsymbol{x}) \circ \boldsymbol{c}_1(\boldsymbol{x}) \\
& = & \lambda_1^{k+1}(\boldsymbol{x})\, \boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\, \boldsymbol{c}_2(\boldsymbol{x}) \;\; = \;\; \boldsymbol{x}^{k+1}.
\end{array}$$

Thus, $\boldsymbol{x}^{(p)} = \boldsymbol{x}^p$ for any p.

Now, we are ready to prove that the power-associativity of $(\mathcal{R}^n, \circ)$. That is, $\boldsymbol{x}^{(p+q)} = \boldsymbol{x}^{(p)} \circ \boldsymbol{x}^{(q)}$ for any nonnegative integers p and q. Due to bilinearity of "$\circ$" and from (8), we get

$$\begin{array}{rcl}
\boldsymbol{x}^{(p+q)} & = & \boldsymbol{x}^{p+q} \\
& = & \lambda_1^{p+q}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{p+q}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) \\
& = & \lambda_1^{p+q}(\boldsymbol{x})\, \boldsymbol{c}_1^2(\boldsymbol{x}) + \lambda_2^{p+q}(\boldsymbol{x})\, \boldsymbol{c}_2^2(\boldsymbol{x}) + \lambda_1^p(\boldsymbol{x})\lambda_2^q(\boldsymbol{x})\, \boldsymbol{c}_1(\boldsymbol{x}) \circ \boldsymbol{c}_2(\boldsymbol{x}) \\
& & +\lambda_2^p(\boldsymbol{x})\lambda_1^q(\boldsymbol{x})\, \boldsymbol{c}_2(\boldsymbol{x}) \circ \boldsymbol{c}_1(\boldsymbol{x}) \\
& = & \left(\lambda_1^p(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \circ \left(\lambda_1^q(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^q(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \\
& = & \boldsymbol{x}^p \circ \boldsymbol{x}^q \;\; = \;\; \boldsymbol{x}^{(p)} \circ \boldsymbol{x}^{(q)}.
\end{array}$$

Thus, $(\mathcal{R}^n, \circ)$ is a power-associative algebra.

Appendix B. Comparison with the algebra of the second-order cone

In this appendix, we compare between the EJA associated with the rotated quadratic cone and that associated with the second-order cone; see Table B.1.

TABLE B.1. Comparing the Jordan algebraic notions and concepts between those associated with the rotated quadratic cone $\mathcal{R}^n_+$ and those associated with the second order cone $\mathcal{E}^n_+$.

Notion	Rotated quadratic cone $\mathcal{R}^n_+$	Second-order cone $\mathcal{E}^n_+$
Constraint	$x_1^{1/2}x_2^{1/2} \geq \lVert\bar{x}\rVert$	$x_0 \geq \lVert\hat{x}\rVert$
Identity	$\boldsymbol{e} \triangleq (1;1;\mathbf{0})$	$\boldsymbol{e} \triangleq (1;\mathbf{0})$
Inner product matrix	$H \triangleq \begin{bmatrix} \frac{1}{2} & 0 & \mathbf{0}^\top \\ 0 & \frac{1}{2} & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & I_{n-2} \end{bmatrix}$	I_n
Reflection matrix	$R \triangleq \begin{bmatrix} 0 & \frac{1}{2} & \mathbf{0}^\top \\ \frac{1}{2} & 0 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & -I_{n-2} \end{bmatrix}$	$R \triangleq \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & -I_{n-1} \end{bmatrix}$
Inner product	$\boldsymbol{x} \bullet \boldsymbol{y} \triangleq \boldsymbol{x}^\top H \boldsymbol{y} = (x_1y_1 + x_2y_2)/2 + \bar{\boldsymbol{x}}^\top \bar{\boldsymbol{y}}$	$\boldsymbol{x} \bullet \boldsymbol{y} \triangleq \boldsymbol{x}^\top \boldsymbol{y} = x_0y_0 + \hat{\boldsymbol{x}}^\top \hat{\boldsymbol{y}}$
Eigenvalues	$\lambda_i(\boldsymbol{x}) \triangleq \frac{1}{2}\left(x_1 + x_2 \pm \sqrt{(x_1-x_2)^2 + 4\bar{\boldsymbol{x}}^\top\bar{\boldsymbol{x}}}\right)$, for $i = 1, 2$	$\lambda_i(x) \triangleq x_0 \pm \lVert\hat{\boldsymbol{x}}\rVert$, for $i = 1, 2$
Eigenvectors	$\boldsymbol{c}_i(\boldsymbol{x}) \triangleq \frac{1}{2}\begin{bmatrix} 1 \pm \frac{x_1-x_2}{\sqrt{(x_1-x_2)^2+4\bar{\boldsymbol{x}}^\top\bar{\boldsymbol{x}}}} \\ 1 \mp \frac{x_1-x_2}{\sqrt{(x_1-x_2)^2+4\bar{\boldsymbol{x}}^\top\bar{\boldsymbol{x}}}} \\ \pm\frac{2\bar{\boldsymbol{x}}}{\sqrt{(x_1-x_2)^2+4\bar{\boldsymbol{x}}^\top\bar{\boldsymbol{x}}}} \end{bmatrix}$ for $i = 1, 2$	$\boldsymbol{c}_i(\boldsymbol{x}) \triangleq \frac{1}{2}\begin{bmatrix} 1 \\ \pm\frac{\hat{\boldsymbol{x}}}{\lVert\hat{\boldsymbol{x}}\rVert} \end{bmatrix}$, for $i = 1, 2$
Spectral decomposition	$\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})$	$\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})$
Trace	$\text{trace}(\boldsymbol{x}) \triangleq \boldsymbol{x}_1 + \boldsymbol{x}_2$	$\text{trace}(\boldsymbol{x}) \triangleq 2x_0$
Determinant	$\det(\boldsymbol{x}) \triangleq \boldsymbol{x}_1\boldsymbol{x}_2 - \lVert\bar{\boldsymbol{x}}\rVert^2$	$\det(\boldsymbol{x}) \triangleq x_0^2 - \lVert\hat{\boldsymbol{x}}\rVert^2$
Jordan multiplication	$\boldsymbol{x} \circ \boldsymbol{y} \triangleq \begin{bmatrix} x_1y_1 + \bar{\boldsymbol{x}}^\top\bar{\boldsymbol{y}} \\ x_2y_2 + \bar{\boldsymbol{x}}^\top\bar{\boldsymbol{y}} \\ \frac{1}{2}(y_1+y_2)\bar{\boldsymbol{x}} + \frac{1}{2}(x_1+x_2)\bar{\boldsymbol{y}} \end{bmatrix}$	$\boldsymbol{x} \circ \boldsymbol{y} \triangleq \begin{bmatrix} x_0y_0 + \hat{\boldsymbol{x}}^\top\hat{\boldsymbol{y}} \\ x_0\hat{\boldsymbol{y}} + y_0\hat{\boldsymbol{x}} \end{bmatrix}$
Inverse	$\boldsymbol{x}^{-1} \triangleq \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_2 \\ x_1 \\ -\bar{\boldsymbol{x}} \end{bmatrix} = \frac{1}{\det(\boldsymbol{x})}H^{-1}R\boldsymbol{x}$	$\boldsymbol{x}^{-1} \triangleq \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_0 \\ -\hat{\boldsymbol{x}} \end{bmatrix} = \frac{1}{\det(\boldsymbol{x})}R\boldsymbol{x}$
Square	$\boldsymbol{x}^2 \triangleq \begin{bmatrix} x_1^2 + \lVert\bar{\boldsymbol{x}}\rVert^2 \\ x_2^2 + \lVert\bar{\boldsymbol{x}}\rVert^2 \\ (x_1+x_2)\bar{\boldsymbol{x}} \end{bmatrix}$	$\boldsymbol{x}^2 \triangleq \begin{bmatrix} x_0^2 + \lVert\hat{\boldsymbol{x}}\rVert^2 \\ 2x_0\hat{\boldsymbol{x}} \end{bmatrix}$
Linear representation	$\mathrm{L}(\boldsymbol{x}) \triangleq \begin{bmatrix} x_1 & 0 & \bar{\boldsymbol{x}}^\top \\ 0 & x_2 & \bar{\boldsymbol{x}}^\top \\ \frac{\bar{\boldsymbol{x}}}{2} & \frac{\bar{\boldsymbol{x}}}{2} & \frac{(x_1+x_2)}{2}I_{n-2} \end{bmatrix}$	$\mathrm{L}(\boldsymbol{x}) \triangleq \begin{bmatrix} x_0 & \hat{\boldsymbol{x}}^\top \\ \hat{\boldsymbol{x}} & x_0 I_{n-1} \end{bmatrix}$
Quadratic representation	$\mathrm{Q}_x \triangleq \begin{bmatrix} x_1^2 & \lVert\bar{\boldsymbol{x}}\rVert^2 & 2x_1\bar{\boldsymbol{x}}^\top \\ \lVert\bar{\boldsymbol{x}}\rVert^2 & x_2^2 & 2x_2\bar{\boldsymbol{x}}^\top \\ x_1\bar{\boldsymbol{x}} & x_2\bar{\boldsymbol{x}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \det(\boldsymbol{x})I_{n-2} \end{bmatrix} = 2\boldsymbol{x}\boldsymbol{x}^\top H - \det(\boldsymbol{x})H^{-1}R$	$\mathrm{Q}_x \triangleq \begin{bmatrix} \lVert\hat{\boldsymbol{x}}\rVert^2 & 2x_0\hat{\boldsymbol{x}}^\top \\ 2x_0\hat{\boldsymbol{x}} & 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top + \det(\boldsymbol{x})I_{n-2} \end{bmatrix} = 2\boldsymbol{x}\boldsymbol{x}^\top - \det(\boldsymbol{x})R$
Logarithmic barrier	$b(\boldsymbol{x}) \triangleq -\ln\left(x_1x_2 - \lVert\bar{\boldsymbol{x}}\rVert^2\right)$	$b(\boldsymbol{x}) \triangleq -\ln\left(x_0^2 - \lVert\hat{\boldsymbol{x}}\rVert^2\right)$
Barrier gradient	$\nabla_x b(\boldsymbol{x}) = -2H\boldsymbol{x}^{-1}$	$\nabla_x b(\boldsymbol{x}) = -2\boldsymbol{x}^{-1}$
Barrier Hessian	$\nabla^2_{xx} b(\boldsymbol{x}) = 2HQ_{x^{-1}}$	$\nabla^2_{xx} b(\boldsymbol{x}) = 2Q_{x^{-1}}$

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