Test Instances for Multiobjective Mixed-Integer Nonlinear Optimization

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Abstract

A suitable set of test instances, also known as benchmark problems, is a key ingredient to systematically evaluate numerical solution algorithms for a given class of optimization problems. While in recent years several solution algorithms for the class of multiobjective mixed-integer nonlinear optimization problems have been proposed, there is a lack of a well-established set of test instances to compare the performance of these algorithms and evaluate their strengths and weaknesses. Hence, in this manuscript we collect and classify test instances that have been presented in the literature so far to obtain a first collection of benchmark problems. In particular, for the classification we give an overview of important properties that potentially influence the performance of solution algorithms for multiobjective mixed-integer optimization problems and investigate these properties for previously presented test instances. Our results form a foundation for the systematic evaluation of solution algorithms as well as the development and classification of corresponding test instances in the future.

Key Words: multiobjective optimization, mixed-integer nonlinear optimization, test instances, nonconvex optimization.

Mathematics subject classifications (MSC 2010): 90C11, 90C26, 90C29, 90C30.

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1 Introduction

One area of research in the field of mathematical optimization is the development of solution algorithms for given classes of optimization problems. Each of these algorithms has its own strengths and weaknesses. While some, for instance, might work especially well for such optimization problems where function evaluations are computationally costly, others might work especially well for large scale optimization problems with a huge number of variables. In order to detect such strengths and weaknesses and to find the best solution algorithm for a specific optimization problem, one typically evaluates their performance on a well-established benchmark set of test instances. For single-objective mixed-integer (nonlinear) optimization, a well-known example for such a set of test instances is the MINLPLib [20] which is used, for instance, in [15] and [18]. Well-known examples for multiobjective optimization problems are the DTLZ [4] and the ZDT [21] test suite, see also [13] for an overview and characterization.

In recent years the first algorithms for multiobjective mixed-integer optimization problems have been presented. These are optimization problems where several conflicting objective functions have to be optimized simultaneously and some of the variables are only allowed to take integer values. In particular, these integrality assumptions immediately make the optimization problem nonconvex and difficult to solve numerically. In the beginning, the focus of research in this area was on multiobjective mixed-integer linear optimization problems, see [12] for a survey of corresponding solution algorithms. Recently also the nonlinear setting has gained more and more attention. This is also the setting in focus of this manuscript. Corresponding solution methods have been presented for instance in [1, 5, 14] for biobjective problems and in [3, 9, 11] for multiobjective mixed-integer convex optimization problems with an arbitrary number of objective functions. Even for the nonconvex setting first algorithms have been proposed in [7, 16]. We remark that open source implementations of some of these algorithms are provided, for instance, in [2] and [8].

Unfortunately, in contrast to single-objective mixed-integer or multiobjective continuous optimization, there currently exists no standard set of test instances for a systematic evaluation and comparison of these algorithms. In fact, most of the papers present their own set of test instances and only some are found in more than one paper. Our manuscript is dedicated to closing this gap. More precisely, in Section 3 we suggest and discuss the properties of test instances that are particularly relevant in the context of multiobjective mixed-integer nonlinear optimization, especially regarding the evaluation and comparison of different solution algorithms. We then collect and categorize 23 convex and nonconvex test instances from the existing literature with regard to these properties in Section 4. As a result, we obtain a benchmark set of well-characterized test instances that can be used for a systematic evaluation of the strengths and weaknesses of solution algorithms for multiobjective mixed-integer nonlinear optimization problems. Moreover, the properties presented in Section 3 also serve as a foundation for the development and classification of new test instances in the future.

2 Notations and Definitions

Throughout this manuscript the inequality $\leq$ between vectors $x, x' \in \mathbb{R}^p$, $p \in \mathbb{N}$ is understood componentwise, i.e., it holds $x \leq x'$ if and only if $x_i \leq x'_i$ is fulfilled for all $i \in [p] := \{1, \dots, p\}$. Based on this we define for $l, u \in \mathbb{R}^p$ with $l \leq u$ by $[l, u] := \{z \in \mathbb{R}^p \mid \ l \leq z \leq u\}$ the p-dimensional box with lower bound l and upper bound u. Finally, for a vector $x \in \mathbb{R}^p$ and for a nonempty set $\Omega \subseteq \mathbb{R}^p$ we denote by $\|x\|_2$ the Euclidean norm of x and by $|\Omega|$ the cardinality of Ω, respectively.

We consider in the following multiobjective optimization problems, i.e., optimization problems given by

$$\begin{aligned} \min_x \ & f(x) \\ \text{s.t.} \ & x \in S \end{aligned} \tag{MOP}$$

with continuous objective functions $f_i \colon \mathbb{R}^\ell \to \mathbb{R}$, $i \in [p]$ where $f = (f_1, \dots, f_p)^\top \colon \mathbb{R}^\ell \to \mathbb{R}^p$, and with a nonempty feasible set $S \subseteq \mathbb{R}^\ell$. Since in general there exists no feasible point $x \in S$ that minimizes all the different objective functions $f_1, \dots, f_p$ at the same time, we use the following optimality concept:

Definition 2.1 *A feasible point $x^\star \in S$ is an* efficient solution of (MOP) *if there exists no $x \in S$ with $f(x) \leq f(x^\star)$ and $f(x) \neq f(x^\star)$. A point $y^\star = f(x^\star)$ with $x^\star \in S$ is a* nondominated point of (MOP) *if $x^\star$ is an efficient solution of* (MOP)*. Moreover, we denote by $\mathcal{E} \subseteq S$ the* efficient set*, i.e., the set of all efficient solutions, and by $\mathcal{N} \subseteq f(S)$ the* nondominated set*, i.e., the set of all nondominated points, of* (MOP)*, respectively.*

Note that by using a set notation one obtains that $x^\star \in S$ is an efficient solution of (MOP) if and only if

$$(\{f(x^\star)\} - \mathbb{R}^p_+) \cap f(S) = \{f(x^\star)\},$$

with $f(S) := \{f(x) \in \mathbb{R}^p \mid \ x \in S\}$ and $\mathbb{R}^p_+ := \{y \in \mathbb{R}^p \mid \ y_i \geq 0 \text{ for all } i \in [p]\}$.

We focus in the following on multiobjective mixed-integer optimization problems, i.e., multiobjective optimization problems of the special type given by

$$\begin{aligned} \min_x \ & f(x) \\ \text{s.t.} \ & x \in S \subseteq \mathbb{R}^n \times X_I \end{aligned} \tag{MOMIP}$$

with continuous objective functions $f_i \colon \mathbb{R}^{n+m} \to \mathbb{R}$ for all $i \in [p]$ where $n, m, p \in \mathbb{N}$, $p \geq 2$, and $f = (f_1, \dots, f_p)^\top \colon \mathbb{R}^{n+m} \to \mathbb{R}^p$. Further, let $X_I := [l_I, u_I] \cap \mathbb{Z}^m$ be a (finite) subset of $\mathbb{Z}^m$ with $l_I, u_I \in \mathbb{Z}^m$, and let $S \subseteq \mathbb{R}^n \times X_I \subseteq \mathbb{R}^n \times \mathbb{Z}^m$ be the nonempty feasible set of (MOMIP).

In order to distinguish between the continuous and the integer variables of $x \in S$ we will write in the following $x = (x_C, x_I)$ with $x_C \in \mathbb{R}^n$ and $x_I \in X_I$. Moreover, we call $x_I \in X_I$ a *feasible integer assignment of* (MOMIP) if there exists $x_C \in \mathbb{R}^n$ such that $x = (x_C, x_I)$ is feasible for (MOMIP). In the same manner, $x_I \in X_I$ is called an *efficient integer assignment of* (MOMIP) if there exists $x_C \in \mathbb{R}^n$ such that $x = (x_C, x_I)$ is an efficient solution of (MOMIP). Based on this, we denote by S_I the *set of all feasible integer assignments* and by $\mathcal{E}_I$ the *set of all efficient integer assignments* of (MOMIP), respectively. Finally, for every feasible integer assignment $\hat{x}_I \in S_I$ we define the corresponding *patch problem* ($\mathrm{P}(\hat{x}_I)$) as

$$\begin{aligned} \min_{x_C} \ & f(x_C, \hat{x}_I) \\ \text{s.t.} \ & (x_C, \hat{x}_I) \in S. \end{aligned} \tag{P($\hat{x}_I$)}$$

3 Useful properties of test instances in multiobjective mixed-integer nonlinear optimization

In the following we suggest some properties which should be taken into account when choosing known or designing new optimization problems to test solution algorithms for multiobjective mixed-integer optimization problems. Thereby we make no claim on completeness in this regard. Depending on the specific techniques that are used by an algorithm to solve (MOMIP), there exist various properties of the considered problem that have an influence on the algorithm's performance. In particular, for different algorithms there will in general exist different properties that influence their performance and not all of these properties will have an effect on all of the algorithms. Nevertheless, there exists at least some properties that influence a huge class of solution approaches and we will present and discuss those properties in the following.

3.1 General properties

First, we consider some of the more general properties of multiobjective mixed-integer optimization problems. More precisely, these properties are not only of relevance for solvers of (MOMIP), but also for solvers of general multiobjective optimization problems (MOP):

(1) number $n \in \mathbb{N}$ of continuous variables

(2) number $m \in \mathbb{N}$ of integer variables

(3) size of the feasible set S

(4) number $p \in \mathbb{N}$ of objective functions

(5) size of the image set (also called attainable set) $f(S)$

(6) size of the nondominated set $\mathcal{N} \subseteq f(S)$ relative to $f(S)$

We remark that the term "size" in properties (3), (5), and (6) typically represents a value associated with the volume of the corresponding sets. In fact, often it refers not to the volume of the set itself, but to the volume of a tight box containing that set. For instance, with regard to decision space based methods such as branch-and-bound algorithms, property (3) is often used to obtain a worst case estimate on the number of branching steps. The same holds basically for property (5) and criterion space based methods, e.g., algorithms that compute an enclosure. Hence, the term "size" is used here to reflect exactly what one would intuitively expect when describing the corresponding sets. However, these two properties are rather hard to determine in practice. For example, the size of the feasible set S is easy to compute if it is only given by some box constraints. But as soon as it becomes more complex, e.g., by adding inequality constraints, it is not clear how exactly one should measure and evaluate the size of S.

This is one of the reasons why properties (1) and (2) are often used to give an impression of the difficulty of an optimization problem (MOP). While the number of variables is not necessarily directly related to the size of the feasible set S, it is at least one of the factors that may contribute to that size. In particular, for test instances that are scalable in the number of variables there often is a close relation between the number of variables and the size of the feasible set. Analogously, property (4) is often used as an estimator for the size of the image set.

Finally, property (6) is usually important for algorithms that rely on image points $y \in f(S)$ to improve a set of upper bounds. The largest improvement of an upper bound set is usually obtained when using some nondominated point $\bar{y} \in \mathcal{N} \subseteq f(S)$ to update the set. If the size of the nondominated set $\mathcal{N}$ is relatively small compared to the size of the image set $f(S)$, then finding such nondominated points (instead of "only" image points) is rather difficult and this can slow down the improvement of upper bound sets significantly.

3.2 Integer properties

In the setting of mixed-integer optimization, there exist some specific properties related to the integer variables that can have an impact on the performance of corresponding solution algorithms:

(7) number $|X_I|$ of possible integer assignments

(8) number $|S_I|$ of feasible integer assignments, in particular relative to $|X_I|$

(9) number $|\mathcal{E}_I|$ of efficient integer assignments, in particular relative to $|S_I|$ and also relative to $|X_I|$

A lot of algorithms for multiobjective mixed-integer optimization problems make use of the finiteness of possible integer assignments. Often, the proof of the finiteness of the algorithms themselves relies on the finiteness of possible integer assignments and some worst case scenario where all these assignments would need to be enumerated. Hence, property (7) is important for various aspects of the algorithmic performance including worst case running time analysis.

As explained for the general properties, a lot of approximation algorithms make use of an upper bound set and need to compute image points in order to improve it. Hence, the number of feasible integer assignments (8) is of great interest for that. If this number is very small (especially in comparison to the number of all possible integer assignments) than it is usually more difficult to find image points. This holds in particular for such algorithms that solve the overall mixed-integer problem (MOMIP) by decomposing it into several purely continuous subproblems that are obtained when fixing the integer variables.

For the same reason property (9) usually has an effect on the performance of these algorithms. Not only that patches ($\mathrm{P}(\hat{x}_I)$) for efficient integer assignments $\hat{x}_I \in \mathcal{E}_I$ lead to the best improvement of upper bound sets, they are often also needed in order to compute valid lower bound sets.

3.3 Algorithm dependent properties

Until now, we considered only such properties of (MOMIP) that affect almost all or at least a huge subclass of solution algorithms for this kind of optimization problems. In the following, we present some properties that are only of interest for very specific types of algorithms:

(10) coupling of decision and criterion space, for example in the sense of Lipschitz continuity of the objective functions

(11) similarity of the patch problems (e.g., if all patch problems lead to the same image set in the criterion space that is "moved around" by the integer variables as in the biobjective example from Figure 1)

(12) distribution/neighborhood of the efficient integer assignments $x_I \in \mathcal{E}_I$ in the decision space

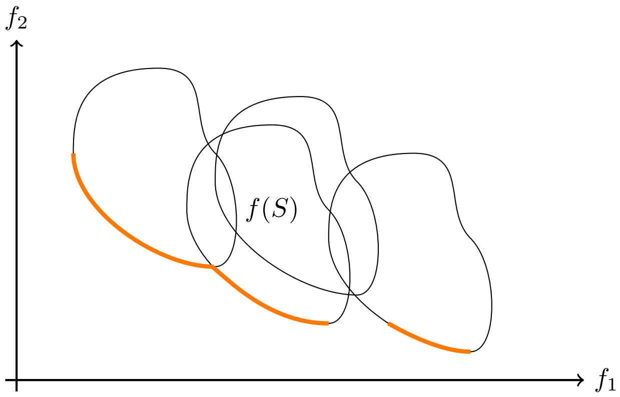

Figure 1: Image set $f(S)$ and nondominated set of a biobjective mixed-integer optimization problem with similar patch problems for each integer assignment $\hat{x}_I \in S_I$

Property (10) is of interest when it comes to the relation of approximations of efficient solutions in the decision space and the approximation of nondominated points in the criterion space. Usually only one of these two is obtained by a certain algorithm. However, if there exists some kind of coupling between decision and criterion space then it is possible to obtain a relation between both kinds of approximations. In particular, algorithms like MOMIX from [3] that work with a termination criterion in the decision space can then also be guaranteed to compute an approximation of the nondominated set (in the criterion space) of certain quality.

For all algorithms that decompose the original problem (MOMIP) into patch problems, property (11) can be particularly relevant. This holds especially when the patches are not treated individually/dynamically but rather a standard procedure is applied to all of them. For example, a patch that consists only of a single isolated point should be treated differently than a patch that contributes to a larger part of the image set $f(S)$. Such a distinction is of course not necessary if the patch problems are all very similar to each other and can hence all be treated in the same way.

Finally, property (12) provides valuable insights for decision space based methods and in particular branch-and-bound approaches. If the efficient integer assignments are all located within close distance to each other, then there is a high chance that this is detected by a branch-and-bound approach and that huge parts of the decision space can be discarded early.

4 Test Instances

In what follows we provide a collection of test instances for multiobjective mixed-integer nonlinear optimization problems used in the literature. In doing so we distinguish between convex problems (see the forthcoming Subsection 4.1) and nonconvex problems (see the forthcoming Subsection 4.2). Recall that (MOMIP) is called a multiobjective mixed-integer convex optimization problem if the relaxed optimization problem, obtained by ignoring the integrality constraints, is a convex optimization problem, i.e., has convex objective functions and a convex feasible set. Otherwise, (MOMIP) is called a multiobjective mixed-integer nonconvex optimization problem. The feasible sets of the test problems are often defined by inequality and equality constraints. Note that in this case the feasible set of the corresponding relaxed optimization problem is convex if all constraint functions $g_j\colon \mathbb{R}^{n+m} \to \mathbb{R}$, $j \in [v]$ which describe inequality constraints $g_j(x) \leq 0$ are convex and if all constraint functions $h_j\colon \mathbb{R}^{n+m} \to \mathbb{R}$, $j \in [w]$ which describe equality constraints $h_j(x) = 0$ are affine linear. Regarding the description and analysis of the presented test instances we restrict ourselves to the general properties (1), (2), and (4), to the integer properties (7), (8), and (9), and, if possible, to the algorithm dependent property (11).

4.1 Test Instances for Multiobjective Mixed-Integer Convex Optimization

In the following we list 14 multiobjective mixed-integer convex optimization problems. We start with 12 quadratic test instances. These are instances where all of the involved objective and constraint functions are quadratic or even affine linear. The first seven quadratic test problems (TI1) – (TI7) contain four biobjective and three triobjective test instances, all with a fixed number of variables. The following two biobjective quadratic test instances (TI8) and (TI9) are scalable only in the number of integer variables. In contrast, the next three biobjective quadratic test instances (TI10) – (TI12) are scalable both in the number of continuous variables as well as in the number of integer variables. Finally, we present with (TI13) and (TI14) two nonquadratic test instances, i.e., multiobjective mixed-integer optimization problems with one nonquadratic but still convex objective or constraint function.

Test instance 1 [3, (T1)]:

$$\begin{aligned} \min_x \ & \begin{pmatrix} x_1 + x_2 \\ x_1^2 + x_2^2 \end{pmatrix} \\ \text{s.t.} \ & (x_1 - 2)^2 + (x_2 - 2)^2 \leq 36, \\ & x \in X = [-2, 2] \times ([-4, 4] \cap \mathbb{Z}) \end{aligned} \tag{TI1}$$

The test instance (TI1) is a biobjective mixed-integer convex optimization problem with a single continuous and a single integer variable. It contains a linear and a quadratic objective function and also a quadratic constraint function. All possible integer assignments are feasible integer assignments, i.e., it holds $X_I = S_I = [-4, 4] \cap \mathbb{Z}$ and $|X_I| = |S_I| = 9$. Moreover, it holds $\mathcal{E}_I = \{-3, -2, -1, 0\}$ and thus $|\mathcal{E}_I| = 4$, i.e., only four of the nine feasible integer assignments are also efficient integer assignments. For an illustration of the image set and the nonconnected nondominated set of (TI1) see Figure 2. Note that there exists an isolated image point $(-2, 20)^\top \in f(S)$ for the feasible integer assignment $x_2 = -4$. All other feasible integer assignments lead to patch problems with a similar image set (quadratic parabolas) in the criterion space (cf. property (11)).

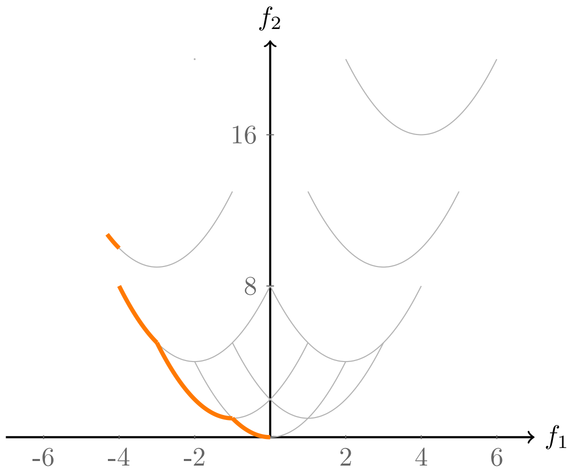

Figure 2: Image set and nondominated set of (TI1)

Test instance 2 [11, (Ex2)]:

$$\begin{array}{rl} \min\limits_{x} & \begin{pmatrix} x_1 + x_3 \\ x_2 + x_4 \end{pmatrix} \\ \text{s.t.} & x_1^2 + x_2^2 \leq 0.25, \\ & x_3^2 + x_4^2 \leq 1, \\ & x \in X = \mathbb{R}^2 \times \left([-1,1]^2 \cap \mathbb{Z}^2\right) \end{array} \tag{TI2}$$

This is an illustrative biobjective example from [11] with two continuous and two integer variables. Originally, it was formulated with $x_3, x_4 \in \mathbb{Z}$, i.e., without a box constraint regarding the integer variables. Here, according to our formulation of (MOMIP), we choose $x_I = (x_3, x_4) \in X_I := [-1, 1]^2 \cap \mathbb{Z}^2$ which allows to calculate the value $|X_I| = 3^2 = 9$ and thus also the ratios between $|X_I|$, $|S_I|$, and $|\mathcal{E}_I|$. Note that the inequality constraint $x_3^2 + x_4^2 \leq 1$ and $x_3, x_4 \in \mathbb{Z}$ imply

$$S_I = \{(0,0), (0,-1), (0,1), (-1,0), (1,0)\} \subseteq X_I,$$

and thus $|S_I| = 5$. Moreover, (TI2) can be decomposed into the purely continuous subproblem

$$\begin{aligned} \min_{(x_1,x_2)} & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 0.25, \\ & (x_1, x_2) \in \mathbb{R}^2 \end{aligned}$$

with efficient and nondominated set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 0.25,\ x_1 \leq 0,\ x_2 \leq 0\}$, and into the purely integer subproblem

$$\begin{aligned} \min_{(x_3,x_4)} & \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \\ \text{s.t.} \quad & x_3^2 + x_4^2 \leq 1, \\ & (x_3, x_4) \in [-1, 1]^2 \cap \mathbb{Z}^2 \end{aligned}$$

with efficient and nondominated set $\{(0, -1), (-1, 0)\}$. According to [6] we refer to problems with such a decomposable structure as *separable multiobjective mixed-integer optimization problems*. For such a separable optimization problem every feasible integer assignment $\hat{x}_I \in S_I$ leads to the same corresponding patch problem (P($\hat{x}_I$)). Hence, every separable (MOMIP), i.e., (TI2) and all following separable test instances, fulfills property (11). Note that for (TI2) the image set of the patch problem (P($\hat{x}_I$)) is a ball with center $\hat{x}_I$ and radius 0.5 for all $\hat{x}_I \in S_I$. Using the separable structure it follows by [6, Theorem 3.4] that for (TI2) the efficient set is given by

$$\mathcal{E} = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 0.25,\ x_1 \leq 0,\ x_2 \leq 0\Big\} \times \Big\{(0, -1), (-1, 0)\Big\}$$

and the nondominated set by

$$\mathcal{N} = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 0.25,\ x_1 \leq 0,\ x_2 \leq 0\Big\} + \Big\{(0, -1), (-1, 0)\Big\}.$$

Thus it holds $\mathcal{E}_I = \{(0, -1), (-1, 0)\}$ and $|\mathcal{E}_I| = 2$. For an illustration of the image set and the nonconnected nondominated set of (TI2) see Figure 3.

Recently, a test instance generator for multiobjective mixed-integer linear and nonlinear optimization problems was presented in [6]. Similarly to (TI2), it generates separable test instances based on well-known subproblems from multiobjective continuous and multiobjective integer optimization. Moreover, the special structure allows to construct instances scalable in the number of variables and objective functions. If the efficient and nondominated sets of the purely continuous and of the purely integer input problems are known, then the generator allows to control the resulting efficient and nondominated sets as well as the number of efficient integer assignments. The following test instances (TI12), (TI19), and (TI21) have been created using this generator.

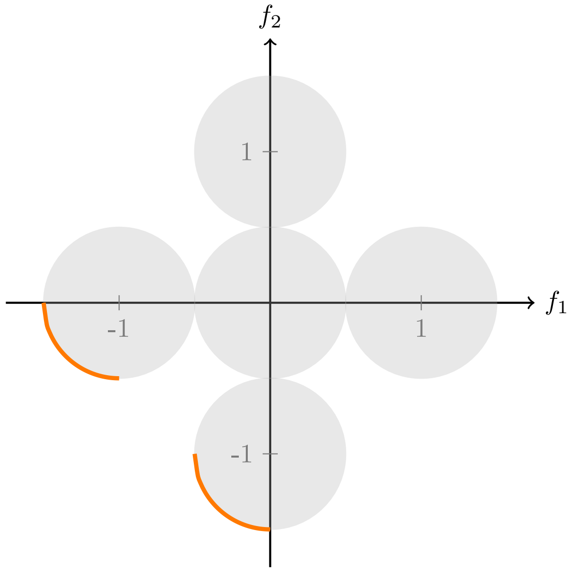

Figure 3: Image set and nondominated set of (TI2)

Test instance 3 [19, Example 1]:

$$\begin{aligned}
\min_x \quad & \begin{pmatrix} x_1^2 - x_2 + x_3 + 3x_4 + 2x_5 + x_6 \\ 2x_1^2 + x_3^2 - 3x_1 + x_2 - 2x_4 + x_5 - 2x_6 \end{pmatrix} \\
\text{s.t.} \quad & 3x_1 - x_2 + x_3 + 2x_4 \leq 0, \\
& 4x_1^2 + 2x_1 + x_2 + x_3 + x_4 + 7x_5 \leq 40, \\
& -x_1 - 2x_2 + 3x_3 + 7x_6 \leq 0, \\
& -x_1 + 12x_4 \leq 10, \\
& x_1 - 2x_4 \leq 5, \\
& -x_2 + x_5 \leq 20, \\
& x_2 - x_5 \leq 40, \\
& -x_3 + x_6 \leq 17, \\
& x_3 - x_6 \leq 25, \\
& x \in X = \mathbb{R}^3 \times \{0,1\}^3
\end{aligned} \tag{TI3}$$

(TI3) is a biobjective mixed-integer convex optimization problem with three continuous and three binary variables. All possible integer assignments are feasible integer assignments, i.e., it holds $X_I = S_I = \{0,1\}^3$ and thus $|X_I| = |S_I| = 2^3 = 8$.

Test instance 4 [10, (T9)]:

$$\begin{aligned}
\min_x \quad & \begin{pmatrix} x_1 + x_3 + x_5 + x_7 \\ x_2 + x_4 + x_6 + x_8 \end{pmatrix} \\
\text{s.t.} \quad & x_1^2 + x_2^2 \leq 1, \\
& x_3^2 + x_4^2 \leq 1, \\
& (x_5 - 2)^2 + (x_6 - 5)^2 \leq 10, \\
& (x_7 - 3)^2 + (x_8 - 8)^2 \leq 10, \\
& x \in X = [-20,20]^4 \times \left([-20,20]^4 \cap \mathbb{Z}^4\right)
\end{aligned} \tag{TI4}$$

(TI4) is a separable biobjective mixed-integer convex optimization problem with four continuous and four integer variables. Regarding property (11) the image set of all patch problems is the Minkowski sum of two (two-dimensional) unit balls, and thus a ball with radius 2. Moreover, it holds $|X_I| = 41^4$, $|S_I| = 37^2$, $|\mathcal{E}_I| = 9$, and

$$\begin{aligned}\mathcal{E}_I &= \{(x_5, x_6, x_7, x_8) \in S_I \mid x_5 + x_6 = 3, x_7 + x_8 = 7\} \\ &= \{(-1, 4), (0, 3), (1, 2)\} \times \{(0, 7), (1, 6), (2, 5)\}.\end{aligned}$$

Thus only 0.048% of the possible integer assignments are feasible, 0.00032% of the possible integer assignments are efficient, and 0.66% of the feasible integer assignments are efficient. For an illustration of the image set and the connected nondominated set of (TI4) we refer to Figure 4. Note that S_I remains the same if $[-20, 20]^4 \cap \mathbb{Z}^4$ is replaced by $([-1, 5] \times [2, 8] \times [0, 6] \times [5, 11]) \cap \mathbb{Z}^4$. In this case 57% of the possible integer assignments are feasible and 0.37% of the possible integer assignments are efficient, respectively. Such a modification can have a significant impact on the performance of an algorithm.

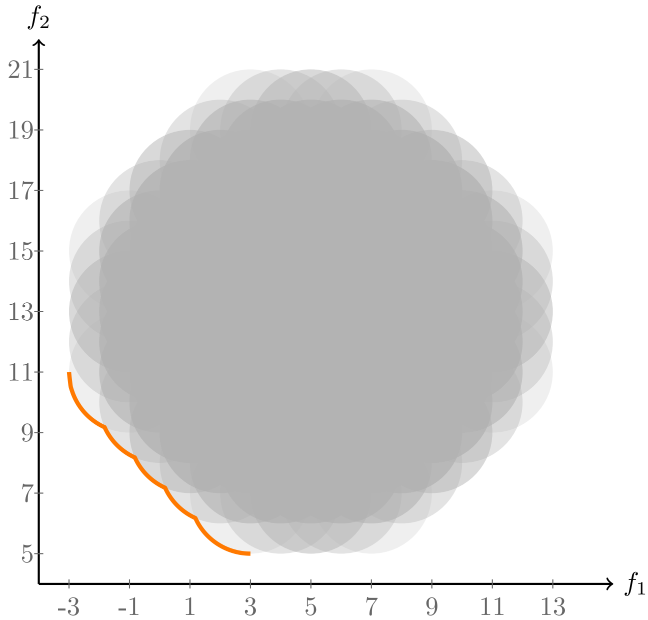

Figure 4: Image set and nondominated set of (TI4)

Test instance 5 [3, (T5)]:

$$\begin{aligned}\min_x \; & \begin{pmatrix} x_1 + x_4 \\ x_2 - x_4 \\ x_3 + x_4^2 \end{pmatrix} \\ \text{s.t.} \; & x_1^2 + x_2^2 + x_3^2 \leq 1, \\ & x \in X = [-2, 2]^3 \times ([-2, 2] \cap \mathbb{Z})\end{aligned} \tag{TI5}$$

Test instance (TI5) is a separable biobjective mixed-integer convex optimization problem with three continuous variables and a single integer variable. Since the integer variable is only box-constrained, all possible integer assignments are feasible. Moreover, all possible/feasible integer assignments are also efficient integer assignments, i.e., it holds $X_I = S_I = \mathcal{E}_I = \{-2, -1, 0, 1, 2\}$ and thus $|X_I| = |S_I| = |\mathcal{E}_I| = 5$. Note that due to the separable structure of (TI5), the image set of all patch problems is basically the same, see also property (11). More precisely, for each integer assignment $\hat{x}_I = x_4 \in \{-2, -1, 0, 1, 2\}$ the image set of the corresponding patch problem ($P(\hat{x}_I)$) is the (three-dimensional) unit ball with center $(x_4, -x_4, x_4^2)$. Hence, the nondominated set is given by

$$\begin{aligned}\mathcal{N} \;=\; & \left\{x \in \mathbb{R}^3 \mid \; x_1^2 + x_2^2 + x_3^2 = 1,\; x_1 \leq 0,\; x_2 \leq 0,\; x_3 \leq 0\right\} \\ & + \left\{(-2, 2, 4), (-1, 1, 1), (0, 0, 0), (1, -1, 1), (2, -2, 4)\right\}.\end{aligned}$$

For an illustration of an approximation of the nondominated set of (TI5) we refer to [3, Figure 13].

Test instance 6 [19, Example 5]:

$$\begin{aligned}\min_x \; & \begin{pmatrix} x_1^2 + x_2^2 - 10x_1 - x_2 - x_3 - 2x_4 \\ \frac{1}{3}(4x_1^2 + 3x_2^2 - x_1 - 5x_2 - x_3 + 10x_4 - 10) \\ \frac{1}{2}(2x_1^2 + 7x_1 - 14x_2 + 2x_3 + 2x_4 - 6) \end{pmatrix} \\ \text{s.t.} \; & -x_1 + 3x_2 - x_3 \;\leq\; -\tfrac{1}{2}, \\ & x \in X = \mathbb{R}^2 \times \{0, 1\}^2 \end{aligned} \tag{TI6}$$

(TI6) is a triobjective mixed-integer convex optimization problem with two continuous and two binary variables. It holds $X_I = S_I = \{0, 1\}^2$ and thus $|X_I| = |S_I| = 2^2 = 4$, i.e., all possible integer assignments are again feasible integer assignments.

Test instance 7 [11, (Ex1)]:

$$\begin{aligned}\min_x \; & \begin{pmatrix} x_1 + x_4 \\ x_2 + x_5 \\ x_3 + x_6 \end{pmatrix} \\ \text{s.t.} \; & x_1^2 + x_2^2 + x_3^2 \;\leq\; 1, \\ & x_4^2 + x_5^2 + x_6^2 \;\leq\; 1, \\ & x \in X = \mathbb{R}^3 \times ([-1, 1]^3 \cap \mathbb{Z}^3) \end{aligned} \tag{TI7}$$

This separable triobjective test instance (TI7) is formulated in [11] with $x_4, x_5, x_6 \in \mathbb{Z}$. Again, in order to calculate the value $|X_I|$ we choose here $[-1, 1]^3 \cap \mathbb{Z}^3$ and obtain $X_I = \{-1, 0, 1\}^3$ and $|X_I| = 3^3 = 27$. Note that the inequality constraint $x_4^2 + x_5^2 + x_6^2 \leq 1$ and $x_4, x_5, x_6 \in \mathbb{Z}$ imply

$$S_I = \{(0, 0, 0), (0, 0, -1), (0, 0, 1), (0, -1, 0), (0, 1, 0), (-1, 0, 0), (1, 0, 0)\} \subseteq X_I$$

and thus $|S_I| = 7$. As for (TI5), each image set of a patch problem is a (three-dimensional) unit ball shifted by one of the seven points $x_I \in S_I$. Finally, it holds $\mathcal{E}_I = \{(0, 0, -1), (0, -1, 0), (-1, 0, 0)\}$, $|\mathcal{E}_I| = 3$, and

$$\mathcal{N} \;=\; \left\{x \in \mathbb{R}^3 \mid \; x_1^2 + x_2^2 + x_3^2 = 1,\; x_1 \leq 0,\; x_2 \leq 0,\; x_3 \leq 0\right\} + \mathcal{E}_I.$$

Test instance 8 [3, (T2)]:

$$\begin{aligned} \min_x & \begin{pmatrix} x^\top Q_1^\top Q_1 x + (1,2,\dots,2,1)x \\ x^\top Q_2^\top Q_2 x + (-1,-2,\dots,-2,5)x \end{pmatrix} \\ \text{s.t. } & x \in X = [-5,5]^2 \times ([-5,5]^m \cap \mathbb{Z}^m) \end{aligned} \tag{TI8}$$

with

$$(Q_1)_{i,j} := \begin{cases} 3 & \text{, if } i = j = 1 \\ 4 & \text{, if } i = j = 2+m \\ 1 & \text{, else} \end{cases}$$

and

$$(Q_2)_{i,j} := \begin{cases} 2 & \text{, if } i = j = 1 \text{ or } i = j = 2+m \\ 4 & \text{, if } i = j \text{ and } i \notin \{1, 2+m\} \\ 1 & \text{, else} \end{cases}$$

This box-constrained test instance (TI8) is scalable in the number $m \in \mathbb{N}$ of integer variables and it holds $|X_I| = |S_I| = 11^m$. Note that $Q_1^\top Q_1$ and $Q_2^\top Q_2$ are positive semidefinite, and hence both objective functions are convex.

Test instance 9 [3, (T3)]:

$$\begin{aligned} \min_x & \begin{pmatrix} x_1 \\ x_2 + \sum_{i=3}^{m+2} 10(x_i - 0.4)^2 \end{pmatrix} \\ \text{s.t. } & \sum_{i=1}^{m+2} x_i^2 \leq 4, \\ & x \in X = [-2,2]^2 \times ([-2,2]^m \cap \mathbb{Z}^m) \end{aligned} \tag{TI9}$$

(TI9) is again scalable in the number $m \in \mathbb{N}$ of integer variables and it holds $|X_I| = 5^m$. The following proposition provides a result regarding the feasible integer assignments.

Proposition 4.1 *For the number $|S_I|$ of feasible integer assignments of the biobjective optimization problem (TI9) with $m \in \mathbb{N}$ it holds*

$$|S_I| = 1 + \frac{2}{3}m + \frac{16}{3}m^2 - \frac{8}{3}m^3 + \frac{2}{3}m^4. \tag{4.1}$$

Proof. For any feasible integer assignment $x_I \in S_I$ of (TI9) we define

$$\begin{aligned} \mathcal{I}_0(x_I) &:= \{i \in [m] \mid x_{I,i} = 0\}, \\ \mathcal{I}_{\pm 1}(x_I) &:= \{i \in [m] \mid x_{I,i} = -1 \text{ or } x_{I,i} = 1\}, \text{ and} \\ \mathcal{I}_{\pm 2}(x_I) &:= \{i \in [m] \mid x_{I,i} = -2 \text{ or } x_{I,i} = 2\}. \end{aligned}$$

Hence, it holds $0 \leq |\mathcal{I}_{\pm 2}(x_I)| \leq 1$ for every $x_I \in S_I$ by the inequality constraint.

We first prove (4.1) for $m \geq 4$. If $|\mathcal{I}_{\pm 2}(x_I)| = 1$, then it follows by the inequality constraint that $|\mathcal{I}_{\pm 1}(x_I)| = 0$ and $|\mathcal{I}_0(x_I)| = m - 1$, and thus there exist

$$|\{-2, 2\}| \cdot \binom{m}{1} = 2m \tag{4.2}$$

different feasible integer assignments in this case.

Hence, we may assume in the following $|\mathcal{I}_{\pm 2}(x_I)| = 0$ and we obtain $0 \le |\mathcal{I}_{\pm 1}(x_I)| \le 4$ by the inequality constraint for every corresponding feasible integer assignment x_I. If now $|\mathcal{I}_{\pm 2}(x_I)| = 0$ and $|\mathcal{I}_{\pm 1}(x_I)| = \ell \in \{0\} \cup [4]$, then it follows $|\mathcal{I}_0(x_I)| = m - \ell$ and

$$|\{-1,1\}|^{\ell} \cdot \binom{m}{\ell}, \; \ell \in \{0\} \cup [4] \tag{4.3}$$

for the number of the corresponding different feasible integer assignments.

By adding up (4.2) and (4.3) for all $\ell \in \{0\} \cup [4]$ it holds

$$|S_I| = 1 + 4m + 2m(m-1) + \frac{4}{3}m(m-1)(m-2) + \frac{2}{3}m(m-1)(m-2)(m-3), \tag{4.4}$$

and (4.1) follows after a short calculation. Finally, it is easy to see that (4.4) holds also in the case $m \in [3]$ and we are done. □

Moreover, it holds $\mathcal{E} = \{x \in [-2,0]^2 \times \{0\}^m \mid \; x_1^2 + x_2^2 = 4\}$, and thus there exists a uniquely determined efficient integer assignment $\mathcal{E}_I = \{0\} \subseteq \mathbb{Z}^m$.

Test instance 10 [3, (T4)]:

$$\begin{aligned} \min_x \; & \begin{pmatrix} \sum_{i=1}^{\frac{n}{2}} x_i + \sum_{i=n+1}^{n+m} x_i \\ \sum_{i=\frac{n}{2}+1}^{n} x_i - \sum_{i=n+1}^{n+m} x_i \end{pmatrix} \\ \text{s.t.} \; & \sum_{i=1}^{n} x_i^2 \le 1, \\ & x \in X = [-2,2]^n \times ([-2,2]^m \cap \mathbb{Z}^m) \end{aligned} \tag{TI10}$$

(TI10) is scalable both in the number $n \in \mathbb{N}$ of continuous variables, which is assumed to be even, as well as in the number $m \in \mathbb{N}$ of integer variables. Since the integer variables are only box-constrained, all possible integer assignments are feasible integer assignments. Moreover, it holds $X_I = S_I = \mathcal{E}_I = \{-2,-1,0,1,2\}^m$ and thus $|X_I| = |S_I| = |\mathcal{E}_I| = 5^m$. Note that here the image set of every patch problems is a shifted (two-dimensional) ball with radius $\sqrt{n/2}$. For an illustration of the image set and the connected nondominated set of the separable test instance (TI10) in the case $n = 4$ and $m = 2$ see Figure 5.

Test instance 11 [9, (H1)]:

$$\begin{aligned} \min_x \; & \begin{pmatrix} \sum_{i=1}^{\frac{n}{2}} x_i + \sum_{i=n+1}^{n+\frac{m}{2}} x_i^2 - \sum_{i=n+\frac{m}{2}+1}^{n+m} x_i \\ \sum_{i=\frac{n}{2}+1}^{n} x_i - \sum_{i=n+1}^{n+\frac{m}{2}} x_i + \sum_{i=n+\frac{m}{2}+1}^{n+m} x_i^2 \end{pmatrix} \\ \text{s.t.} \; & \sum_{i=1}^{n} x_i^2 \le 1, \\ & x \in X = [-2,2]^n \times ([-2,2]^m \cap \mathbb{Z}^m) \end{aligned} \tag{TI11}$$

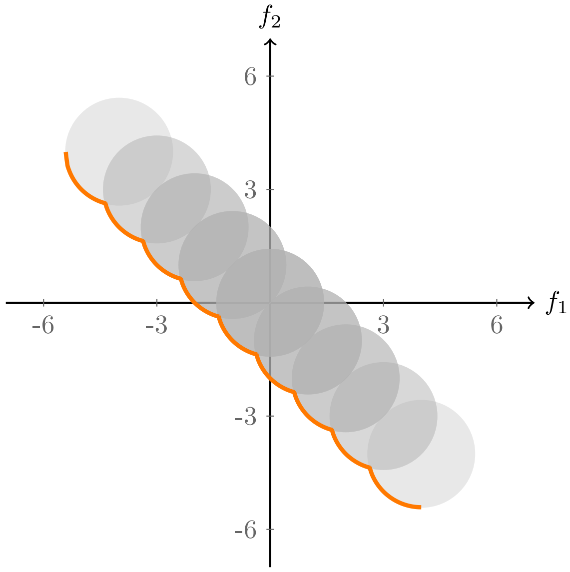

Figure 5: Image set and nondominated set of (TI10) with $n = 4$ and $m = 2$

This scalable and separable problem (TI11), with $n, m \in \mathbb{N}$ and both even, is a slight modification of (TI10) with quadratic objective functions and the same quadratic constraint function. Hence, it holds again $X_I = S_I = \{-2, -1, 0, 1, 2\}^m$ and $|X_I| = |S_I| = 5^m$. Note that, as for (TI10), the image set of every patch problems is a shifted (two-dimensional) ball with radius $\sqrt{n/2}$. For an illustration of the image set and the nonconnected nondominated set of (TI11) in the case $n = 4$ and $m = 2$ see Figure 6. Here we obtain $\mathcal{E}_I = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$ and thus $|\mathcal{E}_I| = 6$.

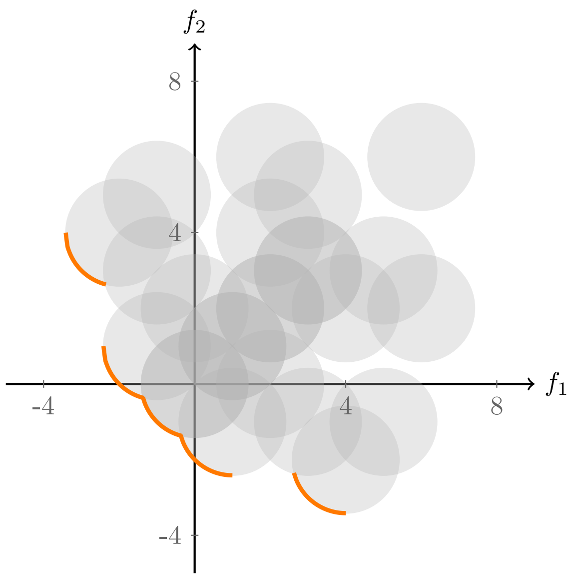

Figure 6: Image set and nondominated set of (TI11) with $n = 4$ and $m = 2$

Test instance 12 [6, Example 4.8 (*ii*)]:

$$\begin{aligned} \min_x & \begin{pmatrix} \frac{\alpha_1}{n}\sum_{i=1}^n x_i^2 + \sum_{i\in J} x_i + \sum_{i\in \bar{J}} x_i \\ \frac{\alpha_2}{n}\sum_{i=1}^n (x_i-2)^2 + \sum_{i\in J} x_i - \sum_{i\in \bar{J}} x_i \end{pmatrix} \\ \text{s.t. } & x \in X = [0,2]^n \times ([-1,1]^m \cap \mathbb{Z}^m) \end{aligned} \tag{TI12}$$

For the box-constrained, scalable, and separable biobjective test instance (TI12) with $n, m \in \mathbb{N}$, $0 < \alpha_i < 0.25$ for all $i \in [2]$, $J \subsetneq \{n+1, \ldots, n+m\}$, and $\bar{J} := \{n+1, \ldots, n+m\} \setminus J$ it is $X_I = S_I = \{-1, 0, 1\}^m$,

$$\begin{aligned} \mathcal{E} &= \{x \in [0,2]^n \mid x_1 = x_2 = \ldots = x_n\} \\ &\quad \times \{x \in [-1,1]^m \cap \mathbb{Z}^m \mid x_i = -1 \text{ for all } i+n \in J\}, \\ \mathcal{N} &= \{(\alpha_1 t^2, \alpha_2(t-2)^2) \mid t \in [0,2]\} \\ &\quad + \{(-m+\delta, m-2|J|-\delta) \in \mathbb{Z}^2 \mid \ \delta \in \{0\} \cup [2(m-|J|)]\}, \text{ and} \\ \mathcal{E}_I &= \{x \in [-1,1]^m \cap \mathbb{Z}^m \mid x_i = -1 \text{ for all } i+n \in J\}. \end{aligned}$$

Consequently, we obtain $|X_I| = |S_I| = 3^m$ and $|\mathcal{E}_I| = 3^{m-|J|}$. Hence, the number of efficient integer assignments is controllable by the choice of m and J, respectively. Test instance (TI12) is obtained by the test instance generator formulated in [6]. For more details see [6, Example 4.8 (*ii*)] and the explanations made there. For an illustration of the nonconnected nondominated set of (TI12) in the case $m = 3$, $J = \{n+1\}$, and $\alpha_1 = \alpha_2 = 0.2$ we refer to [6, Figure 5].

Test instance 13 [3, (T6)]:

$$\begin{aligned} \min_x & \begin{pmatrix} x_1 + x_3 \\ x_2 + \exp(-x_3) \end{pmatrix} \\ \text{s.t. } & x_1^2 + x_2^2 \leq 1, \\ & x \in X = [-2,2]^2 \times ([-2,2] \cap \mathbb{Z}) \end{aligned} \tag{TI13}$$

For the separable test problem (TI13) with a single box-constrained integer variable it holds $X_I = S_I = \mathcal{E}_I = \{-2, -1, 0, 1, 2\}$ and thus $|X_I| = |S_I| = |\mathcal{E}_I| = 5$. For an illustration of the image set and the nonconnected nondominated set of this test instance we refer to Figure 7. The image set of every patch problem is a shifted unit ball. The second objective function is nonquadratic but still convex.

Test instance 14 [10, (T10)]:

$$\begin{aligned} \min_x & \begin{pmatrix} x_1 + x_3 + x_5 + \exp(x_7) - 1 \\ x_2 + x_4 + x_6 + x_8 \end{pmatrix} \\ \text{s.t. } & x_1^2 + x_2^2 \leq 1, \\ & x_3^2 + x_4^2 \leq 1, \\ & (x_5-2)^2 + (x_6-5)^2 \leq 10, \\ & (x_7-3)^2 + (x_8-8)^2 \leq 10, \\ & x \in X = [-20,20]^4 \times \left([-20,20]^4 \cap \mathbb{Z}^4\right) \end{aligned} \tag{TI14}$$

In the last listed convex test problem (TI14) the first objective function is nonquadratic but still convex. It is separable and a slight modification of (TI4) with $|X_I| = 41^4$, $|S_I| = 37^2$, and

$$\mathcal{E}_I = \{(-1,4,0,7), (0,3,0,7), (1,2,0,7), (1,2,1,6), (1,2,2,5)\}.$$

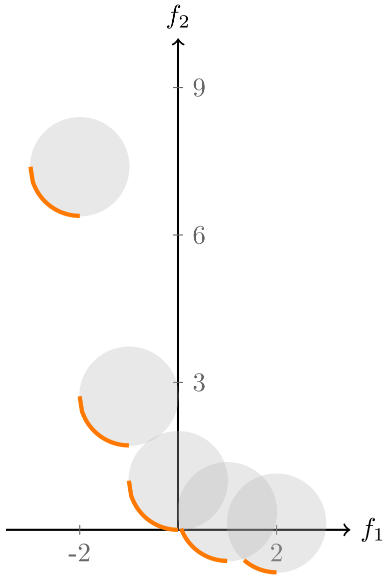

Figure 7: Image set and nondominated set of (TI13)

Thus, as for (TI4), only 0.048% of the possible integer assignments are feasible. Moreover, 0.00018% of the possible integer assignments are efficient and 0.37% of the feasible integer assignments are efficient. For an illustration of the image set and the nonconnected nondominated set of (TI14) see Figure 8. The image sets of the patch problems remain the same as for (TI4). Moreover, the set S_I remains the same if $[-20, 20]^4 \cap \mathbb{Z}^4$ is replaced by $([-1, 5] \times [2, 8] \times [0, 6] \times [5, 11]) \cap \mathbb{Z}^4$.

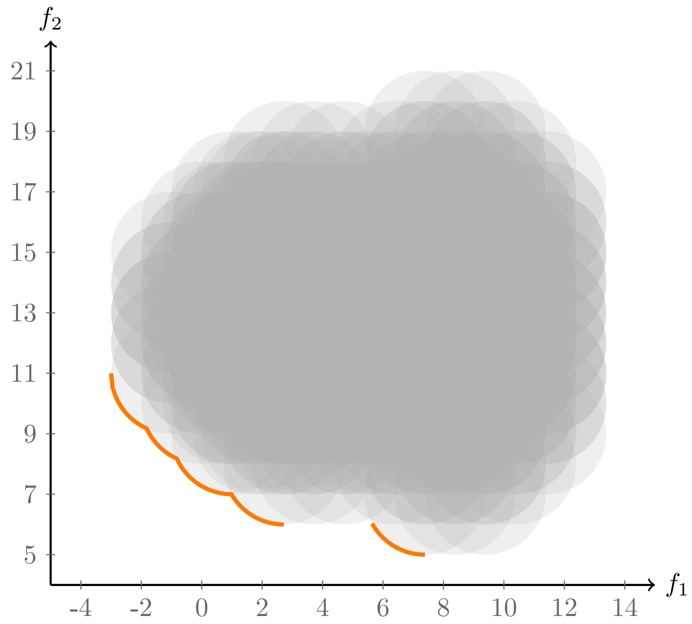

Figure 8: Image set and nondominated set of (TI14) for $x_7 \leq 2$

4.2 Test Instances for Multiobjective Mixed-Integer Nonconvex Optimization

We continue with nine multiobjective mixed-integer nonconvex test instances. The first four test instances (TI15) – (TI18) are problems with a fixed number of variables. The following test problem (TI19) is scalable in the number of continuous variables, and the test problems (TI20) and (TI21) are scalable both in the number of continuous variables as well as in the number of integer variables. Note that (TI15) – (TI21) are all biobjective. In contrast, test instance (TI22) is an example of a triobjective mixed-integer nonconvex test instance with a fixed number of variables. The last test problem (TI23) is scalable in the number of continuous variables, in the number of integer variables, and also in the number of objective functions by using the identity for the continuous variables.

Test instance 15 [1, Example 1]:

$$\begin{aligned} \min_x \; & \begin{pmatrix} x_1 \\ x_2 \cdot \frac{1}{x_1} + x_3 \cdot \left(0.2 + \exp(\frac{1}{x_1})\right) \end{pmatrix} \\ \text{s.t.} \; & x_2 + x_3 = 1, \\ & x \in X = [0.4, 2.5] \times \{0,1\}^2 \end{aligned} \tag{TI15}$$

The first nonconvex test instance (TI15) has a very special structure. It contains a single continuous variable and two binary variables. Hence, it holds $X_I = \{0,1\}^2$, $|X_I| = 2^2 = 4$, and it follows by the equality constraint that exactly one of the binary variables takes the value 1 while the other binary variable takes the value 0, i.e., it holds $S_I = \{(0,1),(1,0)\}$ and $|S_I| = 2$. Based on this we obtain for $x_2 = 0$ and $x_3 = 1$ the corresponding patch problem

$$\begin{aligned} \min_x \; & \begin{pmatrix} x \\ 0.2 + \exp(\frac{1}{x}) \end{pmatrix} \\ \text{s.t.} \; & x \in [0.4, 2.5] \end{aligned} \tag{4.5}$$

and for $x_2 = 1$ and $x_3 = 0$

$$\begin{aligned} \min_x \; & \begin{pmatrix} x \\ \frac{1}{x} \end{pmatrix} \\ \text{s.t.} \; & x \in [0.4, 2.5], \end{aligned} \tag{4.6}$$

respectively. Hence, the binary variables act as a switch for the second objective function. Thus, the image set for this test instance is the union of the two curves given by the image sets of (4.5) and (4.6) (cf. property (11)), and we obtain $\mathcal{E}_I = \{(1,0)\}$, $\mathcal{E} = [0.4, 2.5] \times \{(1,0)\}$, and $\mathcal{N} = \left\{(x, \frac{1}{x}) \mid x \in [0.4, 2.5]\right\}$. For an illustration of the image set and the connected nondominated set of (TI15) see Figure 9.

Obviously, the special approach described here can be used to construct other test instances (see for instance the forthcoming problem (TI23)).

Test instance 16 [11, (P2)]:

$$\begin{aligned} \min_x \; & \begin{pmatrix} x_1 + x_3 \\ x_2 + x_4 \end{pmatrix} \\ \text{s.t.} \; & -x_1^2 - x_2^2 \leq -1, \\ & x_3^2 + x_4^2 \leq 9, \\ & x \in X = [0,1]^2 \times \left([-3,3]^2 \cap \mathbb{Z}^2\right) \end{aligned} \tag{TI16}$$

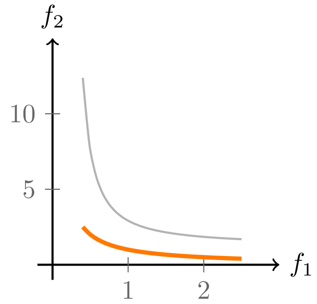

Figure 9: Image set and nondominated set of (TI15)

(TI16) is a separable biobjective mixed-integer nonconvex optimization problem with two continuous variables, two integer variables, and a quadratic but nonconvex inequality constraint for the continuous variables. Here it holds $|X_I| = 7^2 = 49$, $|S_I| = 29$, $\mathcal{E}_I = \{(-3,0),(-2,-2),(0,-3)\}$, and thus $|\mathcal{E}_I| = 3$. For each integer assignment $\hat{x}_I \in S_I$ the image set of (P($\hat{x}_I$)) is given as $\{\hat{x}_I\} + ([0,1]^2 \setminus \operatorname{int}(B(0,1)))$ where $B(0,1) := \{y \in \mathbb{R}^2 \mid y_1^1 + y_2^2 \leq 1\}$. Hence, all these image sets correspond to the set difference between the unit square and the unit ball. The nondominated set is given by

$$\mathcal{N} = \Big(\{x \in [0,1]^2 \mid x_1^2 + x_2^2 = 1\} + \{(-3,0),(-2,-2),(0,-3)\}\Big) \setminus \{(-2,0),(0,-2)\}.$$

An illustration of the image set and the nonconnected nondominated set of (TI16) is presented in Figure 10.

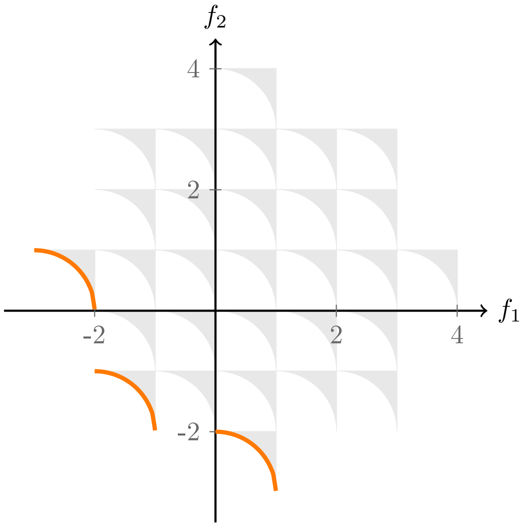

Figure 10: Image set and nondominated set of (TI16)

Test instance 17 [7, (P1)]:

$$\begin{aligned} \min_x \; & \begin{pmatrix} x_1 + x_2 + x_5 \\ x_3 + x_4 - \exp(x_5) \end{pmatrix} \\ \text{s.t.} \; & -\sum_{i=1}^{4} x_i^2 \leq -1, \\ & x \in X = [0,1]^4 \times ([-4,1] \cap \mathbb{Z}) \end{aligned} \tag{TI17}$$

Test instance (TI17) is a separable biobjective mixed-integer nonconvex optimization problem with a single box-constrained integer variable, and a quadratic but nonconvex constraint regarding the four continuous variables. All possible/feasible integer assignments are also efficient integer assignments, i.e., it holds $X_I = S_I = \mathcal{E}_I = \{-4,-3,-2,-1,0,1\}$ and thus $|X_I| = |S_I| = |\mathcal{E}_I| = 6$. Note that here the image set of every patch problem is a shifted set difference between the square $[0,\sqrt{2}]^2$ and the ball with radius $\sqrt{2}$ centered in $(0,0)$.

Test instance 18 [17, (P_1)]:

$$\begin{aligned} \min_x &\begin{pmatrix} \frac{1}{2}x^\top Gx + c^\top x \\ d^\top x \end{pmatrix} \\ \text{s.t. } & x \in X = [-1,1]^2 \times \{0,1\}^8 \end{aligned} \tag{TI18}$$

with

$$G := \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix}, \; c := \begin{pmatrix} -1 \\ -1 \\ 1 \\ -10 \\ 0 \\ 1 \\ -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \text{ and } d := \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 5 \\ -2 \\ 0 \\ 6 \\ 0 \\ 3 \end{pmatrix}$$

(TI18) is a box-constrained biobjective mixed-integer nonconvex optimization problem with two continuous and eight binary variables. All possible integer assignments are feasible integer assignments, i.e., it holds $X_I = S_I = \{0,1\}^8$ and $|X_I| = |S_I| = 2^8 = 256$. Note that $\frac{1}{2}\left(G + G^\top\right)$ is not positive semidefinite, and hence the first objective function is nonconvex. For an illustration of the image set and the nonconnected nondominated set we refer to [17, Figure 6].

Test instance 19 [6, Example 3.1, Example 4.8 (i)]:

$$\begin{aligned} \min_x &\begin{pmatrix} 1 - \exp\left(-\sum_{i=1}^n \left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) + x_{n+1} + x_{n+2} \\ 1 - \exp\left(-\sum_{i=1}^n \left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) - x_{n+1} - x_{n+2} \end{pmatrix} \\ \text{s.t. } & x \in X = [-4,4]^n \times ([-1,1]^2 \cap \mathbb{Z}^2) \end{aligned} \tag{TI19}$$

The box-constrained separable biobjective mixed-integer nonconvex optimization problem (TI19) is scalable in the number $n \in \mathbb{N}$ of continuous variables. Hence, again all possible integer assignments are feasible integer assignments. Moreover, using the separable structure, it follows by [6, Theorem 3.4] that the efficient set is given by

$$\mathcal{E} = \left\{ x \in [-4,4]^n \;\middle|\; x_1 = x_2 = \ldots = x_n \in \left[-\tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}}\right] \right\} \times \{-1,0,1\}^2$$

and the nondominated set by

$$\begin{aligned} \mathcal{N} = &\{(1 - \exp(-4(t-1)^2), 1 - \exp(-4t^2)) \mid t \in [0,1]\} \\ &+ \{(-2,2),(-1,1),(0,0),(1,-1),(2,-2)\}. \end{aligned}$$

Thus it holds $X_I = S_I = \mathcal{E}_I = \{-1, 0, 1\}^2$ and $|X_I| = |S_I| = |\mathcal{E}_I| = 3^2 = 9$.

Similar to (TI12), test instance (TI19) is also obtained by the test instance generator introduced in [6]. For further explanations in this regard we refer to [6, Example 3.1, Example 4.8 (*i*)]. An illustration of the nonconnected nondominated set of (TI19) is provided in [6, Figure 1].

Test instance 20 [7, (P3)]:

$$\begin{aligned} \min_x \ & \begin{pmatrix} \sum_{i=1}^{\frac{n}{2}} x_i + \sum_{i=n+1}^{n+\frac{m}{2}} x_i \\ \sum_{i=\frac{n}{2}+1}^{n} x_i + \sum_{i=n+\frac{m}{2}+1}^{n+m} x_i \end{pmatrix} \\ \text{s.t.} \ & -\sum_{i=1}^{n} x_i^2 \leq -1, \\ & \sum_{i=n+1}^{n+m} x_i^2 \leq 9, \\ & x \in X = [0,1]^n \times ([-3,3]^m \cap \mathbb{Z}^m) \end{aligned} \tag{TI20}$$

The separable biobjective mixed-integer nonconvex test instance (TI20) is scalable both in the number $n \in \mathbb{N}$ of continuous variables as well as in the number $m \in \mathbb{N}$ of integer variables which are both assumed to be even. It holds $|X_I| = 7^m$. The following proposition provides a result regarding the feasible integer assignments.

Proposition 4.2 *For the number $|S_I|$ of feasible integer assignments of the biobjective optimization problem* (TI20) *with $m \in \mathbb{N}$ it holds*

$$\begin{aligned} |S_I| \ = \ & 1 + \tfrac{1126}{315} m - \tfrac{418}{45} m^2 + \tfrac{84668}{2835} m^3 - \tfrac{152}{5} m^4 \\ & + \tfrac{2152}{135} m^5 - \tfrac{64}{15} m^6 + \tfrac{584}{945} m^7 - \tfrac{2}{45} m^8 + \tfrac{4}{2835} m^9. \end{aligned} \tag{4.7}$$

The proof of Proposition 4.2 is similar to the proof of Proposition 4.1 and thus omitted. Note that here the image set of every patch problem is a shifted set difference between the square $[0, \sqrt{n/2}]^2$ and the ball with radius $\sqrt{n/2}$ centered in $(0, 0)$.

Test instance 21 [6, Example 4.8 (*iii*)]:

$$\begin{aligned} \min_x \ & \begin{pmatrix} \alpha_1 \left(1 - \exp\left(-\sum_{i=1}^{n} \left(x_i - \frac{1}{\sqrt{n}}\right)^2\right)\right) + \sum_{i \in J} x_i + \sum_{i \in \bar{J}} x_i \ + 0.75 x_{m+n} \\ \alpha_2 \left(1 - \exp\left(-\sum_{i=1}^{n} \left(x_i + \frac{1}{\sqrt{n}}\right)^2\right)\right) + \sum_{i \in J} x_i - \sum_{i \in \bar{J}} x_i \ - 0.25 x_{m+n} \end{pmatrix} \\ \text{s.t.} \ & x \in X = [-4,4]^n \times \left(([-1,1]^{m-1} \times [0,1]) \cap \mathbb{Z}^m\right) \end{aligned} \tag{TI21}$$

For the box-constrained scalable and separable test instance (TI21) with $n, m \in \mathbb{N}$,

$$0 < \alpha_i < \frac{1}{4 \cdot (1 - \exp(-4))} \quad \text{for all} \quad i \in [2],$$

$J \subsetneq \{n+1,\ldots,n+m-1\}$, $\bar{J} := \{n+1,\ldots,n+m-1\} \setminus J$, and $X_I = S_I = \{-1,0,1\}^{m-1} \times \{0,1\}$ it holds

$$\begin{aligned}
\mathcal{E} &= \left\{x \in [-4,4]^n \,\middle|\, x_1 = x_2 = \ldots = x_n \in \left[-\tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}}\right]\right\} \\
&\quad \times \{x \in ([-1,1]^{m-1} \times [0,1]) \cap \mathbb{Z}^m \mid x_i = -1 \text{ for all } i+n \in J\}, \\
\mathcal{N} &= \{(\alpha_1(1-\exp(-4(t-1)^2)), \alpha_2(1-\exp(-4t^2))) \mid t \in [0,1]\} + (\mathcal{N}_1 \cup \mathcal{N}_2), \\
&\quad \mathcal{N}_1 = \{(-(m-1)+\delta, m-1-2|J|-\delta) \in \mathbb{Z}^2 \mid \ \delta \in \Xi\}, \\
&\quad \mathcal{N}_2 = \{(-(m-1)+0.75+\delta, m-1-2|J|-0.25-\delta) \in \mathbb{Z}^2 \mid \ \delta \in \Xi\}, \\
&\quad \Xi = \{0\} \cup [2(m-1-|J|)], \text{ and} \\
\mathcal{E}_I &= \{x \in ([-1,1]^{m-1} \times [0,1]) \cap \mathbb{Z}^m \mid x_i = -1 \text{ for all } i+n \in J\}.
\end{aligned}$$

Consequently, we obtain $|X_I| = |S_I| = 2 \cdot 3^{m-1}$ and $|\mathcal{E}_I| = 2 \cdot 3^{m-1-|J|}$. Hence, as for (TI12), the number of efficient integer assignments is controllable by the choice of m and J, respectively. As for the test instances (TI12) and (TI19), the test instance (TI21) is obtained by the test instance generator introduced in [6] (see [6, Example 4.8 (*iii*)]). For an illustration of the nonconnected nondominated set of (TI21) in the case $m = 4$, $J = \{n+1\}$, and $\alpha_1 = \alpha_2 = 0.2$ we refer to [6, Figure 6].

Test instance 22 [7, (P2)]:

$$\begin{aligned}
\min_x \ & \begin{pmatrix} x_1 + x_4 \\ x_2 - x_4 \\ x_3 - \exp(x_4) - 3 \end{pmatrix} \\
\text{s.t.} \ & x_1^2 + x_2^2 \leq 1, \\
& \exp(x_3) \leq 1, \\
& x_1 x_2 (1 - x_3) \leq 1, \\
& x \in X = [-2,2]^3 \times ([-2,2] \cap \mathbb{Z})
\end{aligned} \tag{TI22}$$

We continue with the separable triobjective mixed-integer nonconvex test instance (TI22). Since the integer variable is only box-constrained, all possible integer assignments are feasible integer assignments. Moreover, it holds $X_I = S_I = \mathcal{E}_I = \{-2,-1,0,1,2\}$ and thus $|X_I| = |S_I| = |\mathcal{E}_I| = 5$. For an illustration of an enclosure of the nondominated set of (TI22) we refer to [7, Figure 3].

Test instance 23 [16, Example 3.13]:

$$\begin{aligned}
\min_x \ & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
\text{s.t.} \ & \sum_{j=n+1}^{n+m} x_j = 1, \\
& x_j \left(\|(x_1,\ldots,x_n) - c^j\|_2^2 - r^2 \right) = 0 \ \forall j \in [n+m] \setminus [n], \\
& x_j \left(c_i^j - x_i \right) \leq 0 \ \forall j \in [n+m] \setminus [n] \ \forall i \in [n], \\
& x \in X = \mathbb{R}_+^n \times \{0,1\}^m
\end{aligned} \tag{TI23}$$

Test instance (TI23) with $m, n \in \mathbb{N}$, $r > 0$, $c^j \in \mathbb{R}^n$, and $j \in [n+m] \setminus [n]$ has the same structure as (TI15). It is scalable in the number of continuous variables and in the number of binary variables. Thus, it is also scalable in the number of objective functions by using the identity map for the continuous variables. Moreover, it follows by the equality constraint that exactly one of the m binary variables takes the value 1. Hence, for any $x \in S$ there exists exactly one $j \in [m+n] \setminus [n]$ with $x_j = 1$ and all the remaining components of x_I are zero. While it holds $X_I = \{0,1\}^m$, the set of all feasible integer assignments S_I (with a cardinality bounded from above by m) and the set of all efficient integer assignments $\mathcal{E}_I$ depend on the special choice of the parameters $r > 0$ and $c^j \in \mathbb{R}^n$, $j \in [n+m] \setminus [n]$. Note that, with regard to property (11), for all feasible points $\hat{x} \in S$ with $j \in [m+n] \setminus [n]$ as described above such that $\hat{x}_j = 1$ the image set of the corresponding patch problem (P($\hat{x}_I$)) is a subset of $\{c^j\} + \{y \in \mathbb{R}^n_+ \mid \ \|y\|_2 = r\}$.

For instance, cf. [16], we obtain for the special choice $n = 2$, $m = 3$, $r = 1$, $c^3 = (3,0)$, $c^4 = (2,1)$, and $c^5 = (0,3)$ that $X_I = \{0,1\}^3$ and $S_I = \{(1,0,0),(0,1,0),(0,0,1)\}$ with the corresponding patch problems

$$\begin{array}{rrcl} \min\limits_x & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & & \\ \text{s.t.} & \|(x_1,x_2)-(3,0)\|_2^2 & = & 1, \\ & x_1 & \geq & 3, \\ & x_2 & \geq & 0 \end{array}$$

for $x_I = (1,0,0)$,

$$\begin{array}{rrcl} \min\limits_x & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & & \\ \text{s.t.} & \|(x_1,x_2)-(2,1)\|_2^2 & = & 1, \\ & x_1 & \geq & 2, \\ & x_2 & \geq & 1 \end{array}$$

for $x_I = (0,1,0)$, and

$$\begin{array}{rrcl} \min\limits_x & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & & \\ \text{s.t.} & \|(x_1,x_2)-(0,3)\|_2^2 & = & 1, \\ & x_1 & \geq & 0, \\ & x_2 & \geq & 3 \end{array}$$

for $x_I = (0,0,1)$. Thus, we obtain here $\mathcal{E}_I = S_I$ and

$$\begin{array}{rcl} \mathcal{E} & = & \Big(\big(\{x \in \mathbb{R}^2_+ \mid \ x_1^2 + x_2^2 = 1\} + \{(3,0)\}\big) \times \{(1,0,0)\}\Big) \\ & & \cup \Big(\big(\{x \in \mathbb{R}^2_+ \mid \ x_1^2 + x_2^2 = 1\} + \{(2,1)\}\big) \times \{(0,1,0)\}\Big) \\ & & \cup \Big(\big(\{x \in \mathbb{R}^2_+ \mid \ x_1^2 + x_2^2 = 1\} + \{(0,3)\}\big) \times \{(1,0,0)\}\Big) \end{array}$$

and

$$\mathcal{N} = \{x \in \mathbb{R}^2_+ \mid \ x_1^2 + x_2^2 = 1\} + \{(3,0),(2,1),(0,3)\}.$$

For an illustration of the image set and the nonconnected nondominated set of (TI23) with this special choice see Figure 11.

test instance		n	m	p	$\lvert X_I\rvert$	$\lvert S_I\rvert$	$\frac{\lvert S_I\rvert}{\lvert X_I\rvert} \cdot 100\%$	$\lvert \mathcal{E}_I\rvert$	$\frac{\lvert \mathcal{E}_I\rvert}{\lvert X_I\rvert} \cdot 100\%$	$\frac{\lvert \mathcal{E}_I\rvert}{\lvert S_I\rvert} \cdot 100\%$
(TI1)	c, q	1	1	2	9	9	100%	4	44%	44%
(TI2)	c, q	2	2	2	3^2	5	56%	2	22%	40%
(TI3)	c, q	3	3	2	2^3	2^3	100%	–	–	–
(TI4)	c, q	4	4	2	41^4	37^2	0.048%	9	0.00032%	0.66%
(TI5)	c, q	3	1	3	5	5	100%	5	100%	100%
(TI6)	c, q	2	2	3	2^2	2^2	100%	–	–	–
(TI7)	c, q	3	3	3	3^3	7	26%	3	11%	43%
(TI8)	c, q	2	s	2	11^m	11^m	100%	–	–	–
(TI9)	c, q	2	s	2	5^m	see (4.1)	depends on m	1	depends on m	depends on m
(TI10)	c, q	s	s	2	5^m	5^m	100%	5^m	100%	100%
(TI11)	c, q	s	s	2	5^m	5^m	100%	–	–	–
(TI12)	c, q	s	s	2	3^m	3^m	100%	$3^{m-\lvert J\rvert}$	$3^{-\lvert J\rvert} \cdot 100\%$	$3^{-\lvert J\rvert} \cdot 100\%$
(TI13)	c, nq	2	1	2	5	5	100%	5	100%	100%
(TI14)	c, nq	4	4	2	41^4	37^2	0.048%	5	0.00018%	0.37%
(TI15)	nc	1	2	2	4	2	50%	1	25%	50%
(TI16)	nc	2	2	2	7^2	29	59%	3	6.1%	10%
(TI17)	nc	4	1	2	6	6	100%	6	100%	100%
(TI18)	nc	2	8	2	2^8	2^8	100%	–	–	–
(TI19)	nc	s	2	2	3^2	3^2	100%	3^2	100%	100%
(TI20)	nc	s	s	2	7^m	see (4.7)	depends on m	–	–	–
(TI21)	nc	s	s	2	$2 \cdot 3^{m-1}$	$2 \cdot 3^{m-1}$	100%	$2 \cdot 3^{m-1-\lvert J\rvert}$	$3^{-\lvert J\rvert} \cdot 100\%$	$3^{-\lvert J\rvert} \cdot 100\%$
(TI22)	nc	3	1	3	5	5	100%	5	100%	100%
(TI23)	nc	s	s	s	2^m	–	–	–	–	–

Table 1: Test instances of Subsection 4.1 and Subsection 4.2 where 'c' refers to convex, 'nc' to nonconvex, 'q' to quadratic, 'nq' to nonquadratic, 's' to scalable instances, and '–' to not known.

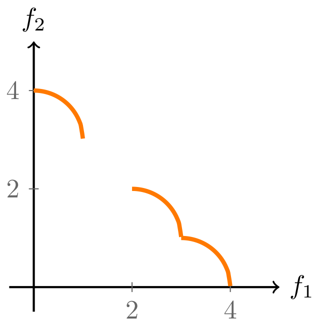

Figure 11: Image set and nondominated set of (TI23) with $n = 2$, $m = 3$, $r = 1$, $c^3 = (3, 0)$, $c^4 = (2, 1)$, and $c^5 = (0, 3)$

Table 1 provides a comprehensive overview of all the considered test instances for multiobjective mixed-integer convex and nonconvex optimization problems in Section 4.

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