A Test Instance Generator for Multiobjective Mixed-integer Optimization

Gabriele Eichfelder* Tobias Gerlach†† Leo Warnow‡‡

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Abstract

Application problems can often not be solved adequately by numerical algorithms as several difficulties might arise at the same time. When developing and improving algorithms which hopefully allow to handle those difficulties in the future, good test instances are required. These can then be used to detect the strengths and weaknesses of different algorithmic approaches. In this paper we present a generator for test instances to evaluate solvers for multiobjective mixed-integer linear and nonlinear optimization problems. Based on test instances for purely continuous and purely integer problems with known efficient solutions and known nondominated points, suitable multiobjective mixed-integer test instances can be generated. The special structure allows to construct instances scalable in the number of variables and objective functions. Moreover, it allows to control the resulting efficient and nondominated sets as well as the number of efficient integer assignments.

Key Words: multiobjective optimization, mixed-integer optimization, test instances, nonconvex optimization.

Mathematics subject classifications (MSC 2010): 90C11, 90C26, 90C29, 90C30.

*Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0002-1938-6316, gabriele.eichfelder@tu-ilmenau.de

††Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0002-5074-5284, tobias.gerlach@tu-ilmenau.de

‡‡Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0002-2177-8466, leo.warnow@tu-ilmenau.de

1 Introduction

Optimization problems that arise within practical applications often turn out to be very challenging for solution algorithms from a numerical point of view. Thereby, the challenges can be caused, for instance, by a high number of variables or objective functions as well as certain properties of the objective and constraint functions including nonlinearity or nonconvexity. Hence, there is a need to evaluate the strengths and weaknesses of different solution algorithms. This is important to decide which of them might perform best for a specific type of optimization problems or how the algorithm can be improved to do so in the future. This is typically done by evaluating the performance of an algorithm on a set of certain test instances that cover the above mentioned challenges. In this paper we present a generator for such test instances that yields multiobjective mixed-integer optimization problems. This means that multiple objective functions have to be optimized at the same time and that some of the variables are continuous while others are only allowed to take integer values.

When it comes to multiobjective mixed-integer optimization, most of the literature focuses on multiobjective mixed-integer linear optimization problems. This includes, for instance, the Triangle Splitting Method from [1] and the Boxed Line Method from [22] for the biobjective setting, as well as the GoNDEF algorithm from [23] for an arbitrary number of objective functions. For a comprehensive overview of algorithmic approaches to solve multiobjective mixed-integer linear optimization problems we refer to [16]. To the best of our knowledge, the first deterministic solution method for multiobjective mixed-integer convex optimization problems is given in [5]. Only recently, also a solution approach for multiobjective mixed-integer nonconvex optimization problems has been presented in [10]. Further solution methods can be found for multiobjective mixed-integer convex optimization problems in [11, 13], for biobjective mixed-integer convex optimization problems in [3, 9], for biobjective mixed-integer quadratic optimization problems in [18], and for multiobjective mixed-integer nonconvex optimization problems in [19], respectively.

While a relatively large number of (even scalable) test instances exists for multiobjective continuous optimization (e.g. [2, 4, 8, 14, 15, 17, 24]), so far only a limited number of test instances for multiobjective mixed-integer optimization problems was introduced and used for numerical testing. Such instances can be found for the convex case in [5, 11, 12, 13, 18, 21] and for the nonconvex case in [3, 10, 19, 20], respectively. Note that these test instances consist of 14 biobjective and four triobjective mixed-integer problems. Only one test problem is scalable in the number of objective functions (by using the identity for the continuous variables).

However, for a systematic evaluation of the strengths and weaknesses of solution algorithms for multiobjective mixed-integer optimization problems more test instances are needed. For instance, there is a demand for test instances that allow to investigate the influence of the number of continuous and integer variables on the performance of the solver, i.e., test instances scalable in the number of variables. This property is especially useful to evaluate the performance of decision space based solution approaches and to compare them with criterion space based solution approaches. Typically, one would expect that decision space based approaches are more influenced by the size of the decision space than criterion space based approaches. In this regard it should be noted that only five of the above mentioned test instances from the literature are scalable in the number of integer variables, and only three of them are additionally scalable in the number of continuous variables. Further important properties of test instances to evaluate the performance of different solution algorithms are for instance the number of so called feasible integer assignments and of efficient integer assignments. Especially for algorithms that decompose the mixed-integer optimization problem into a family of purely continuous optimization problems obtained for certain fixings of the integer variables (see also the forthcoming Remark 2.1) these are of high importance.

Taking all of this into account, the new generator for test instances allows to vary the numbers of variables as well as the number of objective functions depending on its input. The proposed method generates test instances that possess a separable structure. This is also the case for several of the known test instances for multiobjective mixed-integer optimization problems from the literature. Thereby, separable means that these test instances can be decomposed into a multiobjective continuous subproblem and into a multiobjective integer subproblem. We will show that under certain assumptions this provides us with full control over the resulting efficient and nondominated sets of the test instances, as well as their number of efficient integer assignments. This allows to verify the correctness of the results from a solution algorithm for multiobjective mixed-integer optimization problems and to evaluate the quality of its output. Especially for the evaluation of nondeterministic or heuristic approaches this is an important feature.

The remaining paper is structured as follows. In Section 2 we briefly present the notations and definitions that are used within this paper. We then analyze separable multiobjective mixed-integer optimization problems in Section 3. Based on this, in Section 4, we provide the test instance generator for multiobjective mixed-integer optimization problems. Finally, some (scalable) multiobjective continuous and integer optimization problems as possible inputs for the generator are listed and discussed in Subsection 4.1 and Subsection 4.2, respectively.

2 Notations and Definitions

For a positive integer $p \in \mathbb{N}$ and real numbers $a, b \in \mathbb{R}$ with $a \leq b$ we use the notations $[p] := \{1, \ldots, p\}$, $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$, and $]a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$. Moreover, the inequalities $\leq$ and $<$ between vectors are understood componentwise, i.e., for $x, x' \in \mathbb{R}^p$ it holds $x \leq x'$ or $x < x'$ if and only if $x_i \leq x'_i$ or $x_i < x'_i$ is fulfilled for all $i \in [p]$, respectively. Based on this we denote for $l, u \in \mathbb{R}^p$ with $l \leq u$ by $[l, u] := \{y \in \mathbb{R}^p \mid l \leq y \leq u\}$ the box with lower bound l and upper bound u. For two vectors $x, x' \in \mathbb{R}^p$ their *Hadamard product* is defined by $x \circ x' := (x_1 x'_1, \ldots, x_p x'_p)$. Finally, for a nonempty set $\Omega \subseteq \mathbb{R}^p$ we denote its cardinality by $|\Omega|$ and its convex hull by $\operatorname{conv}(\Omega)$.

In the following we consider multiobjective mixed-integer optimization problems, i.e., multiobjective optimization problems defined by

$$\begin{aligned} \min_x \ & f(x) \\ \text{s.t.} \ & g(x) \leq 0_q, \\ & x \in X := X_C \times X_I. \end{aligned} \tag{MOMIP}$$

Thereby, let $n, m \in \mathbb{N}_0$ with $n \geq 1$ or $m \geq 1$ and let $f_i \colon \mathbb{R}^{n+m} \to \mathbb{R}$, $i \in [p]$, $p \geq 2$, $g_j \colon \mathbb{R}^{n+m} \to \mathbb{R}$, $j \in [q]$ be continuous functions, where $f = (f_1, \ldots, f_p) \colon \mathbb{R}^{n+m} \to \mathbb{R}^p$, $g = (g_1, \ldots, g_q) \colon \mathbb{R}^{n+m} \to \mathbb{R}^q$, and $0_q := 0_{\mathbb{R}^q}$. Moreover, let $X_C := [l_C, u_C] \subseteq \mathbb{R}^n$ be a box with $l_C, u_C \in \mathbb{R}^n$, let $X_I := [l_I, u_I] \cap \mathbb{Z}^m$ be a finite subset of $\mathbb{Z}^m$ with $l_I, u_I \in \mathbb{Z}^m$ and let the feasible set $S := \{x \in \mathbb{R}^{n+m} \mid g(x) \leq 0_q,\ x \in X\}$ of (MOMIP) be nonempty.

For the variables $x \in X$ of (MOMIP), we will write in the following $x = (x_C, x_I)$ with $x_C \in X_C$ and $x_I \in X_I$ to distinguish between the continuous and the integer variables. We call (MOMIP) a multiobjective mixed-integer convex optimization problem if all involved functions $f_i, i \in [p]$ and $g_j, j \in [q]$ are convex. Otherwise, we call it a multiobjective mixed-integer nonconvex optimization problem.

Obviously, for $m = 0$ and for $n = 0$ the special cases of a multiobjective continuous optimization problem (MOP) given by

$$\begin{aligned} \min_x\ & f(x) \\ \text{s.t. } & g(x) \leq 0_q, \\ & x \in X_C \end{aligned} \tag{MOP}$$

and of a multiobjective integer optimization problem (MOIP) given by

$$\begin{aligned} \min_x\ & f(x) \\ \text{s.t. } & g(x) \leq 0_q, \\ & x \in X_I \end{aligned} \tag{MOIP}$$

are included in (MOMIP).

Recall that a feasible point $\bar{x} \in S$ is called an *efficient solution* of (MOMIP) if there exists no $x \in S$ with $f(x) \leq f(\bar{x})$ and $f(x) \neq f(\bar{x})$. Moreover, a point $\bar{y} = f(\bar{x})$ with $\bar{x} \in S$ is called a *nondominated point* of (MOMIP) if $\bar{x}$ is an efficient solution. The set $\mathcal{E} \subseteq S$ denotes the set of all efficient solutions (also called the *efficient set*) and the set $\mathcal{N} \subseteq f(S)$ denotes the set of all nondominated points (also called the *nondominated set*) of (MOMIP). If for all $x \in S$ there exists an $\bar{x} \in \mathcal{E}$ such that $f(\bar{x}) \leq f(x)$, then the multiobjective optimization problem is said to satisfy the so called *domination property*. Note that under our assumptions (MOMIP), and thus also the special cases (MOP) and (MOIP), fulfill the domination property and it holds $\mathcal{E} \neq \emptyset$ and $\mathcal{N} \neq \emptyset$.

For the multiobjective mixed-integer optimization problem (MOMIP) we call $x_I \in X_I$ a *feasible integer assignment* if there exists $x_C \in X_C$ such that (x_C, x_I) is feasible for (MOMIP). Analogously, we call $x_I \in X_I$ an *efficient integer assignment* if there exists $x_C \in X_C$ such that $(x_C, x_I) \in \mathcal{E}$, i.e., (x_C, x_I) is an efficient solution of (MOMIP). We denote by S_I the set of all feasible integer assignments and by $\mathcal{E}_I$ the set of all efficient integer assignments of (MOMIP).

Remark 2.1 *The absolute number (and the percentage) of efficient integer assignments within the set of feasible integer assignments is an important characteristic of a multiobjective mixed-integer optimization problem. It might also influence the performance of numerical algorithms depending on how they are constructed. Note that this is a significant difference to the singleobjective setting: in singleobjective mixed-integer optimization the optimal value is unique and hence it is enough to find one optimal integer assignment. For $p \geq 2$ there are in general infinitely many nondominated points. As a consequence, there can be instances with a large number of efficient integer assignments which lead to different nondominated points. Even all feasible integer assignments can be efficient. Hence, algorithms that decompose* (MOMIP) *into a family of purely continuous problems may be forced to a full enumeration in that case.*

3 Separable Multiobjective Mixed-integer Optimization Problems

Many of the known test instances for multiobjective mixed-integer (nonlinear) optimization have a common structure, which we will formalize and study in this section. The results form the basis for our approach regarding the test instance generator.

A multiobjective mixed-integer optimization problem (MOMIP) which can be formulated by a decomposition

$$\begin{aligned} \min_{x=(x_C,x_I)} \ & f_C(x_C) + f_I(x_I) \\ \text{s.t. } & g_C(x_C) \leq 0_{q_C}, \\ & g_I(x_I) \leq 0_{q_I}, \\ & x \in X = X_C \times X_I \end{aligned} \tag{sMOMIP}$$

with continuous objective functions $f_C\colon \mathbb{R}^n \to \mathbb{R}^p$, $f_I\colon \mathbb{R}^m \to \mathbb{R}^p$ with $n, m \in \mathbb{N}$, and continuous constraint functions $g_C\colon \mathbb{R}^n \to \mathbb{R}^{q_C}$, $g_I\colon \mathbb{R}^m \to \mathbb{R}^{q_I}$ is called *separable*. To the separable multiobjective mixed-integer optimization problem (sMOMIP) we formulate the following two subproblems: the multiobjective continuous subproblem

$$\begin{aligned} \min_{x_C} \ & f_C(x_C) \\ \text{s.t. } & g_C(x_C) \leq 0_{q_C}, \\ & x_C \in X_C \end{aligned} \tag{sMOMIP$_C$}$$

and the multiobjective integer subproblem

$$\begin{aligned} \min_{x_I} \ & f_I(x_I) \\ \text{s.t. } & g_I(x_I) \leq 0_{q_I}, \\ & x_I \in X_I. \end{aligned} \tag{sMOMIP$_I$}$$

The efficient solutions and the nondominated points of the subproblems and of the original separable problem are related what we discuss next. We denote by $S_C^{\mathrm{s}}/\mathcal{E}_C^{\mathrm{s}}/\mathcal{N}_C^{\mathrm{s}}$ and by $S_I^{\mathrm{s}}/\mathcal{E}_I^{\mathrm{s}}/\mathcal{N}_I^{\mathrm{s}}$ the feasible set/the set of all efficient solutions/the set of all nondominated points of (sMOMIP$_C$) and of (sMOMIP$_I$), respectively. It is easy to see that for a separable multiobjective mixed-integer optimization problem (sMOMIP) and the corresponding subproblems (sMOMIP$_C$) and (sMOMIP$_I$) it holds $S = S_C^{\mathrm{s}} \times S_I^{\mathrm{s}}$ and $S_I = S_I^{\mathrm{s}}$. Here, S denotes the feasible set and S_I denotes the set of all feasible integer assignments of (sMOMIP). Note that under our assumptions it holds $S_C^{\mathrm{s}} \neq \emptyset$ and $S_I^{\mathrm{s}} \neq \emptyset$, and both subproblems fulfill the domination property. Hence, it holds $\mathcal{E}_C^{\mathrm{s}} \neq \emptyset$, $\mathcal{E}_I^{\mathrm{s}} \neq \emptyset$, $\mathcal{N}_C^{\mathrm{s}} \neq \emptyset$, and $\mathcal{N}_I^{\mathrm{s}} \neq \emptyset$. We illustrate such a decomposable mixed-integer optimization problem with the following example.

Example 3.1 *The separable biobjective mixed-integer nonconvex optimization problem given by*

$$\begin{aligned} \min_x \ & \begin{pmatrix} 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) + x_{n+1} + x_{n+2} \\ 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) - x_{n+1} - x_{n+2} \end{pmatrix} \\ \textit{s.t. } & x \in X = [-4,4]^n \times ([-1,1]^2 \cap \mathbb{Z}^2). \end{aligned} \tag{3.1}$$

can be decomposed into the (well known) biobjective continuous subproblem introduced by Fonseca and Fleming in [14] and into the biobjective integer linear subproblem given by

$$\begin{aligned} &\min_x \begin{pmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix} \\ &\text{s.t. } x \in X_I = \{-1, 0, 1\}^2 \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} \mathcal{E}_I^{\mathrm{s}} &= X_I = \{-1,0,1\}^2, \\ \mathcal{N}_I^{\mathrm{s}} &= \{(\delta, -\delta) \mid \delta \in \{-2,-1,0,1,2\}\} = \{(-2,2),(-1,1),(0,0),(1,-1),(2,-2)\}. \end{aligned}$$

For further explanations regarding the corresponding sets $\mathcal{E}_C^{\mathrm{s}}$, $\mathcal{N}_C^{\mathrm{s}}$, $\mathcal{E}$, $\mathcal{N}$, and $\mathcal{E}_I$ we refer to (4.6), (4.7) and Example 4.8 (i). For an illustration of the nondominated sets $\mathcal{N}_I^{\mathrm{s}}$ and $\mathcal{N}$ see Figure 1.

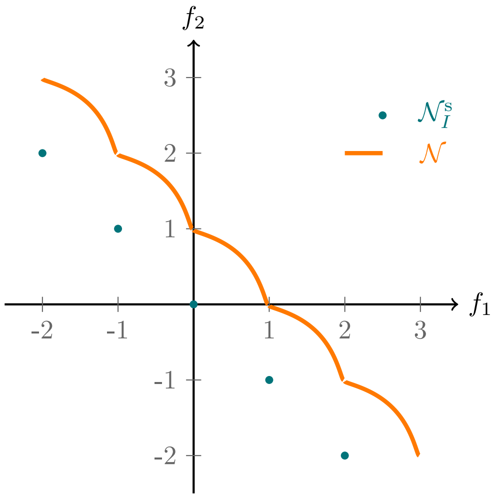

Figure 1: Nondominated set $\mathcal{N}$ of the separable optimization problem (3.1) and nondominated set $\mathcal{N}_I^{\mathrm{s}}$ of the integer subproblem (3.2) from Example 3.1.

With the following lemma we start the examination of the relations between the efficient solutions and the nondominated points regarding the subproblems and the original separable problem.

Lemma 3.2 *Let $\mathcal{E}$ denote the set of all efficient solutions, $\mathcal{N}$ the set of all nondominated points, and $\mathcal{E}_I$ the set of all efficient integer assignments of* (sMOMIP)*. Then for the corresponding subproblems* (sMOMIP$_C$) *and* (sMOMIP$_I$) *it holds:*

(i) $\mathcal{E} \subseteq \mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}}$.

(ii) $\mathcal{N} \subseteq \mathcal{N}_C^{\mathrm{s}} + \mathcal{N}_I^{\mathrm{s}}$.

(iii) $\mathcal{E}_I \subseteq \mathcal{E}_I^{\mathrm{s}}$.

Proof. The relations (ii) and (iii) follow immediately by (i). For the proof of (i) let $\bar{x} = (\bar{x}_C, \bar{x}_I) \in \mathcal{E} \subseteq S = S_C^{\mathrm{s}} \times S_I^{\mathrm{s}}$ and assume that $\bar{x}_C \notin \mathcal{E}_C^{\mathrm{s}}$. Then there exists

$\hat{x}_C \in S_C^{\mathrm{s}}$ such that $f_C(\hat{x}_C) \leq f_C(\bar{x}_C)$ and $f_C(\hat{x}_C) \neq f_C(\bar{x}_C)$. Thus we obtain for $x = (\hat{x}_C, \bar{x}_I) \in S_C^{\mathrm{s}} \times S_I^{\mathrm{s}} = S$ that

$$f(x) = f_C(\hat{x}_C) + f_I(\bar{x}_I) \leq f_C(\bar{x}_C) + f_I(\bar{x}_I) = f(\bar{x}) \text{ and } f(x) \neq f(\bar{x}),$$

which contradicts $\bar{x} \in \mathcal{E}$. The proof for $\bar{x}_I \in \mathcal{E}_I^{\mathrm{s}}$ is analogous. □

Note that equality for (i), (ii) and (iii) from Lemma 3.2 is trivially fulfilled in the scalar-valued setting $p = 1$. In the vector-valued case $p \geq 2$ these equalities do, in general, not hold as the following example shows.

Example 3.3 *We consider the separable biobjective mixed-integer optimization problem*

$$\begin{aligned} \min_x & \begin{pmatrix} x_1 + x_2 & + 0.75x_3 \\ -x_1 - x_2 & - 0.25x_3 \end{pmatrix} \\ s.t. \ & x \in X = [0,1] \times ([-1,1] \times [0,1]) \cap \mathbb{Z}^2. \end{aligned} \tag{3.3}$$

It can be decomposed into the biobjective continuous subproblem

$$\begin{aligned} \min_x & \begin{pmatrix} x \\ -x \end{pmatrix} \\ s.t. \ & x \in X_C = [0,1] \end{aligned} \tag{3.4}$$

and the biobjective integer subproblem

$$\begin{aligned} \min_x & \begin{pmatrix} x_1 & + 0.75x_2 \\ -x_1 & - 0.25x_2 \end{pmatrix} \\ s.t. \ & x \in X_I = \{-1,0,1\} \times \{0,1\}. \end{aligned} \tag{3.5}$$

The efficient and nondominated sets of (3.3), (3.4) *and* (3.5) *are given by*

$$\begin{aligned} \mathcal{E}_C^{\mathrm{s}} &= X_C, \\ \mathcal{E}_I^{\mathrm{s}} &= X_I, \\ \mathcal{E} &= \big([0,1] \times \{(-1,0),(0,0),(1,0)\}\big) \cup \big(]\tfrac{3}{4},1] \times (1,1)\big), \\ \mathcal{N}_C^{\mathrm{s}} &= \operatorname{conv}\{(0,0),(1,-1)\}, \\ \mathcal{N}_I^{\mathrm{s}} &= \{(-1,1),(0,0),(1,-1),(-\tfrac{1}{4},\tfrac{3}{4}),(\tfrac{3}{4},-\tfrac{1}{4}),(\tfrac{7}{4},-\tfrac{5}{4})\}, \ \textit{and} \\ \mathcal{N} &= \big(\operatorname{conv}\{(0,0),(1,-1)\} + \{(-1,1),(0,0),(1,-1)\}\big) \\ & \quad \cup \Big(\big(\operatorname{conv}\{(\tfrac{3}{4},-\tfrac{3}{4}),(1,-1)\} \setminus \{(\tfrac{3}{4},-\tfrac{3}{4})\}\big) + \{(\tfrac{7}{4},-\tfrac{5}{4})\}\Big). \end{aligned}$$

Hence, we obtain $\mathcal{E}_I = \{(-1,0),(0,0),(1,0),(1,1)\} \subsetneq \mathcal{E}_I^{\mathrm{s}}$ *and* $\mathcal{N} \subsetneq \mathcal{N}_C^{\mathrm{s}} + \mathcal{N}_I^{\mathrm{s}}$. *In Figure 2 we provide an illustration of the nondominated sets* $\mathcal{N}$ *and* $\mathcal{N}_I^{\mathrm{s}}$.

For the test instance generator in the forthcoming Section 4 we intend to have (as much as possible) control over the efficient sets and over the nondominated sets of the resulting test instances. Hence, we are interested in additional assumptions to ensure equality for the statements of Lemma 3.2. The following theorem provides a sufficient condition for that.

Theorem 3.4 *Let* (sMOMIP) *be given and let* $\Delta_C \in \mathbb{R}^p, \Delta_I \in (\mathbb{R} \cup \{\infty\})^p$ *be the vectors defined by*

$$\begin{aligned} \Delta_{C,i} &:= \sup\{y_i - \hat{y}_i \mid y, \hat{y} \in \mathcal{N}_C^{\mathrm{s}}\}, \\ \Delta_{I,i} &:= \inf\{|y_i - \hat{y}_i| \mid y, \hat{y} \in \mathcal{N}_I^{\mathrm{s}}, \ y_i \neq \hat{y}_i\} \end{aligned}$$

for all $i \in [p]$. *If* $\Delta_C < \Delta_I$, *then it holds:*

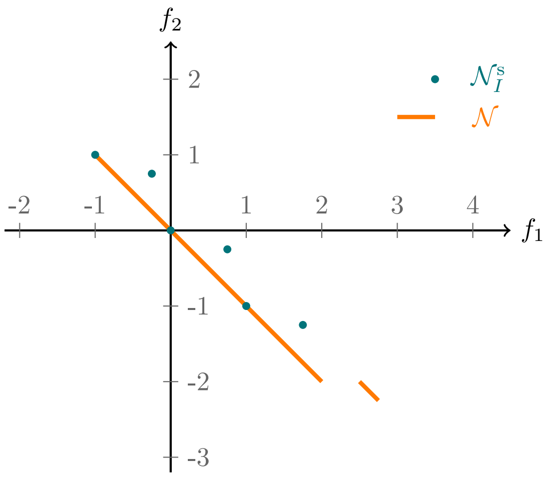

Figure 2: Nondominated set $\mathcal{N}$ of the separable optimization problem (3.3) and nondominated set $\mathcal{N}_I^{\mathrm{s}}$ of the integer subproblem (3.5) from Example 3.3.

(i) $\mathcal{E} = \mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}}$.

(ii) $\mathcal{N} = \mathcal{N}_C^{\mathrm{s}} + \mathcal{N}_I^{\mathrm{s}}$.

(iii) $\mathcal{E}_I = \mathcal{E}_I^{\mathrm{s}}$.

Proof. The equalities (ii) and (iii) follow immediately by (i). Moreover, for the proof of (i), by Lemma 3.2 (i), it only remains to show that $\mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}} \subseteq \mathcal{E}$. Hence, let $\Delta_C < \Delta_I$ and assume that there exist $\hat{x}_C \in \mathcal{E}_C^{\mathrm{s}}$ and $\hat{x}_I \in \mathcal{E}_I^{\mathrm{s}}$ such that $\hat{x} = (\hat{x}_C, \hat{x}_I) \notin \mathcal{E}$. Then it holds $f_C(\hat{x}_C) \in \mathcal{N}_C^{\mathrm{s}}$, $f_I(\hat{x}_I) \in \mathcal{N}_I^{\mathrm{s}}$ and $\hat{x} \in S \setminus \mathcal{E}$. Thus, by the domination property and Lemma 3.2 (i), there exists $\bar{x} = (\bar{x}_C, \bar{x}_I) \in \mathcal{E} \subseteq \mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}}$ such that $f_C(\bar{x}_C) \in \mathcal{N}_C^{\mathrm{s}}$, $f_I(\bar{x}_I) \in \mathcal{N}_I^{\mathrm{s}}$ and

$$f_C(\bar{x}_C) + f_I(\bar{x}_I) = f(\bar{x}) \leq f(\hat{x}) = f_C(\hat{x}_C) + f_I(\hat{x}_I) \text{ and } f(\bar{x}) \neq f(\hat{x}). \tag{3.6}$$

Hence, we obtain by the definition of $\Delta_C \in \mathbb{R}^p$ that

$$f_{I,i}(\bar{x}_I) - f_{I,i}(\hat{x}_I) \leq f_{C,i}(\hat{x}_C) - f_{C,i}(\bar{x}_C) \leq \Delta_{C,i} < \Delta_{I,i} \tag{3.7}$$

for all $i \in [p]$. Assume now that there exists $i \in [p]$ such that $f_{I,i}(\bar{x}_I) > f_{I,i}(\hat{x}_I)$ (and thus $f_I(\bar{x}_I) \neq f_I(\hat{x}_I)$ and $\Delta_{I,i} \in \mathbb{R}$). Then it follows by (3.7) and by the definition of $\Delta_{I,i}$ that

$$0 < f_{I,i}(\bar{x}_I) - f_{I,i}(\hat{x}_I) < \Delta_{I,i} \leq |f_{I,i}(\bar{x}_I) - f_{I,i}(\hat{x}_I)| = f_{I,i}(\bar{x}_I) - f_{I,i}(\hat{x}_I),$$

which is obviously a contradiction. Hence, we obtain $f_I(\bar{x}_I) \leq f_I(\hat{x}_I)$ and by $\hat{x}_I \in \mathcal{E}_I^{\mathrm{s}}$ this implies $f_I(\bar{x}_I) = f_I(\hat{x}_I)$. Thus, by (3.6) it follows $f_C(\bar{x}_C) \leq f_C(\hat{x}_C)$ and by $\hat{x}_C \in \mathcal{E}_C^{\mathrm{s}}$ we conclude $f_C(\bar{x}_C) = f_C(\hat{x}_C)$. As a result, we get $f(\bar{x}) = f(\hat{x})$, which contradicts (3.6), and (i) is proven. □

In Example 3.3 the sufficient condition to ensure equality by Theorem 3.4 is not fulfilled since $\Delta_{C,1} = \Delta_{C,2} = 1$ and $\Delta_{I,1} = \Delta_{I,2} = 0.25$.

Remark 3.5 *We remark that the infimum (instead of the minimum) in the definition of the components of* Δ_I *in Theorem 3.4 is only necessary since it could be the case that for some index* $i \in [p]$ *all nondominated points of* (sMOMIP$_I$) *share the same value. In that case the set* $\{|y_i - \hat{y}_i| \mid y, \hat{y} \in \mathcal{N}_I^{\mathrm{s}},\ y_i \neq \hat{y}_i\}$ *is empty. We then follow the standard convention that* $\inf(\emptyset) := +\infty$. *However, as long as there exist* $y, \hat{y} \in \mathcal{N}_I^{\mathrm{s}}$ *such that* $y_i \neq \hat{y}_i$ *the finiteness of the feasible set* $S_I^{\mathrm{s}} \subseteq \mathbb{Z}^m$, *in particular due to the box constraints* $x_I \in X_I$, *immediately implies that the infimum is actually a minimum and* $\Delta_{I,i} = \min\{|y_i - \hat{y}_i| \mid y, \hat{y} \in \mathcal{N}_I^{\mathrm{s}}, y_i \neq \hat{y}_i\} \in \mathbb{R}$.

For the third equality from Theorem 3.4, i.e. $\mathcal{E}_I = \mathcal{E}_I^{\mathrm{s}}$, it is actually sufficient that $\Delta_{C,i} < \Delta_{I,i}$ holds for $p-1$ indices $i \in [p]$ only. In particular, this means that for biobjective optimization problems (sMOMIP) the inequality $\Delta_{C,i} < \Delta_{I,i}$ needs to hold for only one index $i \in \{1, 2\}$. This is particularly useful for such algorithms as the one from [3] that assume prior knowledge of the set S_I of feasible integer assignments and hence would highly benefit if even the set $\mathcal{E}_I$ of efficient integer assignments would be known.

Lemma 3.6 *Let* (sMOMIP) *be given, let the vectors* Δ_C, Δ_I *be defined as in Theorem 3.4, and let* $j \in [p]$ *be an arbitrary index. If* $\Delta_{C,i} < \Delta_{I,i}$ *is fulfilled for all* $i \in [p] \setminus \{j\}$, *then it holds* $\mathcal{E}_I = \mathcal{E}_I^{\mathrm{s}}$.

Proof. By Lemma 3.2 (*iii*) it only remains to prove that $\mathcal{E}_I^{\mathrm{s}} \subseteq \mathcal{E}_I$. Hence, let $\Delta_{C,i} < \Delta_{I,i}$ for all $i \in [p] \setminus \{j\}$ and assume that there exists $\hat{x}_I \in \mathcal{E}_I^{\mathrm{s}} \setminus \mathcal{E}_I$. Moreover, let

$$\hat{x}_C \in \operatorname{argmin}\{f_{C,j}(x_C) \mid x_C \in S_C^{\mathrm{s}}\} \cap \mathcal{E}_C^{\mathrm{s}}. \tag{3.8}$$

Such a point $\hat{x}_C \in S_C^{\mathrm{s}}$ always exists since $f_{C,j}$ is continuous, S_C^{s} is compact, and the domination property holds for (sMOMIP$_C$). Since $\hat{x} = (\hat{x}_C, \hat{x}_I) \in S \setminus \mathcal{E}$ and the domination property holds for (sMOMIP), there exists $\bar{x} = (\bar{x}_C, \bar{x}_I) \in \mathcal{E}$ such that (3.6) holds. Further, by Lemma 3.2 (*i*) we have that $(\bar{x}_C, \bar{x}_I) \in \mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}}$ and with the same reasoning as in the proof of Theorem 3.4 it holds (3.7) and $f_{I,i}(\bar{x}_I) \leq f_{I,i}(\hat{x}_I)$ but only for all $i \in [p] \setminus \{j\}$. Assume now that also for j-th component it holds $f_{I,j}(\bar{x}_I) \leq f_{I,j}(\hat{x}_I)$. Then we obtain $f_I(\bar{x}_I) \leq f_I(\hat{x}_I)$ and since $\hat{x}_I \in \mathcal{E}_I^{\mathrm{s}}$ this implies $f_I(\bar{x}_I) = f_I(\hat{x}_I)$. Again, with the same reasoning as in the proof of Theorem 3.4, we get $f_C(\bar{x}_C) = f_C(\hat{x}_C)$ and thus $f(\bar{x}) = f(\hat{x})$ which contradicts (3.6). Hence, it holds that $f_{I,j}(\bar{x}_I) > f_{I,j}(\hat{x}_I)$ and by (3.6) it follows

$$0 < f_{I,j}(\bar{x}_I) - f_{I,j}(\hat{x}_I) \leq f_{C,j}(\hat{x}_C) - f_{C,j}(\bar{x}_C).$$

This leads to $f_{C,j}(\bar{x}_C) < f_{C,j}(\hat{x}_C)$ which contradicts the choice of $\hat{x}_C$ by (3.8). □

The biobjective optimization problem in the following Example 3.7 is a slight modification of the optimization problem (3.3) from Example 3.3 with $\Delta_{C,2} < \Delta_{I,2}$ but $\Delta_{C,1} \not< \Delta_{I,1}$. While the equality $\mathcal{E}_I = \mathcal{E}_I^{\mathrm{s}}$ holds in that case due to Lemma 3.6, the example also shows that none of the other equalities $\mathcal{E} = \mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}}$ and $\mathcal{N} = \mathcal{N}_C^{\mathrm{s}} + \mathcal{N}_I^{\mathrm{s}}$ from Theorem 3.4 holds in that scenario. Hence, for those equalities only the stronger assumption $\Delta_{C,i} < \Delta_{I,i}$ for all $i \in [p]$ (without any exception for an index $j \in [p]$) is sufficient.

Example 3.7 *We consider the following slight modification of the separable biobjective mixed-integer optimization problem from Example 3.3 given by*

$$\begin{aligned} \min_x & \begin{pmatrix} x_1 + x_2 \ + 0.75x_3 \\ -0.2x_1 - x_2 \ - 0.25x_3 \end{pmatrix} \\ \text{s.t. } & x \in X = [0,1] \times ([-1,1] \times [0,1]) \cap \mathbb{Z}^2. \end{aligned} \tag{3.9}$$

While the biobjective integer subproblem remains unchanged and is again given by (3.5), we use here the biobjective continuous subproblem

$$\begin{aligned} \min_x & \begin{pmatrix} x \\ -0.2x \end{pmatrix} \\ \text{s.t. } & x \in X_C = [0,1] \end{aligned}$$

with $\mathcal{E}_C^s = X_C$ *and* $\mathcal{N}_C^s = \operatorname{conv}\{(0,0),(1,-0.2)\}$*. Then it holds* $\Delta_{C,1} = 1 > 0.25 = \Delta_{I,1}$ *and* $\Delta_{C,2} = 0.2 < 0.25 = \Delta_{I,2}$*. Thus the sufficient condition to ensure* $\mathcal{E}_I = \mathcal{E}_I^s$ *in Lemma 3.6 is fulfilled. For the efficient and for the nondominated set of (3.9) it holds that*

$$\begin{aligned} \mathcal{E} &= \left([0,\tfrac{3}{4}[\times\{(-1,0),(0,0),(1,0)\}\right) \cup \left([0,\tfrac{1}{4}[\times\{(-1,1),(0,1)\}\right) \cup \left([0,1]\times\{(1,1)\}\right), \\ \mathcal{N} &= \left((\operatorname{conv}\{(0,0),(\tfrac{3}{4},-\tfrac{3}{20})\} \setminus \{(\tfrac{3}{4},-\tfrac{3}{20})\}) + \{(-1,1),(0,0),(1,-1)\}\right) \\ &\quad \cup\left((\operatorname{conv}\{(0,0),(\tfrac{1}{4},-\tfrac{1}{20})\} \setminus \{(\tfrac{1}{4},-\tfrac{1}{20})\}) + \{(-\tfrac{1}{4},\tfrac{3}{4}),(\tfrac{3}{4},-\tfrac{1}{4})\}\right) \\ &\quad \cup\left(\operatorname{conv}\{(0,0),(1,-\tfrac{1}{5})\} + \{(\tfrac{7}{4},-\tfrac{5}{4})\}\right). \end{aligned}$$

We obtain $\mathcal{E}_I = \mathcal{E}_I^s = X_I = \{-1,0,1\} \times \{0,1\}$*, but also* $\mathcal{E} \subsetneq \mathcal{E}_C^s \times \mathcal{E}_I^s$ *and* $\mathcal{N} \subsetneq \mathcal{N}_C^s + \mathcal{N}_I^s$*. We provide an illustration of the nondominated sets* $\mathcal{N}$ *and* $\mathcal{N}_I^s$ *in Figure 3.*

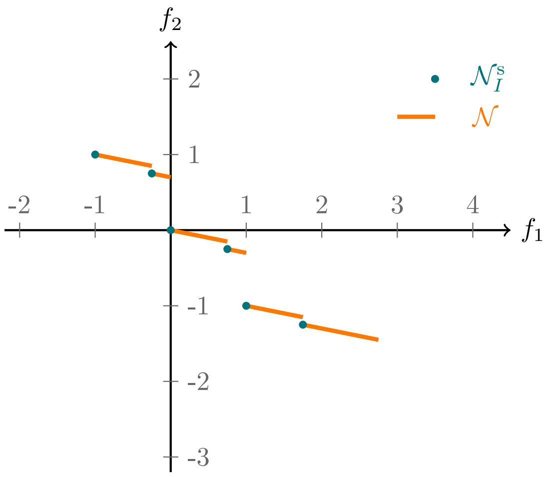

Figure 3: Nondominated set $\mathcal{N}$ of the separable optimization problem (3.9) and nondominated set $\mathcal{N}_I^s$ of the integer subproblem (3.5) from Example 3.7.

4 A Test Instance Generator for Multiobjective Mixed-integer Optimization Problems

Based on the results of the previous section we are now able to formulate the test instance generator for multiobjective mixed-integer optimization problems. As **Input** it requires two optimization problems:

- A multiobjective continuous optimization problem

$$\begin{aligned} \min_{x_C}\ & f_C(x_C) \\ \text{s.t. } & g_C(x_C) \leq 0_{q_C}, \\ & x_C \in X_C, \end{aligned} \tag{4.1}$$

  with a continuous objective function $f_C\colon \mathbb{R}^n \to \mathbb{R}^p$, a continuous constraint function $g_C\colon \mathbb{R}^n \to \mathbb{R}^{q_C}$, bounds on the variables defined by a box $X_C := [l_C, u_C] \subseteq \mathbb{R}^n$, known efficient set $\mathcal{E}_C^{\mathrm{s}}$ and known nondominated set $\mathcal{N}_C^{\mathrm{s}} \neq \emptyset$, and a vector $\blacktriangle_C \in \mathbb{R}^p$ for which it holds $\Delta_C \leq \blacktriangle_C$ (i.e., an upper bound is sufficient). Here Δ_C is defined as in Theorem 3.4, i.e., it holds

$$\blacktriangle_{C,i} \geq \Delta_{C,i} = \sup\{y_i - \hat{y}_i \mid y, \hat{y} \in \mathcal{N}_C^{\mathrm{s}}\}$$

  for all $i \in [p]$.

- A multiobjective integer optimization problem

$$\begin{aligned} \min_{x_I}\ & f_I(x_I) \\ \text{s.t. } & g_I(x_I) \leq 0_{q_I}, \\ & x_I \in X_I, \end{aligned} \tag{4.2}$$

  with a continuous objective function $f_I\colon \mathbb{R}^m \to \mathbb{R}^p$, a continuous constraint function $g_I\colon \mathbb{R}^m \to \mathbb{R}^{q_I}$, bounds on the variables defined by a box $X_I := [l_I, u_I] \cap \mathbb{Z}^m$, known efficient set $\mathcal{E}_I^{\mathrm{s}}$ and known nondominated set $\mathcal{N}_I^{\mathrm{s}}$, and a vector $\blacktriangle_I \in (\mathbb{R} \cup \{\infty\})^p$ for which it holds $0_p < \blacktriangle_I \leq \Delta_I$ (i.e., a positive lower bound is sufficient). Here Δ_I is defined as in Theorem 3.4, i.e., it holds

$$0 < \blacktriangle_{I,i} \leq \Delta_{I,i} = \inf\{|y_i - \hat{y}_i| \mid y, \hat{y} \in \mathcal{N}_I^{\mathrm{s}}, y_i \neq \hat{y}_i\}$$

  for all $i \in [p]$.

Further, one needs to **provide scaling factors** $\alpha_i > 0$ for all $i \in [p]$, such that for the vector $\alpha \in \mathbb{R}^p$ it holds

$$\alpha \circ \blacktriangle_C < \blacktriangle_I.$$

Finally, the **Output** of the generator is the separable multiobjective mixed-integer optimization problem defined by

$$\begin{aligned} \min_{x=(x_C,x_I)}\ & \alpha \circ f_C(x_C) + f_I(x_I) \\ \text{s.t. } & g_C(x_C) \leq 0_{q_C}, \\ & g_I(x_I) \leq 0_{q_I}, \\ & x \in X = X_C \times X_I, \end{aligned}$$

such that the assumptions of Theorem 3.4 are fulfilled. As a consequence, it holds $\mathcal{E} = \mathcal{E}_C^{\mathrm{s}} \times \mathcal{E}_I^{\mathrm{s}}$, $\mathcal{N} = \{\alpha \circ y \mid\ y \in \mathcal{N}_C^{\mathrm{s}}\} + \mathcal{N}_I^{\mathrm{s}}$ and $\mathcal{E}_I = \mathcal{E}_I^{\mathrm{s}}$.

We continue with a first example for the application of the generator using the subproblems given in Example 3.3.

Example 4.1 *We choose as input for the proposed test instance generator for the multiobjective continuous optimization problem and for the multiobjective integer optimization problem the biobjective subproblems* (3.4) *and* (3.5) *of Example 3.3. Then, as already mentioned on page 8, it holds* $\Delta_{C,1} = \Delta_{C,2} = 1$ *and* $\Delta_{I,1} = \Delta_{I,2} = 0.25$*. Further, we choose* $\blacktriangle_C := \Delta_C$ *and* $\blacktriangle_I := \Delta_I$*. To ensure* $\alpha \circ \blacktriangle_C < \blacktriangle_I$*, we set* $\alpha := (0.2, 0.2)$*. Then we obtain as output the separable biobjective mixed-integer optimization problem given by*

$$\begin{aligned} &\min_x \begin{pmatrix} 0.2x_1 + x_2 \ + 0.75x_3 \\ -0.2x_1 - x_2 \ - 0.25x_3 \end{pmatrix} \\ &s.t.\ x \in X = [0,1] \times ([-1,1] \times [0,1]) \cap \mathbb{Z}^2. \end{aligned} \tag{4.3}$$

We derive

$$\begin{aligned} \mathcal{E} &= [0,1] \times \{(-1,0),(0,0),(1,0),(-1,1),(0,1),(1,1)\}, \\ \mathcal{N} &= \operatorname{conv}\{(0,0),(\tfrac{1}{5},-\tfrac{1}{5})\} + \{(-1,1),(0,0),(1,-1),(-\tfrac{1}{4},\tfrac{3}{4}),(\tfrac{3}{4},-\tfrac{1}{4}),(\tfrac{7}{4},-\tfrac{5}{4})\}, \\ \mathcal{E}_I &= \{(-1,0),(0,0),(1,0),(-1,1),(0,1),(1,1)\}. \end{aligned}$$

We provide an illustration of the nondominated sets $\mathcal{N}$ *and* $\mathcal{N}_I^s$ *in Figure 4.*

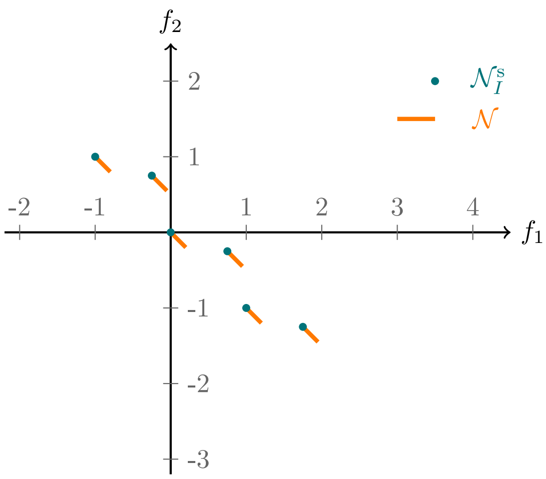

Figure 4: Nondominated set $\mathcal{N}$ of the separable optimization problem (4.3) and nondominated set $\mathcal{N}_I^s$ of the integer subproblem (3.5) from Example 4.1.

If for the chosen input problems (4.1) and (4.2) the exact values of $\Delta_C \in \mathbb{R}^p$ and of $\Delta_I \in (\mathbb{R} \cup \{\infty\})^p$ are not known, then the determination of some vectors $\blacktriangle_C \in \mathbb{R}^p$ and $\blacktriangle_I \in (\mathbb{R} \cup \{\infty\})^p$ with $\Delta_C \leq \blacktriangle_C$ and $0_p < \blacktriangle_I \leq \Delta_I$ is of great importance in terms of applicability of the formulated test instance generator. If, for instance, a set $M \supseteq \mathcal{E}_C^s$ with known ideal point ideal_C of f_C over M defined by

$$\text{ideal}_{C,i} := \inf_{x_C \in M} f_{C,i}(x_C) = \inf_{y \in f_C(M)} y_i \quad \text{for all } i \in [p]$$

and known anti-ideal point a-ideal_C of f_C over M defined by

$$\text{a-ideal}_{C,i} := \sup_{x_C \in M} f_{C,i}(x_C) = \sup_{y \in f_C(M)} y_i \quad \text{for all } i \in [p],$$

is given, then every $\blacktriangle_C$ with $\blacktriangle_C \geq \text{a-ideal}_C - \text{ideal}_C$ is an upper bound of Δ_C.

Example 4.2 *If we choose in the test instance generator for the multiobjective continuous optimization problem* (4.1) *the optimization problem*

$$\min_x \begin{pmatrix} (1+x_3)(x_1^3x_2^2 - 10x_1 - 4x_2) \\ (1+x_3)(x_1^3x_2^2 - 10x_1 + 4x_2) \\ 3(1+x_3)x_1^2 \end{pmatrix}$$
$$s.t.\ \ x \in X_C = [1, 3.5] \times [-2, 2] \times [0, 1]$$

as introduced in [8], then it holds

$$\mathcal{E}_C^{\mathrm{s}} \subseteq \left\{ x \in \mathbb{R}^3 \mid \ 1 \leq x_1 \leq 3.5,\ -2 \leq x_1^3 x_2 \leq 2,\ -2 \leq x_2 \leq 2,\ x_3 = 0 \right\} =: M.$$

By simple calculations we obtain $(-47, -47, 3) \leq \mathrm{ideal}_C$ *and* $\text{a-ideal}_C \leq (2, 2, 37)$*. Thus every* $\blacktriangle_C \in \mathbb{R}^3$ *with* $\blacktriangle_C \geq (49, 49, 34) \geq \text{a-ideal}_C - \mathrm{ideal}_C$ *is an upper bound of* Δ_C*.*

A possibility for the determination of a lower bound $\blacktriangle_I$ of Δ_I is to use the finite cardinality of the set $X_I = [l_I, u_I] \cap \mathbb{Z}^m$. Moreover, to ensure that $\Delta_I > 0_p$ and thus $\blacktriangle_I \leq \Delta_I$ can be chosen such that $\blacktriangle_I > 0_p$ we can use the following property introduced and examined in [6, 7].

Definition 4.3 *[6, Definition 2.3] Let* $\mathcal{X} \subseteq \mathbb{R}^m$ *and* $\gamma > 0$*. A function* $g \colon \mathcal{X} \to \mathbb{R}$ *is called a* positive γ-function *over* $\mathcal{X} \cap \mathbb{Z}^m$ *if it holds* $|g(x) - g(x')| \geq \gamma$ *for all* $x, x' \in \mathcal{X} \cap \mathbb{Z}^m$ *with* $g(x) \neq g(x')$*.*

For instance, every quadratic function $g \colon \mathcal{X} \to \mathbb{R}$ with $g(x) := x^\top Q x + c^\top x$ for all $x \in \mathcal{X}$ with $Q \in \mathbb{Z}^{m \times m}, c \in \mathbb{Z}^m$ is a positive γ-function over $\mathcal{X} \cap \mathbb{Z}^m$ with $\gamma = 1$. For more classes of positive γ-functions and the corresponding values of γ we refer to [7, Section 4.3].

If now $f_{I,i}$, $i \in [p]$ is a positive γ_i-function over $X_I = [l_I, u_I] \cap \mathbb{Z}^m$, then we obtain

$$\begin{aligned} \Delta_{I,i} &= \inf\{|y_i - \hat{y}_i| \mid y, \hat{y} \in \mathcal{N}_I^{\mathrm{s}}, y_i \neq \hat{y}_i\} \\ &= \inf\{|f_{I,i}(x) - f_{I,i}(x')| \mid x, x' \in \mathcal{E}_I^{\mathrm{s}}, f_{I,i}(x) \neq f_{I,i}(x')\} \\ &\geq \inf\{|f_{I,i}(x) - f_{I,i}(x')| \mid x, x' \in X_I, f_{I,i}(x) \neq f_{I,i}(x')\} \\ &\geq \gamma_i > 0. \end{aligned}$$

Hence, if there exists some $\gamma \in \mathbb{R}^p$ such that $\gamma_i > 0$ and $f_{I,i}$ is a positive γ_i-function over X_I for all $i \in [p]$, then every $\blacktriangle_I$ with $0_p < \blacktriangle_I \leq \gamma$ is a lower bound of Δ_I.

In the following we provide some multiobjective continuous optimization problems and some multiobjective integer optimization problems that can be used as input for the formulated test instance generator. What is more, all of these optimization problems are scalable in the number of decision variables.

4.1 Scalable Multiobjective Continuous Problems

The major advantage of Theorem 3.4 is that we can generate test instances for which the efficient set, the nondominated set, and the set of efficient integer assignments are known as long as the nondominated and the efficient sets of the input problems (4.1) and (4.2) are known. Regarding a listing of suitable inputs for the test instance generator we start with two biobjective continuous optimization problems scalable in the number of variables. This scalability is a useful property in order to evaluate and compare the performance of (especially decision space based) solution algorithms.

The following simple convex optimization problem is based on a well known univariate biobjective test instance introduced in [24]:

$$\begin{aligned}
&\min_x \begin{pmatrix} \frac{1}{n}\sum\limits_{i=1}^{n} x_i^2 \\ \frac{1}{n}\sum\limits_{i=1}^{n} (x_i-2)^2 \end{pmatrix} \\
&\text{s.t. } x \in X_C = [0,2]^n.
\end{aligned} \tag{4.4}$$

Besides the efficient and nondominated sets also the vector $\Delta_C \in \mathbb{R}^2$ is known for this optimization problems. More precisely, we have that

$$\begin{aligned}
\mathcal{E} &= \{x \in X_C \mid x_1 = x_2 = \ldots = x_n\}, \\
\mathcal{N} &= \{(t^2, (t-2)^2) \mid t \in [0,2]\}, \text{ and} \\
\Delta_{C,i} &= 4 \text{ for all } i \in [2].
\end{aligned} \tag{4.5}$$

Another possible choice for (4.1) is the biobjective continuous nonconvex optimization problem introduced by Fonseca and Fleming in [14]:

$$\begin{aligned}
&\min_x \begin{pmatrix} 1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) \\ 1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \end{pmatrix} \\
&\text{s.t. } x \in X_C = [-4,4]^n.
\end{aligned} \tag{4.6}$$

For the efficient set, the nondominated set, and $\Delta_C \in \mathbb{R}^2$ we obtain

$$\begin{aligned}
\mathcal{E} &= \left\{x \in X_C \,\middle|\, x_1 = x_2 = \ldots = x_n \in \left[-\tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}}\right]\right\}, \\
\mathcal{N} &= \{(1-\exp(-4(t-1)^2), 1-\exp(-4t^2)) \mid t \in [0,1]\}, \text{ and} \\
\Delta_{C,i} &= 1 - \exp(-4) \text{ for all } i \in [2].
\end{aligned} \tag{4.7}$$

One well-known method to generate other input problems where $\mathcal{E}$, $\mathcal{N}$ and also an overestimator $\blacktriangle_C$ of Δ_C are known is presented in [8, Section 6.4]. The main idea of this approach is to start with a parametric description of the nondominated set (for instance a part of the unit sphere in the forthcoming optimization problem (4.9)) and then to extend this in order to obtain an optimization problem. What is more, all of the multiobjective continuous optimization problems that are generated with that technique are scalable in the number of variables. Besides that, one can also generate problems that are scalable in the number of objective functions. Again, this is a useful property when generating a collection of test instances to evaluate and compare the performance of different (especially criterion space based) solution algorithms.

In the following, we present two examples of continuous optimization problems from [8] that are scalable in both the number $n \in \mathbb{N}$ of (continuous) variables and $p \in \mathbb{N}$ of objective functions. The first one is test problem DTLZ1 given by

$$\begin{aligned}
&\min_x f(x) \\
&\text{s.t. } g(x) \leq 0, \\
&\quad\;\; x \in X_C = [0,1]^n
\end{aligned} \tag{4.8}$$

where $n > p$. The objective functions $f_i \colon \mathbb{R}^n \to \mathbb{R}, i \in [p]$ are defined as

$$\begin{aligned}
f_1(x) &:= 0.5(1-g(x))x_1x_2\cdots x_{p-1}, \\
f_i(x) &:= 0.5(1-g(x))x_1x_2\cdots x_{p-i}(1-x_{p-i+1}) \text{ for all } i \in ([p] \setminus \{1,p\}), \\
f_p(x) &:= 0.5(1-g(x))(1-x_1)
\end{aligned}$$

and $g\colon \mathbb{R}^n \to \mathbb{R}$ only depends on the last $n-p$ variables, i.e., there exists $h\colon \mathbb{R}^{n-p} \to \mathbb{R}$ such that $g(x) = h(x_{p+1}, \dots, x_n)$ for all $x \in [0,1]^n$. Further, we assume that there exists some $x' \in [0,1]^n$ such that $g(x') = 0$. It then holds for (4.8) that

$$\begin{array}{lcl} \mathcal{E} & = & \{x \in [0,1]^n \mid g(x) = 0\}, \\ \mathcal{N} & = & \{y \in [0,1]^p \mid \|y\|_1 = 0.5\}, \text{ and} \\ \Delta_{C,i} & = & 0.5 \text{ for all } i \in [p]. \end{array}$$

The next optimization problem can be found as (6.7) in [8] and is given by

$$\begin{aligned} &\min_x f(x) \\ &\text{s.t. } g(x) \le 0, \\ &\qquad x \in X_C = [0, \pi/2]^n \end{aligned} \tag{4.9}$$

where $n > p$, the objective functions $f_i\colon \mathbb{R}^n \to \mathbb{R}, i \in [p]$ are defined as

$$\begin{aligned} f_1(x) &:= (1 - g(x)) \cos(x_1) \cos(x_2) \cdots \cos(x_{p-1}), \\ f_i(x) &:= (1 - g(x)) \cos(x_1) \cos(x_2) \cdots \cos(x_{p-i}) \sin(x_{p-i+1}) \text{ for all } i \in ([p] \setminus \{1, p\}), \\ f_p(x) &:= (1 - g(x)) \sin(x_1) \end{aligned}$$

and $g\colon \mathbb{R}^n \to \mathbb{R}$ only depends on the last $n-p$ variables. Again, we assume that there exists some $x' \in [0, \pi/2]^n$ such that $g(x') = 0$. Then it holds for (4.9) that

$$\begin{array}{lcl} \mathcal{E} & = & \{x \in [0,\pi/2]^n \mid g(x) = 0\}, \\ \mathcal{N} & = & \{y \in [0,1]^p \mid \|y\|_2 = 1\}, \text{ and} \\ \Delta_{C,i} & = & 1 \text{ for all } i \in [p]. \end{array}$$

We remark that for the specific choice of $g(x) := \sum_{i=p+1}^{n} (x_i - \pi/4)^2,\ x \in [0, \pi/2]^n$ this leads to the test problem DTLZ2 from [8].

4.2 Scalable Multiobjective Integer Problems

Besides the continuous subproblems we also need suitable integer subproblems. In the following we present two such problems that are not only scalable in the number of variables, but for which we are also able to control the number of efficient solutions and nondominated points.

Lemma 4.4 *Let $J \subsetneq [m]$. Then for the scalable biobjective integer linear optimization problem*

$$\begin{aligned} &\min_x \begin{pmatrix} \sum\limits_{i \in J} x_i + \sum\limits_{i \in [m] \setminus J} x_i \\ \sum\limits_{i \in J} x_i - \sum\limits_{i \in [m] \setminus J} x_i \end{pmatrix} \\ &\text{s.t. } x \in X_I = [-1, 1]^m \cap \mathbb{Z}^m \end{aligned} \tag{4.10}$$

it holds:

(i) $\mathcal{E} = \{x \in X_I \mid x_i = -1 \text{ for all } i \in J\}$.

(ii) $|\mathcal{E}| = 3^{m-|J|}$.

(iii) $\mathcal{N} = \{(-m + \delta, m - 2|J| - \delta) \in \mathbb{Z}^2 \mid \ \delta \in \{0\} \cup [2(m - |J|)]\}$.

(iv) $|\mathcal{N}| = 2(m - |J|) + 1$.

(v) $\Delta_{I,i} = 1$ *for all* $i \in [2]$.

Proof. Statement (ii) follows by (i), and the statements (iv) and (v) follow by (iii). We start with the proof of (i). Here, for every $\bar{x} \in \mathcal{E} \subseteq X_I$ it obviously holds that $\bar{x}_i = -1$ for all $i \in J$ and we obtain $\mathcal{E} \subseteq \{x \in X_I \mid x_i = -1 \text{ for all } i \in J\}$. Let now $\bar{x} \in \{x \in X_I \mid x_i = -1 \text{ for all } i \in J\}$ and assume that $\bar{x} \notin \mathcal{E}$. Then by the domination property there exists $x \in \mathcal{E}$ with

$$\begin{pmatrix} \sum\limits_{i \in J} x_i + \sum\limits_{i \in [m] \setminus J} x_i \\ \sum\limits_{i \in J} x_i - \sum\limits_{i \in [m] \setminus J} x_i \end{pmatrix} \leq \begin{pmatrix} \sum\limits_{i \in J} \bar{x}_i + \sum\limits_{i \in [m] \setminus J} \bar{x}_i \\ \sum\limits_{i \in J} \bar{x}_i - \sum\limits_{i \in [m] \setminus J} \bar{x}_i \end{pmatrix}$$

and with strict inequality in one component. By componentwise addition of the inequalities it follows

$$2 \sum_{i \in J} x_i < 2 \sum_{i \in J} \bar{x}_i.$$

In case $J \neq \emptyset$ this contradicts $x \in \mathcal{E} \subseteq X_I$ and $\bar{x}_i = -1$ for all $i \in J$. In case $J = \emptyset$ we obtain $0 < 0$. Hence, it holds $\bar{x} \in \mathcal{E}$ and thus also $\{x \in X_I \mid x_i = -1 \text{ for all } i \in J\} \subseteq \mathcal{E}$.

For the proof of (iii) let at first $\bar{y} \in \mathcal{N}$. Then by definition there exists $\bar{x} \in \mathcal{E}$ such that $f(\bar{x}) = \bar{y}$, and by (i) it holds $\bar{x}_i = -1$ for all $i \in J$. Moreover, let $\bar{J}^{-1} := \{i \in [m] \setminus J \mid \bar{x}_i = -1\}$, $\bar{J}^0 := \{i \in [m] \setminus J \mid \bar{x}_i = 0\}$ and $\bar{J}^1 := \{i \in [m] \setminus J \mid \bar{x}_i = 1\}$. Then we obtain that

$$\begin{aligned} \bar{y}_1 = f_1(\bar{x}) &= \sum_{i \in J} x_i + \sum_{i \in \bar{J}^{-1}} x_i + \sum_{i \in \bar{J}^1} x_i \\ &= -|J| - |\bar{J}^{-1}| + |\bar{J}^1| \\ &= -|J| - |\bar{J}^{-1}| - |\bar{J}^0| - |\bar{J}^1| + |\bar{J}^0| + 2|\bar{J}^1| \\ &= -|J| - (m - |J|) + |\bar{J}^0| + 2|\bar{J}^1| \\ &= -m + |\bar{J}^0| + 2|\bar{J}^1| \end{aligned}$$

and similarly $\bar{y}_2 = m - 2|J| - \left(|\bar{J}^0| + 2|\bar{J}^1|\right)$. Thus, we derive for $\delta := |\bar{J}^0| + 2|\bar{J}^1| \geq 0$ that

$$\begin{aligned} \delta &= |\bar{J}^0| + 2|\bar{J}^1| \\ &= |\bar{J}^{-1}| + |\bar{J}^0| + |\bar{J}^1| - |\bar{J}^{-1}| + |\bar{J}^1| \\ &= m - |J| - |\bar{J}^{-1}| + |\bar{J}^1| \\ &\leq m - |J| + |\bar{J}^1| \\ &\leq 2(m - |J|), \end{aligned}$$

and consequently $\mathcal{N} \subseteq \{(-m + \delta, m - 2|J| - \delta) \in \mathbb{Z}^2 \mid \ \delta \in \{0\} \cup [2(m - |J|)]\}$.

Let now $\delta \in \mathbb{N}_0$ with $0 \leq \delta \leq 2(m - |J|)$ and let $\bar{y} \in \mathbb{R}^2$ with $\bar{y}_1 := -m + \delta$ and $\bar{y}_2 := m - 2|J| - \delta$. Further, let $\bar{J}^{-1}, \bar{J}^1 \subseteq [m] \setminus J$ with

$$\left|\bar{J}^{-1}\right| = \max\{m - |J| - \delta, 0\} \quad \text{and} \quad \left|\bar{J}^1\right| = \max\{\delta - (m - |J|), 0\}.$$

Then at least one of the sets $\bar{J}^{-1}$ or $\bar{J}^1$ is empty (as at least $m - |J| - \delta \leq 0$ or $\delta - (m - |J|) \leq 0$). Moreover, $\left|J \cup \bar{J}^{-1} \cup \bar{J}^1\right| \leq m$. Define $\bar{x} \in X_I$ by $\bar{x}_i = -1$ for all $i \in J \cup \bar{J}^{-1}$, $\bar{x}_i = 1$ for all $i \in \bar{J}^1$, and $\bar{x}_i = 0$ for all $i \in m \setminus (J \cup \bar{J}^{-1} \cup \bar{J}^1)$. Then we obtain $\bar{x} \in \mathcal{E}$ by (i). Moreover, one can verify that $f_1(\bar{x}) = -m + \delta = \bar{y}_1$, $f_2(\bar{x}) = m - 2|J| - \delta = \bar{y}_2$, and thus $\{(-m + \delta, m - 2|J| - \delta) \in \mathbb{Z}^2 \mid \ \delta \in \{0\} \cup [2(m - |J|)]\} \subseteq \mathcal{N}$, which concludes the proof. □

The structure of the nondominated set of (4.10) is quite simple, since all nondominated points are located on a line. In particular, all of the nondominated points are so-called supported nondominated points. This means that they can be found by solving a weighted sum scalarization of (4.10). For this reason, we also present a slight modification of this problem, see (4.11), for which only nearly half of the nondominated set consists of supported nondominated points. The proof of Lemma 4.5 is similar to the proof of Lemma 4.4 and thus omitted.

Lemma 4.5 *Let $J \subsetneq [m-1]$. Then for the scalable biobjective integer linear optimization problem*

$$\begin{aligned} \min_x & \begin{pmatrix} \sum\limits_{i\in J} x_i + \sum\limits_{i\in[m-1]\setminus J} x_i + 0.75x_m \\ \sum\limits_{i\in J} x_i - \sum\limits_{i\in[m-1]\setminus J} x_i - 0.25x_m \end{pmatrix} \\ \text{s.t. } & x \in X_I = ([-1,1]^{m-1} \times [0,1]) \cap \mathbb{Z}^m \end{aligned} \tag{4.11}$$

it holds:

(i) $\mathcal{E} = \{x \in X_I \mid x_i = -1 \text{ for all } i \in J\}$.

(ii) $|\mathcal{E}| = 2 \cdot 3^{m-1-|J|}$.

(iii) $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ *with*

$$\begin{aligned} \mathcal{N}_1 &:= \{(-(m-1)+\delta, m-1-2|J|-\delta) \in \mathbb{Z}^2 \mid \delta \in \Xi\}, \\ \mathcal{N}_2 &:= \left\{(-(m-1)+0.75+\delta, m-1-2|J|-0.25-\delta) \in \mathbb{Z}^2 \middle| \delta \in \Xi\right\}, \\ \Xi &:= \{0\} \cup [2(m-1-|J|)]. \end{aligned}$$

(iv) $|\mathcal{N}| = 4(m-1-|J|)+2$.

(v) $\Delta_{I,i} = 0.25$ *for all* $i \in [2]$.

Remark 4.6 *Note that for the optimization problems (4.10) in Lemma 4.4 and (4.11) in Lemma 4.5 not only the absolute number of efficient solutions but also their share in relation to the feasible set X_I can be controlled by the choice of the set J. We obtain in both cases $\frac{|\mathcal{E}|}{|X_I|} = 3^{-|J|}$. This equals the percentage of efficient integer assignments within the set of feasible integer assignments if one of these problems is chosen as input for the test instance generator.*

Further examples for an integer optimization problem (sMOMIP$_I$) can be obtained from (4.10) and (4.11) and basically any other multiobjective integer optimization problem by replacing the decision variables $x \in X_I$ by functions $\tilde{x}\colon \mathbb{R}^k \to \mathbb{R}^m, k \in \mathbb{N}$ such that for some box $\tilde{X} \subseteq \mathbb{R}^k$ it holds that $\tilde{x}(\tilde{X} \cap \mathbb{Z}^k) = X_I$. The following lemma presents one possible realization of such a replacement of the decision variables $x \in X_I$ for (4.10) and (4.11).

Lemma 4.7 *Let $u_1, u_2, u_3, u_4 \in \mathbb{N}_0$ with $u := u_1 + u_2 + u_3 + u_4 \geq 1$, u odd, and let $x \in \{-1, 0, 1\}$. Then it holds*

$$x = \left[x^{u_1}\right] \cdot \left[\sin^{u_2}\left(x \cdot \frac{\pi}{2}\right)\right] \cdot \left[\cos^{u_3}\left((x-1) \cdot \frac{\pi}{2}\right)\right] \cdot \left[\tan^{u_4}\left(x \cdot \frac{\pi}{4}\right)\right].$$

The idea of replacing decision variables by functions is also mentioned in [8] as one possibility to obtain new optimization problems out of an existing one for which the nondominated set is already known.

Moreover, while in [8] the authors focus on purely continuous optimization problems, see also page 14, their approach to generate (scalable) test problems works in the purely integer case as well. Hence, we can use the exact same approach to also obtain multiobjective integer optimization problems (4.2). In fact, we can even reuse the test problem that we presented on page 14. More precisely, we can modify (4.8) and reduce it to the multiobjective binary optimization problem

$$\begin{aligned} \min_x \ & f(x) \\ \text{s.t.} \ & g(x) \leq 0, \\ & x \in X_I = \{0,1\}^n \end{aligned} \tag{4.12}$$

with the same assumptions, objective functions $f_i \colon \mathbb{R}^n \to \mathbb{R}, i \in [p]$ and constraint function $g \colon \mathbb{R}^n \to \mathbb{R}$. Then it holds for (4.12) that

$$\begin{aligned} \mathcal{E} &= \{x \in \{0,1\}^n \mid g(x) = 0\}, \\ \mathcal{N} &= \{y \in \{0, 0.5\}^p \mid \|y\|_1 = 0.5\}, \text{ and} \\ \Delta_{I,i} &= 0.5 \text{ for all } i \in [p]. \end{aligned}$$

However, one should keep in mind that for the construction of a test instance with the methods from [8] a parametric description of the nondominated set is needed as a starting point. While this is often possible in continuous optimization, for the discrete nondominated set of multiobjective integer optimization problems this is usually much harder or leads to nondominated sets of a very simple structure as in the example above. For the same reason the approach from [8] is not well suited in order to directly obtain test instances for multiobjective mixed-integer optimization problems. However, if there was some (nontrivial) nondominated set that has a parametric description consisting of both continuous and integer parameters then such a construction of test instances would be possible.

To conclude this section, we present some examples for multiobjective mixed-integer optimization problems that are obtained by the proposed test instance generator when using the continuous and integer subproblems from the previous subsections as input.

Example 4.8 (*i*) *We choose as input for the multiobjective continuous optimization problem the biobjective subproblem* (4.6) *and for the multiobjective integer optimization problem the biobjective subproblem* (4.10) *with* $J = \emptyset$ *and* $m = 2$*. Then we obtain by* (4.7) *and Lemma 4.4* (*v*) *that* $\Delta_{C,i} = 1 - \exp(-4) < 1 = \Delta_{I,i}$ *for all* $i \in [2]$*. Thus, we can set* $\blacktriangle_C := \Delta_C$*,* $\blacktriangle_I := \Delta_I$*, and* $\alpha := (1,1)$*. The output of the test instance generator then is the separable biobjective mixed-integer nonconvex optimization problem* (3.1) *of Example 3.1 with*

$$\begin{aligned} \mathcal{E} &= \left\{x \in [-4,4]^n \,\middle|\, x_1 = x_2 = \ldots = x_n \in \left[-\tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}}\right]\right\} \\ &\quad \times \{-1,0,1\}^2, \\ \mathcal{N} &= \{(1-\exp(-4(t-1)^2), 1-\exp(-4t^2)) \mid t \in [0,1]\} \\ &\quad + \{(\delta, -\delta) \mid \delta \in \{-2,-1,0,1,2\}\}, \text{ and} \\ \mathcal{E}_I &= \{-1,0,1\}^2. \end{aligned}$$

(*ii*) *Let* (4.4) *be the input for the multiobjective continuous subproblem and* (4.10) *the input for the multiobjective integer subproblem. Then by* (4.5) *and Lemma 4.4* (*v*) *it holds* $\Delta_{C,i} = 4 > 1 = \Delta_{I,i}$ *for all* $i \in [2]$*. Thus, we can use* $\blacktriangle_C := \Delta_C$*,* $\blacktriangle_I := \Delta_I$*, and* $0 < \alpha_i < 0.25$ *for all* $i \in [2]$*. The resulting test instance is the scalable separable biobjective mixed-integer convex optimization problem given by*

$$\begin{aligned} \min_x & \begin{pmatrix} \frac{\alpha_1}{n} \sum_{i=1}^{n} x_i^2 + \sum_{i \in J} x_i + \sum_{i \in \{n+1,\ldots,n+m\} \setminus J} x_i \\ \frac{\alpha_2}{n} \sum_{i=1}^{n} (x_i - 2)^2 + \sum_{i \in J} x_i - \sum_{i \in \{n+1,\ldots,n+m\} \setminus J} x_i \end{pmatrix} \\ \text{s.t. } & x \in X = [0,2]^n \times ([-1,1]^m \cap \mathbb{Z}^m) \end{aligned} \tag{4.13}$$

with $J \subsetneq \{n+1, \ldots, n+m\}$*. For the efficient set, the nondominated set, and the set of efficient integer assignments we derive*

$$\begin{aligned} \mathcal{E} &= \{x \in [0,2]^n \mid x_1 = x_2 = \ldots = x_n\} \\ &\quad \times \{x \in [-1,1]^m \cap \mathbb{Z}^m \mid x_i = -1 \text{ for all } i+n \in J\}, \\ \mathcal{N} &= \{(\alpha_1 t^2, \alpha_2 (t-2)^2) \mid t \in [0,2]\} \\ &\quad + \{(-m+\delta, m-2|J|-\delta) \in \mathbb{Z}^2 \mid \ \delta \in \{0\} \cup [2(m-|J|)]\}, \text{ and} \\ \mathcal{E}_I &= \{x \in [-1,1]^m \cap \mathbb{Z}^m \mid x_i = -1 \text{ for all } i+n \in J\}. \end{aligned}$$

For an illustration of the nondominated set $\mathcal{N}$ *see Figure 5.*

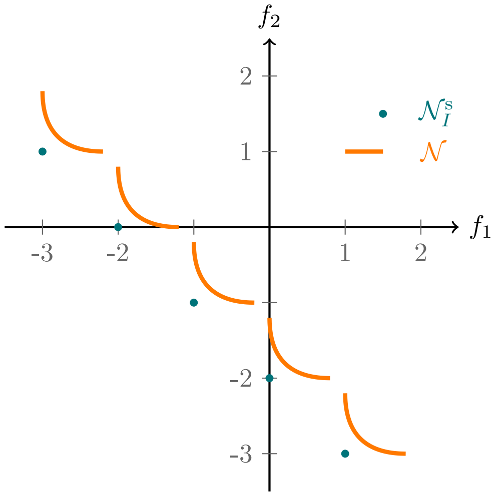

Figure 5: Nondominated set $\mathcal{N}$ of the separable optimization problem (4.13) and nondominated set $\mathcal{N}_I^s$ of the corresponding integer subproblem (4.10) from Example 4.8 (*ii*) for $J = \{n+1\}$, $m = 3$, and $\alpha_1 = \alpha_2 = 0.2$.

(*iii*) *If* (4.6) *and* (4.11) *are chosen as input, then it holds* $\Delta_{C,i} = 1 - \exp(-4) > 0.25 = \Delta_{I,i}$ *for all* $i \in [2]$ *by* (4.7) *and Lemma 4.5* (*v*)*. Thus, we can again choose* $\blacktriangle_C := \Delta_C$ *and* $\blacktriangle_I := \Delta_I$*. For any choice* $0 < \alpha_i < \frac{1}{4 \cdot (1 - \exp(-4))}$ *for all* $i \in [2]$ *we then obtain the scalable separable biobjective mixed-integer nonconvex* *optimization problem*

$$\min_x \begin{pmatrix} \alpha_1\left(1-\exp\left(-\sum_{i=1}^n\left(x_i-\frac{1}{\sqrt{n}}\right)^2\right)\right)+\sum_{i\in J}x_i+\sum_{i\in\{n+1,\ldots,n+m-1\}\setminus J}x_i+0.75x_{m+n} \\ \alpha_2\left(1-\exp\left(-\sum_{i=1}^n\left(x_i+\frac{1}{\sqrt{n}}\right)^2\right)\right)+\sum_{i\in J}x_i-\sum_{i\in\{n+1,\ldots,n+m-1\}\setminus J}x_i-0.25x_{m+n} \end{pmatrix}$$
$$\text{s.t. } x\in X=[-4,4]^n\times\left(([-1,1]^{m-1}\times[0,1])\cap\mathbb{Z}^m\right) \tag{4.14}$$

with $J \subsetneq \{n+1,\ldots,n+m-1\}$. *This leads to the efficient set, nondominated set, and set of efficient integer assignments given by*

$$\begin{aligned}
\mathcal{E} &= \left\{x\in[-4,4]^n \,\middle|\, x_1=x_2=\ldots=x_n\in\left[-\tfrac{1}{\sqrt{n}},\tfrac{1}{\sqrt{n}}\right]\right\} \\
&\quad\times\{x\in([-1,1]^{m-1}\times[0,1])\cap\mathbb{Z}^m \mid x_i=-1 \text{ for all } i+n\in J\}, \\
\mathcal{N} &= \{(\alpha_1(1-\exp(-4(t-1)^2)),\alpha_2(1-\exp(-4t^2))) \mid t\in[0,1]\}+(\mathcal{N}_1\cup\mathcal{N}_2), \\
&\quad \mathcal{N}_1=\{(-(m-1)+\delta, m-1-2|J|-\delta)\in\mathbb{Z}^2 \mid \delta\in\Xi\}, \\
&\quad \mathcal{N}_2=\{(-(m-1)+0.75+\delta, m-1-2|J|-0.25-\delta)\in\mathbb{Z}^2 \mid \delta\in\Xi\}, \\
&\quad \Xi=\{0\}\cup[2(m-1-|J|)], \text{ and} \\
\mathcal{E}_I &= \{x\in([-1,1]^{m-1}\times[0,1])\cap\mathbb{Z}^m \mid x_i=-1 \text{ for all } i+n\in J\}.
\end{aligned}$$

For an illustration of the nondominated set $\mathcal{N}$ *see Figure 6.*

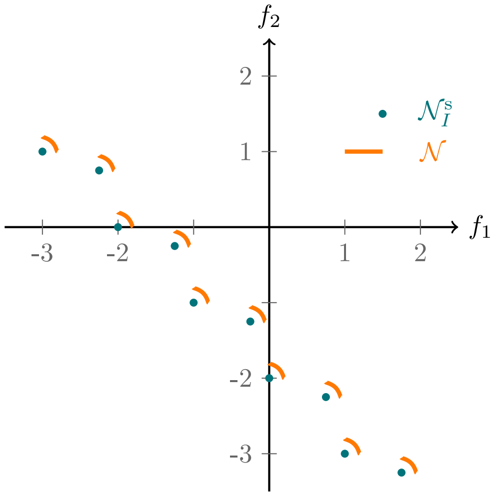

Figure 6: Nondominated set $\mathcal{N}$ of the separable optimization problem (4.14) and nondominated set $\mathcal{N}_I^s$ of the corresponding integer subproblem (4.11) from Example 4.8 (*iii*) for $J=\{n+1\}$, $m=4$, and $\alpha_1=\alpha_2=0.2$.

Declarations

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