Cooperative locker location games

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Abstract

More and more people are ordering products online, having their parcels delivered to their homes. This leads to more congestion, which negatively impacts the environment as well as public health and safety. To reduce these negative impacts, carriers can use parcel lockers to consolidate and serve their customers. The implementation of a locker network can, however, be financially challenging. To overcome this, carriers can decide to collaborate and invest in parcel lockers together. In this paper, we introduce a stylized model in which a group of carriers can decide to position parcel lockers collectively. In this model opening a locker comes at a cost, while serving a customer via close-by lockers generates a customer-specific profit. By introducing and studying the associated cooperative game, we then investigate whether the carriers can allocate the joint profit in a stable way. For this game, we prove that a stable allocation is guaranteed for a particular class of networks. Generating a large set of instances, we furthermore conduct a number of numerical experiments, showing that a stable profit allocation is possible in most situations.

Keywords— last-mile logistics; parcel lockers; facility location; cooperative game theory; stable allocation

1 Introduction

Since 2007, more people have lived in cities than in rural areas and this trend is presumed to continue, with an expected 68% of the world population living in urban areas by 2050 (Ritchie and Roser, 2018). At the same time, more and more people in these cities are ordering products online (i.e., using e-commerce), and getting their parcels delivered to their homes. Statista (2021) states in this context that the revenue of the parcel industry in the Netherlands grew in 2020 from nearly 2,000 million euros in 2016 to 3,600 million euros. Consequently, more delivery vehicles are required to bridge the last mile towards customers within city centers. This leads to more congestion, negatively impacting the environment as well as public health and safety (Savelsbergh and Van Woensel, 2016). In addition, last-mile delivery is often considered the most expensive part of the delivery process. Nabot and Omar (2016) showed, in this context, that last-mile delivery accounts for more than 50% of the overall shipping costs of products and more than 2/3 of the CO_2 emissions associated with the delivery process.

Various novel last-mile logistics concepts have been proposed to overcome these issues in recent years. One such concept is the integration of automated parcel lockers into city logistics systems, where automated parcel lockers are package containers that allow autonomous parcel collection at the most convenient moment for the consumer. These lockers are usually located in key locations where people frequently pass by, such as grocery stores, public transport stops, gas stations, residential buildings, and offices. By allowing customers to combine the parcel collection with their daily routine and consolidating the deliveries of several customers in one location, parcel lockers have the potential to reduce urban traffic, while at the same time preventing/reducing failed deliveries.

*We would like to stress that the authors contributed equally to the development of this paper.

Although parcel lockers already find application in practice (Iwan, Kijewska, and Lemke, 2016; van Duin et al., 2020), various barriers remain, hindering implementation at a larger scale. One of these barriers is the high cost associated with the lockers. Seghezzi, Siragusa, and Mangiaracina (2022) state, for example, that the annual rental cost of one locker, including operating expenses, is around \$5,000 per year in urban areas (\$3,000 in rural areas), while the investment cost of buying a locker is estimated at around \$20,000. Keeping in mind that carriers would need multiple lockers to achieve a certain level of coverage, setting up such a network of lockers can be financially challenging.
One way to overcome this financial barrier is through horizontal collaboration among carriers. A possible example of such a collaboration could be a group of carriers deciding to buy lockers collectively. This type of collaboration has already been proposed in some countries, e.g., in Poland, where multiple carriers have access to the same "open access" parcel locker network (IPC, 2022). This paper aims to investigate how this type of horizontal collaboration can be facilitated to promote the use of parcel lockers in last-mile delivery. For this purpose, we introduce a stylized model in which a group of carriers can decide collectively to position a finite set of parcel lockers. In this model opening a locker comes at a cost while serving customers via close-by lockers generates a customer-specific profit. The carriers collectively decide how many lockers to open and where to position them in order to maximize their profit (i.e., the revenues from serving customers minus the costs of opening the lockers). The problem is formulated as an integer linear programming problem which can be solved efficiently when the network has a tree structure. To promote this horizontal collaboration, we investigate whether the carriers can allocate the profit in a stable way, by introducing an associated cooperative game and studying its corresponding core. In cooperative game theory the core represents the set of allocations which divide the profit among the carriers in such a way that no group of carriers has an incentive to leave the collaboration. We show mathematically that the core of our game can be empty, meaning that no stable allocation of the joint profit exists, which may hinder collaboration. However, at the same time our numerical experiments indicate that for most situations, the core is non-empty, so that a stable profit allocation does exist.

The contributions of this research are threefold:

1. Focusing on the location of parcel lockers, this paper studies a new practical application at the interface of operations research and cooperative game theory within the context of last-mile deliveries. For this purpose, we first introduce the locker location problem and, subsequently, study the associated cooperative game, where multiple carriers collaborate, facing a joint locker location problem together.

2. We investigate the properties of the proposed cooperative game to identify patterns that may lead to allocations inside the core. In this context, we prove that the core of the locker location game is non-empty if the optimal value of the underlying decision problem corresponds to the optimal value of its linear relaxation, i.e., $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$. We further show that this condition holds for all games for which the induced graph has a tree structure.

3. We conduct an extensive numerical analysis to study the impact of different contexts on core emptiness and the likelihood of $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$, considering a total of 403,200 instances with different parameter settings. Based on this analysis, we derive valuable managerial insights regarding the viability of collaboration.

Our paper contributes to the recent literature on the interface of cooperative game theory and facility location problems. Examples of this stream can be found in Goemans and Skutella (2004), Schlicher, Slikker, and van Houtum (2017), Schlicher (2018), and Osicka, Guajardo, and van Oost (2020). The focus of these papers is mostly theoretical, studying sufficient (and necessary) conditions for core non-emptiness based on the features of the underlying optimization problem. Considering a different type of facility location problem, namely the locker location problem, we also study sufficient conditions for core non-emptiness. However, in comparison to most of the other papers we also analyze how our game performs in settings for which the derived conditions do not hold. For this purpose, we conduct an extensive numerical analysis to explore the impact of different instance structures on core emptiness. We show that even in the absence of our conditions, core non-emptiness is often preserved.
The paper is organized as follows. In Section 2, we provide a literature overview of the most relevant contributions to the related research disciplines before introducing our locker location problem in Section 3. Section 4 then introduces the corresponding multi-carrier extension of the locker location problem, called the locker location situation, based on which we introduce the associated cooperative game in Section 5, studying its core theoretically. Presenting several numerical experiments, Section 6 studies

the frequency of core non-emptiness for different instances. Finally, our conclusions are given in Section 7.

2 Literature review

In this section, we review the relevant literature. We identify two main streams of literature: (i) operations research models to improve parcel locker operations and (ii) cooperative game theory approaches to achieve stable cost savings allocations in location types of problems.

2.1 Operations research models for parcel lockers

The growing interest in parcel lockers and their use within logistics applications over recent years can also be observed within the scientific literature. In operations research, this has resulted in the study of many decision problems arising in parcel locker systems (Rohmer and Gendron, 2020). In the following, the focus will be predominantly on models and decision problems that aim to optimize the location decisions relating to the placement of parcel lockers.
One of the first studies addressing this type of problem is the research of Deutsch and Golany (2018), in which a delivery company needs to determine the amount and the location of the locker sites. The authors show that their problem can be formulated as an uncapacitated facility location problem maximizing total profit, which is given by the revenue obtained from serving the customers minus a location-dependent setup cost. In this research, the probability of a customer visiting the lockers and, thus, the corresponding effect on the revenue is assumed to depend on the distance. At the same time, in other studies, this choice is modeled more explicitly using discrete-choice theory. Examples of such problems are found in Lin et al. (2020), Lin et al. (2022) and Lyu and Teo (2022). The research of Lin et al. (2020) integrates, in this context, the location decisions within a multinomial logit model, which predicts customers' preferences and their probability of using a locker based on a distance parameter. In contrast, Lin et al. (2022) predict the use of lockers by customers based on the threshold Luce model, considering sets of selected locker locations for each customer. The research of Lyu and Teo (2022), on the other hand, uses data from a commercial courier company in Singapore to calibrate a locker-choice model considering the distance, location type, and time aspects. The model is combined with a facility location model and then used to estimate the subsequent adoption of customers. For a more detailed overview of facility location problems, not focusing on parcel lockers, we refer to the book of Laporte, Nickel, and Saldanha-da-Gama (2019).

In addition to papers focusing on location decisions only, a few papers related to parcel lockers combine location and routing decisions within a common framework. Focusing on minimizing costs, Zhou et al. (2016), for instance, propose a location-routing problem that incorporates both home deliveries and pickup locations, assuming that customers accept to use lockers if their distance from the locker is within a certain threshold. Another problem proposed by Zhou et al. (2019) is a location-routing model with intermediate distribution centers. For each customer, it is decided whether to deliver at home or let the customer collect the parcel at the distribution center. For a more detailed overview of location-routing problems without focusing on parcel locker applications, we refer to the review of Prodhon and Prins (2014).

2.2 Cooperative games on resource consolidation and facility location

The topic of collaboration between carriers to consolidate resources has been addressed in several studies, applying cooperative game theory to various transportation settings. Examples of such applications can be found in the research of Özener and Ergun (2008); Özener, Ergun, and Savelsbergh (2011); Houghtalen, Ergun, and Sokol (2011); Özener, Ergun, and Savelsbergh (2013); Giudici et al. (2021); van Zon, Spliet, and van den Heuvel (2021) and Lai, Cai, and Hall (2022). Given the focus of the application in this research, the emphasis in the following will be predominantly on studies applying cooperative game theory to location problems. Goemans and Skutella (2004) are among the first to study this type of problem, considering a cooperative game based on the unconstrained facility location problem. The authors show that checking whether an allocation belongs to the core (i.e., the set of cost allocations for which no group of players has reasons to leave the collaboration) is an NP-hard problem. Moreover, they observe that the core is non-empty if and only if there is no integrality gap for the optimal value of a relaxed problem and explore graph patterns that can lead to such situations. The research on specific patterns for core non-emptiness in cooperative games on location problems links to the work of Schlicher, Slikker, and van Houtum (2017), who define a cooperative game based on the maximal covering problem. The authors study some properties of the cooperative game, discuss an example with an empty core, and provide a sufficient condition for core non-emptiness in terms of a relaxation of the maximal covering location problem. Inspired by a coast patrol context, Schlicher (2018) illustrate that maximal covering location games have a non-empty core when the underlying graph structure is a line segment. Similarly to the previous section, we observe cooperative games based on problems where location decisions are integrated into a vehicle routing problem. Osicka, Guajardo, and van Oost (2020) investigate cooperative games based on multiple variants of the location-routing problem. The authors make a theoretical focus on subadditivity, core non-emptiness, and convexity for each variant, finding that the core can be empty in all the proposed variants, but subadditivity is present in some of them. They also compute the frequency of subadditive, non-empty core, and convex game instances with numerical simulations.

3 The locker location problem

We consider a delivery environment with a finite non-empty set of customers $C \subset \mathbb{N}$ and a carrier (e.g., a parcel delivery company) that wants to serve these customers. We assume that, to save costs, the carrier should consider introducing a locker network instead of visiting each potential customer directly. The possible locations of these lockers are given by a finite non-empty set $L \subset \mathbb{N}$, with $L \cap C = \varnothing$, and the cost of opening a locker in location $j \in L$ is $c_j \in \mathbb{R}_+$. The locker network will serve a customer $k \in C$ if at least one locker is located at a distance of at most $D_k \in \mathbb{R}_+$. In such a case, the carrier obtains a profit of $p_k \in \mathbb{R}_+$. This profit can be derived from possible cost savings, such as the consolidation of deliveries and reduced failed deliveries. If no locker is located at a distance of at most D_k, the customer will not be served by the locker network, and consequently, the carrier will not receive an additional profit. We denote the distance between customer k and the locker location j by $d_{kj} \in \mathbb{R}_+$, and define L_k as the set of locker locations that can serve customer k, i.e., $L_k = \{j \in L | \, d_{kj} \leq D_k\}$.
The carrier aims to open those lockers for which the total profit, i.e., the sum of the customer-specific profits of all the served customers minus the cost of opening the lockers, is maximized. We refer to this problem as the strict locker location problem (shortly, SLLP). The SLLP can be defined with the parameters and variables listed in Table 1.

Table 1: Sets, parameters, and variables

Sets	
C	Set of customers
L	Set of locations
Parameters	
p_k	Profit of customer k
c_j	Cost of locker in location j
d_{kj}	Distance between customer k and location j
D_k	Maximum distance for customer k
Variables	
x_k	1 if customer k is served, otherwise 0
y_j	1 if locker is opened at location j, otherwise 0

Formally, the SLLP is given by:

$$SLLP := \max \quad \sum_{k \in C} p_k x_k - \sum_{j \in L} c_j y_j \tag{1a}$$

$$\text{s.t.} \quad x_k \leq \sum_{j \in L_k} y_j \qquad \forall k \in C \tag{1b}$$

$$x_k \in \{0,1\} \qquad \forall k \in C \tag{1c}$$

$$y_j \in \{0,1\} \qquad \forall j \in L \tag{1d}$$

The objective function (1a) maximizes the sum of the customer-specific revenues obtained from serving customers minus the cost of opening the lockers. Constraints (1b) ensure that a customer can only be served if at least one locker in its proximity is open while constraints (1c) and (1d) define the decision variables as binary. Similarly to Deutsch and Golany (2018), the studied problem does not include capacity constraints. This simplification allows us to exploit certain properties when dealing with the cooperative game version of the problem. The lockers' capacity could be partially integrated within the considered cost-revenue structure. Also note that our optimization problem SLLP is upper-bounded by $\sum_{k\in C} p_k$ and always has a feasible solution, namely $x_k = 0$ for all $k \in C$ and $y_j = 0$ for all $j \in L$. As such, the problem is well-defined. In the following, we use opt$(SLLP)$ to refer to the optimal value of the SLLP.

We propose a relaxed formulation of the SLLP, which turns out to be useful when studying the properties of SLLP. In particular, this relaxed formulation will help identify a class of situations for which the problem can be solved efficiently. We refer to this problem as the locker location problem (LLP), which can be formally written as:

$$
\begin{aligned}
LLP := \max \quad & \sum_{k\in C} p_k x_k - \sum_{j\in L} c_j y_j && && \text{(2a)}\\
\text{s.t.} \quad & x_k \le \sum_{j\in L_k} y_j && \forall k \in C && \text{(2b)}\\
& x_k \le 1 && \forall k \in C && \text{(2c)}\\
& x_k \in \mathbb{R}_+ && \forall k \in C && \text{(2d)}\\
& y_j \in \mathbb{N} && \forall j \in L && \text{(2e)}
\end{aligned}
$$

Like the SLLP, we use opt(LLP) to refer to the optimal value of the LLP. Although LLP is a relaxation of SLLP, both problems have the same optimal value. We formalize this result in Lemma 1.

Lemma 1. *The optimal value of SLLP and LLP coincide, i.e., opt(SLLP) = opt(LLP).*

Proof. Since the feasible set of SLLP is included in the feasible set of LLP, and (1a) and (2a) are the same, it remains to show that for every feasible solution of LLP that is not feasible in SLLP, there exists another solution that is feasible in both SLLP and LLP and has the same or higher objective value. Let $\big((x_k)_{k\in C}, (y_j)_{j\in L}\big)$ be a feasible solution of LLP that is not feasible in SLLP.

We distinguish between two situations.

1. *Case 1:* $\exists S \subseteq C$, $S \neq \varnothing$ *with* $x_k \in (0,1)$ *for all* $k \in S$ *and* $x_k \in \{0,1\}$ *for all* $C\backslash S$.
   Let $\bar{x}_k = 1$ for all $k \in S$ and $\bar{x}_k = x_k$ for all $k \in C\backslash S$ and $\bar{y}_j = \min\{1, y_j\}$ for all $j \in L$. Clearly $\bar{x}_k \in \{0,1\}$ for all $k \in C$ and $\bar{y}_j \in \{0,1\}$ for all $j \in L$. Due to constraints (2b) and (2e), there exists for all $k \in S$ a $j(k) \in L_k$ for which $y_{j(k)} \in \{1,2,...\}$ and so $\bar{y}_{j(k)} = 1$. Hence, for all $k \in S$,
$$\bar{x}_k \le 1 \le \bar{y}_{j(k)} \le \sum_{j\in L_k} \bar{y}_j. \tag{3}$$
   Since $\bar{x}_k = 0$ for all $k \in C\backslash S$ and $\bar{y}_j \in \{0,1\}$, we have for all $k \in C\backslash S$
$$\bar{x}_k \le \sum_{j\in L_k} \bar{y}_j.$$
   Consequently, $\big((\bar{x}_k)_{k\in C}, (\bar{y}_j)_{j\in L}\big)$ is a feasible solution of LLP and SLLP. Moreover,
$$\sum_{k\in C} p_k x_k - \sum_{j\in L} c_j y_j \le \sum_{k\in C} p_k \bar{x}_k - \sum_{j\in L} c_j \bar{y}_j, \tag{4}$$
   since $x_k \le \bar{x}_k$ for all $k \in C$ and $y_j \ge \bar{y}_j$ for all $j \in L$.
2. *Case 2:* $x_k \in \{0,1\}$ *for all* $k \in C$.
   By the case condition, there exists a $S \subseteq L, S \neq \varnothing$, for which $y_j \in \{2,3,...\}$ for all $j \in S$ and $y_j \in \{0,1\}$ for all $L\backslash S$. Now, let $\bar{x}_k = x_k$ for all $k \in C$ and $\bar{y}_j = \min\{1, y_j\}$ for all $j \in L$. Then, for every $k \in C$ with $\bar{x}_k = 1$, we know that $\sum_{j\in L_k} y_j \ge 1$ (since $x_k = 1$) and so $\sum_{j\in L_k} \bar{y}_j \ge 1 = \bar{x}_k$. Moreover, for every $k \in C$ with $\bar{x}_k = 0$, we have $\bar{x}_k \le \sum_{j\in L_k} \bar{y}_k$ since $y_k \in \{0,1\}$. So, $\big((\bar{x}_k)_{k\in C}, (\bar{y}_j)_{j\in L}\big)$ is a feasible solution of LLP and SLLP and (4) applies.

Thus, $\big((\bar{x}_k)_{k\in C}, (\bar{y}_j)_{j\in L}\big)$ is a feasible solution of both SLLP and LLP, and the corresponding objective value is greater than or equal to the objective value of solution $\big((x_k)_{k\in C}, (y_j)_{j\in L}\big)$. $\square$

We now present an example of our LLP. Note that parameters are chosen to keep the example calculations easy to follow.

Example 1. Consider a setting with four customers, given by $C = \{4, 5, 6, 7\}$, and three locker locations, given by $L = \{8, 9, 10\}$. The customer-specific profits are $(p_k)_{k\in C} = (5, 4, 3, 4)$, the costs for opening lockers are $(c_j)_{j\in L} = (5, 9, 5)$, the distances are $(d_{kj})_{k\in C, j\in L} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, and the maximal distances are $(D_k)_{k\in C} = (2, 1, 2, 2)$. Consequently, $L_4 = \{8, 9\}$, $L_5 = \{8, 9\}$, $L_6 = \{8, 9, 10\}$, $L_7 = \{9, 10\}$. Figure 1 presents a graphical representation of this setting.

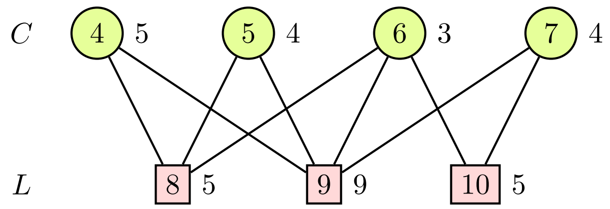


Figure 1: Graphical illustration of setting in Example 1. Circles represent the customers, and rectangles the locations. The internal label is the name, and the external label is the profit or cost. The edges represent whether the customer is willing to visit the locker location.

The optimal solution for this example is $x_4 = x_5 = x_6 = x_7 = 1$, $y_8 = y_{10} = 0$ and $y_9 = 1$. This means we open a locker at location 9 (at a cost of 9) and serve all customers. Consequently, the corresponding optimal total profit of the LLP equals $(5 + 4 + 3 + 4) - 9 = 7$. $\triangle$

As illustrated in Example 1, we can visualize our setting by a bipartite graph. A formal definition of such a graph, for a given LLP, is provided by an LLP-induced graph $G^{LLP} = (V, E)$, where $V = C \cup L$ denotes the vertex set, consisting of customers and locations, and $E = ((k, j))_{k\in C, j\in L_k}$ the edge set, representing which customers can be served by a locker.

Our LLP is a mixed-integer linear programming problem. Typically, these problems can be hard to solve. However, under certain conditions, our LLP can be solved quite efficiently. In particular, this holds if the LLP-induced graph G^{LLP} is a tree, i.e., a connected graph with no cycles. In that case, our LLP problem can be tackled by solving a linear programming relaxation. We formalize this statement in the following. For this purpose, we first introduce some definitions.

A *square submatrix* of a matrix $A \in \mathbb{R}^{w\times z}$ (with w the number of rows and z the number of columns) is a matrix $A' \in \mathbb{R}^{q\times q}$ formed by selecting q rows and q columns from A. Moreover, a matrix A is *totally unimodular* if every square submatrix of A has a determinant equal to 1, -1, or 0.

Lemma 2. *Let A be a totally unimodular $w \times z$ matrix and let $b \in \mathbb{Z}^w$ and $c \in \mathbb{R}^z$. Then the linear programming problem* $\max\{cx | x \geq 0, Ax \leq b\}$ *has integer optimal solutions whenever it has a finite optimum.*

Proof. See e.g. Wolsey (1998, p.40). $\square$

Lemma 3. *Given a totally unimodular matrix, if we add a row or a column with a unique non-zero element, that is -1 or 1, then the matrix is still totally unimodular.*

Proof. See Schrijver (1998, p.280) $\square$

We now introduce the corresponding linear programming relaxation RLLP by relaxing the integrality constraint of $(y_j)_{j\in L}$. RLLP reads as follows:

$$RLLP := \max \quad \sum_{k\in C} p_k x_k - \sum_{j\in L} c_j y_j \tag{5a}$$

$$\text{s.t.} \quad x_k \le \sum_{j\in L_k} y_j \qquad \forall k \in C \tag{5b}$$

$$x_k \le 1 \qquad \forall k \in C \tag{5c}$$

$$x_k \in \mathbb{R}_+ \qquad \forall k \in C \tag{5d}$$

$$y_j \in \mathbb{R}_+ \qquad \forall j \in L \tag{5e}$$

Please, note that in standard linear programming form, i.e., in the form $Az \le b$ with $z \ge 0$, our RLLP can be represented by a matrix

$$A^{LLP} = \left[\begin{array}{c|c} I_{|C|} & M^{LLP} \\ \hline I_{|C|} & \text{O} \end{array}\right], \tag{6}$$

where $I_{|C|}$ is the identity matrix of size $|C|$, O is the zero-matrix of size $|C| \times |L|$, M^{LLP} is a negative binary matrix of size $|C| \times |L|$ whose entries are

$$m_{kj}^{LLP} = \begin{cases} -1 & \text{if } j \in L_k \\ 0 & \text{otherwise} \end{cases} \quad \forall\, k \in C, j \in L, \tag{7}$$

and a vector $b^{LLP} = ((0)_{k\in C}, (1)_{k\in C})$. Since matrix M^{LLP} will play an important role later in various proofs, we refer to this matrix as the *connection matrix*.

Lemma 2 provides a sufficient condition for linear programming problems to have integral optimal solutions. Our RLLP meets this condition, given that the connection matrix M is totally unimodular. This has an important consequence, which we prove in the next theorem.

Theorem 1. *If the connection matrix M^{LLP} of A^{LLP} is totally unimodular, then RLLP finds an optimal solution of LLP.*

Proof. Let M^{LLP} be totally unimodular. First, we show that A^{LLP} is also totally unimodular. Please, observe that $[I_{|C|} \,|\, M^{LLP}]$ can be constructed by adding columns of $[I_{|C|}]$ one-by-one. Please, note that each column of $[I_{|C|}]$ is totally unimodular since it is a column with one unique non-zero element "1". By Lemma 3, these operations preserve the totally unimodular property. Consequently, matrix $[I_{|C|} \,|\, M]$ is totally unimodular. Now, observe that A^{LLP} can be constructed by adding rows of $[I_{|C|} \,|\, O]$, one-by-one. Please, note that each row of $[I_{|C|} \,|\, O]$ is totally unimodular since it is a row with one unique non-zero element "1". By Lemma 3, these operations also preserve the totally unimodular property. As a consequence, A^{LLP} is totally unimodular.

Second, observe that RLLP is bounded above by $\sum_{k\in C} p_k$ and has a feasible integral solution $x_k = 0$ for all $k \in C$ and $y_j = 0$ for all $j \in L$. Hence, RLLP has a finite optimum. Vector b^{LLP} has all integer values (0 or 1). Now, by Lemma 2, we know that RLLP has an integral optimal solution, and so, it solves the corresponding LLP. □

By using Theorem 1 above, we will prove that every LLP, for which the induced LLP graph G^{LLP} is a tree, can be solved by the RLLP, i.e., efficiently.

Theorem 2. *Every LLP for which the induced LLP graph G^{LLP} is a tree can be solved by RLLP.*

Proof. Let the LLP with induced LLP graph G^{LLP} be a tree. Moreover, let $M_{p,q}$ be the corresponding connection matrix of LLP with $p \in \mathbb{N}_+$ the number of customers and $q \in \mathbb{N}_+$ the number of lockers. Hence, our tree G consists of $p+q$ nodes and $p+q-1$ edges. Since $L, C \ne \emptyset$, $p+q \ge 2$, i.e., the tree consists of at least two nodes. It is well known that every tree, with at least two nodes, has at least two nodes with degree one (see, e.g., Deo (2017)). Consider such a node with degree one in our graph G^{LLP}. If we remove this node and its corresponding edge, we end up with a graph with $p+q-1$ nodes and $p+q-2$ edges. This is a tree again, whose connection matrix can be constructed as follows:

- *i) The removing node is a customer*
  Remove the row corresponding to the node from matrix $M_{p,q}$. This row consists of zeros and exactly one element "-1" since the node has degree one, i.e., it is connected to exactly one location. We end up with matrix $M_{p-1,q}$, i.e., a matrix with $p-1$ rows and q columns.
- *ii) The removing node is a location*
  Remove the column corresponding to the node from matrix $M_{p,q}$. This column consists of zeros and exactly one element "-1" since the node has degree one, i.e., it is connected to exactly one customer. This leads to matrix $M_{p,q-1}$, i.e., a matrix with p rows and $q-1$ columns.

We can repeat the process of removing a node with degree one until we end up with a graph tree with two nodes and one linking edge. This is possible since, after each removal, we end up with a tree again. Similarly, we can construct the corresponding connection matrix of each (intermediate) tree by removing a row or column, as suggested in i) and (ii) one by one. Finally, we end up with the connection matrix $M_{1,1}$.

A connection matrix with one row and one column, i.e., $M_{1,1}$, is totally unimodular. By working backward, we can rebuild $M_{p,q}$ by adding the rows and columns oppositely to how we removed them, according to i) and ii). Since these rows and columns consist of zeros and exactly one element "-1", adding them will preserve the totally unimodular property (Lemma 3). Hence, the connection matrix $M_{p,q}$ is totally unimodular, and by Theorem 1, LLP can be solved by RLLP. □

We conclude this section with an example for which the induced LLP graph has a tree structure.

Example 2. Consider a setting with six customers, given by $C = \{1, 2, 3, 4, 5, 6\}$, and four locker locations, given by $L = \{7, 8, 9, 10\}$. The customer-specific profits are $(p_k)_{k\in C} = (10)_{k\in C}$, the costs for opening lockers are $(c_j)_{j\in L} = (11, 20, 25, 12)$, the distances are

$$(d_{kj})_{k\in C, j\in L} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 1 & 3 & 5 \\ 3 & 1 & 3 & 5 \\ 3 & 1 & 1 & 3 \\ 5 & 3 & 1 & 3 \\ 5 & 3 & 1 & 1 \end{bmatrix},$$

and the maximal distances are $(D_k)_{k\in C} = (2)_{k\in C}$. Consequently, $L_1 = \{10\}$, $L_2 = \{10, 11\}$, $L_3 = \{11\}$, $L_4 = \{11, 12\}$, $L_5 = \{12\}$, $L_6 = \{13\}$. In Figure 2, one can find the graphical representation of the corresponding LLP-induced graph. Note that this graph has a tree structure.

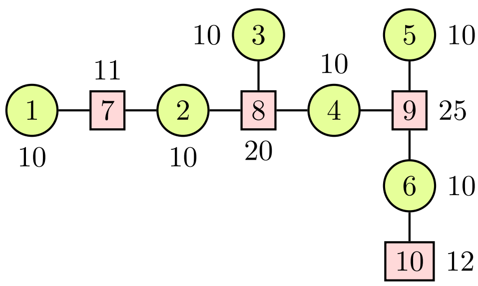


Figure 2: Graphical illustration of the LLP-induced graph of Example 2.

Based on Theorem 2, an optimal solution of the LLP can be found by solving the RLLP, which reads $x_4 = x_5 = x_7 = x_8 = x_9 = y_{10} = y_{12} = 1$ and $x_6 = y_{11} = y_{13} = 0$ with objective value 14. △

4 The locker location situation

Consider a context with a finite non-empty set of $N \subset \mathbb{N}$ carriers such that each faces an individual locker location problem as introduced in Section 3. We denote the finite non-empty set of customers by C, such that $C \subset \mathbb{N}$, $C \cap N = \varnothing$, and denote the customer set of carrier $i \in N$ by $C_i \subset C$ such that $C_i \cap C_j = \varnothing$ for all $i, j \in N$ with $i \neq j$, i.e., every customer is served by only one carrier, and $C = \bigcup_{i\in N} C_i$.

We denote the finite non-empty set of possible locker locations by $L \subset \mathbb{N}$, with $C \cap L = \varnothing$ and $L \cap N = \varnothing$, and the cost of opening these lockers by $c = (c_j)_{j \in N}$. Note that these locations and costs are the same for all carriers. Moreover, we introduce the distance matrix $d = (d_{kj})_{j \in L, k \in C}$, with $d_{kj} \in \mathbb{R}_+$ the distance between customer $k \in C$ and location $j \in L$, and the vector of maximal customer distances by $D = (D_k)_{k \in C}$. Finally, the vector of profits is denoted by $p = (p_k)_{k \in C}$, with p_k the profit of customer k, obtained if at least one locker is located within distance $D_k \in \mathbb{R}_+$.
In this context, the carriers can also decide to collaborate instead of facing $|N|$ individual locker location problems. If they do so, they will face one single LLP. That means they need to decide together where to open lockers while considering C as the set of customers, L as the set of locker locations, c as the vector with costs for opening lockers, d as the distance matrix, D as the vector of maximal distances, and p as the profit vector of customers. We refer to this setting as a locker location (LL) situation and denote it by tuple $\theta = (N, L, (C_i)_{i \in N}, p, c, d, D)$. Moreover, we define the set of all LL situations by Θ.

Example 3. Consider an LL situation $\theta \in \Theta$ with $N = \{1, 2, 3\}$, $C = \{4, 5, 6, 7\}$, $L = \{8, 9, 10\}$ $C_1 = \{4\}$, $C_2 = \{5\}$, $C_3 = \{6, 7\}$, $p = (5, 4, 3, 4)$, $c = (5, 9, 5)$, $d = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, and $D = (2, 1, 2, 2)$. A graphical representation of this setting is given in Figure 3, where the customers of carrier 1 are shown in orange, the customer of carrier 2 in blue, and the customers of carrier 3 in green.

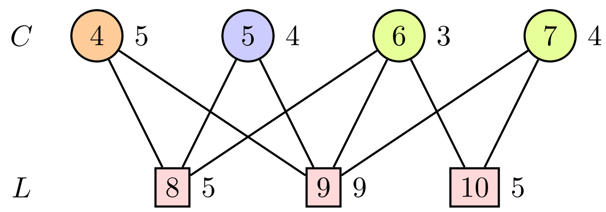


Figure 3: Graphical representation of the LL situation of Example 3.

In this situation, each carrier can solve its locker location problem individually. For instance, for carrier 1 the solution is opening no locker (with profit 0), opening location 8 (with a profit of $5 - 5 = 0$), or opening Location 9 (with a profit of $5 - 9 = -4$). As such, the best decision for carrier 1 would be to not open a locker. Following the same reasoning, we can conclude that carrier 2 should not open a locker either, while carrier 3 should open locker 10 and generate a profit of $7 - 5 = 2$. Hence, the sum of the individual profits equals 2. If the carriers decide to collaborate, they solve the joint LLP as presented in Example 1 in Section 3, which resulted in a profit of $7 (> 2)$. $\triangle$

These examples show that carriers can indeed increase their overall profit by collaborating. However, although this form of collaborating leads to a high overall profit, it is not clear a priori how to allocate this additional profit of $5 (= 7 - 2)$ to the three carriers. In the next section, we use cooperative game theory to address this profit allocation problem.

5 The locker location game

We now introduce the locker location (LL) game. Formally, we can associate each LL situation $\theta \in \Theta$ to a transferable utility LL game given by (N, v^θ). In this game, N denotes the set of carriers which we refer to as players, and $v^\theta : 2^N \to \mathbb{R}$ the characteristic value function associating a real value to any coalition $S \subseteq N$, i.e., a subset of carriers. This value function $v^\theta(S)$ reflects the optimal profit that can be achieved when the players in S decide to cooperate, i.e., the optimal value of the LLP restricted to the customers that are served by coalition S. This problem is given by:

$$
\begin{aligned}
LLP_S^\theta := \max \quad & \sum_{k\in C_S} p_k x_k - \sum_{j\in L} c_j y_j && && \text{(8a)}\\
\text{s.t.} \quad & x_k \le \sum_{j\in L_k} y_j && \forall\, k \in C_S && \text{(8b)}\\
& x_k \le 1 && \forall\, k \in C_S && \text{(8c)}\\
& x_k \in \mathbb{R}_+ && \forall\, k \in C_S && \text{(8d)}\\
& y_j \in \mathbb{N} && \forall\, j \in L, && \text{(8e)}
\end{aligned}
$$

.

where $C_S = \bigcup_{i\in S} C_i$ denotes the set of customers of coalition S. N is also called the grand coalition. We now formalize the LL game.

Definition 1. *For each LL situation $\theta \in \Theta$, the associated LL game is defined by:*

$$v^\theta(S) = \begin{cases} opt(LLP_S^\theta) & \text{if } S \subseteq N \\ 0 & \text{if } S = \varnothing. \end{cases} \tag{9}$$

Note that $v^\theta(N)$ represents the profit of the grand coalition, i.e., when all players collaborate. The central question in this section is whether $v^\theta(N)$ can be allocated in a stable way amongst the players. In the literature, it is common to address this question by investigating the core of the associated game, as the core represents the set of allocations for which no individual player, nor a group of players (i.e., coalition) has incentives to break from the grand coalition (Guajardo and Rönnqvist, 2016). The core is defined as the following:

$$\mathscr{C}(N, v^\theta) = \Big\{ u \in \mathbb{R}^N \mid \sum_{i\in N} u_i = v^\theta(N), \quad \sum_{i\in S} u_i \ge v^\theta(S)\ \forall\, S \subseteq N,\, S \ne \varnothing \Big\}.$$

If such allocation exists, the core is considered to be non-empty. Otherwise, the core is empty. The following example illustrates a case with a non-empty core.

Example 4. Reconsider the LL situation θ of Example 3. The values of the characteristic function v^θ are represented in Table 2.

Table 2: Coalitional values of the cooperative game

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v^\theta(S)$	0	0	(3+4)-5 = 2	(5+4)-5 = 4	(5+3+4)-9 = 3	(4+3+4)-9 = 2	7

For this example, we can verify that $x = (1, 3.5, 2.5)$ is a core allocation. Hence, the core is non-empty. △

There also exist LL games for which the core is empty, as illustrated in example 5.

Example 5. Let $\theta \in \Theta$ with $N = \{1,2,3\}$, $C = \{4,5,6\}$, $L = \{7,8,9\}$, $C_1 = \{4\}$, $C_2 = \{5\}$, $C_3 = \{6\}$, $p = (1,1,1)$, $c = (1,1,1)$, $d = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, and $D = (1,1,1)$. A graphical representation of the corresponding LLP-induced graph is presented in Figure 4.
The coalitional values for (N, v^θ) are presented in Table 3.

Table 3: Coalitional values of the cooperative game of Figure 4.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v^\theta(S)$	0	0	0	1	1	1	1

Suppose $\mathscr{C}(N, v^\theta) \ne \emptyset$. Let $u \in \mathscr{C}(N, v^\theta)$. By stability and efficiency, we know that $u_i \le v^\theta(N) - v^\theta(N\backslash\{i\})$ for all $i \in N$. Consequently, we have $u_i \le 0$ for all $i \in N$. This conflicts with efficiency, i.e., $u_1 + u_2 + u_3 \le 0 < 1 = v^\theta(N)$. Hence, $\mathscr{C}(N, v^\theta) = \varnothing$. △

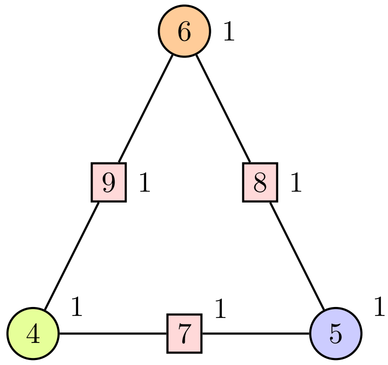


Figure 4: Graphical illustration of the LLP-induced graph of Example 5.

We now investigate under which conditions the core of our LL game is non-empty. As a first step, we will explore in this context whether LL games are superadditive. Superadditivity states that two groups of players together can always generate at least the profit they could generate without collaborating. For 2-player games, this property is a sufficient condition for core non-emptiness. The next proof shows that this property holds for our LL game.

Theorem 3. *LL games are superadditive.*

Proof. Let $\theta = (N, L, (C_i)_{i\in N}, p, c, d, D)$ be a LL situation and (N, v^θ) be the associated LL game. Moreover, let $S, T \in 2^N \backslash \{\varnothing\}$ be two coalitions such that $S \cap T = \varnothing$. Finally, let (x^S, y^S) be an optimal solution for LLP_S, and (x^T, y^T) be an optimal solution for LLP_T. For coalition $S \cup T$, we propose the merged strategy $(x^{S\cup T}, y^{S\cup T})$ where

$$x_k^{S\cup T} = \begin{cases} x_k^S & \text{if } k \in C_S \\ x_k^T & \text{if } k \in C_T \end{cases} \quad , \quad y_j^{S\cup T} = \max\{y_j^S, y_j^T\}.$$

We now show that $\big((x^{S\cup T}), (y^{S\cup T})\big)$ is a feasible solution of $LLP_{S\cup T}$. First, observe that

$$x_k^{S\cup T} = x_k^S \leq \sum_{j\in L_k} y_j^S \leq \sum_{j\in L_k} y_j^{S\cup T}\ \forall\, k \in C_S, \text{ and } x_k^{S\cup T} = x_k^T \leq \sum_{j\in L_k} y_j^T \leq \sum_{j\in L_k} y_j^{S\cup T}\ \forall\, k \in C_T.$$

Consequently, we have

$$x_k^{S\cup T} \leq \sum_{j\in L_k} y_j^{S\cup T} \quad \forall\, k \in C_{S\cup T}.$$

Secondly,

$$x_k^{S\cup T} = x_k^S \leq 1\ \forall\, k \in C_S, \text{ and } x_k^{S\cup T} = x_k^T \leq 1\ \forall\, k \in C_T.$$

Consequently, we have

$$x_k^{S\cup T} \leq 1 \quad \forall\, k \in C_{S\cup T}.$$

Thus, $(x^{S\cup T}, y^{S\cup T})$ is a feasible solution of $LLP_{S\cup T}$. With $\sum_{k\in C_{S\cup T}} p_k x_k^{S\cup T} - \sum_{j\in L} c_j y_j^{S\cup T}$ being the value of the merged solution $(x^{S\cup T}, y^{S\cup T})$, we observe the following inequalities:

$$\begin{aligned} v(S) + v(T) &= \sum_{k\in C_S} p_k x_k^S - \sum_{j\in L} c_j y_j^S + \sum_{k\in C_T} p_k x_k^T - \sum_{j\in L} c_j y_j^T = \\ &= \sum_{k\in C_S} p_k x_k^S + \sum_{k\in C_T} p_k x_k^T - \sum_{j\in L} c_j (y_j^S + y_j^T) \leq \sum_{k\in C_{S\cup T}} p_k x_k^{S\cup T} - \sum_{j\in L} c_j \max\{y_j^S, y_j^T\} = \\ &= \sum_{k\in C_{S\cup T}} p_k x_k^{S\cup T} - \sum_{j\in L} c_j y_j^{S\cup T} \leq \mathrm{opt}(LLP_{S\cup T}) = v(S\cup T). \end{aligned}$$

We conclude that the LL game is superadditive. □

A sufficient condition for core non-emptiness is when LLP_N can be solved efficiently. To prove this result, we first need to introduce some new definitions. Let $S \subseteq N, S \neq \varnothing$, and let $RLLP_S$ be the linear relaxation of LLP_S with $y_j \in \mathbb{R}_+\ \forall\, j \in L$. We use $RLLP_S$ to prove that the core is non-empty if $\mathrm{opt}(LLP_N) = \mathrm{opt}(RLLP_N)$, i.e., LLP_N can be solved efficiently.

Theorem 4. *LL games have a non-empty core if* $opt(LLP_N) = opt(RLLP_N)$.

Proof. Let $\theta = (N, L, (C_i)_{i\in N}, p, c, d, D)$ be an LL situation for which $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$, and (N, v^θ) be the associated LL game. Let $S \subseteq N$, $S \neq \varnothing$. Now, we introduce the dual of $RLLP_S$, referred to as $DRLLP_S$, given by the following:

$$
\begin{aligned}
DRLLP_S := \min \quad & \sum_{k\in C_S} \beta_k && && \text{(10a)}\\
\text{s.t.} \quad & \alpha_k + \beta_k \geq p_k && \forall k \in C_S && \text{(10b)}\\
& \sum_{k\in \hat{C}_j \cap C_S} \alpha_k \leq c_j && \forall j \in L && \text{(10c)}\\
& \alpha_k, \beta_k \geq 0 && \forall k \in C_S && \text{(10d)}
\end{aligned}
$$

where $\hat{C}_j = \{k \in C \mid d_{kj} \leq D_k\}$ is the set of customers that can be served by locker $j \in L$. We notice that $DRLLP_S$ is feasible for $\alpha_k = 0$ and $\beta_k = p_k$ for all $k \in C_S$. The problem is also bounded from below by 0. Hence, $RLLP_S$ and $DRLLP_S$ are both feasible and bounded. Then, by the strong duality theorem (see, Schrijver (1998, p. 90)), we have $\text{opt}(RLLP_S) = \text{opt}(DRLLP_S)$.
Let $\alpha = (\alpha_k)_{k\in C}$, $\beta = (\beta_k)_{k\in C}$ be the variables such that (α, β) is an optimal solution of $DRLLP_N$. For coalition S and an optimal solution (α, β) of $DRLLP_N$, we construct the restricted solution $((\alpha_k^S)_{k\in C_S}, (\beta_k^S)_{k\in C_S})$, or, shortly, (α^S, β^S), such that

$$
\begin{aligned}
\alpha_k^S &= \alpha_k \quad \forall\, k \in C_S,\\
\beta_k^S &= \beta_k \quad \forall\, k \in C_S.
\end{aligned}
$$

We will prove that (α^S, β^S) is feasible for $DRLLP_S$. Recall that for the $DRLLP_N$, it holds that

$$\alpha_k + \beta_k \geq p_k \quad \forall\, k \in C.$$

Consequently, for all $k \in C_S \subseteq C$, we have

$$\alpha_k^S + \beta_k^S = \alpha_k + \beta_k \geq p_k.$$

Moreover, for all $j \in L$,

$$\sum_{k\in \hat{C}_j \cap C_S} \alpha_k^S \leq \sum_{k\in \hat{C}_j} \alpha_k^S = \sum_{k\in \hat{C}_j} \alpha_k \leq c_j \quad \forall j \in L,$$

where the inequality holds since $\alpha_k^S \geq 0$ for all $k \in S$. Finally, $\alpha_k^S, \beta_k^S \geq 0$ for all $k \in C_S$, since $\alpha_k, \beta_k \geq 0$ for all $k \in C$. Hence, (α^S, β^S) is a feasible solution of $DRLLP_S$.
Now, consider an allocation vector $u = (u_i)_{i\in N}$, with

$$u_i = \sum_{k\in C_i} \beta_k \quad \forall\, i \in N.$$

We will show that vector u belongs to the core. Let coalition $S \subseteq N, S \neq \varnothing$. Then,

$$
\begin{aligned}
\sum_{i\in S} u_i &= \sum_{i\in S}\sum_{k\in C_i} \beta_k && \text{by definition of } u_i\\
&= \sum_{k\in C_S} \beta_k && C_i \text{ being a partition of } C_S\\
&= \sum_{k\in C_S} \beta_k^S && \text{by definition of } (\alpha^S, \beta^S)\\
&\geq \text{opt}(DRLLP_S) && \text{by feasibility of } (\alpha^S, \beta^S) \text{ for } DRLLP_S\\
&= \text{opt}(RLLP_S) && \text{by strong duality theorem}\\
&\geq \text{opt}(LLP_S) && \text{being } RLLP_S \text{ a relaxation of } LLP_S\\
&= v(S) && \text{by definition of } v.
\end{aligned}
\tag{11}
$$

Second,

$$
\begin{aligned}
\sum_{i\in N} u_i &= \sum_{i\in N}\sum_{k\in C_i} \beta_k && \text{by definition of } u_i \\
&= \sum_{k\in C} \beta_k && C_i \text{ being a partition of } C \\
&= \operatorname{opt}(DRLLP_N) && \text{by optimality of } (\alpha,\beta) \text{ for } DRLLP_N \\
&= \operatorname{opt}(RLLP_N) && \text{by strong duality theorem} \\
&= \operatorname{opt}(LLP_N) && \text{by hypothesis} \\
&= v(N) && \text{by definition of } v.
\end{aligned} \tag{12}
$$

So, u is an efficient and stable vector. Hence, $u \in \mathscr{C}(N, v^\theta)$, which concludes the proof. □

Recall from Theorem 1 that if the connection matrix M^{LLP} of LLP is totally unimodular, then RLLP finds an optimal solution of the LLP. Thus, an LL game for which LLP_N has a totally unimodular connection matrix has a non-empty core.

Corollary 1. *LL games for which the connection matrix of LLP_N is totally unimodular have a non-empty core.*

Also, recall from Theorem 2 that every LLP for which the induced graph G^{LLP} is a tree can be solved by RLLP. Thus, we also formulate the following corollary that merges Theorems 2 and 4.

Corollary 2. *LL games for which the induced graph G^{LLP} is a tree have a non-empty core.*

6 Numerical experiment

In the previous section, we provided a sufficient condition for core non-emptiness. In particular, we showed that if the optimal value of an LLP and its linear programming relaxation coincide, then the core of the associated LL game is non-empty. By using Theorem 2, this condition applies to LL games for which the induced graph of the grand coalition is a tree. This section will investigate how much we can generalize core non-emptiness when we relax the tree structure. For this purpose, we conduct several numerical experiments to study the effect of different LL situations on core non-emptiness. In the following, we first explain how we construct the instances used for our experiments, specifying the parameters and the related LL situations before presenting the results of these experiments in section 6.2.

6.1 Instance design

The instances for our experiments are generated considering a 100×100 square, in which we randomly position customers, using two different types of probability distributions; a uniform distribution and a triangular distribution. For the uniform distribution, each coordinate of the customer's position follows a uniform distribution between 0 and 100. This distribution reflects a Manhattan-style neighborhood, where the population is equally distributed over the neighborhood. In contrast, in the case of the triangular distribution, each coordinate of the customer's position follows a triangular distribution from 0 to 100 with mode 50, representing a setting where the population is concentrated in the city center and gets sparser around it. In Figure 5, we provide an example visualizing the difference between the two distributions for instances of 20 customers.

After positioning the customers in the square, each customer is assigned to exactly one carrier. The number of carriers is set to a value between three and six, as this seems reasonable for the number of delivery companies working in a city. Two types of customer assignments to carriers are considered:

- **Random assignment:** In this case, every customer is assigned to one carrier randomly with equal probability.
- **Clustered assignment:** Every customer gets assigned according to a clustering approach. For this purpose, we run a K-means clustering algorithm (with the standard parameters proposed by *scikit-learn* library in Python, without fine-tuning the parameters) splitting the square of 100×100 into sub-areas corresponding to the number of carriers, allocating each sub-area to one carrier. We

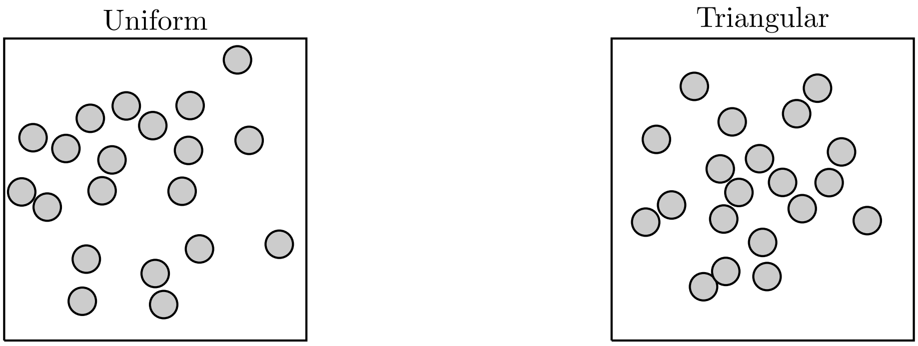


Figure 5: Two examples of distribution of customers: uniform to the left, triangular to the right.

then assign customers to carriers according to a predefined cluster density. In doing so, each customer gets a random integer between 0 and 100. If this number is less or equal to the cluster density, the customer is assigned to the carrier of its sub-area. Otherwise, the customer is allocated to one of the other carriers with equal probability. If the cluster density is 100%, all customers of a certain sub-area are assigned to the same carrier. Note that a 0% clustered assignment is not equal to a random assignment, as in the former the carrier associated with the sub-area is not serving any customer.

The number of locker locations is then set proportional to the number of customers. For this purpose, we consider proportions of 5%, 7.5%, or 10%, setting the number of locker locations equal to the closest integer from the product of this proportion parameter and the number of customers. This results in five locker locations for an instance with 100 customers and a proportion of 5%. These locker locations are then positioned randomly in the square with uniform probability.
Distances between customers and locker locations are calculated based on the Manhattan distance, while the customer-related maximum distance threshold, to identify whether the customer wants to use a locker at a certain location, is randomly extracted from a uniform distribution between $0.75A$ and $1.25A$, where A denotes the mean of the distribution. For example, if $A = 10$, a customer gets a maximum distance parameter with uniform probability in the $[7.5, 12.5]$ range. If a customer uses a locker, we add an edge between this customer and the specific locker. In Figure 6, we visualize the assignment of customers to both carriers (using colors) and lockers (using connecting arcs) for a given instance. We present an instance with 20 customers which have been located using a uniform distribution, three carriers, and five locker locations. Note that each customer is assigned to only one carrier (indicated by blue, green, or orange) and that customers are only connected to lockers in their neighborhood.

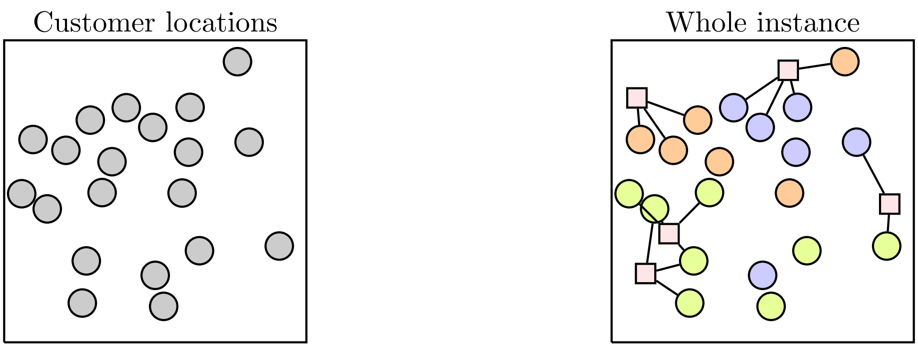


Figure 6: Example of an instance after assigning carriers to customers and showing lockers locations.

The profits for serving customers are randomly extracted for every customer from a normal distribution with a mean of 10 and a standard deviation of 1. Similarly, the cost of locating a locker in a given position follows a normal distribution, with mean $10r_{c/p}$ and standard deviation $r_{c/p}$, where $r_{c/p}$ represents the cost-to-profit ratio. This cost-to-profit ratio corresponds to the average number of customers a locker should serve to recover the cost of the locker, i.e. if the cost of the locker can be recovered by serving 20

customers, this corresponds to a cost-to-profit ratio of 20.
Table 4 presents an overview of the configurations used to generate the instances for our experiments. For each combination of these configurations, we generate 20 unique instances. The total number of instances we consider in all experiments together is $403,200$.

Table 4: Parameter configurations used for our experiments.

Experimental Parameter	*Values*
Customer distribution	Triangular, Uniform
Number of customers	150, 300, 450
Number of carriers	3, 4, 5, 6
Proportion of lockers	5%, 7.5%,10%
Cost-to-profit ratio	1, 2, 5, 10, 15, 25, 35
Average maximal distance	30, 60, 90, 120, 150
Cluster density	Random, 20%, 40%, 60%, 80%, 100%

6.2 Results

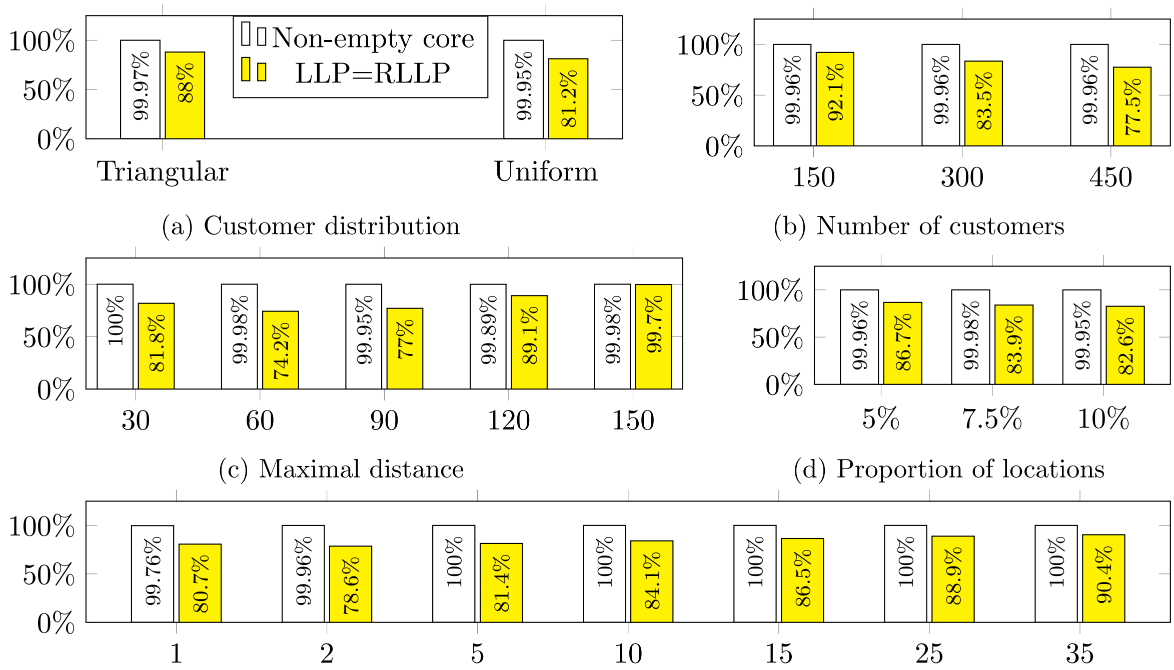
This section presents the results from our experiments using the instances described in Section 6.1, focusing in particular on the impact of different cluster densities and cost-to-profit ratios on the emptiness of the core. In this context, we also study the likelihood of $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$, i.e., the condition for which we have proven that the core is non-empty as shown in Theorem 4.

6.2.1 General case

In this first experiment, we consider the 50,400 (i.e., $20 \times (2 \times 3 \times 4 \times 3 \times 7 \times 5 \times 1)$ instances with random customer assignments. The results from these experiments are shown in Figure 7, where the white bars represent the frequency of non-empty core instances, while the yellow bars represent the frequency of $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$ instances.
The figure shows that almost all instances have a non-empty core, while only 20 out of the 50,400 instances have an empty core. From a practical perspective, this is a positive sign for collaboration as it indicates that it is possible to find a stable profit allocation amongst the carriers in most cases. Most of these empty core instances relate to a cost-to-profit ratio of 1 or 2 (see Figure 7e), which suggests that collaboration is less sensible in situations where the cost of lockers can be recovered easily, i.e., where serving only a few customers is sufficient to open a locker.
Analyzing the results with regards to $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$, we can see that the frequency of $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$ is lower for the uniform distribution. Moreover, we observe that the frequency of $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$ instances decreases as the number of customers and locations increases. This is logical as the likelihood of having only binary decision variables, which is a requirement for $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$, decreases as the number of variables (corresponding to customers and lockers) increases. In addition, we observe that the frequency of $\text{opt}(LLP_N) = \text{opt}(RLLP_N)$ generally increases with higher values of the maximal distance as well as the cost-to-profit ratio. A possible explanation may be that in the case of a high maximal distance, opening only the cheapest locker to cover all customers seems to suffice. In contrast, in the case of a high cost-to-profit ratio, it may be best to open no locker at all. Overall, these findings indicate that a core allocation can be identified (efficiently) for most instances.

6.2.2 Impact of clustered customer assignment

In practice, carriers may dominate certain areas, operating more or less locally. In this section, we conduct several experiments with clustered customer assignment, varying the cluster density. In this context, we compare the random assignment with multiple clustered assignments with different values for cluster density, namely 20%, 40%, 60%, 80%, and 100%. As a result, we analyze $20 \times (2 \times 3 \times 4 \times 3 \times 7 \times 5 \times 6) = 302,400$



(a) Customer distribution

(b) Number of customers

(c) Maximal distance

(d) Proportion of locations

(e) Cost-to-profit ratio

Figure 7: Results for instances with random customer assignment, where the white bar represents the frequency of non-empty core instances, and the yellow bar the frequency of opt(LLP)=opt($RLLP$) instances.

instances in this experiment. In Figure 8, we present the results for these instances, focusing again on the frequencies of non-empty core instances and opt(LLP_N) = opt($RLLP_N$).

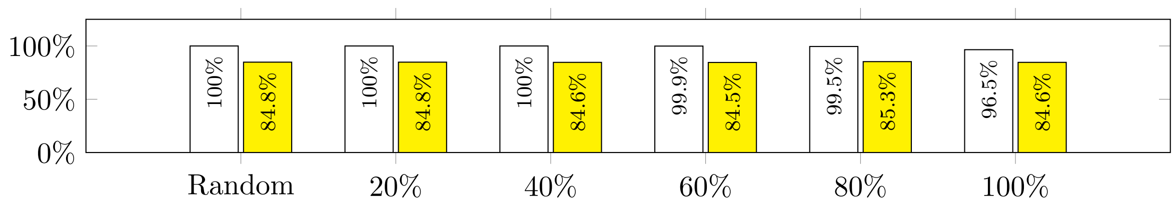
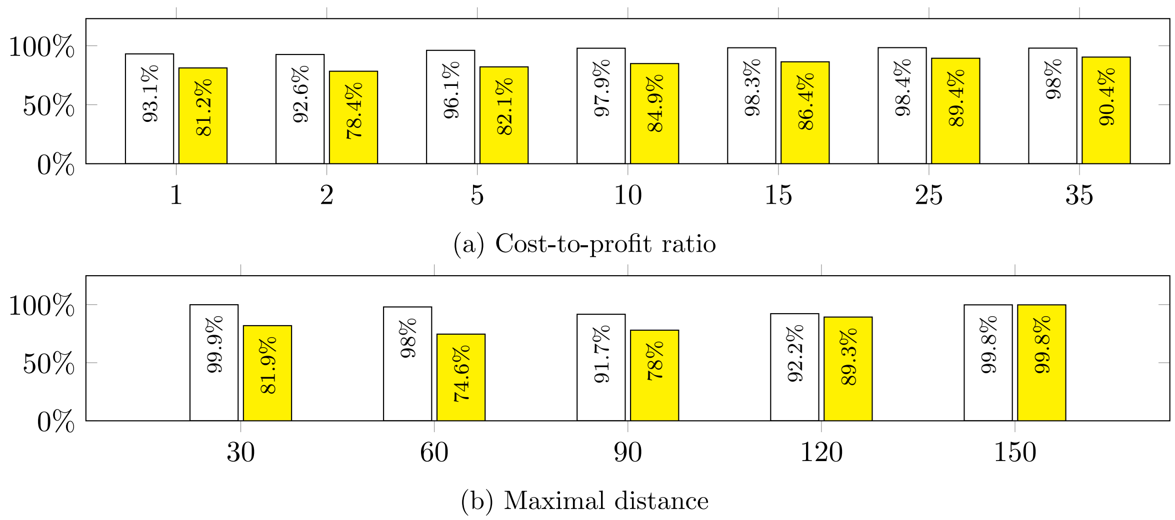


Figure 8: Results on the influence of cluster density in the second experiment.

The figure shows that a high cluster density leads to a lower frequency of non-empty core instances. More specifically, we observe that for cluster density values lower than or equal to 40%, which includes the random customer assignment, the frequency of non-empty core instances is equal to 100%, while for a cluster density of 100%, the frequency of non-empty core instances reduces to 96.5%. This might indicate that a geographically clustered assignment of customers to carriers may harm collaboration. At the same time, the frequency of opt(LLP_N) = opt($RLLP_N$) seems relatively stable over the various cluster densities, suggesting that cluster density does not have a strong effect on the occurrence of this condition.

6.2.3 Impact of high cluster density in combination with a low cost-to-profit ratio

In the first two experiments, we see that a high cluster density and a low cost-to-profit ratio may negatively affect core non-emptiness. In this final part of our analysis, we aim to investigate this in greater detail by focusing only on instances with 100% cluster density while still observing the values of the cost-to-profit ratio proposed in the previous experiments. Having only 100% cluster density reduces the number of instances again to 50,400 instances, as in the first experiment.



(a) Cost-to-profit ratio

(b) Maximal distance

Figure 9: Results on cost-to-profit ratio and maximal distance with 100% cluster density.

The results from this analysis (shown in Figure 9) indicate that a combination of a 100% cluster density and low cost-to-profit ratios reduces the number of instances with a non-empty core. For very low values of the cost-to-profit ratio (i.e., 1 or 2), this results in around 7% of the instances having an empty core. Analyzing the results for these instances for different values of the maximal distance, we can observe a similar trend for maximal distance values between 90 and 120. In this case, about 8% of the instances have an empty core. Despite this significant increase in empty core instances, these experiments indicate that a stable profit allocation amongst carriers is still possible in more than 90% of the considered instances. As such, our analysis demonstrates that a stable profit allocation is possible for instances with an underlying tree structure, and in most instances violating the tree-structure property. This is a promising indication for practice carriers with plans to jointly open lockers.

7 Conclusion

In this paper, we studied a context where carriers intend to collaborate and together position parcel lockers to serve their customers. For this purpose, we introduced the locker location problem (LLP), which involves positioning parcel lockers to generate a profit from serving the customers while facing location-dependent opening costs. Studying the properties of this problem, we have observed that the optimal value of the LLP coincides with a linear programming relaxation when the LLP-induced graph is a tree.

Based on the LLP, we formulate the LL game using cooperative game theory, where multiple carriers seek to collaborate to improve the total joint profit, i.e., the profits from the customers minus the costs of opening lockers. Focusing on profit allocations in the core, representing the set of allocations for which it is unprofitable for any group of carriers to leave the collaboration, we prove that the core is non-empty when the optimal value of the LLP and a linear programming relaxation coincide.

In addition, we generate a large set of instances representing different LL situations to conduct a series of numerical experiments, studying the frequency of LL games with a non-empty core. The results of these experiments show that with generic parameter initialization, the core is non-empty in more than 99.9% of the cases. Analyzing the impact of specific instance configurations, we see that a geographical separation of the carriers, low locker costs, and the considered maximal distance interval can harm collaborations.

Acknowledgments

The authors thank the Eindhoven Artificial Intelligence Systems Institute (EAISI) for supporting and funding the project. The first author also mentions the colleague and friend Riccardo Lo Bianco who gave inspiration on the geographical separation of carriers as a cause of no collaboration.

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