A successive centralized circumcentered-reflection method for the convex feasibility problem

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Abstract

In this paper, we present a successive centralization process for the circumcentered-reflection scheme with several control sequences for solving the convex feasibility problem in Euclidean space. Assuming that a standard error bound holds, we prove the linear convergence of the method with the most violated constraint control sequence. Moreover, under additional smoothness assumptions on the target sets, we establish the superlinear convergence. Numerical experiments confirm the efficiency of our method.

Keywords: Convex Feasibility Problem, Superlinear convergence, Circumcentered-reflection method, Projection methods.

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1 Introduction

The convex feasibility problem (CFP) aims at finding a point in the intersection of m closed and convex sets:

$$\text{find } x^* \in C := \bigcap_{i=1}^{m} C_i, \tag{1.1}$$

where $C_i \subset \mathbb{R}^n$ is closed and convex for $i = 1, 2, \ldots, m$. Convex feasibility represents a modeling paradigm for solving many engineering and physics problems, *e.g.*, image recovery [1], wireless sensor networks localization [2], and gene regulatory network inference [3].

Based on the orthogonal projections, a broad class of methods is available for solving problem (1.1); see, for instance, [4]. Two well-known algorithms among them are the Sequential Projection Method (SePM) and the Simultaneous Projection Method (SiPM), which only use the individual projections onto C_i's, P_{C_i}. The projection operator for each C_i, $P_{C_i} : \mathbb{R}^n \to C_i$, is given by

$$P_{C_i}(x) = \operatorname*{arg\,min}_{s \in C_i} \|x - s\|. \tag{1.2}$$

The SePM and the SiPM operators are defined as $\bar{P} = P_{C_m} \circ \cdots \circ P_{C_1}$ and $\hat{P} = \frac{1}{m} \sum_{i=1}^{m} P_{C_i}$, respectively. Given $s^0, y^0 \in \mathbb{R}^n$, we set $s^{k+1} = \bar{P}(s^k)$ and $y^{k+1} = \hat{P}(y^k)$ with $k \in \mathbb{N}$ the sequences generated by SePM and SiPM, respectively. These two iterations converge to a solution of problem (1.1) if $\cap_{i=1}^{m} C_i \neq \emptyset$. Moreover, it is well-known that under some error bound conditions, to be discussed later, SePM and SiPM have linear convergence rates. Further study of these two algorithms can be found in [4, 5].

In this paper we are going to use the a circumcentered-reflection scheme for solving (1.1). The circumcentered-reflection method (CRM) has been proposed in [6] to solve the CFP (1.1) with two closed convex sets $A, B \subset \mathbb{R}^n$. CRM was first proposed to accelerate the Douglas-Rachford method (DRM) [7], and Method of Alternating projections (MAP) [8, 9] (which coincides with SePM introduced above, if only two sets are considered). During the past four years, CRM has fascinated researchers in the field of continuous optimization, resulting in an avalanche of surprising results and improvements for the circumcenter scheme; see, for instance, [10–29].

The circumcenter of three points $x, y, z \in \mathbb{R}^n$, noted as $\operatorname{circ}(x, y, z)$, is the point in $\mathbb{R}^n$ that lies in the affine space defined by x, y and z and is equidistant to these three points. The circumcentered-reflection method iterates by means of the operator $\mathscr{C}_{A,B}$ with respect to A, B defined as

$$\mathscr{C}_{A,B}(x) = \operatorname{circ}(x, R_A(x), R_B(R_A(x))),$$

where $R_A = 2P_A - \operatorname{Id}$ and $R_B = 2P_B - \operatorname{Id}$, and Id is the identity operator in $\mathbb{R}^n$.

One of the limitations of CRM is that its convergence theory requires one of the sets to be a linear manifold. A counter-example for which CRM does not converge for two general convex sets was found in [30]. It is worth noting that CRM can be used for solving the CFP with m general arbitrary closed convex sets by using Pierra's product space reformulation [31]. Define $\mathbf{W} = C_1 \times C_2 \times \cdots \times C_m \subset \mathbb{R}^{nm}$ and $\mathbf{D} \coloneqq \{(x, x, \ldots, x) \in \mathbb{R}^{nm} \mid x \in \mathbb{R}^n\}$. One can easily see that

$$x^* \in C \Leftrightarrow (x^*, x^*, \ldots, x^*) \in \mathbf{W} \cap \mathbf{D}. \tag{1.3}$$

Due to (1.3), solving problem (1.1) corresponds to solve

$$\text{find } z^* \in \mathbf{W} \cap \mathbf{D}. \tag{1.4}$$

Since $\mathbf{D}$ is an affine manifold, CRM can be applied to the convex sets $\mathbf{D}$ and $\mathbf{W}$ in the product space $\mathbb{R}^{nm}$. On the other hand, the numerical evidence in [13] showed that the cost of introducing the product space is expensive.

More recently, an extension of CRM, called the centralized circumcentered-reflection method (cCRM), was introduced in [13] for overcoming the drawback of CRM, namely the request that one of the sets be an affine manifold. For describing cCRM, we need some notation. Suppose that $A, B \subset \mathbb{R}^n$ are both closed convex sets, and define the alternating projection operator $Z_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$Z_{A,B} = P_A \circ P_B,$$

and the simultaneous projection operator $\tilde{Z}_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\tilde{Z}_{A,B} = \frac{1}{2}(P_A + P_B), \tag{1.5}$$

with P_A, P_B as in (1.2). Finally, we define operator $\bar{Z}_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\bar{Z}_{A,B} = \frac{1}{2}(Z_{A,B} + P_B \circ Z_{A,B}) = \tilde{Z}_{A,B} \circ Z_{A,B}.$$

With this notation, we will define the cCRM operator $T_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ for sets A and B at $z \in \mathbb{R}^n$ as

$$T_{A,B}(z) = \mathscr{C}_{A,B}(\bar{Z}_{A,B}(z)) = \operatorname{circ}(\bar{Z}_{A,B}(z), R_A(\bar{Z}_{A,B}(z)), R_B(\bar{Z}_{A,B}(z))). \tag{1.6}$$

Then, given $z^0 \in \mathbb{R}^n$, the cCRM method is defined by the iteration

$$z^{k+1} = T_{A,B}(z^k). \tag{1.7}$$

It has been proved in [13] that the sequence defined by cCRM converges to point $x^* \in A \cap B$ whenever $A \cap B \neq \emptyset$. Under an error bound assumption the sequence converges linearly. Moreover, under some additional smoothness hypotheses and an error bound condition, the sequence generated by cCRM was proved to have superlinear convergence [13, Thm. 3.13].

In this paper, we will extend the cCRM to the case of CFP with m sets. The natural way to generalize it is to choose a pair of sets among $\{C_1, C_2, \ldots, C_m\}$ at the iteration k and apply cCRM to this pair of sets. We define this pair as $\ell(k)$ and $r(k)$, with $\ell(k), r(k) \in \{1, \ldots, m\}$, and then apply the operator in (1.7) to this pair. Therefore, cCRM for CFP with m sets is defined as

$$z^{k+1} = T_{C_{r(k)}, C_{\ell(k)}}(z^k), \tag{1.8}$$

where $T_{C_{r(k)}, C_{\ell(k)}}$ is cCRM operator defined in (1.6) w.r.t. $C_{\ell(k)}$ and $C_{r(k)}$. The sequences $\{\ell(k)\}, \{r(k)\}$, determining which sets are used at the k-th iteration, are called control sequences.

In all successive methods with orthogonal projections, the control sequences considerably impact the algorithms' performance. A natural one is the *cyclic control sequence*, *e.g.*, $\ell(k) = 1, 2, 3, \ldots m-1, m, 1, 2, \ldots$, and $r(k) = 2, 3, 4, \ldots m-1, m, 1, 2, 3, \ldots$. The problem is that this cyclic control sequence is fully exogenous since it doesn't use the information available at iteration k, *e.g.*, the point z^k distances to the target sets C_i's. An alternative option is the *most violated constraint control sequence (distance version)*, which chooses $\ell(k)$ as the set which lies the farthest away from z^k, with the goal of getting closer to the intersection set C. The drawback is that the distance from z^k to all sets C_i must be calculated to determine $\ell(k)$, which in general, is computationally expensive. If the sets C_i are represented as sublevel sets of easily computable convex functions, as in (1.10) below, which happens very frequently in applications, we have the option of the *most violated constraint control sequence (functional value version)*, where $\ell(k)$ is chosen so that $f_{\ell(k)}(z^k) \geq f_i(z^k)$ for all i, again with the expectation of getting closer to C. Next, we formally define these three control sequences.

Definition 1.1 (control sequences) We say that the sequence $\{\ell(k)\}$ is:

(i) *almost cyclic*, if $1 \leq \ell(k) \leq m$ and there exists an integer $Q \geq m$ such that, for all $k \geq 0$ and $\{1, 2, \ldots, m\} \subset \{\ell(k+1), \ell(k+2), \ldots, \ell(k+Q)\}$. An almost cyclic control with $Q = m$ is called *cyclic*;

(ii) *most violated constraint* (distance version), if

$$\begin{aligned} \ell(k) &:= \operatorname*{arg\,max}_{1 \leq i \leq m} \{\operatorname{dist}(z^k, C_i)\}, \\ r(k) &:= \operatorname*{arg\,max}_{1 \leq i \leq m} \{\operatorname{dist}(P_{C_{\ell(k)}}(z^k), C_i)\}; \end{aligned} \tag{1.9}$$

(iii) *most violated constraint* (function value version), if we assume that the sets C_i in problem (1.1) are in the form

$$C_i := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}, \tag{1.10}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex for all $i = 1, 2, \dots, m$, then the most violated constraint control sequence in the function value version is given by

$$\begin{aligned} \ell(k) &:= \arg\max_{1\le i\le m}\{f_i(z^k)\}, \\ r(k) &:= \arg\max_{1\le i\le m}\{f_i(P_{C_{\ell(k)}}(z^k))\}. \end{aligned} \tag{1.11}$$

When the sets in CFP are presented as in (1.10), the performance of Algorithm 3 will depend not only on the sets themselves but also on the specific functions f_i used to represent them, which of course are not unique. The control sequence will change if we change the functions f_i (keeping the same sets C_i). Here, we assume that the f_i's are fixed from the onset, so that the original problem can be seen as that of finding a point $\bar{x}$ which satisfies $f_i(\bar{x}) \le 0$ for all $i \in \{1, 2, \dots m\}$.

We note that Borwein and Tam, in [32, 33], introduced and analyzed a cyclic Douglas-Rachford iteration scheme. In this paper, in addition to studying the cyclic version of cCRM, we are going to employ the other two control sequences above. We call cCRM with the almost cyclic control sequence (Algorithm 1), cCRM with the most violated constraint control sequence (distance version) Algorithm 2, and cCRM with the most violated constraint control sequence (function value version) Algorithm 3.

The paper is organized as follows: In Section 2, we give definitions and preliminaries. In Section 3, we introduce the successive cCRM and we prove the global convergence for the cCRM under the three control sequences defined above. In Section 4, we prove linear convergence for both versions of the most violated constraint control sequence under a standard *error bound* assumption, and superlinear convergence under some additional smoothness assumptions. Section 5 presents numerical experiments comparing Algorithms 1-3 to CRM in the product space (CRM-Prod) and the SeMP for solving problem (1.1).

2 Preliminaries

Throughout this paper, we work in $\mathbb{R}^n$, and the norm $\|\cdot\|$ is the norm induced by the Euclidean scalar product $\langle \cdot, \cdot \rangle$. In this section we recall several basic results needed in our convergence analysis.

First, we introduce some orthogonal projection properties.

Lemma 2.1 (orthogonal projection properties [34, Prop. 4.16]) *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set and P_C be the orthogonal projection onto set C defined in (1.2). Then, the following hold:*

(i) *For all $x, y \in \mathbb{R}^n$, $\|P_C(x) - P_C(y)\|^2 \le \|x - y\|^2 - \|(P_C(x) - x) - (P_C(y) - y)\|^2$.*

(ii) *For all $x \in \mathbb{R}^n$ and $s \in C$, $\|P_C(x) - s\| \le \|x - s\|$.*

(iii) *For all $x \in \mathbb{R}^n$ and $s \in C$, $\|P_C(x) - x\|^2 + \|P_C(x) - s\|^2 \le \|x - s\|^2$.*

(iv) *For all $x \in \mathbb{R}^n$, $\operatorname{dist}(P_C(x), C) \le \operatorname{dist}(x, C)$.*

Lemma 2.1(ii) means that projections onto convex sets are *firmly nonexpansive* [34, Def. 4.1]. Note that this property is stronger than the well-known nonexpansiveness of projections, that is, $\|P_C(x) - P_C(y)\| \leq \|x - y\|$, for all $x, y \in \mathbb{R}^n$.

We continue with several definitions and facts, beginning with the notion of *Fejér monotonicity.*

Definition 2.2 Suppose that $M \subset \mathbb{R}^n$ is nonempty. Let $\{x^k\}$ be a sequence in $\mathbb{R}^n$. We say $\{x^k\}$ is *Fejér monotone* with respect to M if $\|x^{k+1} - s\| \leq \|x^k - s\|$, for all $s \in M$, and for all $k \in \mathbb{N}$.

The following lemma gives the properties of Fejér monotone sequences.

Lemma 2.3 (Fejér monotonicity properties [4, Thm. 2.16]) *Suppose that $M \subset \mathbb{R}^n$ is nonemtpy, and the sequence $\{x^k\}$ is Fejér monotone with respect to M. Then,*

(i) *$\{x^k\}$ is bounded.*
(ii) *For every $s \in M$, $\{\|x^k - s\|\}$ converges.*
(iii) *If there exists a cluster point x^* of $\{x^k\}$ such that $x^* \in M$, then $\{x^k\}$ converges to x^*.*

Next, we are going to present some results about the boundedness and approximation property of convex functions.

Definition 2.4 Take $f : \mathbb{R}^n \to \mathbb{R}$ and $U \in \mathbb{R}^n$. We say that f is *locally Lipschitz continuous* in U if for each $z \in U$, there exists a neighborhood V of z and an $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|,$$

for all $x, y \in V$.

It is well-know that convex functions are *locally Lipschitz continuous*; see [35, Thm. 2.1.12]. We now recall some properties on subgradients.

Definition 2.5 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. We say that a vector $v \in \mathbb{R}^n$ is a *subgradient* of f at a point $x \in \mathbb{R}^n$ if, for all $y \in \mathbb{R}^n$,

$$f(y) \geq f(x) + \langle v, y - x \rangle .$$

The set of all *subgradients* of a convex function at $x \in \mathbb{R}^n$ is called the *subdifferential* of f at x, and is denoted by $\partial f(x)$. We present now a version of the mean value theorem for convex functions.

Lemma 2.6 (mean value theorem for convex functions [36]) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let x and y be vectors in $\mathbb{R}^n$. Then, there exists a vector $u \in \mathbb{R}^n$*

and a subgradient $v(u) \in \partial f(u)$ *such that*

$$f(y) = f(x) + \langle v(u), y - x \rangle ,$$

where $u = \alpha x + (1-\alpha)y$*, and* $\alpha \in (0,1)$*.*

We end this section recalling that the subdifferential operator of the convex functions f, $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, is *maximal monotone* [37, Cor. 31.5.2] and *locally bounded* [38, Thm. 3].

3 Convergence analysis of cCRM for the multiset case

We proceed to the convergence analysis of cCRM applied to the Convex Feasibility Problem with m convex sets.

3.1 The cCRM for two convex sets

In this subsection we are going to present some results of [13] in order to allow us to define the successive cCRM iteration to solve problem (1.1).

First, we present the notion of *centralized point* in connection with CRM, which was introduced in [13]. Note that circumcenter steps taken from centralized points move towards the solution of the convex feasibility problem involving two intersecting sets, as showed in the aforementioned work.

Definition 3.1 (centralized point) Let $A, B \subset \mathbb{R}^n$ be two nonempty closed convex sets, and a point $z \in \mathbb{R}^n$ is said to be *centralized* with respect to A, B if

$$\langle R_A(z) - z, R_B(z) - z \rangle \leq 0.$$

First, we show that if we apply the operator $\tilde{Z}_{A,B}(Z_{A,B})$ to a point $z \in \mathbb{R}^n$, then the resulting point will be centralized with respect to A and B.

Lemma 3.2 ([13, Lem. 2.2]) *Let* $A, B \subset \mathbb{R}^n$ *be two nonempty closed convex sets with nonempty intersection. For any* $z \in \mathbb{R}^n$*, then* $\tilde{Z}_{A,B}(Z_{A,B}(z))$ *is centralized w.r.t.* A *and* B*.*

We next state the firmly nonexpansiveness of a parallel circumcenter when the initial point is centralized.

Lemma 3.3 ([13, Lem. 2.5]) *Let* $A, B \subset \mathbb{R}^n$ *be two nonempty closed convex sets with nonempty intersection. Assume that* $z \in \mathbb{R}^n$ *is a centralized point with respect to* A, B*. Then,* $\mathscr{C}_{A,B}(z) := \operatorname{circ}(z, R_A(z), R_B(z))$ *satisfies*

$$\|\mathscr{C}_{A,B}(z) - s\|^2 \leq \|z - s\|^2 - \|z - \mathscr{C}_{A,B}(z)\|^2,$$

for all $s \in A \cap B$*.*

Lemma 3.4 ([13, Lem. 2.6]) *Let $A, B \subset \mathbb{R}^n$ be two nonempty closed convex sets with nonempty intersection. Let $z \in \mathbb{R}^n$. Then,*

$$\|T_{A,B}(z) - s\|^2 \le \|z - s\|^2 - \frac{1}{8}\|z - T_{A,B}(z)\|^2,$$

where for all $s \in A \cap B$.

3.2 Successive cCRM for the multiset case

We are ready to present now the results that extend the cCRM for the case multiset case, that is, to solve the CFM problem (1.1). We will consider three options for the control sequence, as explained in the Introduction, namely almost cyclic, most violated constraint (distance version) and most violated constraint (function value version), giving rise to Algorithms 1, 2 and 3.

We initially extend Lemma 3.3 result to the multiset case.

Corollary 3.5 *Let $C_1, \dots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets with nonempty intersection. Assume that $z \in \mathbb{R}^n$ is a centralized point with respect to $C_i \cap C_j$ for any $i, j \in \{1, 2, \dots, m\}$. Then, we have*

$$\left\|\mathscr{C}_{C_i,C_j}(z) - s\right\| = \|\operatorname{circ}(z, R_{C_i}(z), R_{C_j}(z)) - s\| \le \|z - s\|,$$

for all $s \in C := \cap_{i=1}^m C_i$ and any $i, j \in \{1, 2, \dots, m\}$.

Proof Since $C \subset C_i \cap C_j$ for any $i, j \in \{1, 2, \dots, m\}$, the result is a direct consequence of Lemma 3.3. □

In sequel, the nonexpansiveness of the SiPM iteration (1.5) is established.

Lemma 3.6 *Let $C_1, \dots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets with C being their nonempty intersection. Assume that $z \in \mathbb{R}^n$ is an arbitrary point. Then, we have*

$$\|\tilde{Z}_{C_i,C_j}(z) - s\| \le \|z - s\|,$$

for each $s \in C$ and for any $i, j \in \{1, \dots, m\}$.

Proof Observe that, for $z \in \mathbb{R}^n$ and $s \in C$,

$$\begin{aligned} \|\tilde{Z}_{C_i,C_j}(z) - s\| &= \|\frac{1}{2}(P_{C_i} + P_{C_j})(z) - s\| \\ &\le \frac{1}{2}\|P_{C_i}(z) - s\| + \frac{1}{2}\|P_{C_j}(z) - s\| \\ &\le \|z - s\|, \end{aligned}$$

where the second inequality holds by Lemma 2.1(ii). □

Next, the firmly nonexpansiveness of the cCRM iteration (1.8) is stated as a corollary.

Corollary 3.7 *Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex sets with nonempty intersection. Then,*

$$\|T_{C_i,C_j}(z) - s\|^2 \leq \|z - s\|^2 - \frac{1}{8}\|z - T_{C_i,C_j}(z)\|^2, \tag{3.1}$$

for all $s \in C := \cap_{i=1}^m C_i$, for all $z \in \mathbb{R}^n$, and for arbitrary $i, j \in \{1, 2, \ldots, m\}$.

Proof Since $C \subset C_i \cap C_j$ for any $i, j \in \{1, 2, \ldots, m\}$, the result is a direct consequence of Lemma 3.4.

□

We state now the Fejér monotonicity of successive cCRM, which follows from (3.1) immediately.

Corollary 3.8 (Fejér monotonicity of successive cCRM) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets with nonempty intersection C, and suppose that the sequence $\{z^k\} \subset \mathbb{R}^n$ is generated by cCRM defined in (1.8) with any of the control sequences given in Definition 1.1. Then $\{z^k\}$ is Fejér monotone with respect to C.*

Now, the asymptotic convergence of successive cCRM is stated and proved.

Corollary 3.9 (asyptotic convergence of successive cCRM) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \cap_{i=1}^m C_i \neq \emptyset$. Suppose that the sequence $\{z^k\} \subset \mathbb{R}^n$ is generated by cCRM defined in (1.8) with any of the control sequences given in Definition 1.1. Then,*

$$\|z^{k+1} - z^k\| \to 0.$$

Proof By using (3.1) at z^k, we get

$$\frac{1}{8}\|T_{C_{r(k)},C_{\ell(k)}}(z^k) - z^k\|^2 \leq \|z - s\|^2 - \|T_{C_{r(k)},C_{\ell(k)}}(z^k) - s\|^2,$$

for any $s \in C$. By the definition of cCRM for CFP with m sets,

$$\frac{1}{8}\|z^{k+1} - z^k\|^2 \leq \|z^k - s\|^2 - \|z^{k+1} - s\|^2.$$

Using Lemma 3.7 and Lemma 2.3(iii), we establish the result. □

Lemma 3.10 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \cap_{i=1}^m C_i \neq \emptyset$. Suppose that the sequence $\{z^k\}$ is generated by cCRM as defined in (1.8) with any of the control sequences given in Definition 1.1. Then, we have*

$$\|z^{k+1} - s\| \leq \|P_{C_{\ell(k)}}(z^k) - s\|, \tag{3.2}$$

for all $s \in C$.

Proof Note that, for $s \in C$,

$$\begin{aligned}\|z^{k+1} - s\| = \|T_{C_{r(k)},C_{\ell(k)}}(z^k) - s\| &\leq \|\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k) - s\| \\ &\leq \|Z_{C_{r(k)},C_{\ell(k)}}(z^k) - s\| \leq \|P_{C_{\ell(k)}}(z^k) - s\|, \end{aligned} \tag{3.3}$$

where the first inequality follows from Lemma 3.2 and Corollary 3.5, the second inequality holds by Lemma 3.6, and the last one Lemma 2.1(ii). □

According to the last results, we only need to prove that there exists a cluster point lying in the intersection of the underlying sets, in order to achieve the convergence of a successive cCRM sequence. We establish this in sequel, for Algorithms 1, 2 and 3.

Theorem 3.11 (convergence of Algorithm 1) *Let $C_1, \dots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \cap_{i=1}^m C_i \neq \emptyset$. Suppose that z^0 is an arbitrary point in $\mathbb{R}^n$ and the sequence $\{z^k\}$ is generated by cCRM operator defined in (1.8) with almost cyclic control sequence. Then, there exists some point $x^* \in C := \cap_{i=1}^m C_i$ such that $z^k \to x^*$.*

Proof By Corollary 3.8 and Lemma 2.3(ii), we get that $\{z^k\}$ is a Fejér monotone sequence w.r.t. C, so it is bounded. Hence, there exist a subsequence $\{z^{k_j}\}$ of $\{z^k\}$ and a point $x^* \in \mathbb{R}^n$ such that $z^{k_j} \to x^*$. By the definition of almost cyclic control sequence, for each $q \in \{1, \dots, m\}$, there exists a sequence $\{h_j\}$ which satisfies $k_j \leq h_j \leq k_j + Q$ such that $\ell(h_j) = q$, for each $j \geq 1$. By the triangular inequality,

$$\|z^{h_j} - z^{k_j}\| \leq \sum_{i=0}^{h_j - k_j - 1} \|z^{k_j+i+1} - z^{k_j+i}\| \leq \sum_{i=0}^{Q-1} \|z^{k_j+i+1} - z^{k_j+i}\|. \tag{3.4}$$

Note that the rightmost side of (3.4) is a sum of Q terms, and each one of them converges to 0 as $j \to \infty$ by Corollary 3.9. Hence, the whole summation goes to 0, so that

$$\lim_{j \to \infty} z^{h_j} - z^{k_j} = 0. \tag{3.5}$$

Since $z^{h_j} = z^{k_j} + (z^{h_j} - z^{k_j})$ and $z^{k_j} \to x^*$, by assumption, we conclude from (3.5) that $\lim_{j\to\infty} z^{h_j} - x^* = 0$. Note that

$$\begin{aligned}\|z^{h_j+1} - s\|^2 &\leq \|P_{C_{\ell(h_j)}}(z^{h_j}) - s\|^2 \\ &\leq \|z^{h_j} - s\|^2 - \|P_{C_{\ell(h_j)}}(z^{h_j}) - z^{h_j}\|^2,\end{aligned}$$

using Lemma 3.10 in the first inequality, and Lemma 2.1(iii) in the second one. Therefore,

$$\|P_{C_{\ell(h_j)}}(z^{h_j}) - z^{h_j}\|^2 \leq \|z^{h_j} - s\|^2 - \|z^{h_j+1} - s\|^2. \tag{3.6}$$

Since for all $\ell(h_j) = q$, in view of Lemma 2.3(ii), taking limit with $j \to \infty$ in (3.6), we get that the right side of (3.6) goes to 0. Consequently,

$$\|P_{C_q}(x^*) - x^*\|^2 \leq 0.$$

Hence, we obtain that $P_{C_q}(x^*) = x^*$. Since q is an arbitrary index, we have that $x^* \in C_q$ for all $q \in \{1, 2, \dots, m\}$. Consequently, $x^* \in C$. By Lemma 2.3(iii), we get that $z^k \to x^* \in C$.

□

Remark 3.12 We know that the cyclic control sequence is a special case of the almost cyclic control sequence, so we conclude that the sequence $\{z^k\}$ generated by cCRM defined in (1.8) will converge to some point in C if we use a cyclic control sequence.

Theorem 3.13 (convergence of Algorithm 2) *Suppose that z^0 is an arbitrary point in $\mathbb{R}^n$ and the sequence $\{z^k\}$ is generated by the cCRM defined in (1.8) with the most violated constraint control sequence in distance version (1.9). Then there exists some point $x^* \in C \coloneqq \cap_{i=1}^m C_i$ such that $z^k \to x^*$.*

Proof By Corollary 3.8 and Lemma 2.3(ii), there exists a subsequence $\{z^{k_j}\}$ and a point $x^* \in \mathbb{R}^n$ such that $z^{k_j} \to x^*$. Take any $i \in \{1, \dots, m\}$, then

$$\begin{aligned} \operatorname{dist}^2(z^{k_j}, C_i) &\leq \operatorname{dist}^2(z^{k_j}, C_{\ell(k_j)}) = \|z^{k_j} - P_{C_{\ell(k_j)}}(z^{k_j})\|^2 \\ &\leq \|z^{k_j} - s\|^2 - \|P_{C_{\ell(k_j)}}(z^{k_j}) - s\|^2 \\ &\leq \|z^{k_j} - s\|^2 - \|z^{k_j+1} - s\|^2, \end{aligned} \tag{3.7}$$

using the definition of the most violated constraint control sequence in the first inequality, Lemma 2.1(iii) in the second one and Lemma 3.10 in the third one. Take $j \to \infty$, and use Lemma 2.1(iii) for proving that the rightmost expression in (3.7) converges to 0. Hence,

$$\operatorname{dist}(x^*, C_i) \leq 0, \ \forall i \in \{1, 2, \dots, m\}.$$

Since i is an arbitrary index, we conclude that $x^* \in C_i$ for all $i \in \{1, \dots, m\}$, so that $x^* \in C$. The result follows immediately by Lemma 2.3(iii) □

Theorem 3.14 (convergence of Algorithm 3) *Consider the CFP problem (1.1), suppose z^0 is an arbitrary point in $\mathbb{R}^n$ and assume that sequence $\{z^k\}$ is generated by the cCRM defined in (1.8) with the most violated constraint control sequence in function value version (1.11). Then there exists some point $x^* \in C \coloneqq \cap_{i=1}^m C_i$ such that $z^k \to x^*$.*

Proof By Corollary 3.8 and Lemma 2.3(ii), there exists a subsequence $\{z^{k_j}\}$ and a point $x^* \in \mathbb{R}^n$ such that $z^{k_j} \to x^*$. Using the locally Lipschitz continuity of the convex functions f_i's, for $i \in \{1, \dots, m\}$, there exists a neighborhood V_i of x^* such that f_i is Lipschitz continuous in V_i with constant L_i. Take $V = \cap_{i=1}^m V_i$ and $L = \max_{1 \leq i \leq m} L_i$, so f_i is Lipschitz continuous in V with constant L for each $i = 1, 2, \dots, m$.

For large enough k_j, we have

$$\begin{aligned} f_i(z^{k_j}) &\leq f_{\ell(k_j)}(z^{k_j}) \leq f_{\ell(k_j)}(z^{k_j}) - f_{\ell(k_j)}(P_{C_{\ell(k_j)}}(z^{k_j})) \\ &\leq L\|z^{k_j} - P_{C_{\ell(k_j)}}(z^{k_j})\| \\ &\leq L(\|z^{k_j} - s\|^2 - \|P_{C_{\ell(k_j)}}(z^{k_j}) - s\|^2)^{\frac{1}{2}} \\ &\leq L(\|z^{k_j} - s\|^2 - \|z^{k_j+1} - s\|^2)^{\frac{1}{2}}, \end{aligned} \tag{3.8}$$

using the definition of the most violated control sequence in the first inequality, the definition of orthogonal projection and (1.10) in the second, the Lipschitz continuity in the third one, and Lemma 2.1(iii) and (3.2) in the fifth one. Taking $j \to \infty$, we have from Lemma 2.3(ii) that the rightmost expression in (3.8) converges to 0. Hence,

$$f_i(x^*) \leq 0.$$

Since i is an arbitrary index, we get that $f_i(x^*) \leq 0$ for all $i \in \{1, \dots, m\}$, and hence $x^* \in C$. In view of Lemma 2.3(iii) and Corollary 3.8, $z^k \to x^* \in C$ □

4 Linear and superlinear convergence rate

In this section, we first introduce two options of error bounds for CFP. Then, we review the proofs of the linear convergence of SiMP and SePM under these error bounds. Next, we prove the linear convergence of Algorithms 2 and 3, also under these error bounds. Finally, we prove that under a smoothness assumption on the sets C_i and a Slater condition, Algorithms 2 and 3 achieve a superlinear convergence rate. These results are in agreement with those established in [13] for cCRM applied to CFP with two sets.

4.1 Error bounds for CFP

Definition 4.1 Let $A, B \subset \mathbb{R}^n$ be closed convex and assume that $A \cap B \neq \emptyset$. We say that A and B *satisfy a local error bound condition* if for some point $\bar{z} \in X \cap Y$, there exist a real number $\omega \in (0,1)$, and a neighborhood V of $\bar{z}$ such that

$$\omega \operatorname{dist}(z, A \cap B) \leq \max\{\operatorname{dist}(z, A), \operatorname{dist}(z, B)\},$$

for all $z \in V$.

Under this condition, a point in V cannot be too close to both A and B, and at the same time, far from $A \cap B$. This condition was used in [10] and [13] to prove the linear convergence rate for CRM and cCRM. Now we extend this concept to the case of m sets.

Definition 4.2 (EB 1) Let $C_1, C_2, \dots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C \coloneqq \cap_{i=1}^m C_i \neq \emptyset$. We say that $C_1, C_2, \dots, C_m$ satisfy *the local error bound condition (EB1) at a point* $\bar{z} \in C$, if there exists a real number $\omega \in (0,1)$, and a neighborhood V of $\bar{z}$ such that

$$\omega \operatorname{dist}(z, C) \leq \max_{1 \leq i \leq m} \operatorname{dist}(z, C_i),$$

for all $z \in V$.

Definition 4.3 (EB 2) Let $C_1, C_2, \dots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C \coloneqq \cap_{i=1}^m C_i \neq \emptyset$. We say that $C_1, C_2, \dots, C_m$ satisfy *the local error bound condition 2 (EB2) at a point* $\bar{z} \in C$, if there exists a real number $\omega \in (0,1)$, and a neighborhood V of $\bar{z}$ such that

$$\omega \operatorname{dist}(z, C) \leq \max_{1 \leq i \leq m} f_i(z), \tag{4.1}$$

for all $z \in V$.

Remark 4.4 Using the definition of the most violated constraint control sequence (1.11), EB2 equation (4.1) becomes, for all $z \in V$,

$$\omega \operatorname{dist}(z, C) \leq f_\ell(z),$$

where $\ell := \underset{1 \leq i \leq m}{\arg\max}\{f_i(z)\}$.

Definition 4.5 Let $\{x^k\} \subset \mathbb{R}^n$ be a sequence converging to some point $\bar{x} \in \mathbb{R}^n$. Assume that $x^k \neq \bar{x}$ for all $k \in \mathbb{N}$. Define

$$\xi := \lim_{k \to \infty} \sup \frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|}, \text{ and } \rho := \lim_{k \to \infty} \sup \|x^k - \bar{x}\|^{\frac{1}{k}}.$$

Then, the convergence of $\{x^k\}$ is
(i) *Q-linear* if $\xi \in (0, 1)$;
(ii) *Q-superlinear* if $\xi = 0$,
(iii) *R-linear* if $\rho \in (0, 1)$.

It is long-familiar that Q-linear convergence is a sufficient condition for R-linear convergence (with the same asymptotic constant), but the converse statement does not hold true [39].

Next lemma claims the linear convergence of Fejér monotone sequences.

Lemma 4.6 ([13, Prop. 3.8]) *If the sequence $\{x^k\} \subset \mathbb{R}^n$ is Fejér Monotone with respect to a set $M \subset \mathbb{R}^n$, and the scalar sequence $\{\operatorname{dist}(x^k, M)\}$ converges Q-linearly to 0, then $\{x^k\}$ converges R-linearly to a point $\bar{x} \in M$.*

4.2 Linear convergence rate of Algorithm 2

Before proving the linear convergence rate of Algorithm 2, we recall the proof of the linear convergence for SeMP and SiPM with the most violated constraint control related to (1.9).

Lemma 4.7 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose $C := \cap_{i=1}^m C_i \neq \emptyset$ Assume that EB1 at $\bar{z} \in C$. Let B be a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1 - \omega^2}$. Then,*

$$\operatorname{dist}(Z_{C_r, C_\ell}(z), C) \leq \beta^2 \operatorname{dist}(z, C), \tag{4.2}$$

and

$$\operatorname{dist}(\tilde{Z}_{C_r, C_\ell}(z), C) \leq \left(\frac{1 + \beta}{2}\right) \operatorname{dist}(z, C), \tag{4.3}$$

where $\ell := \underset{1 \leq i \leq m}{\arg\max}\{\operatorname{dist}(z, C_i)\}$ and $r := \underset{1 \leq i \leq m}{\arg\max}\{\operatorname{dist}(P_{C_\ell}(z), C_i)\}$.

Proof We start with the SeMP case. Note that

$$\operatorname{dist}^2(z, C) = \|z - P_C(z)\|^2 \geq \|P_{C_\ell}(z) - P_C(z)\|^2 + \|z - P_{C_\ell}(z)\|^2$$

$$
\begin{aligned}
&\geq \operatorname{dist}^2(P_{C_\ell}(z), C) + \operatorname{dist}^2(z, C_\ell) \\
&= \operatorname{dist}^2(P_{C_\ell}(z), C) + \max_{1\leq i\leq m} \operatorname{dist}^2(z, C_i) \\
&\geq \operatorname{dist}^2(P_{C_\ell}(z), C) + \omega^2 \operatorname{dist}^2(z, C),
\end{aligned}
$$

where the first inequality holds by Lemma 2.1(iii), in the second one we use the definition of P_{C_ℓ}, the last equality follows from the definition of ℓ, and the third inequality follows by EB1. Hence,

$$
\operatorname{dist}(P_{C_\ell}(z), C) \leq \sqrt{1-\omega^2}\operatorname{dist}(z, C) = \beta \operatorname{dist}(z, C). \tag{4.4}
$$

Since $\bar{z} \in C$, by the nonexpansiveness of P_{C_ℓ} and P_{C_r}, we have

$$
\|P_{C_r}(P_{C_\ell}(z)) - \bar{z}\| \leq \|P_{C_\ell}(z) - \bar{z}\| \leq \|z - \bar{z}\|, \tag{4.5}
$$

hence, $P_{C_r}(P_{C_\ell}(z)) \in B$. Consequently,

$$
\begin{aligned}
\operatorname{dist}^2(P_{C_\ell}(z), C) &= \|P_{C_\ell}(z) - P_C(P_{C_\ell}(z))\|^2 \\
&\geq \|P_{C_r}(P_{C_\ell}(z)) - P_C(P_{C_\ell}(z))\|^2 + \|P_{C_\ell}(z) - P_{C_r}(P_{C_\ell}(z))\|^2 \\
&\geq \operatorname{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \max_{1\leq i\leq m} \operatorname{dist}^2(P_{C_i}(z), C_r) \\
&\geq \operatorname{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \omega^2 \operatorname{dist}^2(P_{C_\ell}(z), C),
\end{aligned} \tag{4.6}
$$

where the first inequality follows from Lemma 2.1(iii), the second follows from the definition of r, and the third one from EB1. From (4.6), we obtain

$$
\operatorname{dist}(P_{C_r}(P_{C_\ell}(z)), C) \leq \sqrt{1-\omega^2}\operatorname{dist}(P_{C_\ell}(z), C) = \beta \operatorname{dist}(P_{C_\ell}(z), C). \tag{4.7}
$$

Now combining (4.5) and (4.7), we get

$$
\begin{aligned}
\operatorname{dist}(Z_{C_r,C_\ell}(z), C) = \operatorname{dist}(P_{C_r}(P_{C_\ell}(z)), C) &\leq \sqrt{1-\omega^2}\operatorname{dist}(P_{C_\ell}(z), C) \\
&\leq (1-\omega^2)\operatorname{dist}(z, C) = \beta^2 \operatorname{dist}(z, C),
\end{aligned}
$$

which establishes (4.2).

Next, we will establish the linear convergence for SiPM with EB1. By the nonexpansiveness of P_{C_r}, we have that

$$
\operatorname{dist}(P_{C_r}(z), C) \leq \operatorname{dist}(z, C). \tag{4.8}
$$

Note that

$$
\begin{aligned}
\operatorname{dist}(\tilde{Z}_{C_r,C_\ell}(z), C) &= \operatorname{dist}\left(\frac{1}{2}[P_{C_\ell}(z) + P_{C_r}(z)], C\right) \\
&\leq \frac{1}{2}\left[\operatorname{dist}(P_{C_\ell}(z), C) + \operatorname{dist}(P_{C_r}(z), C)\right] \\
&\leq \left(\frac{1+\beta}{2}\right)\operatorname{dist}(z, C),
\end{aligned}
$$

using the convexity of the distance function to C in the first inequality, and (4.4) and (4.8) in the second one. Hence, we get (4.3), and the results hold. □

Corollary 4.8 *Suppose $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ nonempty closed convex sets and assume $C := \cap_{i=1}^m C_i \neq \emptyset$. Let $\{s^k\}$, $\{y^k\}$ be generated by SePM and SiPM starting from some $s^0 \in \mathbb{R}^n$, and some $y^0 \in \mathbb{R}^n$, respectively. Assume also that $\{s^k\}$*

and $\{y^k\}$ are both infinite sequences. If EB1 holds at the limit point $\bar{s}$ of $\{s^k\}$, $\bar{y}$ of $\{y^k\}$, then the sequences $\{\operatorname{dist}(s^k, C)\}$ and $\{\operatorname{dist}(y^k, C)\}$ converge Q-linearly to 0, with asymptotic constants given by β^2, $\frac{1+\beta}{2}$, respectively, where $\beta = \sqrt{1-\omega^2}$, and ω is the constant in EB1.

Proof Convergence of $\{s^k\}$ and $\{y^k\}$ to points $\bar{s} \in C$ and $\bar{y} \in C$ follows from [4, Corollary 3.3(i)] and in [5, Theorem 3], respectively. Hence, for large enough k, y^k belongs to a ball centered at $\bar{y}$ contained in V and s^k belongs to a ball centered at $\bar{s}$ contained in V. In view of the definitions of the SeMP and SiMP sequences, we get from Lemma 4.7,

$$\frac{\operatorname{dist}(s^{k+1}, C)}{\operatorname{dist}(s^k, C)} \leq \beta^2, \quad \frac{\operatorname{dist}(y^{k+1}, C)}{\operatorname{dist}(y^k, C)} \leq \frac{1+\beta}{2},$$

and the results follow from Definition 4.5, noting that

$$\operatorname{dist}(y^k, C) \leq \beta^{2k} \operatorname{dist}(y^0, C), \quad \operatorname{dist}(s^k, C) \leq \left(\frac{1+\beta}{2}\right)^k \operatorname{dist}(s^0, C).$$

Hence, both distance sequences converge to 0, since $\beta \in (0,1)$, because $\omega \in (0,1)$. □

Corollary 4.9 *Suppose $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C := \cap_{i=1}^m C_i \neq \emptyset$. Let $\{s^k\}$ and $\{y^k\}$ be sequences generated by SeMP and SiMP starting from some $s^0 \in \mathbb{R}^n$ and $y^0 \in \mathbb{R}^n$, respectively. Assume also $\{s^k\}$ and $\{y^k\}$ are both infinite sequences. If EB1 holds at the limit point $\bar{s}$ of $\{s^k\}$, $\bar{y}$ of $\{y^k\}$, then the sequences $\{s^k\}$, $\{y^k\}$ converge R-linearly, with asymptotic constants bounded above by β^2, $\frac{1+\beta}{2}$ respectively, where $\beta = \sqrt{1-\omega^2}$ and ω is the constant in Assumption EB1.*

Proof The fact $\{s^k\}$ and $\{y^k\}$ are Fejér monotone with respect to C is immediate consequence of Lemma 2.1(ii) and Lemma 3.6, respectively. Then the result follows then from Lemma 4.6 and Corollary 4.8. □

We now proceed to prove the linear convergence of cCRM for the multiset case. First, we show the linear rate for the most violated constraint control related to (1.9), in view of EB 1.

Lemma 4.10 *Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex sets, and assume that $C := \cap_{i=1}^m C_i \neq \emptyset$. Assume that EB1 at $\bar{z} \in C$. Let B be a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1-\omega^2}$. Then,*

$$\operatorname{dist}(T_{C_r, C_\ell}(z), C) \leq \beta^2 \operatorname{dist}(z, C),$$

where $\ell := \operatorname*{arg\,max}_{1 \leq i \leq m}\{\operatorname{dist}(z, C_i)\}$ and $r := \operatorname*{arg\,max}_{1 \leq i \leq m}\{\operatorname{dist}(P_{C_\ell}(z), C_i)\}$.

Proof By (4.5), we know that $Z_{C_r,C_\ell}(z) \in B$. Using the definition of $\bar{Z}_{C_r,C_\ell}$, we have

$$\begin{aligned}\operatorname{dist}(\bar{Z}_{C_r,C_\ell}(z), C) &= \operatorname{dist}\left(\frac{1}{2}\Big(Z_{C_r,C_\ell}(z) + P_{C_r}(Z_{C_r,C_\ell}(z))\Big), C\right)\\ &\le \frac{1}{2}\operatorname{dist}(Z_{C_r,C_\ell}(z), C) + \frac{1}{2}\operatorname{dist}(P_{C_r}(Z_{C_r,C_\ell}(z)), C)\\ &\le \frac{1}{2}\beta^2 \operatorname{dist}(z, C) + \frac{1}{2}\beta^2 \operatorname{dist}(z, C)\\ &= \beta^2 \operatorname{dist}(z, C), \end{aligned} \tag{4.9}$$

where the first inequality follows from the convexity of the distance function, and the second from Lemma 2.1(iv) and (4.2).

By Corollary 3.5, we have

$$\|\mathscr{C}_{C_r,C_\ell}(z) - s\| \le \|z - s\|,$$

for any centralized point $z \in B$ and for any $s \in C$. By Lemma 3.2 and (4.5), $\bar{Z}_{C_r,C_\ell}(z)$ is centralized with respect to C_ℓ and C_r and belongs to B. So, taking $\bar{Z}_{C_r,C_\ell}(z)$, we get

$$\|T_{C_r,C_\ell}(z) - s\| = \left\|\mathscr{C}_{C_r,C_\ell}(\bar{Z}_{C_r,C_\ell}(z)) - s\right\| \le \|\bar{Z}_{C_r,C_\ell}(z) - s\|.$$

Using the definition of distance between $T_{C_r,C_\ell}(z)$ and C, we get

$$\operatorname{dist}(T_{C_r,C_\ell}(z), C) \le \|\bar{Z}_{C_r,C_\ell}(z) - s\|.$$

Take $s = P_C(\bar{Z}_{C_r,C_\ell}(z))$, then

$$\operatorname{dist}(T_{C_r,C_\ell}(z), C) \le \operatorname{dist}(\bar{Z}_{C_r,C_\ell}(z), C), \tag{4.10}$$

for all $z \in B$. Combining (4.9) and (4.10), we obtain

$$\operatorname{dist}(T_{C_r,C_\ell}(z), C) \le \beta^2 \operatorname{dist}(z, C),$$

which establishes the result. □

Now, we are going to establish linear convergence of Algorithm 2, under EB1.

Theorem 4.11 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i \neq \emptyset$. Assume that sequence $\{z^k\}$ is generated by cCRM with the most violated constraint control sequence (distance version) as in (1.9), starting from some $z^0 \in \mathbb{R}^n$. Assume also that $\{z^k\}$ is an infinite sequence. If EB1 holds at the limit $\bar{z}$ of $\{z^k\}$, then $\{z^k\}$ converges to $\bar{z} \in C$ R-linearly, with asymptotic constant bounded above β^2, where $\beta = \sqrt{1-\omega^2}$ and ω is the constant defined in EB1.*

Proof The convergence of $\{z^k\}$ to a point $\bar{z} \in C$ follows from Theorem 3.13. Hence, for large enough k, z^k belongs to the ball centered at $\bar{z}$ and contained in V, whose existence is ensured in EB1.

We recall that the cCRM sequence is defined as $z^{k+1} = T_{C_{r(k)},C_{\ell(k)}}(z^k)$, so that it follows from Lemma 4.10 that

$$\frac{\operatorname{dist}(z^{k+1}, C)}{\operatorname{dist}(z^k, C)} \le \beta^2. \tag{4.11}$$

Since $\omega \in (0,1)$ implies that $\beta^2 \in (0,1)$, it follows immediately from (4.11) that the scalar sequence $\{\operatorname{dist}(z^k, C)\}$ converges Q-linearly to 0 with asymptotic constant bounded above by β^2.

Finally, recall that sequence $\{z^k\}$ is Fejér monotone with respect to C, due to Corollary 3.8. The R-linear convergence of $\{z^k\}$ to some point in C and the value of the upper bound of the asymptotic follow from Lemma 4.6. □

4.3 Linear convergence of Algorithm 3

Lemma 4.12 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i \neq \emptyset$. Assume that EB2 holds at a point $\bar{z} \in C$. Then, there exists a ball B centered at $\bar{z}$ and a constant $\epsilon > 0$ such that*

$$\|v_i(u)\| \leq \epsilon,$$

for all $u \in B$, all $v_i(u) \in \partial f_i(u)$, and all $i \in \{1, \ldots, m\}$.

Proof Take an arbitrary point $u \in \mathbb{R}^n$, and let $v_i(u) \in \mathbb{R}^n$ be any subgradient of f_i at u. Now, recall that the convex function f_i is locally Lipschitz continuous in $\mathbb{R}^n$ for all $i \in \{1, \ldots, m\}$. Using the fact that the subdifferentials of f_i's are locally bounded in $\mathbb{R}^n$, we establish the result. □

Before proving the linear convergence of cCRM under EB2, we need some lemmas about the relationship between EB1 and EB2.

Lemma 4.13 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i \neq \emptyset$. Assume that EB2 holds at point $\bar{z} \in C$. Then there exists $\epsilon > 0$ such*

$$f_i(z) \leq \epsilon \operatorname{dist}(z, C_i),$$

for all $i \in \{1, \ldots, m\}$, and all z in the neighborhood V of $\bar{z}$ as defined in EB2. Moreover,

$$\operatorname{dist}(z, C) \leq \frac{\epsilon}{\omega} \operatorname{dist}(z, C_\ell), \tag{4.12}$$

where $\ell := \displaystyle\arg\max_{1 \leq i \leq m}\{f_i(z)\}$.

Proof By Lemma 4.12, there exists a ball B contained in V, centered at $\bar{z}$, and a constant ϵ such that

$$\|v_i(u)\| \leq \epsilon, \tag{4.13}$$

for all $v_i(u) \in \partial f_i(u)$, and for all $u \in B$, and for each index $i \in \{1, \ldots, m\}$. Take any $z \in B$ and let $z_i := P_{C_i}(z)$. Using Lemma 2.6, we have

$$f_i(z) = f_i(z_i) + \langle v_i(u_i), z - z_i \rangle,$$

for some u_i in the line segment between z and z_i and, $v_i(u_i) \in \partial f(u_i)$. Since $z_i \in C_i$, we have $f_i(z_i) = 0$, and it follows from (4.13) that

$$f_i(z) \leq \|v_i(u_i)\| \|z - z_i\| = \|v_i(u_i)\| \operatorname{dist}(z, C_i). \tag{4.14}$$

By the nonexpansiveness of orthogonal projection, we have

$$\|P_{C_i}(z) - P_{C_i}(\bar{z})\| \leq \|z - \bar{z}\|.$$

From the definition of z_i and $\bar{z} \in C$, we get

$$\|z_i - \bar{z}\| \leq \|z - \bar{z}\|,$$

which shows that $z_i \in B$. Hence, $u_i \in B$, by the convexity of the ball. In view of (4.13) and (4.14), we have

$$f_i(z) \leq \epsilon \operatorname{dist}(z, C_i),$$

for each $i = 1, \ldots, m$ and each $z \in B$. Therefore,

$$f_\ell(z) \leq \epsilon \operatorname{dist}(z, C_\ell),$$

so that, in view of EB2, we get

$$\operatorname{dist}(z, C) \leq \frac{\epsilon}{\omega} f_\ell(z) \leq \frac{\epsilon}{\omega} \operatorname{dist}(z, C_\ell),$$

which establishes the result. □

Remark 4.14 With arguments very similar to those used in the previous lemma, together with the nonexpansiveness of projection operator, we can easily get

$$\operatorname{dist}(P_{C_\ell}(z), C) \leq \frac{\epsilon}{\omega} \operatorname{dist}(P_{C_\ell}(z), C_r), \tag{4.15}$$

where $r := \underset{1 \leq i \leq m}{\arg\max}\{f_i(P_{C_\ell}(z))\}$.

Lemma 4.15 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \cap_{i=1}^m C_i \neq \emptyset$ Assume that EB2 at $\bar{z} \in C$, and that B is a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1 - (\frac{\omega}{\epsilon})^2}$, with ω being the constant in EB2, and ϵ being the constant in Lemma 4.13. Then,*

$$\operatorname{dist}(Z_{C_r, C_\ell}(z), C) \leq \beta^2 \operatorname{dist}(z, C), \tag{4.16}$$

for all $z \in B$, and

$$\operatorname{dist}(\tilde{Z}_{C_r, C_\ell}(z), C) \leq \frac{1 + \beta}{2} \operatorname{dist}(z, C), \tag{4.17}$$

for all $z \in B$, where $\ell := \underset{1 \leq i \leq m}{\arg\max}\{f_i(z)\}$ and $r := \underset{1 \leq i \leq m}{\arg\max}\{f_i(P_{C_\ell}(z))\}$.

Proof We start with proving (4.16). Take $z \in B$, and note that

$$\begin{aligned}
\operatorname{dist}^2(z, C) &= \|z - P_C(z)\|^2 \\
&\geq \|P_{C_\ell}(z) - z\|^2 + \|P_{C_\ell}(z) - P_C(z)\|^2 \\
&\geq \operatorname{dist}^2(z, C_\ell) + \operatorname{dist}^2(P_{C_\ell}(z), C) \\
&\geq \operatorname{dist}^2(P_{C_\ell}(z), C) + \frac{\omega^2}{\epsilon^2} \operatorname{dist}^2(z, C),
\end{aligned}$$

using Lemma 2.1(iii) in the first inequality, the definition of orthogonal projection in the second one, and (4.12) in the third one. Hence,

$$\operatorname{dist}(P_{C_\ell}(z), C) \leq \sqrt{1 - \frac{\omega}{\epsilon}} \operatorname{dist}(z, C) = \beta \operatorname{dist}(z, C). \tag{4.18}$$

By (4.5), we get $P_{C_r}(P_{C_\ell}(z)) \in B$. Therefore,

$$\begin{aligned}\operatorname{dist}^2(P_{C_\ell}(z), C) &= \|P_{C_\ell}(z) - P_C(P_{C_\ell}(z))\|^2 \\ &\geq \|P_{C_r}(P_{C_\ell}(z)) - P_C(P_{C_\ell}(z))\|^2 \\ &+ \|P_{C_r}(P_{C_\ell}(z)) - P_{C_\ell}(z)\|^2 \\ &\geq \operatorname{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \operatorname{dist}^2(P_{C_\ell}(z), C_r) \\ &\geq \operatorname{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \frac{\omega^2}{\epsilon^2}\operatorname{dist}^2(P_{C_r}(z), C),\end{aligned}$$

using Lemma 2.1(iii) in the first inequality, the definition of orthogonal projection in the second one, and (4.15) in the third one. Thus, we obtain

$$\operatorname{dist}(P_{C_r}(P_{C_\ell}(z)), C) \leq \beta \operatorname{dist}(P_{C_\ell}(z)), C).$$

Together with (4.18), we have

$$\operatorname{dist}(Z_{C_r,C_\ell}(z), C) = \operatorname{dist}(P_{C_r}(P_{C_\ell}(z)), C) \leq \beta^2 \operatorname{dist}(z, C).$$

Next, we prove (4.17). By the nonexpansiveness of P_{C_r}, we have

$$\operatorname{dist}(P_{C_r}(z), C) \leq \operatorname{dist}(z, C). \tag{4.19}$$

Note that

$$\begin{aligned}\operatorname{dist}(\tilde{Z}_{C_r,C_\ell}(z), C) &= \operatorname{dist}(\frac{1}{2}[P_{C_\ell}(z) + P_{C_r}(z)], C) \\ &\leq \frac{1}{2}[\operatorname{dist}(P_{C_\ell}(z), C) + \operatorname{dist}(P_{C_r}(z), C)] \\ &\leq \left(\frac{1+\beta}{2}\right) \operatorname{dist}(z, C).\end{aligned}$$

The first inequality holds by the convexity of the distance function, and the second one follows by (4.19) and (4.18). □

Corollary 4.16 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \cap_{i=1}^m C_i \neq \emptyset$. Let $\{s^k\}$, $\{y^k\}$ be generated by SePM and SiPM starting from some $s^0 \in \mathbb{R}^n$, and some $y^0 \in \mathbb{R}^n$, respectively. Assume also that $\{s^k\}$ and $\{y^k\}$ are both infinite sequences. If EB2 holds at the limit point $\bar{s}$ of $\{s^k\}$, $\bar{y}$ of $\{y^k\}$, then the sequences $\{\operatorname{dist}(s^k, C)\}$ and $\{\operatorname{dist}(y^k, C)\}$ converge Q-linearly to 0, with asymptotic constants given by β^2 and $\frac{1+\beta}{2}$, respectively, where $\beta = \sqrt{1 - (\frac{\omega}{\epsilon})^2}$, ω is the constant in EB2, and ϵ is the constant in Lemma 4.13.*

Proof We invoke again [4, Corollary 3.3(i)] and [5, Theorem 3], to get the convergence of $\{s^k\}$ and $\{y^k\}$ to points $\bar{s} \in C$ and $\bar{y} \in C$, respectively. Hence, y^k belongs to a ball centered at $\bar{y}$ contained in V and s^k belongs to a ball centered at $\bar{s}$ contained in V, for large enough k.

In view of the definitions of the SeMP and SiMP sequences, we get from Lemma 4.15,

$$\frac{\operatorname{dist}(s^{k+1}, C)}{\operatorname{dist}(s^k, C)} \leq \beta^2, \quad \frac{\operatorname{dist}(y^{k+1}, C)}{\operatorname{dist}(y^k, C)} \leq \frac{1+\beta}{2},$$

and the results follow from Definition 4.5, noting that

$$\operatorname{dist}(y^k, C) \leq \beta^{2k} \operatorname{dist}(y^0, C), \quad \operatorname{dist}(s^k, C) \leq \left(\frac{1+\beta}{2}\right)^k \operatorname{dist}(s^0, C),$$

hence, both sequences converge to 0, since $\beta \in (0, 1)$. □

Corollary 4.17 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C := \cap_{i=1}^m C_i \neq \emptyset$. Let $\{s^k\}$ and $\{y^k\}$ be sequences generated by SeMP and SiMP, starting from some $s^0 \in \mathbb{R}^n$ and $y^0 \in \mathbb{R}^n$, respectively. Assume also $\{s^k\}$ and $\{y^k\}$ are both infinite sequences. If EB2 holds at the limit point $\bar{s}$ of $\{s^k\}$, $\bar{y}$ of $\{y^k\}$, then the sequences $\{s^k\}$, $\{y^k\}$ converge R-linearly, with asymptotic constants bounded above by β^2, $\frac{1+\beta}{2}$ respectively, where $\beta = \sqrt{1-(\frac{\omega}{\epsilon})^2}$, ω is the constant in EB2, and ϵ is the constant in Lemma 4.13.*

Proof The facts that $\{s^k\}$ and $\{y^k\}$ are Fejér monotone with respect to C are immediate consequences of Lemma 2.1(ii) and Lemma 3.6, respectively. Then, the result follows from Lemma 4.6 and Corollary 4.16. □

Lemma 4.18 *Suppose $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ nonempty closed convex sets and assume $C := \cap_{i=1}^m C_i \neq \emptyset$ Assume that EB2 at $\bar{z} \in C$. Let B be a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1-(\frac{\omega}{\epsilon})^2}$, with ω being the constant in EB2, and ϵ being the constant in Lemma 4.13. Then,*

$$\operatorname{dist}(T_{C_r,C_\ell}(z), C) \leq \beta^2 \operatorname{dist}(z, C),$$

for all $z \in B$.

Proof By (4.5), we know that $Z_{C_r,C_\ell}(z) \in B$. Using the definition of $\bar{Z}_{C_r,C_\ell}$, we have

$$\begin{aligned}
\operatorname{dist}(\bar{Z}_{C_r,C_\ell}(z), C) &= \operatorname{dist}\left(\frac{1}{2}[Z_{C_r,C_\ell}(z) + P_{C_\ell}(Z_{C_r,C_\ell}(z))], C\right) \\
&\leq \frac{1}{2}\operatorname{dist}(Z_{C_r,C_\ell}(z), C) + \frac{1}{2}\operatorname{dist}(P_{C_\ell}(Z_{C_r,C_\ell}(z)), C) \\
&\leq \frac{1}{2}\beta^2 \operatorname{dist}(z, C) + \frac{1}{2}\beta^2 \operatorname{dist}(z, C) \\
&= \beta^2 \operatorname{dist}(z, C), \qquad (4.20)
\end{aligned}$$

where the first inequality follows from the convexity of the distance function, and the second from Lemma 2.1(iv) and (4.16). Combining (4.20) and (4.10), we establish the result. □

Next, we will state and prove the linear convergence of Algorithm 3 under EB2.

Theorem 4.19 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i \neq \emptyset$. Let the sequence $\{z^k\}$ be generated by cCRM with the most violated constraint control sequence (function value version) as in (1.11), starting from some $z^0 \in \mathbb{R}^n$. Assume also that $\{z^k\}$ is an infinite sequence. If EB2 holds at the limit $\bar{z}$ of $\{z^k\}$, then $\{z^k\}$ converges to $\bar{z} \in C$ R-linearly, with asymptotic constant bounded above β^2, where $\beta = \sqrt{1-(\frac{\omega}{\epsilon})^2}$, ω is the constant in EB2, and ϵ is the constant in Lemma 4.13.*

Proof Convergence of $\{z^k\}$ to a point $\bar{z} \in C$ follows from Theorem 3.14. Hence, for large enough k, z^k belongs to the ball centered at $\bar{z}$ and contained in V, whose existence is ensured in EB2.

We recall that the cCRM sequence is defined as $z^{k+1} = T_{C_{r(k)},C_{\ell(k)}}(z^k)$, so that it follows from Lemma 4.18 that

$$\frac{\operatorname{dist}(z^{k+1}, C)}{\operatorname{dist}(z^k, C)} \leq \beta^2. \tag{4.21}$$

Since $\beta^2 \in (0,1)$, it follows immediately from (4.21) that the scalar sequence $\{\operatorname{dist}(z^k, C)\}$ converges Q-linearly to zero with asymptotic constant bounded above by β^2.

Finally, recall that the sequence $\{z^k\}$ is Fejér monotone with respect to C, due to Corollary 3.8. The R-linear convergence of $\{z^k\}$ to some point in C and the value of the upper bound of the asymptotic follow from Lemma 4.6. ☐

4.4 Superlinear convergence of Algorithms 2 and 3

In this subsection we prove superlinear convergence of Algorithms 2 and 3, assuming a Slater condition and a smoothness assumption: the boundaries of the sets C_i are differentiable manifolds (of codimension 1, due to the Slater condition) near the limit of the sequence.

First, we need a lemma about differentiable manifolds.

Lemma 4.20 ([40, Thm. 24.3]) *Let M be a k-dimensional manifold in $\mathbb{R}^n$, of class $\mathcal{C}^p$. If the boundary of M, $\operatorname{bd}(M)$, is nonempty, then $\operatorname{bd}(M)$ is a $(k-1)$-dimensional manifold without boundary in $\mathbb{R}^n$, of class $\mathcal{C}^p$.*

Lemma 4.21 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i \neq \emptyset$. Let sequence the $\{z^k\}$ be generated by cCRM with the most violated constraint control sequence (distance version) as in (1.9), starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty and that the boundaries of C_i are differentiable manifolds in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Then, the scalar sequence $\{\operatorname{dist}(z^k, C)\}$ converges to zero superlinearly.*

Proof It is trivial if sequence $\{z^k\}$ is finite. Hence, let's assume that it is infinite.

In order to prove the superlinear convergence rate, *i.e.*,

$$\lim_{k\to\infty} \frac{\operatorname{dist}(z^{k+1}, C)}{\operatorname{dist}(z^k, C)} = \lim_{k\to\infty} \frac{\operatorname{dist}(T_{C_{r(k)},C_{\ell(k)}}(z^k), C)}{\operatorname{dist}(z^k, C)} = 0 \tag{4.22}$$

it suffices to show that

$$\lim_{k\to\infty} \frac{\operatorname{dist}(T_{C_{r(k)},C_{\ell(k)}}(z^k), C)}{\operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)} = 0, \tag{4.23}$$

because, by (3.3) and the nonexpansiveness of orthogonal projection, we know that $\operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C) \leq \operatorname{dist}(z^k, C)$, so that (4.22) follows from (4.23) immediately.

We claim that the assumption $\operatorname{int}(C) \neq \emptyset$ implies EB1. Indeed, by Corollary 5.14 in[4], there exists $\hat{k} \in N$ such that

$$\omega \operatorname{dist}(z^k, C) \leq \max_{1 \leq i \leq m} \{\operatorname{dist}(z^k, C_i)\} = \operatorname{dist}(z^k, C_{\ell(k)}), \tag{4.24}$$

for all $\hat{k} \geq k$. In addition, the nonemptyness of the interior of C, together with the hypothesis that the boundaries of C_i are locally differentiable manifolds, proves that these manifolds have dimension $n-1$, in view of Lemma 4.20. Consider the $(n-1)$ dimensional hyperplanes $H^k_{C_{\ell(k)}}$ and $H^k_{C_{r(k)}}$, which are tangent to the manifolds, respectively at

$$P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)) \text{ and } P_{C_{r(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)).$$

As z^{k+1} lies in both $H^k_{C_{\ell(k)}}$ and $H^k_{C_{r(k)}}$, we have

$$\lim_{k \to \infty} \frac{\operatorname{dist}(z^{k+1}, C_{\ell(k)})}{\|z^{k+1} - P_{C_{\ell(k)}}(\bar{Z}_{\ell(k),r(k)}(z^k))\|} = 0. \tag{4.25}$$

Now, using the nonexpansiveness of projections onto $H_{C_{\ell(k)}}$ and $H_{C_{r(k)}}$, we get

$$\|P_{H_{C_{\ell(k)}}}(z^{k+1}) - P_{H_{C_{\ell(k)}}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))\| \leq \|z^{k+1} - \bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)\|.$$

Since $z^{k+1} \in H_{C_{\ell(k)}}$ and $P_{H_{C_{\ell(k)}}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)) = P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))$, we get

$$\|z^{k+1} - P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))\| \leq \|z^{k+1} - \bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)\|. \tag{4.26}$$

By Lemma 3.3, it holds that $\|z^{k+1} - \bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)\| \leq \operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)$, which combined with (4.26), implies that

$$\|z^{k+1} - P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))\| \leq \operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C).$$

Thus, from (4.25), it follows that

$$\lim_{k \to \infty} \frac{\operatorname{dist}(z^{k+1}, C_{\ell(k)})}{\operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)} = 0. \tag{4.27}$$

Note that (4.24) yields

$$\omega \frac{\operatorname{dist}(z^{k+1}, C)}{\operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)} \leq \frac{\operatorname{dist}(z^{k+1}, C_{\ell(k)})}{\operatorname{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)}.$$

Taking limits now as $k \to \infty$, we get (4.23) and the proof is completed. □

The next key result allows us to use the superlinear rate of a scalar distance sequence to prove the superlinear rate of the underlying sequence.

Lemma 4.22 ([13, Prop. 3.12]) *Take a sequence $\{z^k\} \subset \mathbb{R}^n$ which is Fejér monotone with respect to the closed convex set $M \subset \mathbb{R}^n$. If the scalar sequence $\{\operatorname{dist}(z^k, M)\}$ converges superlinearly to 0, then $\{z^k\}$ converges superlinearly to a point $\bar{z} \in M$.*

Next, we present the superlinear convergence result for Algorithm 2.

Theorem 4.23 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i$ is nonempty. Let $\{z^k\}$ be generated by cCRM with the most violated control sequence (distance version) as in (1.9), starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty, and that the boundary of C_i is a differentiable manifold in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Then, $\{z^k\}$ converges to $\bar{z}$ superlinearly.*

Proof The result is a direct consequence of the Fejér monotonicity of $\{z^k\}$ with respect to C given in Corollary 3.8, together with Lemma 4.21 and Lemma 4.22. □

Now, we prove the superlinear convergence of Algorithm 3 under the same assumptions.

Lemma 4.24 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i \neq \emptyset$. Let sequence $\{z^k\}$ be generated by cCRM with the most violated constraint control sequence (function value version) as in (1.11), starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty and that the boundaries of C_i are differentiable manifolds in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Also, assume that EB2 holds at $\bar{z}$. Then, the scalar sequence $\{\mathrm{dist}(z^k, C)\}$ converges to zero superlinearly.*

Proof Assume that $\{z^k\}$ is infinite. Using the definition of EB2 and the fact that $z^k \to \bar{z}$, we conclude that there exists a positive integer $\hat{k}$ such that

$$\mathrm{dist}(z^k, C) \leq \frac{\omega}{\epsilon} \mathrm{dist}(z^k, C_{\ell(k)}), \tag{4.28}$$

for all $k \geq \hat{k}$, where ω and ϵ are the constants in EB2. By (4.28), we get

$$\frac{\mathrm{dist}(z^{k+1}, C)}{\mathrm{dist}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k), C)} \leq \frac{\omega\, \mathrm{dist}(z^{k+1}, C_{\ell(k)})}{\epsilon\, \mathrm{dist}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k), C)}. \tag{4.29}$$

Combining (4.27) and (4.29), we get (4.23) which is sufficient to guarantee (4.22). Then the result holds. □

Theorem 4.25 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \cap_{i=1}^m C_i$ is nonempty. Let $\{z^k\}$ be generated by cCRM with the most violated control sequence (1.11) starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty, and that the boundary of C_i is a differentiable manifold in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Also, assume that EB2 holds at $\bar{z}$. Then, $\{z^k\}$ converges to $\bar{z}$ superlinearly.*

Proof The result is a direct consequence of the Fejér monotonicity of $\{z^k\}$ with respect to C given in Corollary 3.8, together with Lemma 4.22 and Lemma 4.24. □

5 Numerical Experiments

In this section, we present the results of computational experiments comparing Algorithm 1, Algorithm 2, CRM in the product space with the Pierra's reformulation, denoted as CRM-P, and SeMP applied to the problem of finding a point in the intersection of m ellipsoids, *i.e.*,

$$\text{find } x^* \in \bigcap_{i=1}^{m} \xi_i,$$

where each ellipsoid ξ_i is the set as

$$\xi_i := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}, \text{ for } i = 1, 2, \ldots, m$$

with $f_i : \mathbb{R}^n \to \mathbb{R}$ given by

$$f_i(x) = \langle x, A_i x \rangle + 2 \langle x, b^i \rangle - c_i,$$

where A_i is a symmetric positive definite matrix, b_i is an e-vector, and c_i is a positive scalar for each $i = 1, \ldots, m$.

We first form the ellipsoid ξ_1 by generating a matrix A_1 of in the form of $A_1 = \gamma Id + B_1^\top B_1$ with $B_1 \in \mathbb{R}^{n \times n}$, $\gamma \in \mathbb{R}_{++}$. Matrix B_1 is sparse with sparsity density $p = 2n^{-1}$ and its components are sampled from the standard normal distribution. Vector b^1 is sampled from the uniform distribution in $[0, 1]$, then choose each $c_1 > (b^1)^T A_1 b^1$ which ensures that 0 belongs to ξ_1.

Then the remaining ellipsoids, $\xi_2, \ldots, \xi_m$, are constructed in the following form:

$$\xi_i = \{x \in \mathbb{R}^n \mid \langle x - x_c^i, (A_i^\top A_i)^{-1}(x - x_c^i)\rangle \leq 1\},$$

where A_i is a positive definite matrix, and $x_c^i \in \mathbb{R}^n$ is the center of ξ_i for $i = 2, 3, \ldots, m$. To form ξ_2, first randomly generate x_c^2 of ξ_2 outside ξ_1. Define $d_2 := \lambda(P_{\xi_1}(c_2) - c_2)$ as the norm of the longest principal semi-axis of ξ_2, where $\lambda > 1$ is a constant which can decide the intersection is big or small. For ensuring this, we form a diagonal matrix $\Lambda_2 = \operatorname{diag}(\|d_2\|, u)$ where $u \in \mathbb{R}^{n-1}$ is a vector whose components are positive and have values less than $\|d\|$, and orthogonal matrix Q_2 where the first row or column is $\frac{d}{\|d\|}$. Define $A_2 = Q_2 \Lambda_2 Q_2^\top$, and then the ellipsoid ξ_2 is complete.

Before forming the remaining ellipsoids, we need to find a fixed point that lies in the intersections of the m ellipsoids. Take point $p = x_c^2 + d_2 \in \xi_1 \cap \xi_2$, and we will guarantee $p \in \xi_i$ for each $i = 3, 4, \ldots, m$ in the following steps.

Before constructing the remaining ellipsoids, we need to find a fixed point that will lie in all the ellipsoids. We take $p = x_c^2 + d_2 \in \xi_1 \cap \xi_2$, and we will guarantee that $p \in \xi_i$ for each $i = 3, 4, \ldots, m$.

Choose an arbitrary point $x_c^i \in \mathbb{R}^n$ which doesn't belong to $\cup_{i=1}^{i-1} \xi_i$, define $d_i = \lambda(p - x_c^i)$ as the norm of the longest semi-axis of ξ_i, and generate A_i similarly as was done for ξ_2 for each $i = 3, \ldots, m$. Repeat this process until we get all m ellipsoids.

For the comparison, CRM was applied to the equivalent feasibility problem (1.4), which in formulated in the product space $\mathbb{R}^{nm}$.

Computations were performed using Matlab 2015a on an Intel i7-8099G at 3.10GHz (4 threaded), 16GB running 64-but Red Hat Linux 6.4. The following conditions were used:

(i) Choose a random $x^0 \in [-100, 100]^n$. Initial Algorithm 1, Algorithm 2, SeMP with x^0, and CRM with $(x^0, x^0, \dots, x^0) \in \mathbb{R}^{nm}$;

(ii) An iteration limit of 3000 is enforced.

(iii) After each iteration, calculate the current error by

$$e_k = \sum_{i=1}^{m} \|P_{C_i}(x^k) - x^k\|;$$

(iv) Stopping criterion is given by

$$\|x^{k+1} - x^k\| \leq \epsilon \ or \ e_k \leq \epsilon$$

where $\epsilon = 10^{-6}$;

(v) For each pair (n, m) we repeat the experiment 30 times and then calculate the median (max) of CPU running time, error and iteration number.

Results are tabulated in Tables 2, 3 and 1, and are shown in Figures 1, 2, 3 and 4.

We make some comments on the results.

- The results which are summarized in Figure 1-4 using the performance profile in [41]. Performance profiles allow one to benchmark different methods on a set of problems with respect to a performance measure (in our case, CPU Time). The vertical axis indicates the percentage of problems solved, while the horizontal axis indicates the corresponding factor of the performance index used by the best solver. All pictures clearly show that Algorithm 3 does much better than others no matter which pair (n, m) is used.
- Note that the number of orthogonal projections on for iteration differs among these algorithms. The number is $4m$ for Algorithm 1 $m + 4$ for Algorithm 3, $2mn$ for CRM-P and m for SeMP. Even taking this into account, the performance of Algorithm 3 is superior to others.
- As the dimension of the problem, *i.e.*, n increases, CRM-P performs worst among these 4 methods. Except Algorithm 2, other methods can't solve most of the problems with tolerance $\epsilon = 10^{-6}$.
- Algorithm 1 performs better than CRM-P, and is similar to SeMP. Its CUP time increases as the dimension or the number of the sets become larger obviously, and SeMP has the same phenomenon. However, the CPU time of Algorithm 2 keeps very low even when the dimension of the problems increases.

Table 1: The median (max) CPU time comparison among Algorithm 1, Algorithm 2, CRM-Prod and SeMP.

n	m	CPU Time(s)			
		A1	A2	MAP	CRM-P
10	3	0.59(136)	0.21(1.19)	2.34(163)	0.33 (38.8)
10	10	3.25(485)	0.461(34.2)	2.18(536)	140(259)
20	3	0.35(710)	0.20(1.91)	0.52(707)	0.68(69.4)
20	5	1.82(615)	0.25(1.69)	1.65(467)	13.2(479)
20	10	15.9(3181)	1.98(1162)	269(1650)	128(851)
20	20	2675 (6777)	6.43(253)	961(2384)	896(5746)
50	5	261(4597)	1.98(16.8)	389(2512)	7.06(1335)
50	10	484(5240)	4.32(67.6)	1081(2809)	351(1896)
50	20	589(7740)	10(161)	2470(4888)	2244(7020)

Table 2: The median (max) of error comparison among Algorithm 1, Algorithm 2, CRM-Prod and SeMP.

n	m	Errors			
		A1	A2	SeMP	CRM-P
10	3	4.22e-07(1.95e-06)	5.55e-08(5.49e-07)	2.55e-06(1.83e-05)	3.92e-07(1.64e-06)
10	10	5.56e-05(1-e03)	0(0)	6.94-e05(1.3e-03)	2.34e-06(9.42e-06)
20	5	1.6e-05(1.6e-04)	5.74e-08(8.72e-07)	1.02e-05(.107e-04)	3.32e-07(9.77e-07)
20	5	1.19e-06(5.75e-06)	8.10e-08(8.09e-07)	1.55e-04(1.5e-03)	8.07e-07(1.53e-06)
20	10	3.4e-05 (5.2e-04)	0(0)	6.4e-04(6.5e-03)	2.5e-03(5e-02)
20	20	1.34e-03 (1.53e-02)	0(0)	1.57e-03 (1.38e-02)	2.21e-03(4.24e-02)
50	5	1.21e-05 (1.2e-04)	4.7e-08(9.4e-06)	8.73e-06(4.94e-05)	6.36e-06(1.18e-04)
50	10	50(1000)	0(0)	50(1000)	50(1000)
50	20	5.93e-05(4.24e-04)	4.90e-09(9.24e-07)	3.77e-03(7.46e-e02)	50(1000)

Table 3: The median (max) of iteration comparison among Algorithm 1, Algorithm 2, CRM-P and SeMP.

n	m	Iterations			
		A1	A2	SeMP	CRM-P
10	10	9.5(1000)	1(194)	11.5(1000)	996(1000)
20	3	3(3000)	2(37)	10(3000)	22.5(1780)
20	5	8(3000)	1.5(3)	13(3000)	60.5(3000)
20	10	14.5(3000)	2(380)	1815(3000)	679(3000)
20	20	2416(3000)	2(92)	2292(3000)	2466(3000)
50	5	98.5 (3000)	2(5)	1190(3000)	41(3000)
50	10	240(3000)	3.5(7)	3000(3000)	1026(3000)
50	20	200.5(3000)	2(19)	3000(3000)	2765(3000)

6 Concluding remarks

In this paper we extend recent results on the centralized circumcentered-reflection method (cCRM) in [13] for the Convex Feasibility Problem. cCRM was formulated for two convex sets. In the present work, we broaden the scope of cCRM to multi-set convex feasibility problems. This was done by applying cCRM to pairs of sets chosen at each iteration by means of suitable

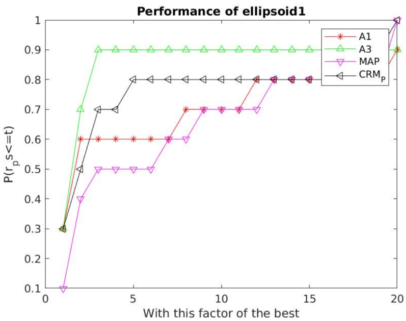

Fig. 1: CPU Time comparison with $n = 10, m = 3$

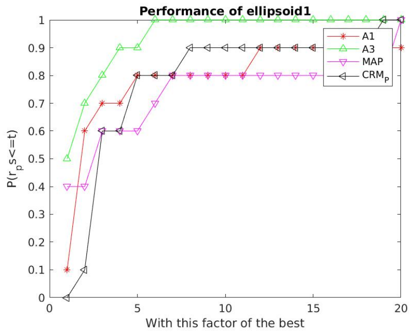

Fig. 2: CPU Time comparison with $n = 20, m = 3$.

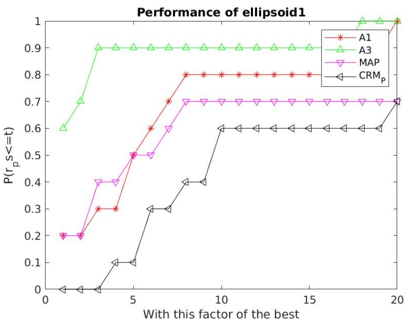

Fig. 3: CPU Time comparison with $n = 20, m = 10$

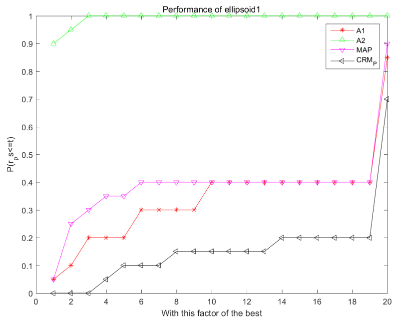

Fig. 4: CPU Time comparison where $n = 20, m = 20$

control sequences. Besides global convergence, we also get linear and superlinear convergence under suitable conditions. Preliminary numerical experiments indicate very good performance of the algorithm with the control sequence given by the most violated constraint in functional value sense.

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