MGPROX: A NONSMOOTH MULTIGRID PROXIMAL GRADIENT METHOD WITH ADAPTIVE RESTRICTION FOR STRONGLY CONVEX OPTIMIZATION *

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Abstract. We study the combination of proximal gradient descent with multigrid for solving a class of possibly nonsmooth strongly convex optimization problems. We propose a multigrid proximal gradient method called MG-Prox, which accelerates the proximal gradient method by multigrid, based on using hierarchical information of the optimization problem. MGProx applies a newly introduced adaptive restriction operator to simplify the Minkowski sum of subdifferentials of the nondifferentiable objective function across different levels. We provide a theoretical characterization of MGProx. First we show that the MGProx update operator exhibits a fixed-point property. Next, we show that the coarse correction is a descent direction for the fine variable of the original fine level problem in the general nonsmooth case. Lastly, under some assumptions we provide the convergence rate for the algorithm. In the numerical tests on the Elastic Obstacle Problem, which is an example of nonsmooth convex optimization problem where multigrid method can be applied, we show that MGProx has a faster convergence speed than competing methods.

Key words. multigrid, restriction, proximal gradient, subdifferential, convex optimization, obstacle problem

MSC codes. 49J52, 49M37, 65K05, 65N55, 90C25, 90C30, 90C90

1. Introduction. We study the combination of two iterative algorithms: proximal gradient descent and multigrid, to solve the following class of optimization problems

$$(1.1) \qquad \underset{x}{\mathrm{argmin}} \ F_0(x) := f_0(x) + g_0(x).$$

(Assumption) We assume $f_0 : \mathbb{R}^n \to \mathbb{R}$ is L_0-smooth and μ_0-strongly convex, and $g_0 : \mathbb{R}^n \to \mathbb{R}$ is proper, possibly nonsmooth, convex, and separable. Regarding the non-smoothness (nondifferentiability) of g_0, we further assume a single point of non-differentiability for g_0. We recall that a function $\zeta(x) : \mathbb{R}^n \to \mathbb{R}$ is separable if $\zeta(x) = \sum_i \zeta_i(x_i)$, and is μ-strongly convex (and thus coercive) with $\mu > 0$ if $\zeta(x) - \frac{\mu}{2}\|x\|_2^2$ is convex; and lastly is L-smooth if ζ is $C_L^{1,1}$ and $\nabla\zeta$ is L-Lipschitz; i.e., for all x, y in $\mathbb{R}^n$, $\nabla\zeta(x)$ exists and

$$(1.2) \qquad \zeta(y) \leq \zeta(x) + \langle \nabla\zeta(x), y - x \rangle + \frac{L}{2}\|y - x\|_2^2.$$

Modern models are nonsmooth. Advancements in nonsmooth (i.e., nondifferentiable) optimization since the 60s [31] enable the use of nonsmooth g_0 in (1.1). The standard textbooks are [42, 46, 3, 4]. Then (1.1) captures many models in machine learning [10, 39], where f_0 is a data fitting term and g_0 models the constraint(s) and/or regularization(s) of the application. A popular tool for solving (1.1) is the proximal gradient method [41, 12], to be reviewed in subsection 1.3.

Classical problems in scientific computing are smooth. Setting $g \equiv 0$ in (1.1) gives

$$(1.3) \qquad \min_{x \in \mathbb{R}^n} \ f_0(x),$$

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in which this problem class of smooth strongly convex optimization subsumes many problems in scientific computing. If problem (1.3) comes from the discretization of certain classes of partial differential equation (PDE) problems, multigrid methods [11, 6, 18, 19, 32], to be reviewed in subsection 1.1, are among the fastest known method for solving (1.3).

This work: bridging smoothness and nonsmoothness. Multigrid and nonsmooth optimization are two communities that seldom interact. In this work we link the two fields and develop a method that can handle nonsmooth problems while enjoying the fast convergence from multigrid. We propose `MGProx` that accelerates the proximal-gradient method by multigrid to solve Problem (1.1). Below we review multigrid and the proximal gradient method.

1.1. Classical multigrid and notation. Multigrid dates back to the 1960s with works by Fedorenko [11] on solving the Poisson equation and was then further developed by Brandt [6] and Hackbusch [18]. There are many multigrid frameworks; in this work we focus on `MGOPT`: a full approximation scheme [6] nonlinear multigrid method which was applied and extended to optimization problems by Nash [32]. `MGOPT` speeds up the convergence of an iterative algorithm (called smoothing or relaxation) by using a hierarchy of coarse discretizations of f_0: it first constructs a series of coarse auxiliary problems of the form

$$(1.4) \qquad \min_{x_\ell} f_\ell(x_\ell) - \langle \tau_{\ell-1\to\ell}, x_\ell \rangle, \qquad \ell \in \{1, 2, \dots, L\},$$

where $\tau_{\ell-1\to\ell}$ carries information from level $\ell-1$ to level ℓ, and f_ℓ, x_ℓ denote the function f and the variable x, at the level ℓ, respectively. `MGOPT` then makes use of the solution of (1.4) to solve (1.3). The convergence of the overall algorithm is sped up by the correction from the coarse levels and by the fact that the coarse problems are designed to be "less expensive" to solve than the given ones.

Notation. The symbol x_0 (or x) is called the fine variable. The symbol x_ℓ in (1.4) with $\ell \geq 1$ is called coarse variable. The subscript $\ell \in \{0, 1, \dots, L\}$ denotes the level. A larger ℓ means a coarser level with lower resolution (fewer variables). For the remainder of the paper, L without a subscript stands for the number of levels, whereas L_ℓ denotes the smoothness parameter for the level-ℓ problem. At a level ℓ, the coarse version of the vector $x_\ell \in \mathbb{R}^{n_\ell}$ is $x_{\ell+1} = \overline{\mathcal{R}}(x_\ell) := \overline{R}x_\ell$ where $\overline{R} \in \mathbb{R}^{n_{\ell+1}\times n_\ell}$ with $n_{\ell+1} \leq n_\ell$ is called a restriction matrix. Similarly, given $x_{\ell+1} \in \mathbb{R}^{n_{\ell+1}}$ and a prolongation matrix $\overline{P} \in \mathbb{R}^{n_\ell \times n_{\ell+1}}$, we obtain the level-$\ell$ version of $x_{\ell+1}$ as $x_\ell = \overline{\mathcal{P}}(x_{\ell+1}) := \overline{P}x_{\ell+1}$. We let $\overline{P} = c\overline{R}^\top$ given scaling factor $c > 0$ and $\overline{R}$. In multigrid, choosing $(\overline{R}, \overline{P})$ depends on the application. In this work, for the applications we consider commonly chosen $(\overline{R}, \overline{P})$, see Section 5.2.1 for our choice of $(\overline{R}, \overline{P})$ used in the experiment and see [8] for an introduction on how to select $(\overline{R}, \overline{P})$.

1.2. MGOPT. Let x_ℓ^k be the level-ℓ variable at iteration k. Algorithm 1.1 shows a 2-level ($\ell \in \{0, 1\}$) `MGOPT` [32] for solving (1.3), with the steps in the algorithm explained as follows:
- (i): $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ denotes an update iteration called pre-smoothing. In this work we focus on σ being the proximal gradient operator.
- (ii): the restriction step.
- (iii): the vector $\tau_{0\to1}^{k+1}$ carries the information at level $\ell = 0$ to level $\ell = 1$.
- (iv): the coarse problem (1.4) is a "smaller version" of the original fine problem. The function $f_1 = \overline{\mathcal{R}}(f_0)$ is the coarsening of f_0 and the linear term $\langle \tau_{0\to1}^{k+1}, \xi \rangle$ links the coarse variable with the τ-correction information from the fine variable.
- (v): the updated coarse variable x_1^{k+1} is used to update the fine variable y_0^{k+1}.
- (vi): this step is the same as (i).

In the algorithm, $\alpha > 0$ is a stepsize. The τ-correction is designed in a way that the iteration has a fixed-point that corresponds to a solution.

Algorithm 1.1 2-level `MGOPT` [32] for an approximate solution of (1.3)

Initialize x_0^1, $\overline{R}$ and $\overline{P}$
for $k = 1, 2, \ldots$ **do**
  (i) $y_0^{k+1} = \sigma(x_0^k)$ pre-smoothing
  (ii) $y_1^{k+1} = \overline{R}y_0^{k+1}$ coarse variable
  (iii) $\tau_{0\to1}^{k+1} = \nabla f_1(y_1^{k+1}) - \overline{R}\nabla f_0(y_0^{k+1})$ create the tau vector
  (iv) $x_1^{k+1} = \underset{\xi}{\mathrm{argmin}}\ f_1(\xi) - \langle \tau_{0\to1}^{k+1}, \xi \rangle$ solve the coarse problem (exactly)
  (v) $z_0^{k+1} = y_0^{k+1} + \alpha\overline{P}(x_1^{k+1} - y_1^{k+1})$ coarse correction
  (vi) $x_0^{k+1} = \sigma(z_0^{k+1})$ post-smoothing
end for

Remark 1.1 (`MGOPT` has no theoretical convergence guarantee). The proof of [32, Theorem 1] on the convergence of `MGOPT` requires additional assumptions. In short the proof states the following: on solving (1.3) with an iterative algorithm $x^{k+1} := \sigma(x^k)$ where the update map $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be converging from any starting point x^1, now suppose $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is some other operator with the descending property that $f_0(\rho(x)) \le f_0(x)$. Then [32, Theorem 1] claimed that an algorithm consisting of interlacing σ with ρ repeatedly is also convergent. This is generally not true without further assumptions. Here is a counterexample for [32, Theorem 1]. Consider minimizing a scalar function $f(x) = x^2 \exp(-x^2)$.
- This f has a unique global minimum at $x = 0$, two global maxima at $x = \pm 1$.
- We decrease f by ρ being the gradient descent step.
- f is differentiable, and its slope is $f'(x) = 2\exp(-x^2)x(1 - x^2)$ with a Lipschitz constant about 0.58, thus we can pick $\alpha = 1 < 1/0.58$ for the gradient stepsize.
- If we initialize at $x^0 = 2$, we have $x^1 = \rho(x^0) = 2 + 12e^{-4}$.
- Take the operator $\sigma : x \mapsto \beta x$ with $\beta = 1/(1+6e^{-4}) = x^0/x^1$. For any x, the sequence $x, \sigma(x), \sigma(\sigma(x)), \ldots$ converges to $x^* = 0$.

Now, if we interlace σ and ρ, then the sequence $\{x^k\}$ does not converge, it alternates between x^0 and x^1 indefinitely: $x^0 = 2 \xrightarrow{\rho} x^1 = 2 + 12e^{-4} \xrightarrow{\sigma} x^0 = 2 \xrightarrow{\rho} x^1 = 2 + 12e^{-4}$. Thus, the convergence analysis of `MGOPT` is incomplete, and as a side product, the method we propose establishes the convergence of `MGOPT` as a special case (if gradient descent is used as the update step and our other assumptions hold), to be discussed in the contribution section. We remark that the issue of non-convergence of `MGOPT` has also been addressed using both line-search [50] and trust region methods [17].

1.3. Proximal gradient method. Nowadays subgradient [42] and proximal operator [31] are standard tools for designing first-order algorithms to solve nonsmooth optimization problems [10, 39, 4], especially for large-scale optimization where computing higher-order derivatives (e.g. the Hessian) is not feasible. Here we give a quick review of the proximal operator and the proximal gradient operator. We review subgradients in subsection 2.2.

Rooted in the concept of Moreau's envelope [31], the proximal gradient method was first introduced in the 1980s in [12, Eq. (4)] as a generalization of the proximal point method [43]. Under the abstraction of monotone operators, the proximal gradient method is understood as a forward-backward algorithm [41], and it was later popularized by [10] as the proximal forward-backward splitting. Nowadays proximal gradient method is ubiquitous in machine learning [39].

The proximal gradient method solves problems of the form (1.1) as follows. Starting from an initial guess x^1, the method updates the variable by a gradient descent step (with a

stepsize $\alpha > 0$) followed by a proximal step associated with g_0:

$$(1.5a) \qquad x^{k+1} = \text{prox}_{\alpha g_0}\big(x^k - \alpha \nabla f_0(x^k)\big),$$

$$(1.5b) \qquad \text{prox}_{\alpha g_0}(x) := \underset{u}{\text{argmin}} \left\{ \alpha g_0(u) + \frac{1}{2}\|u - x\|_2^2 \right\} = \underset{u}{\text{argmin}} \left\{ g_0(u) + \frac{1}{2\alpha}\|u - x\|_2^2 \right\}.$$

If f_0 is L_0-smooth (1.2), we can set stepsize α in (1.5a) as $\alpha \in (0, \frac{2}{L_0})$ because such a stepsize brings strict functional decrease, and convergence to critical points [4]. The proximal operator (1.5b) itself is also an optimization problem, and in practice many commonly used g_0 are "proximable" in that (1.5b) has an efficiently computable closed-form solution. The proximal gradient method has many useful properties. To keep the introduction short, we introduce these properties later when needed.

1.4. Contributions. In this work, our contributions are:
1. We propose `MGProx` (multigrid proximal gradient method) to solve (1.1). It generalizes `MGOPT` on smooth problems to nonsmooth problems using proximal gradient as the smoothing method. A key ingredient in `MGProx` is a newly introduced adaptive restriction operator in the multigrid process to handle the Minkowski sum of subdifferentials. The key idea is about collapsing a set-valued vector into a singleton vector to ease computation, more to be explained in section 2.
2. We provide theoretical results for 2-level `MGProx`: we show that
   - `MGProx` exhibits a fixed-point property, see Theorem 2.4;
   - the coarse correction update (in step (v) of Algorithm 2.1) is a descent direction for the fine variable of the original fine level problem, see the subdifferential obtuse angle condition in Theorem 2.5 and Lemma 2.7 for the existence of a coarse correction stepsize that provides a descent condition;
   - the sequence $\{f(x_0^k)\}_{k \in \mathbb{N}}$ at the finest level converges to the optimal value with rate $1/k$ and $(1 - \frac{\mu_0}{L_0})^k$, see Theorems 2.13 and 2.18; this result also establishes the convergence of `MGOPT` (for σ being the gradient update) in the convex case (see remark 1.1).
   - if we combine `MGProx` with Nesterov's acceleration, the sequence $\{f(x_0^k)\}_{k \in \mathbb{N}}$ at the finest level converges to the optimal value with rate $1/k^2$, see Theorem 3.1.
3. On the elastic obstacle problem, we show that multigrid accelerates the proximal gradient method; we show that `MGProx` runs faster than other methods. See section 5.

1.5. Literature review. The idea of multigrid is natural when handling large-scale elliptic PDE problems.

1.5.1. Early works. Early multigrid methods for non-smooth problems like (1.1) pertain to the case of constrained optimization problems where g_0 is an indicator function on the feasible set. For example, [7] and [30] develop multigrid methods for a symmetric positive definite (SPD) quadratic optimization problem with a bound constraint, which is equivalent to a linear complementarity problem. This applies, for example, to linear Elastic Obstacle Problems where g_0 is a box indicator function that models non-penetration constraints. In [20] this is extended to more general constrained nonlinear variational problems with SPD Fréchet derivatives, and to their associated nonlinear variational inequalities. Later [14] developed a Newton-MG (see below) method for an SPD quadratic optimization problem with more general but separable nonsmooth g_0. This is extended in [15] to a nonlinear objective function with nonsmooth g_0.

1.5.2. Two families of multigrid. We emphasize that there are at least two different approaches to perform multigrid in optimization. The 2-level MGOPT algorithm (Algorithm

1.1) is an example of a *full approximation scheme* (FAS) multigrid method for nonlinear problems. The FAS approach, which was first described in [6], adds a τ-correction term to the coarse nonlinear problem to ensure that the multigrid cycle satisfies a fixed-point property. And because of the τ, FAS is also called tau-correction method.

There is an alternative multigrid approach for solving nonlinear problems, which is the so-called Newton-multigrid method (Newton-MG, [8, Ch. 6]), where the fine-level problem is first linearized using Newton's method and the linear systems in each Newton iteration are solved approximately using a linear multigrid method. In other words, Newton-MG applies multigrid on solving the linear system $\nabla^2 f(x)d = \nabla f(x)$. Newton-MG includes the works [26, 14, 16, 25, 15].

In the context of optimization problems, nonlinear multigrid methods can be devised to either work directly on the optimization problem and coarse versions of the optimization problem (as `MGOPT` does), or they can be designed to work on the fine-level optimality conditions and coarse versions of them.

1.5.3. How our approach differs.
Our method is a FAS approach like [7, 20, 30] but our approach applies to general g_0 functions that go beyond indicator functions and include nonsmooth regularizations. While [7, 30] deal with linear problems, our approach applies to general nonlinear f_0. In contrast to [7, 20], we don't use injection for the restriction operation, which often leads to slow multigrid convergence, but instead we use an adaptive restriction and interpolation mechanism that precludes coarse-grid updates to active points.

Our adaptive restriction and interpolation mechanism is similar to the truncation process used in [14, 15], but our approach uses a FAS framework while [14, 15] use Newton-MG, and, most important we provide a convergence proof with convergence rates $1/k$, $(1 - \mu_0/L_0)^k$ and $1/k^2$, while [15] has no result on convergence rate. Furthermore, Newton-MG requires the computation of 2nd-order information (the Hessian), while MGProx is a 1st-order method.

To sum up, our approach **is a first-order method** that avoids computing second derivative, and the method **is a FAS** that does not require solving the equations $\nabla^2 f(x)d = \nabla f(x)$. While existing multigrid methods in optimization are problem specific, our approach is **general for a class of non-smooth functions**.

1.5.4. Multigrid outside PDEs.
Multigrid in image processing. Besides PDEs, multigrid was used in the 1990s in image processing for solving problems with a nondifferentiable total variation semi-norm in image recovery (e.g., [49, 9]). Note that these works bypassed the non-smoothness by *smoothing* the total variation term, making them technically only solving (1.3) but not (1.1).

Multigrid in machine learning. In the 2010s multigrid started to appear in machine learning, e.g., ℓ_1-regularized least squares [48] and Nonnegative Matrix Factorization [13]. We remark that these works are not true multigrid method as there is no τ in the schemes, nor is the information of the fine variable carried to the coarse variable when solving the problem.

Recent work. Recently [40] proposed a multilevel proximal gradient method with a FAS structure, however it bypassed the technically challenging part of nonsmoothness by using smoothing, making it similar to [49, 9] in that they are only solving (1.3) but not (1.1).

The table below summarizes the comparison. In the table, "1st-order" means the method discussed in the paper is a 1st-order method,

Work	1st-order	FAS	convergence theory	general nonsmooth g_0
[26]	no	no	yes but no rate	no, box constraints only
MGOPT [32]	yes and no	yes	no (Remark1.1)	no
[14]	no	no	no	yes
[16]	no	no	no	no, box constraints only
[25]	yes	yes	yes but no rate	no, box constraints only
[40]	yes	yes	yes but smoothing	yes
[15]	no	no	yes but no rate	yes
This work	yes	yes	yes with rates	yes

1.6. Organization. In section 2 we present a 2-level `MGProx` and discuss its theoretical properties. Then we present an accelerated `MGProx` in section 3 and a multi-level `MGProx` in section 4. In section 5 we demonstrate the performance of `MGProx` compared with other methods. We conclude the paper in section 6.

2. A two-level multigrid proximal gradient method. In subsection 2.1-2.4, we review subgradients for nonsmooth functions, discuss their interaction with restriction (the coarsening operator), introduce the notion of adaptive restriction, and define the τ vector that carries the cross-level information. We introduce a 2-level `MGProx` method in subsection 2.5 and we provide theoretical results about the algorithm: fixed-point property (Theorem 2.4), descent property (Theorem 2.5), existence of coarse correction stepsize (Lemma 2.7) and convergence rates (Theorems 2.13 and 2.18). In subsection 2.5.7 we discuss further details of τ.

2.1. Functions at different levels. Following subsection 1.1, we use $f_\ell : \mathbb{R}^{n_\ell} \to \mathbb{R}$ to denote functions at different coarse levels. In this section we will focus on $\ell \in \{0, 1\}$ but we remark that all the notations and definitions are generalized to $\ell \in \{0, 1, \ldots, L\}$ in Section 4. We denote the restriction of the fine objective function F_0 in (1.1) as $F_1 := \mathcal{R}(F_0) = \mathcal{R}(f_0 + g_0)$, where $\mathcal{R}$ is defined below.

DEFINITION 2.1 (Restriction). *At a level $\ell \in \mathbb{N}$, given a function $f_\ell : \mathbb{R}^{n_\ell} \to \mathbb{R}$, the restriction $\mathcal{R}$ of f_ℓ, denoted as $f_{\ell+1} := \mathcal{R}(f_\ell)$, is defined as $f_{\ell+1}(x_{\ell+1}) := f_\ell(Rx_\ell)$, where $R : \mathbb{R}^{n_\ell} \to \mathbb{R}^{n_{\ell+1}}$ is a restriction matrix . We also define the associated prolongation matrix $P : \mathbb{R}^{n_{\ell+1}} \to \mathbb{R}^{n_\ell}$ as $P = cR^\top$ where $c > 0$ is a predefined constant.*

Adaptive restriction and non-adaptive restriction. We recall that a contribution of this work is the introduction of the adaptive restriction, to be discussed in subsection 2.4 (see Definition 2.3). To differentiate the classical non-adaptive restriction (and the associated prolongation) from the adaptive version, we denote the non-adaptive restriction by $\overline{\mathcal{R}}, \overline{R}, \overline{\mathcal{P}}, \overline{P}$, and denote the adaptive one by $\mathcal{R}, R, \mathcal{P}, P$. We remark that Definition 2.1 can be used for both versions of restriction and prolongation. We give an example of $\overline{R}, \overline{P}$ in section 5.

2.2. Review of subdifferential of nonsmooth functions. The subdifferential [42] is a standard framework used in convex analysis to deal with nondifferentiable functions. A convex function $g(x) : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called nonsmooth if it is not differentiable for some x in $\mathbb{R}^n$. A point $q \in \mathbb{R}^n$ is called a subgradient of g at x if for all $y \in \mathbb{R}^n$ the inequality $g(y) \geq g(x) + \langle q, y - x \rangle$ holds. The subdifferential of g at a point x is defined as the set of all subgradients of g at x, i.e.,

$$(2.1) \qquad \partial g(x) := \left\{ q \in \mathbb{R}^n \;\middle|\; g(y) \geq g(x) + \langle q, y - x \rangle \; \forall y \in \mathbb{R}^n \right\} \subset \mathbb{R}^n,$$

so $\partial g(x)$ is generally set-valued. If g is differentiable at x, then $\nabla g(x)$ exists and the set $\partial g(x)$ reduces to the singleton $\{\nabla g(x)\}$.

Subdifferential sum rule (Moreau–Rockafellar theorem). Let $\oplus$ denote the Minkowski sum. Since subdifferentials are generally set-valued, hence for two functions f, g, generally $\partial(f + g) \neq \partial f \oplus \partial g$ but $\partial(f + g) \supset \partial f \oplus \partial g$. The sum rule $\partial(f + g) = \partial f \oplus \partial g$ holds if f, g satisfy a qualification condition (e.g. [4, Theorem 3.36]): the relative interior of the domain of f has a non-empty intersection with the relative interior of domain of g, i.e.,

$$(2.2) \quad \mathrm{ri}(\mathrm{dom}\, f) \cap \mathrm{ri}(\mathrm{dom}\, g) \neq \varnothing \implies \partial\big(f(x) + g(x)\big) = \partial f(x) \oplus \partial g(x), \quad \forall x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g.$$

The right-hand side (RHS) of (2.2), which is the subdifferential sum rule, is known as the Moreau–Rockafellar theorem [27]. We now discuss the fact that the functions f_ℓ, g_ℓ for all levels ℓ in this work satisfy the Moreau–Rockafellar theorem.

- At level $\ell = 0$, we have the left-hand side (LHS) of (2.2) for f_0, g_0 by assumption.
- At levels $\ell > 0$, by Definition 2.1, the LHS of (2.2) holds for the coarse functions. To see this, take the coarse function as composition of the fine function with a linear map, and recognize the fact that the domain of a function is preserved under composition with a linear map. To be explicit, we have

$$\mathrm{ri}\big(\mathrm{dom}\, f_1\big) \cap \mathrm{ri}\big(\mathrm{dom}\, g_1\big) \;=\; \mathrm{ri}\big(\underbrace{\mathrm{dom}\,(f_0 \circ R)}_{\text{whole } \mathbb{R}^{n_1}}\big) \cap \mathrm{ri}\big(\mathrm{dom}\,(g_0 \circ R)\big) \;\neq\; \varnothing.$$

To sum up, in this work (2.2) holds for all levels ℓ:

$$(2.3) \qquad \partial F_\ell(x_\ell) := \partial\big(f_\ell(x_\ell) + g_\ell(x_\ell)\big) = \partial f_\ell(x_\ell) \oplus \partial g_\ell(x_\ell) = \nabla f_\ell(x_\ell) + \partial g_\ell(x_\ell),$$

where $+$ is used instead of $\oplus$ in $\nabla f_\ell(x_\ell) + \partial g_\ell(x_\ell)$ because $\nabla f_\ell(x_\ell)$ is a singleton.

2.3. Convexity and subdifferential of coarse function. From Definition 2.1, the coarse function can be written as $F_{\ell+1} := F_\ell \circ R$, meaning that we can see the restriction process as the fine function taking composition with the linear map R. Such composition view point gives us a series of useful properties for this work. First, we have a closed-form expression for the subdifferential of the coarse function in terms of the the subdifferential of the fine function. I.e., by [4, Theorem 3.43], we have that

$$(2.4) \qquad \partial F_{\ell+1}(x) = \partial(F_\ell \circ R)(x) = R^\top \partial F_\ell(Rx).$$

Then, by the fact that convexity is preserved under linear map, we have that "restriction preserves convexity". In other words, $f_{\ell+1}$ is convex if f_ℓ is convex. Furthermore, if R is full-rank (which is the case in this paper), restriction is submultiplicative on the modulus of convexity, as illustrated in the following lemma.

LEMMA 2.2 (Composition with full-rank matrix preserves convexity). *Given a function $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ that is μ-strongly convex and a rank-n matrix $R \in \mathbb{R}^{m \times n}$, the function $F \circ R$ is $\mu\sigma_n^2$-strongly convex, where σ_n is the nth singular value of R.*

Proof. F is strongly convex so ∂F is strongly monotone [3]: for all $x, y \in \mathrm{dom}\, F$,

$$(2.5) \qquad \langle x - y,\ \partial F(x) - \partial F(y) \rangle \;\geq\; \mu \|x - y\|_2^2.$$

Now we show $\partial(F \circ R)$ is also strongly monotone. For all x, y in $\mathrm{dom}\,(F \circ R)$, we have

$$
\begin{aligned}
\langle x - y,\ \partial(F \circ R)(x) - \partial(F \circ R)(y) \rangle \;&\overset{(2.4)}{=}\; \langle x - y,\ R^\top \partial F(Rx) - R^\top \partial F(Ry) \rangle \\
&=\; \langle Rx - Ry,\ \partial F(Rx) - \partial F(Ry) \rangle \\
&\overset{(2.5)}{\geq}\; \mu \|Rx - Ry\|_2^2 \;\geq\; \mu\sigma_n^2 \|x - y\|_2^2,
\end{aligned}
$$

where $\sigma_n > 0$ is the nth singular value of the full-rank matrix R. Thus the subdifferential $\partial(F \circ R)$ is is $\mu\sigma_n^2$-strongly monotone, and thus the function $F \circ R$ is $\mu\sigma_n^2$-strongly convex. $\square$

Notation for sets. From now on, when we encounter an expression containing both set-valued vector(s) and singleton vector(s), we underline the set-valued term(s) for visual clarity.

2.4. Adaptive restriction and the τ vector. Since subdifferentials are set-valued, we define τ in MGProx as an element of a set. At a level ℓ, we define a set $\underline{\tau_{\ell \to \ell+1}} := \underline{\partial F_{\ell+1}(x_{\ell+1})} \oplus (-R)\underline{\partial F_\ell(x_\ell)}$, where $\ell \to \ell + 1$ specifies that τ connects level ℓ to level $\ell + 1$, and the matrix R here is an adaptive restriction operator that we will define soon. In MGProx we choose an element of $\underline{\tau_{\ell \to \ell+1}}$ as the tau vector. That is, at level $\ell = 0$,

$$\text{(2.6a)} \qquad \tau_{0 \to 1} \in \underline{\tau_{0 \to 1}} := \underline{\partial F_1(x_1)} \oplus (-R)\underline{\partial F_0(x_0)}$$

$$\text{(2.6b)} \qquad \overset{(2.3)}{=} \nabla f_1(x_1) - R\nabla f_0(x_0) + \underline{\partial g_1(x_1)} \oplus (-R)\underline{\partial g_0(x_0)}.$$

Note that $\tau_{0 \to 1}$ is a function of two points at two different levels. In (2.6a) $\underline{\tau_{0 \to 1}}$ is the Minkowski sum of two subdifferentials $\partial F_1(x_1)$ and $-R\partial F_0(x_0)$ which are generally set-valued. To obtain a tractable coarse-grid optimization problem (corresponding to line (iv) in Algorithm 1.1) we need to avoid complications coming from the Minkowski sum, and we do this by modifying the standard restriction (and prolongation) by zeroing out columns in $\overline{R}$ to form R such that the second subdifferential $R\partial g_0(x_0)$ in (2.6b) is a singleton vector. Similarly, we zero out the corresponding rows in $\overline{P}$ to form P for the coarse correction step, such that non-differentiable fine points are not corrected by the coarse grid. This zeroing out process is adapted to the current point x_0, so we call this R an adaptive restriction operator. In other words, the purpose of the adaptive restriction is to reduce a generally set-valued subdifferential $R\partial g_0(x_0)$ to a singleton. We denote the adaptive operator R corresponding to a point x as $R(x)$ and thereby the adaptive restriction of x is denoted as $R(x)x$. Sometimes we just write Rx if the meaning is clear from the context. Based on the above discussion, we now formally define adaptive restriction, and we give an example in section 5.

DEFINITION 2.3 (Adaptive restriction operator for separable g). *For a possibly nonsmooth function $g : \mathbb{R}^n \to \mathbb{R}$ that is separable, i.e., with $x = [x_1, x_2, \ldots, x_n]$, $g(x) = \sum g_i(x_i)$ where g_i is a function only of component x_i, given a full restriction operator $\overline{R}$ and a vector x, the adaptive restriction operator R with respect to a function g at x is defined by zeroing out the columns of $\overline{R}$ corresponding to the elements in ∂g that are set-valued.*

The coarse problem is nonsmooth and τ is an element of a set. Now it is clear that the subdifferential $R\partial g_0(x_0)$ in (2.6b) is a singleton. From the fact that the coarse problem is nonsmooth (where the function g_1 is nonsmooth), there are two consequences:
- it makes the coarse problems difficult to solve as well as the original problem. This makes our approach differ from works such as [40] where the coarse problems are replaced by a smooth approximation; and
- the first subdifferential $\partial g_1(x_1)$ in (2.6b) is possibly set-valued, thus the RHS of (2.6b) is generally set-valued and so is $\underline{\tau_{0 \to 1}}$, and we define $\tau_{0 \to 1}$ to be a member of the set $\underline{\tau_{0 \to 1}}$. We emphasize that in the algorithm to be discussed below we can pick any value for $\tau_{0 \to 1}$ in the set. We will explore the choice of τ after we have given a complete picture of MGProx.

Adaptive restriction differs from Kornhuber's basis truncation. In the PDE literature there is a multigrid method called Kornhuber's basis truncation [26], in which at first glance looks similar to MGProx. We remark that Kornhuber's basis truncation is designed only for box-constrained optimization. The truncation zeros out the basis of the optimization variable, while adaptive restriction zeros the subdifferential vector (see (2.6b)). Also, the truncation is only applied to the finest level [25, Section 2.2], while MGProx applies adaptive restriction applies to all the levels.

We are now ready to present `MGProx`. Here we present a 2-level `MGProx` method for illustration, and we move to a general multi-level version in section 4. For adaptive R, now all the Minkowski additions are trivial addition so we use + instead of $\oplus$.

2.5. A 2-level `MGProx` algorithm. Similar to the 2-level `MGOPT` method for solving Problem (1.3), we propose a 2-level `MGProx` method (Algorithm 2.1) that solves Problem (1.1) by utilizing a coarse problem defined as

$$(2.7) \qquad \underset{\xi \in \mathbb{R}^{n_1}}{\operatorname{argmin}} \left\{ F_1^\tau(\xi) := F_1(\xi) - \langle \tau_{0 \to 1}^{k+1}, \xi \rangle = f_1(\xi) + g_1(\xi) - \langle \tau_{0 \to 1}^{k+1}, \xi \rangle \right\}.$$

Algorithm 2.1 2-level `MGProx` for an approximate solution of (1.1)

Initialize x_0^1, R and P
for $k = 1, 2, \ldots$ **do**

   (i) $y_0^{k+1} = \operatorname{prox}_{\frac{1}{L_0} g_0}\left(x_0^k - \frac{1}{L_0}\nabla f(x_0^k)\right)$ level-0 proximal gradient step

   (ii) $y_1^{k+1} = R(y_0^{k+1})y_0^{k+1}$ construct the level-1 coarse variable

   (iii) $\tau_{0 \to 1}^{k+1} \in \ \overline{\partial F_1(y_1^{k+1})} - R(y_0^{k+1})\overline{\partial F_0(y_0^{k+1})}$ construct the tau vector

   (iv) $x_1^{k+1} = \underset{\xi}{\operatorname{argmin}}\left\{F_1^\tau(\xi) := F_1(\xi) - \langle \tau_{0 \to 1}^{k+1}, \xi \rangle \right\}$ solve the level-1 coarse problem

   (v) $z_0^{k+1} = y_0^{k+1} + \alpha P(x_1^{k+1} - y_1^{k+1})$ coarse correction

   (vi) $x_0^{k+1} = \operatorname{prox}_{\frac{1}{L_0} g_0}\left(z_0^{k+1} - \frac{1}{L_0}\nabla f(z_0^{k+1})\right)$ level-0 proximal gradient step

end for

Here are remarks for the steps in Algorithm 2.1.
- (i): we perform one or more proximal gradient iterations on the fine variable with a constant stepsize $1/L_0$, where L_0 is the Lipschitz constant of ∇f_0.
- (iii): we pick a value within the set to define τ; as we are now using adaptive R, we use + instead of $\oplus$ in the expression of τ.
- (v): $\alpha > 0$ is a stepsize; for its selection see subsection 2.5.3.
- The restriction for variable y_0^{k+1} and the restriction for the subdifferential $\partial F_0(y_0^{k+1})$ can be slightly different. On y_0^{k+1} the restriction is the full restriction, on $\overline{\partial F_0(y_0^{k+1})}$ is the adaptive one. The explanation is as follows. For the particular cases of g_0 such as ℓ_1, $\max\{\cdot, 0\}$ (element-wise maximum) and $\iota_{[0,\infty)}$ (indicator of nonnegative orthant), when we zero out column i of R, the corresponding entry of y_0^{k+1} is already 0. Note that this conclusion does not hold in general for other nonsmooth functions or when the non-differentiability occur at another point (say at $x = 1$). Generally in those cases we will need to specify which restriction matrix (the full R or the adaptive R) to be used to define P. However, to simplify the presentation we always assume the non-differentiability occurs at $x = 0$, by performing a translation to shift the non-differentiability to occur at $x = 0$.

2.5.1. Fixed-point property. Algorithm 2.1 exhibits the following fixed-point property.

THEOREM 2.4 (Fixed-point). *In Algorithm 2.1, if x_0^k solves (1.1), then we have the fixed-point properties $x_0^{k+1} = y_0^{k+1} = x_0^k$ and $x_1^{k+1} = y_1^{k+1}$.*

Proof. The fixed-point property of the proximal gradient operator [39, page 150] gives

$$(2.8) \qquad\qquad\qquad y_0^{k+1} \overset{\text{fixed-point}}{=} x_0^k \overset{\text{assumption}}{=} \operatorname{argmin} F_0(x).$$

As a result, the coarse variable satisfies $y_1^{k+1} := Ry_0^{k+1} \overset{(2.8)}{=} Rx_0^k$. The 1st-order optimality of $y_0^{k+1} \overset{(2.8)}{=}$ argmin F_0 gives $0 \in \partial F_0(y_0^{k+1})$. Multiplying by $-R$ (which reduces the set $\underline{\partial F_0(x_0^k)}$ to a singleton) gives

$$(2.9) \qquad\qquad 0 = -R\underline{\partial F_0(x_0^k)}.$$

Then adding $\partial F_1(y_1^{k+1})$ on both sides of (2.9) gives

$$(2.10a) \qquad\qquad \underline{\partial F_1(y_1^{k+1})} = \underline{\partial F_1(y_1^{k+1})} - R(x_0^k)\underline{\partial F_0(x_0^k)}$$

$$(2.10b) \qquad\qquad\qquad \overset{(2.6a)}{\ni} \tau_{0\to 1}^{k+1}$$

In (2.9), $-R\underline{\partial F_0(x_0^k)}$ is the zero vector, so the equality in (2.10a) holds since we are adding zero to a (non-empty) set. The inclusion (2.10b) follows from (2.6a) as the expression $\underline{\partial F_1(y_1^{k+1})} - R(x_0^k)\partial F_0(x_0^k)$ is the set $\tau_{0\to 1}^{k+1}$ by definition.

Now rearranging (2.10b) gives $0 \in \underline{\partial F_1(y_1^{k+1})} - \tau_{0\to 1}^{k+1}$, which is exactly the 1st-order optimality condition for the coarse problem $\underset{\xi}{\text{argmin}}\, F_1(\xi) - \langle \tau_{0\to 1}^{k+1}, \xi \rangle$. By strong convexity of $F_1(\xi) - \langle \tau_{0\to 1}^{k+1}, \xi \rangle$, the point y_1^{k+1} is the unique minimizer of the coarse problem, so $x_1^{k+1} = y_1^{k+1}$ by step (iv) of the algorithm and $x_0^{k+1} = y_0^{k+1} \overset{(2.8)}{=} x_0^k$ by steps (v) and (vi). □

Theorem 2.4 shows that at convergence, we have fixed-point $x_0^{k+1} = y_0^{k+1}$ at fine level and also $x_1^{k+1} = y_1^{k+1}$ at the coarse level. Next we show that when $x_1^{k+1} \neq y_1^{k+1}$, the objective function value sequence is converging.

2.5.2. Coarse correction descent: angle condition. In nonsmooth optimization, descent direction properties are drastically different from smooth optimization [37]. For example for the subgradient method, the classical angle condition no longer describes a useful set of search directions for the subgradient. In `MGProx` the coarse correction direction $P(x_1^{k+1} - y_1^{k+1})$ is a nonsmooth descent direction, and we will show that $P(x_1^{k+1} - y_1^{k+1})$ decreases the objective function value, based on the theorem below and Lemma 2.7.

THEOREM 2.5 (Angle condition of coarse correction). *If $P(x_1^{k+1} - y_1^{k+1}) \neq 0$, then*

$$(2.11) \qquad\qquad \langle \underline{\partial F_0(y_0^{k+1})}, P(x_1^{k+1} - y_1^{k+1}) \rangle < 0.$$

Before we prove the theorem we emphasize that (2.11) applies for any subgradient in the set $\underline{\partial F_0(y_0^{k+1})}$. Furthermore,

$$(2.11) \iff \langle P^\top \underline{\partial F_0(y_0^{k+1})}, x_1^{k+1} - y_1^{k+1} \rangle < 0 \overset{P^\top = cR}{\iff} c\langle R\underline{\partial F_0(y_0^{k+1})}, x_1^{k+1} - y_1^{k+1} \rangle < 0.$$

As $c > 0$, showing (2.11) is equivalent to showing

$$(2.12) \qquad\qquad \langle R\underline{\partial F_0(y_0^{k+1})}, x_1^{k+1} - y_1^{k+1} \rangle < 0,$$

where $R\underline{\partial F_0(y_0^{k+1})}$ is a singleton vector for all subgradients in $\underline{\partial F_0(y_0^{k+1})}$ due to the adaptive R.

Proof. By definition $\tau_{0\to 1}^{k+1} \overset{(2.6a)}{\in} \underline{\partial F_1(y_1^{k+1})} - R\underline{\partial F_0(y_0^{k+1})}$ and the fact that $R\underline{\partial F_0(y_0^{k+1})}$ is a singleton, we have $R\underline{\partial F_0(y_0^{k+1})} \in \underline{\partial F_1(y_1^{k+1})} - \tau_{0\to 1}^{k+1} \overset{(2.7)}{=} \underline{\partial F_1^\tau(y_1^{k+1})}$, showing that $R\underline{\partial F_0(y_0^{k+1})}$

is a subgradient of F_1^τ at y_1^{k+1}. For any subgradient in the subdifferential $\partial F_1^\tau(y_1^{k+1})$, we have the following which implies (2.12): $\langle \partial F_1^\tau(y_1^{k+1}), x_1^{k+1} - y_1^{k+1} \rangle < F_1^\tau(x_1^{k+1}) - F_1^\tau(y_1^{k+1}) < 0$, where the first strict inequality is due to F_1^τ being a strongly convex function (which implies strict convexity); the second inequality is by $x_1^{k+1} := \underset{\xi}{\arg\min}\, F_1^\tau(\xi)$ and the assumption that $x_1^{k+1} \neq y_1^{k+1}$. $\qquad\square$

Remark 2.6. Theorem 2.5 holds for convex (not strongly convex) f_0 by changing $<$ to $\leq$.

2.5.3. Existence of coarse correction stepsize α_k. Based on Theorem 2.5, we now show that there exists a stepsize $\alpha_k > 0$ such that

$$(2.13) \qquad F_0(z_0^{k+1}) := F_0(y_0^{k+1} + \alpha_k P(x_1^{k+1} - y_1^{k+1})) < F_0(y_0^{k+1}).$$

LEMMA 2.7 (Existence of stepsize). *There exists $\alpha_k > 0$ such that* (2.13) *is satisfied for* $P(x_1^{k+1} - y_1^{k+1}) \neq 0$.

To prove the lemma, we make use of a fact of the subdifferentials of finite convex functions [21, Def.1.1.4, p.165]: $\partial g(x)$ is a nonempty compact convex set $\mathcal{S} \subset \mathbb{R}^n$ whose support function $\underset{s}{\sup}\{\langle s, x \rangle \mid s \in \mathcal{S}\}$ is the directional derivative of g at x. By this, the subdifferential $\partial F_0(y_0^{k+1})$ is a compact convex set whose support function is the directional derivative of F_0 at y_0^{k+1}. We emphasize that here $F_0(y_0^{k+1})$ is finite, so we can make use of the result on directional derivative in [21, Def. 1.1.4, p.165], which only applies for finite convex functions, associated with the subdifferential.

Proof. We prove the lemma in 3 steps.
  1. (Half-space) The strict inequality in Theorem 2.5 means that $\partial F_0(y_0^{k+1})$ is strictly inside a half-space with normal vector $p = P(x_1^{k+1} - y_1^{k+1})$.
  2. (Strict separation) Being a compact convex set, $\partial F_0(y_0^{k+1}0)$ lying strictly on one side of the hyperplane must be a positive distance (say $\alpha_k > 0$) from that hyperplane.
  3. (Support and directional derivative) Evaluating the support function of $\partial F_0(y_0^{k+1})$, i.e., the directional derivative of F_0 at y_0^{k+1} in the direction p, we have (2.13). $\qquad\square$

Remark 2.8 (On the compactness of subdifferential). For a function ϕ, the set $\partial\phi$ is compact on int dom ϕ. Note that $\partial\phi$ is not compact for indicator functions at the boundary. So, for Lemma 2.7 to hold for function g_0, we assume dom $g_0 := \mathbb{R}^n$ with $g_0 : \mathbb{R}^n \to \mathbb{R}$ as in the **Assumption** in the introduction. Such an assumption is needed for the proof that a positive α exists satisfying the line-search condition. The impact of this assumption is that we are not allowing g_0 to be an indicator function, and thus reducing the scope of the applicability of the theory of `MGProx`. However,
  - Empirically, we have observed that `MGProx` works for g_0 being an indicator function (such as box constraints).
  - The convergence proof of `MGProx` does not require a positive coarse correction stepsize α (i.e., allowing $\alpha < 0$), so in principle we can relax the condition of $g_0 : \mathbb{R}^n \to \mathbb{R}$ to $g_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$.
  - It will be an interesting future work to generalize Lemma 2.7 for $g_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$. There is a different proof showing that a positive α exists in the case of certain indicator functions. For example, the following is a theorem. If H is a hyperplane and P is a polyhedral set (closed but possibly non-compact) lying strictly on one side of H, then there is a positive distance between H and P.

Now we see that Theorem 2.5 implies Lemma 2.7 which then implies the descent condition (2.13). Later in Theorem 2.13 and Theorem 2.18, by using (2.13) together with the sufficient descent property of proximal gradient (Lemma 2.9 below), we prove that the sequence $\{F_0(x_0^k)\}_{k \in \mathbb{N}}$ produced by Algorithm 2.1 converges to the optimal value $F^* := \min F$. Before that, in the next paragraph we first discuss about tuning the coarse correction stepsize.

LEMMA 2.9. *(Sufficient descent property of proximal gradient [4, Lemma 10.4]) Let L_0 be the Lipschitz constant of ∇f_0. For step (i) in Algorithm 2.1, we have*

$$(2.14) \quad F_0(y_0^{k+1}) \leq F_0(x_0^k) - \|G_0(x_0^k)\|_2^2/(2L_0), \quad G_0(x_0^k) = L_0[x_0^k - prox_{\frac{1}{L}g_0}(x_0^k - \nabla f_0(x_0^k)/L_0)].$$

$G_0(x_0^k)$ is called the proximal gradient map of F_0 at x_0^k. The inequality also holds for step (vi).

Remark 2.10 (Theorem 2.5 holds for inexact coarse update). Step (iv) in Algorithm 2.1 can be expensive. If we replace (iv) by an iteration of (coarse level) proximal gradient step, then by (2.14) we have $F_1^\tau(x_1^{k+1}) \leq F_1^\tau(y_1^{k+1}) - \|G_1^\tau(y_1^{k+1})\|_2^2/(2L_1)$. For $x_1^{k+1} \neq y_1^{k+1}$ we have $F_1^\tau(x_1^{k+1}) - F_1^\tau(y_1^{k+1}) < 0$. Thus the descent condition holds for an inexact coarse update.

2.5.4. Tuning the coarse correction stepsize α_k. First, exact line search is impractical: finding $\alpha_k := \underset{\alpha \geq 0}{\operatorname{argmin}} F_0(y_0^{k+1} + \alpha P(x_1^{k+1} - y_1^{k+1}))$ is generally expensive. Next, classical inexact line searches such as the Wolfe conditions, Armijo rule, Goldstein line search (e.g., see [36, Chapter 3]) cannot be used here as they were developed for smooth functions. While it is possible to develop nonsmooth version of these methods, such as a nonsmooth Armijio rule in tandem with backtracking on functions that satisfy the Kurdyka-Łojasiewicz inequality with other additional conditions in [37], this is out of the scope of this work. Precisely, consider the condition $F_0(y_0^{k+1} + \alpha P(x_1^{k+1} - y_1^{k+1})) \leq F_0(y_0^{k+1}) + c_1 \langle \partial F_0(y_0^{k+1}), \alpha P(x_1^{k+1} - y_1^{k+1}) \rangle$ where $c_1 \in (0, 1)$. Due to the strict inequality (2.11), the value c_1 is possibly fluctuating and there is no simple-and-efficient way to determine its value. Thus, for this paper, we use simple naive backtracking as shown in Algorithm 2.2, which just enforces (2.13) without any sufficient descent condition. While we acknowledge that the traditional wisdom in optimization tells that naive descent conditions such as (2.13) are generally not enough to obtain convergence to the optimal point, we note that `MGProx` is not solely using the coarse correction to update the variable; instead it is a chain of interlaced iterations of proximal gradient descent and coarse correction, and we will show next that the sufficient descent property of proximal gradient descent (2.14) alone provides enough descending power for the function value F_0 to convergence to the optimal value.

Algorithm 2.2 Naive line search

Set $\alpha > 0$ and select a tolerance $\epsilon > 0$ (e.g. 10^{-15})
while true **do**
    If $F_0(y_0^{k+1} + \alpha P(x_1^{k+1} - y_1^{k+1})) \leq F_0(y_0^{k+1})$ **then** return $z_0^{k+1} = y_0^{k+1} + \alpha P(x_1^{k+1} - y_1^{k+1})$.
    else if $\alpha > \epsilon$ **then** $\alpha = \alpha/2$.
        else return $z_0^{k+1} = y_0^{k+1}$.
end while

Remark 2.11 (Algorithm 2.2 has a fixed complexity). Suppose the algorithm runs N iterations, then with a tolerance $\epsilon = 10^{-15}$, the coarse stepsize $\alpha = 1/2^N > \epsilon$ gives $N = 50$. I.e., Algorithm 2.2 will run at most 50 iterations to search for α.

Remark 2.12 (Infinitesimal α may not be computable). Lemma 2.7 on the existence of $\alpha > 0$ does not exclude the possibility that α is too close to zero so that z_0^{k+1} cannot be distinguished from y_0^{k+1}. In this case Algorithm 2.2 simply gives $\alpha = 0$.

2.5.5. $O(1/k)$ **convergence rate.** Inequality (2.13) implies that in the worst case the coarse correction $P(x_1^{k+1} - y_1^{k+1})$ in the multigrid process is "doing nothing" on y_0^{k+1}, which occurs when $P(x_1^{k+1} - y_1^{k+1}) = 0$ or $x_1^{k+1} = y_1^{k+1}$. We now show that the descent inequality $F_0(x_0^{k+1}) \leq F_0(z_0^{k+1}) \leq F_0(y_0^{k+1})$ implies that the convergence rate of the sequence $\{F_0(x_0^k)\}_{k\in\mathbb{N}}$ for $\{x_0^k\}_{k\in\mathbb{N}}$ generated by `MGProx` (Algorithm 2.1) follows the convergence rate of the proximal gradient method, which is $O(1/k)$ [4]. In Theorem 2.13 we show that $\{F_0(x_0^k)\}_{k\in\mathbb{N}}$ converges to $F_0^* := \inf F_0(x)$ with such a classical rate.

THEOREM 2.13. *The sequence* $\{x_0^k\}_{k\in\mathbb{N}}$ *generated by* `MGProx` *(Algorithm 2.1) for solving Problem* (1.1) *satisfies* $F_0(x_0^{k+1}) - F_0^* \leq \max\{8\delta^2 L_0, F_0(x_0^1) - F_0^*\}/k$, *where* $F_0^* = F_0(x^*)$ *for* $x^* = argmin\, F_0$, *the point* $x_0^1 \in \mathbb{R}^n$ *is the initial guess,* L_0 *is the Lipschitz constant of* ∇f_0, *and* δ *is the diameter of the sublevel set* $\mathcal{L}_{\leq F_0(x_0^1)}$ *defined in Lemma 2.16.*

Note that we cannot invoke a standard theorem about the convergence of proximal gradient descent such as [4, Theorem 10.21], because we interlace proximal gradient steps with coarse corrections. Also, we note that all the functions and variables in this subsubsection are at level $\ell = 0$ so we omit the subscript. The constant L should be understood as the Lipschitz constant of $\nabla f(x)$. The proof is based on standard techniques in first-order methods. To make the proof more accessible, we divide the proof into four lemmas:
- Lemma 2.14: we derive a sufficient descent inequality for the `MGProx` iteration.
- Lemma 2.15: we derive a quadratic under-estimator of F.
- Lemma 2.16: we give an upper bound for $\|x^k - x^*\|_2$ and $\|y^{k+1} - x^*\|_2$ for all k.
- Lemma 2.17: we recall a convergence rate for a certain a monotonic sequence.

Using these lemmas, we follow the strategy used in [24] to prove Theorem 2.13.

LEMMA 2.14 (Sufficient descent of `MGProx` iteration). *For all iterations k, we have*

$$(2.15) \qquad F(x^{k+1}) - F^* \ \leq \ L(\|x^k - x^*\|_2^2 - \|y^{k+1} - x^*\|_2^2)/2.$$

We put the proof in the appendix. We name the inequality (2.15) sufficient descent because it resembles the sufficient descent property of the proximal gradient iteration (2.14). Also, by definition, $F(x^{k+1}) \geq F^*$, hence (2.15) implies $\|x^k - x^*\|_2^2 \geq \|y^{k+1} - x^*\|_2^2$.

The following lemma is similar to [5, Lemma 2.3] and [24, Lemma 3, Eq.(5.9)].

LEMMA 2.15 (A quadratic function). *For all x, we have*

$$(2.16) \qquad F(x) - F(x^{k+1}) \ \geq \ L\langle x^k - y^{k+1}, x - x^k\rangle + L\|y^{k+1} - x^k\|_2^2/2.$$

We put the proof in the appendix.

LEMMA 2.16 (Diameter of sublevel set). *At initial guess $x^1 \in \mathbb{R}^n$, define*

$$\mathcal{L}_{\leq F(x^1)} := \{x \in \mathbb{R}^n \mid F(x) \leq F(x^1)\}, \qquad \text{(sublevel set of x^1)}$$
$$\delta = diam\, \mathcal{L}_{\leq F(x^1)} := \sup\{\|x - y\|_2 \mid F(x) \leq F(x^1), F(y) \leq F(y^1)\}. \quad \text{(diameter of $\mathcal{L}_{\leq F(x^1)}$)}$$

Then for $x^* := argmin\, F(x)$, we have $\|x^k - x^*\|_2 \leq \delta$ and $\|y^k - x^*\|_2 \leq \delta$ for all k.

Proof. By definition $F(x^*) \leq F(x^1)$. By the descent property of the coarse correction and proximal gradient updates, we have $F(x^k) \leq F(x^1)$ and $F(y^k) \leq F(x^1)$ for all k. These results mean that x^k, y^{k+1} and x^* are inside $\mathcal{L}_{\leq F(x^1)}$, therefore both $\|x^k - x^*\|_2$ and $\|y^{k+1} - x^*\|_2$ are bounded above by δ. Lastly, F is strongly convex so $\mathcal{L}_{\leq F(x^1)}$ is bounded and $\delta < +\infty$. $\square$

LEMMA 2.17 (Monotone sequence). *For a nonnegative sequence $\{\omega_k\}_{k\in\mathbb{N}} \to \omega^*$ that is monotonically decreasing with $\omega_1 - \omega^* \leq 4\mu$ and $\omega_k - \omega_{k+1} \geq (\omega_{k+1} - \omega^*)^2/\mu$, it holds that $\omega_k - \omega^* \leq 4\mu/k$ for all k.*

Proof. By induction. See proof in [24, Lemma 4]. □

Now we are ready to prove Theorem 2.13.

Proof. Rearranging the sufficient descent inequality in Lemma 2.14 gives

$$
\begin{aligned}
F^* - F(x^{k+1}) &\geq L(\|y^{k+1} - x^*\|_2^2 - \|x^k - x^*\|_2^2)/2 \\
&= L(\|y^{k+1} - x^*\|_2 - \|x^k - x^*\|_2)(\|y^{k+1} - x^*\|_2 + \|x^k - x^*\|_2)/2 \\
&\geq -L\|x^k - y^{k+1}\|_2(\|x^k - x^*\|_2 + \|y^{k+1} - x^*\|_2)/2,
\end{aligned}
$$

where the last inequality is by the triangle inequality $\|y^{k+1} - x^*\|_2 + \|x^k - y^{k+1}\|_2 \geq \|x^* - x^k\|_2$. Rearranging the inequality gives

$$
(2.17) \qquad \|x^k - y^{k+1}\|_2 \;\geq\; \frac{-2}{L}\frac{F^* - F(x^{k+1})}{\|x^k - x^*\|_2 + \|y^{k+1} - x^*\|_2} \;=\; \frac{2}{L}\frac{F(x^{k+1}) - F^*}{\|x^k - x^*\|_2 + \|y^{k+1} - x^*\|_2}.
$$

Applying Lemma 2.16 to (2.17) gives

$$
(2.18) \qquad \|x^k - y^{k+1}\|_2 \geq (F(x^{k+1}) - F^*)/(\delta L).
$$

Note that (2.18) implies that if the fine sequence converges ($x^k = y^{k+1}$), then $F(x^{k+1}) = F^*$.

Applying Lemma 2.15 with $x = x^k$ gives $F(x^k) - F(x^{k+1}) \geq L\|y^{k+1} - x^k\|_2^2/2 \overset{(2.18)}{\geq} (F(x^{k+1}) - F^*)^2/(2\delta^2 L)$. This inequality shows that the sequence $\{\omega_k\}_{k\in\mathbb{N}}$ with $\omega_k := F(x^k)$ satisfies the condition $\omega_k - \omega_{k+1} \geq (\omega_k - \omega^*)^2/\mu$ in Lemma 2.17. To complete the proof, applying Lemma 2.17 to the monotonically decreasing sequence $\{F(x^k)\}_{k\in\mathbb{N}}$ with $\mu = 2\delta^2 L$ gives $F_0(x_0^{k+1}) - F_0^* \leq \max\{8\delta^2 L_0, F_0(x_0^1) - F_0^*\}/k$, where we put back the subscript 0. □

Theorem 2.13 shows that $\{F_0(x_0^k)\}_{k\in\mathbb{N}}$ for solving Problem (1.1) satisfies a sublinear convergence bound of $O(1/k)$. Below we show that $\{F_0(x_0^k)\}_{k\in\mathbb{N}}$ satisfies a linear convergence bound.

2.5.6. Linear convergence rate. All the functions and variables here are at level 0 so we omit the subscripts. Now we show that $\{F(x^k)\}_{k\in\mathbb{N}}$ converges to F^* with a linear rate using the *Proximal Polyak-Łojasiewicz inequality* [23, Section 4]. The function F in Problem (1.1) is called ProxPL, if there exists $\mu > 0$, called the ProxPL constant, such that

$$
\text{(ProxPL)} \qquad \mathcal{D}_g(x, L) \;\geq\; 2\mu(F(x) - F^*) \qquad \forall x,
$$

$$
(2.19) \qquad \mathcal{D}_g(x, \alpha) := -2\alpha \min_z \left\{ \tfrac{\alpha}{2}\|z - x\|_2^2 + \langle z - x, \nabla f(x)\rangle + g(z) - g(x) \right\}.
$$

Intuitively, $\mathcal{D}_g$ is defined based on the proximal gradient operator:

$$
\text{prox}_{\frac{1}{L}g}\left(x - \frac{\nabla f(x)}{L}\right) = \underset{z}{\text{argmin}}\; \frac{L}{2}\|z - x\|_2^2 + \langle z - x, \nabla f(x)\rangle + g(z) - g(x).
$$

It has been shown in [23] that if f in (1.1) is μ-strongly convex, then F is μ-ProxPL. Now we prove the linear convergence rate of Algorithm 2.1. Note that a standard result such as [4, Theorem 10.29] on convergence of proximal gradient for strongly convex functions is not directly applicable because, as mentioned above, we interleave proximal gradient steps with coarse correction steps.

THEOREM 2.18. *Let x_0^1 be the initial guess of the algorithm, $x_0^* = \text{argmin } F_0(x)$ and $F_0^* := F_0(x_0^*)$. The sequence $\{x_0^k\}_{k\in\mathbb{N}}$ generated by* MGProx *(Algorithm 2.1) for solving Problem* (1.1) *satisfies $F_0(x_0^{k+1}) - F_0^* \leq \left(1 - \frac{\mu_0}{L_0}\right)^k (F_0(x_0^1) - F_0^*)$.*

Proof. First, by assumption F is strongly convex, so F is μ-ProxPŁ with $\mu > 0$ and

$$
\begin{aligned}
F(x^{k+1}) \quad &\overset{(2.14)}{\leq} \quad F(z^{k+1}) \overset{(2.13)}{\leq} F(y^{k+1}) = f(y^{k+1}) + g(y^{k+1}) + g(x^k) - g(x^k) \\
&\overset{(1.2)}{\leq} \quad f(x^k) + \langle \nabla f(x^k), y^{k+1} - x^k \rangle + \tfrac{L}{2}\|y^{k+1} - x^k\|_2^2 + g(x^k) + g(y^{k+1}) - g(x^k) \\
&= \quad F(x^k) + \langle \nabla f(x^k), y^{k+1} - x^k \rangle + \tfrac{L}{2}\|y^{k+1} - x^k\|_2^2 + g(y^{k+1}) - g(x^k) \\
&= \quad F(x^k) - \tfrac{1}{2L} \underbrace{(-2L)\left(\langle \nabla f(x^k), y^{k+1} - x^k \rangle + \tfrac{L}{2}\|y^{k+1} - x^k\|_2^2 + g(y^{k+1}) - g(x^k) \right)}_{\overset{(2.19)}{=} \mathcal{D}_g(x^k, L), \text{ since } y^{k+1} := \arg\min_z \frac{L}{2}\|z - x^k\|_2^2 + \langle z - x^k, \nabla f(x^k) \rangle + g(z) - g(x^k).} \\
&\overset{(\text{ProxPL})}{\leq} \quad F(x^k) - \tfrac{\mu}{L}(F(x^k) - F^*).
\end{aligned}
$$

Adding $-F^*$ on both sides of the inequality gives $F(x^{k+1}) - F^* \leq (1 - \mu/L)(F(x^k) - F^*)$. Applying this inequality recursively completes the proof. □

Remark 2.19. We now give several remarks about the result.
- Convergence rate: for a μ_0-strongly convex and L_0-smooth f_0, we have $0 < \mu_0 \leq L_0$ and $0 \leq 1 - \mu_0/L_0 < 1$. Depending on the value of μ_0, for k not too large, the sublinear convergence rate $1/k$ from Theorem 2.13 gives a better bound than the linear rate $(1 - \mu_0/L_0)^k$ from Theorem 2.18. This is the case when $\mu_0 \ll L_0$.
- Since x^* is unique, we also conclude that the sequence $\{x_0^k\}_{k\in\mathbb{N}}$ converges to x^*.

Lastly, we emphasize that the bounds in Theorem 2.13 and Theorem 2.18 are loose bounds for Algorithm 2.1 since we only show that the coarse correction can guarantee the descent condition $F_0(z_0^{k+1}) \leq F_0(y_0^{k+1})$ but not a stronger sufficient descent condition.

2.5.7. On the selection of τ. Recall that the tau vector comes from a set:

$$
\tau_{0\to1}^{k+1} \overset{(2.6b)}{\in} \underline{\tau_{0\to1}^{k+1}} := \underbrace{\nabla f_1(y_1^{k+1}) - R\partial F_0(y_0^{k+1})}_{\text{singleton}} + \underbrace{\partial g_1(y_1^{k+1})}_{\text{set-valued}}.
$$

We emphasize that for our theoretical results to hold we can choose any value in the set $\underline{\tau_{0\to1}^{k+1}}$ to define $\tau_{0\to1}^{k+1}$ in Algorithm 2.1. First, the results of in Theorems 2.4 and 2.5 hold for any τ in the set. Second, all the convergence bounds (Theorems 2.13 and 2.18) only contain constants at level $\ell = 0$ and are independent of the choice of tau vector.

Optimal tau selection seems difficult. Recall the two steps in the algorithm related to the coarse correction,

$$
\begin{aligned}
x_1^{k+1}(\tau) &= \arg\min_\xi F_1(\xi) - \Big\langle \underbrace{\nabla f_1(y_1^{k+1}) - R\partial F_0(y_0^{k+1}) + \partial g_1(y_1^{k+1})}_{\ni \tau}, \xi \Big\rangle, \\
x_0^{k+1}(\tau) &= y_0^{k+1} + \alpha(\tau) P(x_1^{k+1}(\tau) - y_1^{k+1}),
\end{aligned}
$$

where x_0^{k+1}, x_1^{k+1} and α are all a function of τ. Now it seems tempting to "optimally tune" $\tau_{0\to1}^{k+1}$ so that it maximizes the gap $F_0(y_0^{k+1}) - F_0(x_0^{k+1})$:

$$
\tau_{0\to1}^{k+1} \in \arg\max_{\tau \in \underline{\tau_{0\to1}^{k+1}}} F(y_0^{k+1}) - F_0(x_0^{k+1}(\tau)) = \arg\min_\tau F_0(x_0^{k+1}(\tau)).
$$

However, this problem generally has no closed-form solution and it is intractable to solve numerically. In the experiments we verified that the sequence produced by MGProx converges for different values of τ confirming the theory.

3. FastMGProx: MGProx with Nesterov's acceleration. In this section we show that, by treating the `MGProx` update as a single iteration, we can embed this update within Nesterov's estimating sequence framework to derive an accelerated method with the optimal $O(1/k^2)$ convergence. In this section we are focusing on level $\ell = 0$ and we hide some of the subscripts 0 for clarity.

Algorithm 3.1 shows the Nesterov's accelerated `MGProx` that we call `FastMGProx`. First we introduce some compact notation. We denote $\big(\text{MGProx} \circ \text{prox}\big)(w)$ as applying a proximal gradient update on F_0 at the point w and then followed by a MGProx update process (i.e., steps (i)-(vi) in Algorithm 2.1). The algorithm introduces two scalar sequences α^k, γ^k and two auxiliary vector sequences y^k, z^k. It is important not to confuse these symbols with those presented in Section 2.

Algorithm 3.1 `FastMGProx` for an approximate solution of (1.1)

Initialize R and P, $z^0 = x^0$ (auxiliary sequence), $\gamma^0 > 0$ (extrapolation parameter)
for $k = 1, 2, \ldots$ **do**

  (i)   Solve $\alpha^k \in \,]0, 1[$ from $L_0(\alpha^k)^2 = (1 - \alpha^k)\gamma^k$, $\gamma^{k+1} = (1 - \alpha^k)\gamma^k$ parameter

  (ii) $y^k \;\; = \alpha^k z^k + (1 - \alpha^k)x^k$ Nesterov's extrapolation

  (iii) $x^{k+1} = (\text{MGProx} \circ \text{prox})(y^k - \frac{1}{L_0}\nabla f_0(y^k))$ prox-grad step with MGProx

  (iv) $z^{k+1} = z^k - \frac{\alpha^k}{\gamma^{k+1}}\frac{y^k - x^{k+1}}{L_0}$ updating the auxiliary sequence

end for

By the sufficient descent lemma of proximal gradient update (Lemma 2.9) and the descent lemma of MGProx update (inequality (2.13)), we can guarantee that for x^k, y^k in Algorithm 3.1 we have $F(x^{k+1}) \leq F(y^k) - \|G(y^k)\|_2^2/(2L)$, where $G(y^k)$ is the proximal gradient map of F at y^k, see (2.14). This inequality forms the basis of the `FastMGProx`: using Nesterov's framework of estimating sequence [35], we have the following theorem

THEOREM 3.1. *For the sequence $\{x^k\}$ produced by Algorithm 3.1, we have the convergence rate $F(x^k) - F^* \leq O(1/k^2)$. To be exact,*

$$F(x^k) - F^* \;\leq\; \frac{4L_0\Big(F(x^0) - F^* + \frac{\gamma^0}{2}\|x^0 - x^*\|_2^2\Big)}{\Big(2\sqrt{L_0} - \sqrt{\gamma^0}\Big)^2 + 2\gamma^0\Big(2\sqrt{L_0} - 1\Big)k + \gamma^0 k^2}.$$

We present the proof of Theorem 3.1 in Appendix 7. Below we give some key points about Algorithm 3.1. First, the convergence rate $F(x^k) - F^* \leq O(1/k^2)$ is optimal in the function-gradient model of [35]. Next, the work [40] also achieved the same optimal rate. However, inspecting the algorithm of [40], it relies on two things:

- It bypassed all the technical challenges caused by the non-smoothness of the coarse problem by using a smooth approximation. In contrast, `MGProx` and `FastMGProx` do not use smoothing.
- There is a safe-guarding IF statement in the algorithm. It means that when the combined effect of Nesterov's extrapolation and multigrid process produces a bad iterate, that whole iteration will be discarded and replaced by a simple proximal gradient iteration. This gives a sign of possible bad interaction between Nesterov's acceleration and multigrid. In contrast, `MGProx` and `FastMGProx` have no such IF statement.

4. A multi-level MGProx. Now we generalize the 2-level `MGProx` to multiple levels. The 2-level `MGProx` method constructs a coarse problem at level ($\ell = 1$), and uses the solution of such problem to help solve the original fine-level problem ($\ell = 0$). If the fine problem has

a large problem size, solving the coarse problem exactly is generally expensive. Hence it is natural to consider applying multigrid recursively until the coarse problem on the coarsest level is no longer expensive to solve. An L-level `MGProx` cycle with a V-cycle structure is shown in Algorithm 4.1. We clarify the naming of the variables in the algorithm as follows: at each iteration k, we have x_ℓ^k: variable before pre-smoothing on level ℓ; y_ℓ^{k+1}: variable after pre-smoothing on level ℓ; z_ℓ^{k+1}: variable after coarse-grid correction on level ℓ; and w_ℓ^{k+1}: variable after post-smoothing on level ℓ. Note that, to obtain a well-defined recursion in Algorithm 4.1, we choose the superscript for the x variables equal to k on all levels. In the 2-level algorithm we chose a different convention, writing the x variable on level 1 as x_1^{k+1}.

Algorithm 4.1 L-level `MGProx` with V-cycle structure for an approximate solution of (1.1)

Initialize x_0^1 and the full version of $R_{\ell\to\ell+1}, P_{\ell+1\to\ell}$ for $\ell \in \{0, 1, \ldots, L-1\}$
for $k = 1, 2, \ldots$ **do**
  Set $\tau_{-1\to0}^{k+1} = 0$
  for $\ell = 0, 1, \ldots, L-1$ **do**

$$y_\ell^{k+1} = \text{prox}_{\frac{1}{L_\ell} g_\ell}\left(x_\ell^k - \frac{\nabla f_\ell(x_\ell^k) - \tau_{\ell-1\to\ell}^{k+1}}{L_\ell}\right) \qquad \text{pre-smoothing}$$

$$x_{\ell+1}^k = R_{\ell\to\ell+1}(y_\ell^{k+1})\, y_\ell^{k+1} \qquad \text{restriction to next level}$$

$$\tau_{\ell\to\ell+1}^{k+1} \in \partial F_{\ell+1}(x_{\ell+1}^k) - R_{\ell\to\ell+1}(y_\ell^{k+1})\, \partial F_\ell(y_\ell^{k+1}) \qquad \text{create tau vector}$$

  end for

$$w_L^{k+1} = \underset{\xi}{\text{argmin}}\left\{ F_L^\tau(\xi) := F_L(\xi) - \langle \tau_{L-1\to L}^{k+1}, \xi\rangle \right\} \qquad \text{solve the level-}L\text{ coarse problem}$$

  for $\ell = L-1, L-2, \ldots, 0$ **do**

$$z_\ell^{k+1} = y_\ell^{k+1} + \alpha P_{\ell+1\to\ell}(w_{\ell+1}^{k+1} - x_{\ell+1}^k) \qquad \text{coarse correction}$$

$$w_\ell^{k+1} = \text{prox}_{\frac{1}{L_\ell} g_\ell}\left(z_\ell^{k+1} - \frac{\nabla f_\ell(z_\ell^{k+1}) - \tau_{\ell-1\to\ell}^{k+1}}{L_\ell}\right) \qquad \text{post-smoothing}$$

  end for

$$x_0^{k+1} = w_0^{k+1} \qquad \text{update the fine variable}$$

end for

Here are remarks about Algorithm 4.1. First, L_ℓ is the Lipschitz constant of ∇f_ℓ. Then,
- At level $\ell \neq L$, we are essentially performing two proximal gradient iterations (pre-smoothing + post-smoothing) and a coarse correction. At the coarsest level $\ell = L$, we perform an exact update by solving the coarse problem exactly.
- From the traditional wisdom of classical multigrid, more than one pre-smoothing and post-smoothing steps can be beneficial to accelerate the overall convergence. We implemented such multiple smoothing steps in the numerical tests.

Remark 4.1 (Convergence of Algorithm 4.1). Regarding the finest level function value $\{F_0(x_0^k)\}_{k\in\mathbb{N}}$, Theorem 2.13 and Theorems 2.18 and 2.18 all hold for the multilevel Algorithm 4.1, since the angle condition of the coarse correction (Theorem 2.5) also holds for multilevel `MGProx` when the coarse problem is solved inexactly. To be specific, the last inequality $F_1^\tau(x_1^{k+1}) - F_1^\tau(y_1^{k+1}) < 0$ in the proof of Theorem 2.5 holds when the coarse problem on x_1^{k+1} is solved inexactly by a combination of proximal gradient iterations and a coarse-grid correction with line search.

Approximate per-iteration computational complexity. The cost of one prox-grad update on the finest level scales with n_0. The cost of all prox-grad operations in one iteration of a L-level `MGProx` is $2(1 + r + \cdots + r^{L-1})n_0 = 2\frac{1-r^L}{1-r}n_0$, where r is a reduction factor. In the experiment $r = \frac{1}{4}$, and all the prox-grad steps per V-cycle iteration amount to at most $\frac{8}{3}(1 - \frac{1}{4^L}) \leq 2.67$ times the cost of performing one fine prox-grad update.

Multi-level FastMGProx. Similar to the *L*-level `MGProx`, we can also propose a *L*-level `FastMGProx`, which is not shown here.

5. Numerical results. We test `MGProx` on an Elastic Obstacle Problem (EOP), to be reviewed below, then we present the test results in subsection 5.3.

5.1. Elastic Obstacle Problem (EOP). The EOP [7, 30] was motivated by physics [44], see [47, Section 4] for related problems. Here we solely focus on solving EOP as a nonsmooth optimization problem in the form of (1.1) and use it to demonstrate the capability of `MGProx`.

As elastic potential energy is proportional to the surface area, we have the problem

$$\text{(continuum EOP)} \quad \min_{u} \iint_{\Omega} \sqrt{1 + \|\nabla u\|_{L^2}^2} dxdy + \lambda \iint_{\Omega} \|(\phi - u)_+\|_{L^1} dxdy \quad \text{s.t.} \quad u = 0 \text{ on } \partial\Omega,$$

where $\nabla u : \Omega \to \mathbb{R}^2$ is the gradient field of u, the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^1}$ are the L^2 and L^1 norm for functions, resp.. We take $\phi \le 0$ on $\partial\Omega$, so the boundary condition $u = 0$ on $\partial\Omega$ makes sense. Here the non-penetration constraint is represented by the penalty term, where $(\cdot)_+ := \max\{\cdot, 0\}$ is taken element-wise, and $\lambda > 0$ is a penalty parameter.

Most multigrid algorithms solve an approximated version of EOP. Many methods have been proposed to solve EOP: adaptive finite element methods [22], penalty methods [45, 47] and level set methods [29]. For the multigrid methods mentioned in the introduction, many of them do not solve (continuum EOP) efficiently.

- They only solve an approximate version of EOP: near $x = 0$ the function $\sqrt{1 + x^2}$ has a Taylor series $1 + x^2/2 + o(x^3)$, ignoring the 1 and higher order terms, such linearization replace the integral of (continuum EOP) by $\frac{1}{2} \int_{\Omega} \|\nabla u\|_{L^2}^2 dx$.
- They only solve the box-constrained version of EOP, i.e., they are not designed to handle the nonsmooth L^1 penalty. In fact [25] pointed out it is general hard to extend multigrid from dealing with box constraint to general nonsmooth functions.
- They are not FAS but Newton-MG (see subsection 1.5).

5.2. The optimization problem. We now discuss how to derive the optimization problem by discretizing the integral in (continuum EOP) to obtain a problem in the form of (1.1). On a grid of $N \times N$ internal points with $\Delta x = \Delta y = h = \frac{1}{N+1}$ on Ω, we let $\mathbf{U} \in \mathbb{R}^{N \times N}$ with $\mathbf{U}(i, j) = u(ih, jh)$ with i, j ranging from 1 to N. Here small italic symbol $u(x, y)$ denotes the continuum variable in the infinite dimensional space, and the capital bold symbol $\mathbf{U}$ denotes a N-by-N matrix obtained by finite discretization of u. Let vec denotes vectorization, then $\mathbf{u} := \text{vec}(\mathbf{U}) \in \mathbb{R}^{N^2}$ is the optimization variable. Discretizing the integral in (continuum EOP) gives

$$\text{(EOP-0)} \quad \underset{\mathbf{u} \in \mathbb{R}^{N^2}}{\arg\min} h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \sqrt{1 + (\mathbf{D}_{(i,j),:}\mathbf{u})^2 + (\mathbf{E}_{(i,j),:}\mathbf{u})^2} + h^2 \lambda \|(\phi - \mathbf{u})_+\|_1$$

where the two matrices $\mathbf{D} \in \mathbb{R}^{N^2 \times N^2}, \mathbf{E} \in \mathbb{R}^{N^2 \times N^2}$ are the first-order forward difference operators that approximate $\partial/\partial x$ and $\partial/\partial y$ respectively, defined using the coordinate index (i, j) of $\mathbf{D}$ and the coordinate index (k, l) of $\mathbf{U}$ as

$$\mathbf{D}_{(i,j),(k,l)} = \begin{cases} 1/h & i = k \text{ and } j = l + 1, \\ -1/h & i = k \text{ and } j = l, \\ 0 & \text{else,} \end{cases} \qquad \mathbf{E}_{(i,j),(k,l)} = \begin{cases} 1/h & i = k + 1 \text{ and } j = l, \\ -1/h & i = k \quad \text{ and } j = l, \\ 0 & \text{else.} \end{cases}$$

The notation $\mathbf{D}_{(i,j),:}$ refers to the (i, j)-th row of $\mathbf{D}$. The vector $\phi \in \mathbb{R}^{N^2}$ is the discretization of ϕ in (continuum EOP), and $\|\cdot\|_1$ is the ℓ_1-norm.

Define $\mathbf{F} \in \mathbb{R}^{2N^2 \times N^2}$ by stacking rows of $\mathbf{D}$ and $\mathbf{E}$ together as in (5.1). Ignoring the constant factor h^2 in (EOP-1) yields the problem

(EOP-1)
$$\underset{\mathbf{u} \in \mathbb{R}^{N^2}}{\operatorname{argmin}} f_0(\mathbf{u}) + g_0(\mathbf{u}) := \sum_{i=1}^{N} \sum_{j=1}^{N} \psi\left(\mathbf{F}_{(i,j),:}\mathbf{u}\right) + \lambda \|(\phi - \mathbf{u})_+\|_1,$$

where

(5.1)
$$\psi : \mathbb{R}^2 \to \mathbb{R} : (s,t) \mapsto \sqrt{1 + s^2 + t^2}, \qquad \mathbf{F}_{(i,j),:} := \begin{bmatrix} \mathbf{D}_{(i,j),:} \\ \mathbf{E}_{(i,j),:} \end{bmatrix}.$$

To avoid confusing the notion of gradient in the finite dimensional Euclidean space with the gradient operator ∇ in (continuum EOP), from now on we denote $\mathbf{grad}\, f_0$ the gradient of f_0 in (EOP-1). We denote $\mathbf{Hess}\, f_0$ the Hessian of f_0. By chain rule and the fact that

(5.2)
$$\mathbf{grad}\,\psi(s,t) = \begin{bmatrix} \dfrac{\partial}{\partial x}\psi(x,y) \\ \dfrac{\partial}{\partial y}\psi(x,y) \end{bmatrix}\Bigg|_{x=s,y=t} = \begin{bmatrix} \dfrac{s}{\sqrt{1 + s^2 + t^2}} \\ \dfrac{t}{\sqrt{1 + s^2 + t^2}} \end{bmatrix} \in \mathbb{R}^2,$$

so $\mathbf{grad}\, f_0(\mathbf{u}) = \sum_{i,j} \underbrace{\mathbf{F}_{(i,j),:}^{\top}}_{N^2\text{-by-2}} \underbrace{\mathbf{grad}\,\psi\left(\mathbf{F}_{(i,j),:}\mathbf{u}\right)}_{2\text{-by-1}} = \sum_{i,j} \begin{bmatrix} \mathbf{D}_{(i,j),:} \\ \mathbf{E}_{(i,j),:} \end{bmatrix}^{\top} \mathbf{grad}\,\psi\left(\mathbf{D}_{(i,j),:}\mathbf{u},\ \mathbf{E}_{(i,j),:}\mathbf{u}\right) \in \mathbb{R}^{N^2}$ is

(5.3)
$$\mathbf{grad}\, f_0(\mathbf{u}) \overset{(5.1)}{=} \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{\psi}_{ij}^{D}\mathbf{D}_{(i,j),:}^{\top} + \tilde{\psi}_{ij}^{E}\mathbf{E}_{(i,j),:}^{\top} = \mathbf{D}^{\top}\bar{\psi}^{D} + \mathbf{E}^{\top}\bar{\psi}^{E}$$

where $\tilde{\psi}_{ij}^{D} := \dfrac{[\mathbf{Du}]_{ij}}{\sqrt{1 + ([\mathbf{Du}]_{ij})^2 + ([\mathbf{Eu}]_{ij})^2}}$ and $\tilde{\psi}_{ij}^{E} := \dfrac{[\mathbf{Eu}]_{ij}}{\sqrt{1 + ([\mathbf{Du}]_{ij})^2 + ([\mathbf{Eu}]_{ij})^2}}$.

By proposition 5.1, $\mathbf{grad}\, f_0$ is Lipschitz and problem (EOP-1) is strongly convex and thus: 1. it has a unique minimizer, and 2. it is within the framework of Problem (1.1) so `MGProx` can be used.

PROPOSITION 5.1. *For $\mathbf{u}$ on a bounded domain, the function f_0 in (EOP-1) is L_0-smooth (i.e., the gradient $\mathbf{grad}\, f_0$ is L_0-Lipschitz) and strongly convex, with L_0 provided below.*

Proof. Consider $\psi(s,t)$ defined by (5.1), $\mathbf{grad}\,\psi(t)$ is given by (5.2) and the Hessian is

$$\mathbf{Hess}\,\psi(s,t) = \frac{1}{(1 + s^2 + t^2)^{3/2}} \begin{pmatrix} 1 + t^2 & -st \\ -st & 1 + s^2 \end{pmatrix}.$$

This shows that ψ is 1-smooth and strongly convex on any bounded domain.

Now recall the function $f_0(\mathbf{u})$ can be written as $f_0(\mathbf{u}) = \sum_{i,j} \psi\left(\mathbf{F}_{(i,j),:}\mathbf{u}\right)$ as in (EOP-1) with $\mathbf{F}_{(i,j),:}$ defined by (5.1). Since a composition of a convex function and linear function is convex, and the sum of convex functions is convex, this shows f_0 is convex. Furthermore,

$$\mathbf{Hess}\, f_0(\mathbf{u}) = \sum_{i,j=1}^{n} \mathbf{Hess}\,\psi\left(\mathbf{F}_{(i,j),:}\mathbf{u}\right) \cdot \mathbf{F}_{(i,j),:}^{\top}\mathbf{F}_{(i,j),:}.$$

Then we compute that the Lipschitz constant L_0 of $\mathbf{grad}\, f_0$ is at most $\sum_{i,j} \|\mathbf{F}_{(i,j),:}^{\top}\mathbf{F}_{(i,j),:}\|_2$. Since $\|\mathbf{F}_{(i,j),:}\|_2 = \sqrt{3}/h$, we can take $L_0 = \sqrt{3}N^2/h$.

For strong convexity, we see that the Hessian is a weighted sum of rank-2 matrices with positive weights. Suppose $\mathbf{z}$ lies in the null space of all the rank-2 matrices. In this case, $\mathbf{z}$ must be identically $\mathbf{0}$ as the following argument shows. If for some (i, j) it holds that $\mathbf{D}_{(i,j),:}\mathbf{z} = \mathbf{0}$, then $z_{i+1,j} = z_{i,j}$. So, if $\mathbf{z}$ is in the null space of all the $\mathbf{D}_{(i,j),:}$'s, then $\mathbf{z}$ is constant in the x-direction. Similarly, if $\mathbf{z}$ is in the null space of all the $\mathbf{E}_{(i,j),:}$'s, then $\mathbf{z}$ is constant in the y-direction. The operators $\mathbf{D}, \mathbf{E}$ applied at the boundaries yield that $\mathbf{z}$ is identically zero.

Thus, $\mathbf{Hess}\,f_0(\mathbf{u})$ is positive definite for all $\mathbf{u}$. Since the Hessian is continuous, this means that its smallest eigenvalue has a positive lower bound over any bounded domain, i.e., f_0 is strongly convex on any bounded domain.   □

Remark 5.2. It is expensive to evaluate f_0 and $\mathbf{grad}\,f_0$ for (EOP-1), since they are sum of N^2 nonlinear terms involving vectors in $\mathbb{R}^{N^2}$. Furthermore, based on Proposition 5.1,
- a tight closed-form expression of the global Lipschitz constant of $\mathbf{grad}\,f_0$ is unknown to us,
- f_0 is μ_0-strongly convex but the strong convexity parameter μ_0 is unknown to us.

5.2.1. MGProx on EOP. The subdifferential and proximal operator of $\lambda\|(\mathbf{c} - \mathbf{v})_+\|_1$ are

$$(5.4) \quad \left[\partial\|(\mathbf{c} - \mathbf{v})_+\|_1\right]_i = \begin{cases} -1 & v_i < c_i \\ [-1, 0] & v_i = c_i \\ 0 & v_i > c_i \end{cases}, \quad \left[\mathrm{prox}_{\lambda\|(\mathbf{c}-\cdot)_+\|_1}(\mathbf{v})\right]_i = \begin{cases} v_i + \lambda & v_i + \lambda < c_i \\ c_i & v_i \le c_i \le v_i + \lambda \\ v_i & v_i > c_i \end{cases}.$$

For a matrix $\mathbf{U}$ at a resolution level ℓ, the (full) restriction of $\mathbf{U}$, denoted as $\hat{\mathbf{U}} := \overline{\mathcal{R}}(\mathbf{U})$, can be defined by a full weighting operator with the kernel defined by $\frac{1}{8}[1\,2\,1]^\top \otimes [1\,2\,1]$ where $\otimes$ is tensor product. Using a block-tridiagonal matrix $\overline{\mathbf{R}}$ and vectorization vec, the expression $\hat{\mathbf{U}} := \overline{\mathcal{R}}(\mathbf{U})$ can be written as $\hat{\mathbf{u}} := \mathrm{vec}(\hat{\mathbf{U}}) = \overline{\mathbf{R}}\mathbf{u} := \overline{\mathbf{R}}\mathrm{vec}(\mathbf{U})$. For the adaptive version of $\mathbf{R}$, we follow Definition 2.3. For the prolongation matrix $\mathbf{P} := c\mathbf{R}^\top$, we take $c = 2$.

5.3. Test results. We now present the test results.

Experimental setup. We take $\phi(x, y) = \max\{0, \sin(x)\}\max\{0, \sin(y)\}$ where $x, y \in [0, 3\pi]$. We initialize x_0^1 as a random nonnegative vector and compute the initial function value $F_0(x_0^1)$ and the initial norm of the proximal gradient map $\|G_0(x_0^{\mathrm{ini}})\|_2$. We stop the algorithm using the proximal first-order optimality condition: we stop the algorithm if $\|G_0(x_0^k)\|_2/\|G_0(x_0^{\mathrm{ini}})\|_2 \le 10^{-16}$. The experiments are conducted in MATLAB R2023a on a MacBook Pro (M2 2022) running macOS 14.0 with 16GB memory[1]. We report the value $(F_0(x_0^k) - F_0^{\min})/F_0(x_0^{\mathrm{ini}})$, where $F_0^{\min}$ is the lowest objective function value achieved among all the tested methods.

MGProx setup. We run a multilevel `MGProx` with N_s number of pre-smoothing and post-smoothing steps on all levels. We take enough levels to make the coarsest problem sufficiently small. At the coarsest level, we do not solve the subproblem exactly, instead we simply just run N_s iterations of proximal gradient update. In the test we take N_s iterations of (accelerated) proximal gradient update for all the pre-smoothing and post-smoothing step. On the τ vector, we do no tuning, on all the levels we simply just take 0 for the subdifferential in (5.4). For the line search of the coarse stepsize, we simply run naive line search (Algorithm 2.2, see Section 2.5.4 for the discussion) up to machine accuracy 10^{-16}.

Benchmark methods. We compare `MGProx` with the followings.

$\boxed{\texttt{ProxGrad}}$: the standard proximal gradient update $x_0^{k+1} = \mathrm{prox}_{\hat{L}^{-1}g_0}\left(x_0^k - \hat{L}^{-1}\nabla f_0(x_0^k)\right)$, where $\hat{L}$ (initialized as $\sqrt{3}n^2/h$ based on Proposition 5.1) is an estimate of the true Lipschitz parameter L_0 of ∇f_0 (i.e., $\mathbf{grad}\,f_0$) at x^k, obtained by backtracking line search [4, page 283]. The convergence rates of `ProxGrad` is $F(x^k) - F^* \le O\left(\min\left\{1/k, \exp\left(-\frac{\mu_0}{L_0}k\right)\right\}\right)$.

---

[1]The code is available at https://angms.science/research.html

FISTA [34]: it is ProxGrad with Nesterov's acceleration. With an additional auxiliary sequence $\{y_0^k\}$, the update iteration is $x_0^{k+1} = \text{prox}_{\hat{L}^{-1}g_0}(y_0^k - \hat{L}^{-1}\nabla f_0(y_0^k))$ together with extrapolation $y_0^{k+1} = x_0^{k+1} + \beta_k(x_0^{k+1} - x_0^k)$. The stepsize α^k is selected using backtracking line search [4, page 291] and the extrapolation parameter β_k follows the standard FISTA setup. It is proved [5] that FISTA has the the optimal first-order convergence rate [34] $F(x^k) - F^* \le O\left(\min\left\{1/k^2, \exp\left(-\sqrt{\frac{\mu_0}{L_0}}k\right)\right\}\right)$.

FISTA-r Restarts can improve the convergence of FISTA [38]. There are various restarting schemes: Necoara's restart [33], adaptive restart [28], and parameter-free restart [1]. See [2] for a survey. We note that by Remark (5.2), both L_0 and μ_0 of f_0 in (EOP-1) are unknown, hence schemes [33] and [2] do not work. While there are parameter-free (without knowing L_0 and μ_0) methods [28], [1]. However these methods potentially requires several evaluations of F_0 and ∂F, with a cost of $O(N^2)$ for (EOP-1). In the test, we implemented a simple function value restarting FISTA [38] that only requires one evaluation of F_0 at every iteration, and thus is inexpensive to implement. The convergence of FISTA-r sheds the light on the convergence of other restarting FISTA since these methods have a similar convergence rates [2, Table 1].

Furthermore, there is another reason that parameter-free restarting FISTA such as [28] and [1] are not applicable for EOP. The improvement in convergence rate from knowledge of μ_0 is insubstantial if μ_0 is close to 0. And indeed this is true for (EOP-1): we have μ_0 tends to 0 as the number of elements in the mesh increases.

Kocvara3 [25, Algorithm 3]: a multigrid method with convergence guarantee but no convergence rate. It is a FAS version of Kornhuber's truncation [26], which is originally a Newton-MG (see subsection 1.5). We pick the "non-truncated version" as it is shown to have a better convergence. We remark that the method is designed for box-constrained quadratic program, and does not apply directly to (EOP-1). Thus, in our implementation, we adapt the structure of the algorithm (which is similar to that of MGProx), but with a small change: instead of projection step we perform a proximal step, since the proximal operator generalizes the projection operator. Also, we remark that such implementation of Kocvara3 is a special case of MGProx that the tau vector τ is defined as (2.6b) with the set-valued subdifferentials $\partial g_0, \partial g_1$ all set to zero.

The results. Table 1 shows the results on three problems with different number of variables, and Fig.1 shows the typical convergence curves of the algorithms. The x-axis in the plot is in terms of time and not the number of iterations, since each method has a different per-iteration cost due to the uncertainty in the number of iterations taken in the backtracking line search.

- In the three tests, MGProx has the fastest convergence. We remark that the test problem is strongly convex but the parameter μ_0 is unknown and not used in any tuning in the MGProx algorithm. This shows case that MGProx empirically works for convex but not strongly convex problem.
- In general MGProx has a higher per-iteration cost than other methods, see the discussion in Section 4. However, note that the per-iteration cost of ProxGrad and FISTA is possibly expensive due to the backtracking line search.
- We remark that we have a similar conclusion about the convergence (i.e., MGProx has the best convergence performance in the test) if we use a larger penalty parameter in the experiment, or we if change the problem from the ℓ_1-penalized form (EOP-1) to the box-constrained form and/or the surface tension cost function is replaced by its Taylor's approximation.
- How the multi-level process enhances convergence speed remains an open problem.
  - We remark that the convergence rate of MGProx is a loose bound since we only showed a descent condition but not a stronger sufficient descent condition.

– An educated guess is that the multigrid process acts as a variable metric method or pre-conditioner that reduces the number of iterations needed. For the EOP test problem, the evaluation of function and gradients is expensive, hence the effect of reducing number of iterations by the multi-grid process out-weights the heavier per-iteration cost of the multi-grid process.
• As `MGProx` already has the best performance among the algorithms, we do not implement `FastMGProx`.

TABLE 1
Convergence results for (EOP-1). *The regularization parameter is* 10^{-7} *for* $N^2 = 225$ *and* 10^{-6} *for* $N^2 = 65025$.

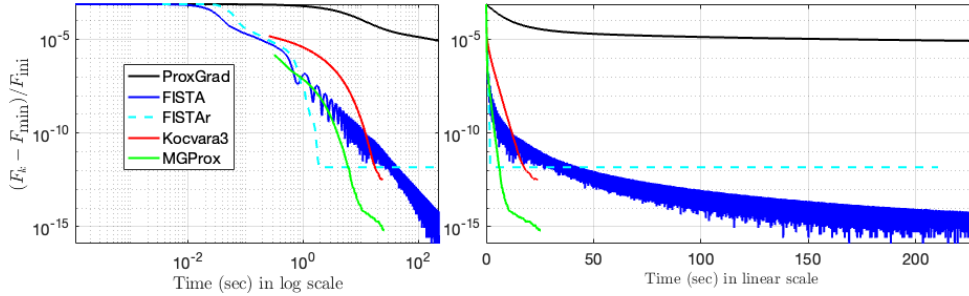| N^2	Method	iterations k	time (sec.)	$\left(F_0(x_0^k) - F_0^{min}\right)/F_0(x_0^{\text{ini}})$
$(2^4 - 1)^2 = 225$	ProxGrad	$> 10^7$	232	8.14×10^{-6}
	FISTA	9.67×10^6	228	1.26×10^{-16}
	FISTA-r	7.77×10^5	210	1.48×10^{-12}
	Kocvara3 $N_s = 3000$	71	24	3.30×10^{-13}
	MGProx $N_s = 3000$	50	25	6.31×10^{-16}
$(2^8 - 1)^2 = 65025$	ProxGrad	$> 10^5$	335.9	8.33×10^{-8}
	FISTA	$> 10^5$	332.62	6.64×10^{-8}
	FISTA-r	$> 10^5$	364.89	6.64×10^{-8}
	Kocvara3 $N_s = 100$	$> 10^3$	986.53	8.07×10^{-8}
	MGProx $N_s = 100$	50	48.37	1.32×10^{-10}

FIGURE 1. *The convergence pattern of the algorithms for the case* $N^2 = 225$. *Left: x-axis in log scale. Right: x-axis in linear scale. FISTAr has the fastest convergence in the first 10 seconds, then MGProx has the fastest convergence over all.*

6. Conclusion. In this work we study the combination of proximal gradient descent and multigrid method for solving a class of possibly non-smooth strongly convex optimization problems. We propose the `MGProx` method, introduce the adaptive restriction operator and provide theoretical convergence results. We also combine `MGProx` with Nesterov's acceleration, together with the optimal convergence rate with respect to the first-order methods in the function-gradient model. Numerical results confirm the efficiency of `MGProx` compared with other methods for solving Elastic Obstacle Problems.

Future works. There are several problems remaining open.
• The convergence rate of `MGProx` is probably not tight; we conjectture that there is a tighter bound.

- How the multi-level process enhances the convergence speed remains open.
- The assumption of strong convexity of `MGProx` may be relaxed.
- On the compactness of subdifferential, it is interesting to generalize Lemma 2.7 for $g_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$. See Remark 2.8.
- Efficient tuning strategy for the τ selection in the subdifferential could be developed.

Acknowledgments. We thank the three referees for their helpful comments.

7. Appendix.

7.1. Convergence of MGProx. We present the proofs of the convergence of `MGProx`.

7.1.1. The proof of Lemma 2.14.

Proof. By convexity and L-smoothness of f, for all y^{k+1}, x^k, ξ we have

$$
\begin{aligned}
f(y^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), y^{k+1} - x^k \rangle + \tfrac{L}{2}\|y^{k+1} - x^k\|_2^2 & f \text{ is } L\text{-smooth} \dots (i) \\
f(x^k) &\leq f(\xi) - \langle \nabla f(x^k), \xi - x^k \rangle & f \text{ is convex} \dots (ii) \\
f(y^{k+1}) &\leq f(\xi) - \langle \nabla f(x^k), \xi - y^{k+1} \rangle + \tfrac{L}{2}\|y^{k+1} - x^k\|_2^2 & (i) + (ii) \\
&= f(\xi) - \langle \nabla f(x^k), \xi - x^k + \tfrac{1}{L}G(x^k) \rangle + \tfrac{1}{2L}\|G(x^k)\|_2^2 & y^{k+1} = x^k - \tfrac{1}{L}G(x^k)
\end{aligned}
$$

where $G(x^k)$ is the proximal gradient map of F at x^k, see (2.14).

Next, adding $g(y^{k+1}) = g(x^k - \tfrac{1}{L}G(x^k))$ on the both sides of the last inequality gives

$$
(7.1) \qquad F(y^{k+1}) \leq f(\xi) - \left\langle \nabla f(x^k), \xi - x^k + \frac{1}{L}G(x^k) \right\rangle + \frac{1}{2L}\|G(x^k)\|_2^2 + g\left(x^k - \frac{1}{L}G(x^k)\right).
$$

Based on the properties of the coarse correction (Theorem 2.5 and Lemma 2.7) and the sufficient descent property of the proximal gradient update (2.14), we have

$$
F(x^{k+1}) \overset{(2.14)}{\leq} F(z^{k+1}) \overset{Theorem\ 2.5,\ Lemma\ 2.7}{\leq} F(y^{k+1}),
$$

so we can replace $F(y^{k+1})$ in (7.1) by $F(x^{k+1})$ and obtain

$$
(7.2) \qquad F(x^{k+1}) \leq f(\xi) - \left\langle \nabla f(x^k), \xi - x^k + \frac{1}{L}G(x^k) \right\rangle + \frac{1}{2L}\|G(x^k)\|_2^2 + g\left(x^k - \frac{1}{L}G(x^k)\right).
$$

In the following we deal with the term $g\left(x^k - \tfrac{1}{L}G(x^k)\right)$ in (7.2). First, by the convexity of g, for all ξ we have

$$
\begin{aligned}
g(\xi) &\geq g\left(x^k - \frac{1}{L}G(x^k)\right) + \left\langle \partial g(x^k - \frac{1}{L}G(x^k)), \xi - (x^k - \frac{1}{L}G(x^k)) \right\rangle \\
\iff \quad g\left(x^k - \frac{1}{L}G(x^k)\right) &\leq g(\xi) - \left\langle \partial g(x^k - \frac{1}{L}G(x^k)), \xi - (x^k - \frac{1}{L}G(x^k)) \right\rangle.
\end{aligned}
$$

By the subgradient optimality of the proximal subproblem associated with g, we can show that $G(x^k) - \nabla f(x^k) \in \partial g(x^k - \tfrac{1}{L}G(x^k))$, hence

$$
(7.3) \qquad g\left(x^k - \frac{1}{L}G(x^k)\right) \leq g(\xi) - \left\langle G(x^k) - \nabla f(x^k), \xi - (x^k - \frac{1}{L}G(x^k)) \right\rangle.
$$

Combining (7.2) and (7.3) with $\xi = x^* := \operatorname{argmin} F$ gives

$$
\begin{aligned}
F(x^{k+1}) &\leq F^* - \left\langle G(x^k), x^* - x^k + \frac{1}{L}G(x^k) \right\rangle + \frac{1}{2L}\|G(x^k)\|_2^2 \\
&= F^* - \left\langle G(x^k), x^* - x^k \right\rangle - \frac{1}{2L}\|G(x^k)\|_2^2 \\
&= F^* + \frac{L}{2}\left(\|x^k - x^*\|_2^2 - \|x^k - x^* - \frac{1}{L}G(x^k)\|_2^2\right) \overset{x^k - \frac{1}{L}G(x^k) =: y^{k+1}}{\iff} \quad (2.15)
\end{aligned}
$$

where completing the squares is used in the second equal sign. $\qquad \square$

7.2. The proof of Lemma 2.15.

Proof. By the convexity of f and g,

$$f(x) \geq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle \qquad\qquad f \text{ is convex} \dots (i)$$

$$g(x) \geq g(y^{k+1}) + \langle \partial g(y^{k+1}), x - y^{k+1} \rangle \qquad\qquad g \text{ is convex} \dots (ii)$$

$$(7.4) \quad F(x) \geq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + g(y^{k+1}) + \langle \partial g(y^{k+1}), x - y^{k+1} \rangle \qquad (i) + (ii)$$

By definitions (1.5a), (1.5b), the proximal gradient iteration is a majorization-minimization process that updates x^k based on minimizing a local quadratic overestimator Q of x^k, i.e., $y^{k+1} = \text{prox}_{\frac{1}{L}g}(x^k - \frac{1}{L}\nabla f(x^k))$ is equivalent to

$$(7.5) \qquad y^{k+1} \;=\; \underset{\xi}{\text{argmin}}\left\{ Q(\xi; x^k) := f(x^k) + \langle \nabla f(x^k), \xi - x^k \rangle + \frac{L}{2}\|\xi - x^k\|_2^2 + g(\xi) \right\}.$$

Being an overestimator (which comes from the majorization-minimization interpretation [39, Section 4.2.1]), we have $F(x) \leq Q(x; x^k)$, which implies for all x

$$
\begin{aligned}
& F(x) - F(y^{k+1}) \\
\geq\; & F(x) - Q(y^{k+1}; x^k) \\
\overset{(7.5)}{=}\; & F(x) - f(x^k) - \langle \nabla f(x^k), y^{k+1} - x^k \rangle - \frac{L}{2}\|y^{k+1} - x^k\|_2^2 - g(y^{k+1}) \\
\overset{(7.4)}{\geq}\; & \langle \nabla f(x^k), x - x^k \rangle + \langle \partial g(y^{k+1}), x - y^{k+1} \rangle - \langle \nabla f(x^k), y^{k+1} - x^k \rangle - \frac{L}{2}\|y^{k+1} - x^k\|_2^2 \\
=\; & \langle \nabla f(x^k) + \partial g(y^{k+1}), x - y^{k+1} \rangle - \frac{L}{2}\|x^k - y^{k+1}\|_2^2.
\end{aligned}
$$

(7.6)

Applying the subgradient optimality condition to (7.5) at y^{k+1} gives

$$0 \in \nabla f(x^k) + L(y^{k+1} - x^k) + \partial g(y^{k+1}) \quad\Longleftrightarrow\quad L(x^k - y^{k+1}) \in \nabla f(x^k) + \partial g(y^{k+1}),$$

so $L(x^k - y^{k+1})$ can be substituted in the first term of the last line of (7.6) and we have

$$
\begin{aligned}
F(x) - F(y^{k+1}) \;\geq\; & L\langle x^k - y^{k+1}, x - y^{k+1} \rangle - \frac{L}{2}\|x^k - y^{k+1}\|_2^2 \\
=\; & L\langle x^k - y^{k+1}, x - x^k + x^k - y^{k+1} \rangle - \frac{L}{2}\|x^k - y^{k+1}\|_2^2 \quad\Longleftrightarrow\quad (2.16).
\end{aligned}
$$

$\square$

7.3. Convergence of FastMGProx.

We make use of Nesterov's estimate sequence to prove the convergence rate on the minimization problem

$$(7.7) \qquad\qquad \underset{x}{\text{argmin}}\left\{ F(x) := f(x) + g(x) \right\},$$

where $f = f_0$ and $g = g_0$ as defined in (1.1). Since this subsection is long, to ease notation we dropped all the subscript. Everything in this subsection refers to the fine problem.

Some terminology and Nesterov's estimate sequence.

- A quadratic over-estimator model of f at y^k

$$(7.8) \qquad m_k(x; y^k) := f(y^k) + \langle \nabla f(y^k), x - y^k \rangle + \frac{1}{2L}\|x - y^k\|_2^2.$$

- A nonsmooth over-estimator model of F at y^k

$$(7.9) \qquad M_k(x; y^k) := f(y^k) + \langle \nabla f(y^k), x - y^k \rangle + \frac{1}{2L}\|x - y^k\|_2^2 + g(x).$$

- Minimizer of M_k at y^k

  (7.10)
  $$x_k^*(y^k) := \underset{x}{\operatorname{argmin}}\, M_k(x; y^k)$$

  with the subgradient 1st-order optimality condition as

  $$0 \in \partial M_k(x_k^*(y^k); y^k) = \nabla f(y^k) + \frac{x_k^*(y^k) - y^k}{L} + \partial g(x_k^*(y^k)),$$

  which implies a "gradient" from $x_k^*(y^k)$ and y^k can be defined as

  (7.11)
  $$g^k := \frac{y^k - x_k^*(y^k)}{L} \in \nabla f(y^k) + \partial g(x_k^*(y^k))$$

- **Definition (Nesterov's estimate sequence)** [35, Def. 2.2.1] A pair of sequences $\{\phi^k(x), \lambda^k\}$ is an *estimate sequence* of an function f if for all $k \in \mathbb{N}$ we have

  (Def 0)
  $$\lambda^k \geq 0,$$

  (Def 1)
  $$\lambda^k \to 0,$$

  (Def 2)
  $$\phi^k(x) \leq (1 - \lambda^k)f(x) + \lambda^k \phi^0(x),$$

  where ϕ^0 is free and λ^k denotes λ at the kth iteration, not λ to the kth power.

How to construct an estimate sequence. The following lemma is analogous to [35, Lemma 2.2.2], summarizing how to construct an estimate sequence for F in (7.7). Compared with [35, Lemma 2.2.2] which was proposed for smooth convex optimization, the new thing here is the introduction of (A0) and the modification in (A7).

LEMMA 7.1. *Assuming*

(A0)
$$F(x_k^*(y^k)) \leq M_k(x_k^*(y^k); y^k).$$

(A1)
$$f \text{ is L-smooth and convex (possibly not strongly convex).}$$

(A2)
$$\phi^0(x) \text{ is a convex function.}$$

(A3)
$$\{y^k\} \text{ is an arbitrary sequence.}$$

(A4a)
$$\{\alpha^k\} \text{ is a sequence that } \alpha^k \in\,]0, 1[.$$

(A4b)
$$\{\alpha^k\} \text{ is a sequence that } \sum_{k=0}^{\infty} \alpha^k = \infty.$$

(A5)
$$\lambda^0 := 1.$$

(A6)
$$\lambda^{k+1} := (1 - \alpha^k)\lambda^k.$$

(A7)
$$\phi^{k+1}(x) := (1 - \alpha^k)\phi^k(x) + \alpha^k\left[F(x_k^*(y^k)) + \langle g^k, x - y^k \rangle + \frac{1}{2L}\|g^k\|_2^2\right].$$

Then the sequences $\{\phi^k(x), \lambda^k\}$ generated as in (A6), (A7) is an estimate sequence of F.

Proof. The proof has 3 parts: showing $\{\lambda^k\}_{k\in\mathbb{N}}$ satisfies (Def 0) and (Def 1), and showing $\{\phi^k(x)\}_{k\in\mathbb{N}}$ satisfies (Def 2).

(Showing $\lambda^k > 0$). We have $\lambda^{k+1} \stackrel{(A6)}{=} \prod_{k=0}^{\infty}(1-\alpha^k)\lambda^0 \stackrel{(A5)}{=} \prod_{k=0}^{\infty}(1-\alpha^k) \stackrel{(A4a)}{>} 0$ and we showed the sequence $\{\lambda^k\}_{k\in\mathbb{N}}$ is lower bounded.

(Showing $\lambda^k \to 0$). By $\lambda^{k+1} \stackrel{(A6)}{=} (1-\alpha^k)\lambda^k$ we have $\dfrac{\lambda^{k+1}}{\lambda^k} = 1 - \alpha^k \stackrel{(A4a)}{<} 1$, so $\{\lambda^k\}_{k\in\mathbb{N}}$ is a monotonic decreasing sequence. By the monotone convergence theorem, we have $\{\lambda^k\}_{k\in\mathbb{N}}$ converges to $\inf\{\lambda^k\}_{k\in\mathbb{N}} = 0$. I.e., $\lambda^k \to 0$.

(Showing $\{\phi^k(x)\}_{k\in\mathbb{N}}$ satisfies (Def 2)*).* We prove by induction. The base case is true by definition and (A5). Assume the induction hypothesis $\phi^k(x) \leq (1 - \lambda^k)F(x) + \lambda^k\phi^0(x)$. Now for the case $k + 1$:

$$
\begin{aligned}
\phi^{k+1}(x) \;&:=\; (1-\alpha^k)\phi^k(x) + \alpha^k\Big[F(x_k^*(y^k)) + \langle g^k, x - y^k\rangle + \frac{1}{2L}\|g^k\|_2^2\Big] && \text{by } (A7)\\
&\leq\; (1-\alpha^k)\phi^k(x) + \alpha^k F(x) && (*)\\
&=\; (1-\alpha^k)\big(\phi^k(x) + (1-\lambda^k)F(x) - (1-\lambda^k)F(x)\big) + \alpha^k F(x)\\
&=\; (1-\alpha^k)\big(\phi^k(x) - (1-\lambda^k)F(x)\big) + \big((1-\alpha^k)(1-\lambda^k) + \alpha^k\big)F(x)\\
&\leq\; (1-\alpha^k)\big(\lambda^k\phi^0(x)\big) + \big(1 - (1-\alpha^k)\lambda^k\big)F(x) && \text{by induction}\\
&=\; (1-\alpha^k)\lambda^k\phi^0(x) + (1-\lambda^{k+1})F(x) && \text{by } (A6)\\
&<\; \lambda^k\phi^0(x) + (1-\lambda^{k+1})F(x) && \text{by } (A4a)
\end{aligned}
$$

where $(*)$ is true because we have

$$
F(x) \;\geq\; F(x_y^*(y^k)) + L\langle y^k - x_k^*(y^k), x - y^*\rangle + \frac{L}{2}\|y^k - x_k^*(y^k)\|_2^2
$$

which is basically (7.2) in Lemma 2.14. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As Nesterov stated in [35], up to here we are free to select the function $\phi^0(x)$. The next lemma is about canonical closed-form expression of $\phi^{k+1}(x)$, which resembles [24, Lemma 6] that generalized [35, Lemma 2.2.3].

LEMMA 7.2. *Let $\phi^0(x) := F(x^0) + \dfrac{\gamma^0}{2}\|x - z^0\|_2^2$. Then ϕ^{k+1} generated recursively as* (A7) *in Lemma 7.1 has a closed-form expression*

$$
\tag{7.14} \phi^{k+1}(x) \;=\; \overline{\phi}^{k+1} + \frac{\gamma^{k+1}}{2}\|x - z^{k+1}\|_2^2,
$$

where

$$
\tag{7.15a} \gamma^{k+1} = (1-\alpha^k)\gamma^k,
$$

$$
\tag{7.15b} z^{k+1} = z^k - \frac{\alpha^k}{\gamma^{k+1}}g^k,
$$

$$
\tag{7.15c} \overline{\phi}^{k+1} = (1-\alpha^k)\overline{\phi}^k + \alpha^k F(x_k^*(y^k)) + \frac{\alpha^k}{2}\Big(\frac{1}{L} - \frac{\alpha^k}{\gamma^{k+1}}\Big)\|g^k\|_2^2 + \alpha^k\langle g^k, z^k - y^k\rangle.
$$

Proof. A 4-part proof.

Part 1. Proving the sequence $\{\phi^k(x)\}$ satisfies (7.14). We first prove $\nabla^2 \phi^k(x) = \gamma^k I$ for all k by induction. The base case at $k = 0$ holds by taking the Hessian of $\phi^0(x) :=$ $F(x^0) + \frac{\gamma^0}{2} \|x - z^0\|_2^2$ from the assumption. Assuming the induction hypothesis $\nabla^2 \phi^k(x) = \gamma^k I$, now for the case $k + 1$, taking the Hessian of ϕ^{k+1} in (A7) gives

$$(7.16) \qquad \nabla^2 \phi^{k+1}(x) \overset{\text{(A7)}}{:=} (1 - \alpha^k)\nabla^2 \phi^k(x) \overset{\text{induction hypothesis}}{=} (1 - \alpha^k)\gamma^k I \overset{\text{(7.15a)}}{=} \gamma^{k+1} I.$$

Eq.(7.16) implies that for all k the function ϕ^k is a quadratic function in the form of (7.14), for some constants $\overline{\phi}^k, \gamma^k, z^k$.

Part 2. Proving the sequence $\{\gamma^k\}$ satisfies (7.15a). This is proved by the last equality in (7.16).

Part 3. Proving the sequence $\{z^k\}$ satisfies (7.15b). First we consider the gradient and Hessian of $\phi^{k+1}(x)$.

$$(7.17a) \quad \phi^{k+1}(x) \overset{\text{(A7),(7.14)}}{:=} (1 - \alpha^k)\left[\overline{\phi}^k + \frac{\gamma^k}{2}\|x - z^k\|_2^2\right] + \alpha^k\left[F(x_k^*(y^k)) + \langle g^k, x - y^k\rangle + \frac{1}{2L}\|g^k\|_2^2\right],$$

$$(7.17b) \qquad\qquad \nabla \phi^{k+1}(x) \overset{\text{(7.17a)}}{=} (1 - \alpha^k)\gamma^k(x - z^k) + \alpha^k g^k.$$

Setting $\nabla \phi^{k+1}(x) = 0$ in (7.17b) gives an expression of z^k

$$(7.18) \qquad\qquad \nabla \phi^{k+1}(x) \overset{\text{(7.17b)}}{=} (1 - \alpha^k)\gamma^k(x - z^k) + \alpha^k g^k = 0.$$

Now we take the gradient of ϕ^{k+1} using (7.14) gives

$$(7.19) \qquad\qquad \nabla \phi^{k+1}(x) = \gamma^{k+1}(x - z^{k+1}) = 0.$$

Combine (7.18), (7.19) and use (7.15a) gives (7.15b).

Part 4. Proving the sequence $\{\overline{\phi}^{k+1}\}$ satisfies (7.15c). Equating (7.14) and (7.17a) at $x = y^k$ gives

(7.20)
$$\overline{\phi}^{k+1} + \frac{\gamma^{k+1}}{2}\|y^k - z^{k+1}\|_2^2 \overset{\text{(7.14)}}{=} \phi^{k+1}(y^k) \overset{\text{(7.17a)}}{=} (1 - \alpha^k)\left[\overline{\phi}^k + \frac{\gamma^k}{2}\|y^k - z^k\|_2^2\right] + \alpha^k\left[F(x_k^*(y^k)) + \frac{1}{2L}\|g^k\|_2^2\right].$$

Note that

$$(7.21) \quad \frac{\gamma^{k+1}}{2}\|z^{k+1} - y^k\|_2^2 \overset{\text{(7.15b)}}{=} \frac{\gamma^{k+1}}{2}\|z^k - y^k\|_2^2 - \gamma^{k+1}\langle z^k - y^k, \frac{\alpha^k}{\gamma^{k+1}}g^k\rangle + \frac{\gamma^{k+1}}{2}\|\frac{\alpha^k g^k}{\gamma^{k+1}}\|_2^2.$$

Combine (7.20), (7.21) and using (7.15a) gives (7.15c). □

At this stage we recall that by (A3) in Lemma 7.1, the sequence $\{y^k\}_{k\in\mathbb{N}}$ is "free". In view of Lemma 7.2, we would like to construct y^k such that $\overline{\phi}^{k+1} \geq F(x_k^*(y^k)) =: F(x^{k+1})$, which holds for $k = 0$ that $\overline{\phi}^0 = F(x^0)$. That is, we define $x^{k+1} = x_k^*(y^k) \overset{\text{(7.10)}}{:=}$ argmin $M_k(x; y^k)$

The following lemma resembles [24, Theorem 3] that reveals the importance of estimate sequence and also the condition $\overline{\phi}^{k+1} \geq F(x_k^*(y^k)) =: F(x^{k+1})$. The lemma differs from [35, Lemma 2.2.1] on smooth convex optimization.

LEMMA 7.3. *For minimization problem (7.7), assume $x^* \in X^* := \text{argmin } F(x)$ exists and denote $F^* := F(x^*)$. Suppose $F(x^k) \leq \overline{\phi}^k := \min_x \phi_k(x)$ holds for a sequence $\{x^k\}_{k\in\mathbb{N}}$, where*

$\{\phi^k, \lambda^k\}_{k\in\mathbb{N}}$ *is an estimate sequence of F, and we define* $\phi^0 := F(x^0) + \dfrac{\gamma^0}{2}\|x^0 - x^*\|_2^2$, *then we have for all* $k \in \mathbb{N}$ *that*

$$F(x^k) - F^* \leq \lambda^k\Big[F(x^0) + \frac{\gamma^0}{2}\|x^0 - x^*\|_2^2 - F^*\Big].$$

Proof. Starting from the assumption $F(x^k) \leq \overline{\phi}^k := \min_x \phi_k(x)$,

$$F(x^k) \leq \overline{\phi}^k =: \min_x \phi^k \overset{\text{(Def 2)}}{\leq} \min_x (1 - \lambda^k)F(x) + \lambda^k\phi^0(x) \leq (1 - \lambda^k)F^* + \lambda^k\phi^0(x^*),$$

$$\implies F(x^k) - F^* \leq \lambda^k\Big[\phi^0(x^*) - F^*\Big] = \lambda^k\Big[F(x^0) + \frac{\gamma^0}{2}\|x^0 - x^*\|_2^2 - F^*\Big]. \qquad \square$$

Lemma 7.3 tells that the convergence rate of $\{F(x^k) - F^*\}_{k\in\mathbb{N}}$ follows the convergence rate of $\{\lambda^k\}_{k\in\mathbb{N}}$ with $\{\lambda^k\}_{k\in\mathbb{N}} \overset{\text{(Def 0)}}{>} 0$ and $\{\lambda^k\}_{k\in\mathbb{N}} \overset{\text{(Def 1)}}{\to} 0$, and thus finding the convergence rate of $\{\lambda^k\}_{k\in\mathbb{N}}$ gives the convergence rate of $\{F(x^k) - F^*\}_{k\in\mathbb{N}}$.

The following lemma resembles [35, Lemma 2.2.1].

THEOREM 7.4. *Suppose* $F(x^k) \leq \overline{\phi}^k := \min_x \phi_k(x)$ *holds for a sequence* $\{x^k\}_{k\in\mathbb{N}}$, *where* $\{\phi^k, \lambda^k\}_{k\in\mathbb{N}}$ *is an estimate sequence of F. Define* $\phi^0 := F(x^0) + \dfrac{\gamma^0}{2}\|x^0 - x^*\|_2^2$. *Assuming all the conditions in Lemma 7.1, Lemma 7.2 and Lemma 7.3. Then we have*

$$0 < \lambda^k < \frac{4L}{(2\sqrt{L} - \sqrt{\gamma^0})^2 + 2\big(2\sqrt{L}\sqrt{\gamma^0} - \sqrt{\gamma^0}\big)\sqrt{\gamma^0}k + (\sqrt{\gamma^0}k)^2}.$$

Proof. A long, highly-involved tedious mechanical proof. We start with γ^{k+1} in (7.15a):

$$(7.22) \qquad \gamma^{k+1} \overset{(7.15a)}{=} (1 - \alpha^k)\gamma^k \overset{(7.15a)}{=} \cdots \overset{(7.15a)}{=} \prod_{i=0}^{k}(1 - \alpha^i)\gamma^0 \overset{(A5),(A6)}{=} \lambda^{k+1}\gamma^0.$$

Next, by the first step in Algorithm FMGProx that $L(\alpha^k)^2 = (1 - \alpha^k)\gamma^k \overset{(7.15a)}{=:} \gamma^{k+1}$, solving for the root α^k gives

$$(7.23) \qquad \alpha^k = \pm\sqrt{\frac{\gamma^{k+1}}{L}} \overset{(7.22)}{=} \pm\sqrt{\frac{\lambda^{k+1}\gamma^0}{L}}.$$

Now consider λ^k in (A6):

$$(7.24) \qquad 1 - \frac{\lambda^{k+1}}{\lambda^k} \overset{(A6)}{=} \alpha^k \overset{(7.23)}{=} \pm\sqrt{\frac{\gamma^{k+1}}{L}} \overset{(7.22)}{=} \pm\sqrt{\frac{\lambda^{k+1}\gamma^0}{L}}.$$

Now by (A4a),(A6) and (A7) we have for all k that $\{\lambda^k\}_{k\in\mathbb{N}}$ is a *strictly* nonnegative sequence and a *strictly* monotonic decreasing sequence, i.e.:

$$(7.25a) \qquad \lambda^k \overset{(A4a),(A6)}{=} \prod_{i=0}^{k}(1 - \alpha^k) \overset{(A4a)}{>} 0,$$

$$(7.25b) \qquad \lambda^{k+1} \overset{(A4a),(A6)}{<} \lambda^{k} \overset{(7.25a)}{\Longleftrightarrow} \frac{1}{\sqrt{\lambda^{k}}} < \frac{1}{\sqrt{\lambda^{k+1}}}.$$

Hence we can divide (7.24) by $\lambda^{k+1} \overset{(7.25a)}{>} 0$:

$$(7.26) \qquad \frac{1}{\lambda^{k+1}} - \frac{1}{\lambda^{k}} = \pm \frac{1}{\sqrt{\lambda^{k+1}}} \sqrt{\frac{\gamma^{0}}{L}}.$$

On the left hand side of (7.26), by the fact that $\lambda^{k} \overset{(7.25a)}{>} 0$ for all k, we have

$$(7.27) \quad \frac{1}{\lambda^{k+1}} - \frac{1}{\lambda^{k}} = \left(\frac{1}{\sqrt{\lambda^{k+1}}} + \frac{1}{\sqrt{\lambda^{k}}} \right) \left(\frac{1}{\sqrt{\lambda^{k+1}}} - \frac{1}{\sqrt{\lambda^{k}}} \right) \overset{(7.25b)}{<} \frac{2}{\sqrt{\lambda^{k+1}}} \left(\frac{1}{\sqrt{\lambda^{k+1}}} - \frac{1}{\sqrt{\lambda^{k}}} \right).$$

Now combine (7.26) and (7.27) with $\gamma^{0} > 0$ and $L > 0$ gives

$$\pm \frac{1}{\sqrt{\lambda^{k+1}}} \sqrt{\frac{\gamma^{0}}{L}} < \frac{2}{\sqrt{\lambda^{k+1}}} \left(\frac{1}{\sqrt{\lambda^{k+1}}} - \frac{1}{\sqrt{\lambda^{k}}} \right) \implies -\frac{1}{2}\sqrt{\frac{\gamma^{0}}{L}} < \frac{1}{2}\sqrt{\frac{\gamma^{0}}{L}} < \frac{1}{\sqrt{\lambda^{k+1}}} - \frac{1}{\sqrt{\lambda^{k}}}.$$

Let $\theta^{k} := \dfrac{1}{\sqrt{\lambda^{k}}}$ be an increasing sequence (since λ^{k} is decreasing), we have

$$\theta^{k+1} > \theta^{k} + \frac{1}{2}\sqrt{\frac{\gamma^{0}}{L}} > \theta^{k-1} + \frac{1}{2}\sqrt{\frac{\gamma^{0}}{L}} + \frac{1}{2}\sqrt{\frac{\gamma^{0}}{L}} > \cdots \overset{\lambda^{0}=1}{>} 1 + \frac{k}{2}\sqrt{\frac{\gamma^{0}}{L}}.$$

Thus $\dfrac{1}{\sqrt{\lambda^{k+1}}} = \theta^{k+1} > 1 + \dfrac{k}{2}\sqrt{\dfrac{\gamma^{0}}{L}}$ gives $0 \overset{(Def\ 0)}{<} \sqrt{\lambda^{k+1}} < \dfrac{1}{1 + \dfrac{k}{2}\sqrt{\dfrac{\gamma^{0}}{L}}}$ and hence

$$0 < \lambda^{k+1} < \frac{1}{1 + k\sqrt{\dfrac{\gamma^{0}}{L}} + \dfrac{k^{2}}{4}\dfrac{\gamma^{0}}{L}} \implies 0 < \lambda^{k} < \frac{1}{1 + (k-1)\sqrt{\dfrac{\gamma^{0}}{L}} + \dfrac{(k-1)^{2}}{4}\dfrac{\gamma^{0}}{L}}$$

$$\implies 0 < \lambda^{k} < \frac{4L}{(2\sqrt{L} - \sqrt{\gamma^{0}})^{2} + 2\left(2\sqrt{L}\sqrt{\gamma^{0}} - \sqrt{\gamma^{0}}\right)\sqrt{\gamma^{0}}k + (\sqrt{\gamma^{0}}k)^{2}}. \qquad \square$$

Lastly, the following corollary gives the proof of Theorem 3.1 in the main text.

COROLLARY 7.5. *For the sequence $\{x^{k}\}$ produced by Algorithm FMGProx, we have*

$$F(x^{k}) - F^{*} \leq \frac{4L\left(F(x^{0}) + \dfrac{\gamma^{0}}{2}\|x^{0} - x^{*}\|_{2}^{2} - F^{*}\right)}{(2\sqrt{L} - \sqrt{\gamma^{0}})^{2} + 2\gamma^{0}\left(2\sqrt{L} - 1\right)k + (\sqrt{\gamma^{0}}k)^{2}}.$$

Proof. Combine Lemma 7.3 and Theorem 7.4. $\qquad \square$

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