(ϵ-)EFFICIENCY IN FRACTIONAL VECTOR OPTIMIZATION

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Abstract. The issue of characterizing completely efficient (Pareto) solutions to a fractional vector (multiobjective or multicriteria) minimization problem, where the involved functions are convex, has not been addressed previously. Thanks to an earlier characterization of weak efficiency in difference vector optimization by El Maghri, we get a vectorial necessary and sufficient condition given in terms of both strong (Fenchel) and weak (Pareto) vector ϵ-subdifferentials that completely characterizes the exact or approximate weak efficiency in fractional multiobjective optimization. Moreover, this result applies not only for unconstrained problems but also for convex constrained problems, where in the first case no assumption of convexity is required, while in the second case only the numerators need to be convex. When the fractional problem is to minimize the ratios of convex functions by concave functions, simpler vectorial characterizations for exact or approximate proper or weak efficiency are also developed. Finally, application to the particular case of linear fractional multiobjective programs also provides new results.

Keywords. Vector optimization. Fractional objective. Pareto ϵ-solutions. Efficiency conditions. Vector ϵ-subdifferentials.

1 Introduction

In the field of global scalar optimization for nonconvex problems, significant results have been obtained by taking advantage of special structures of objectives as well as constraints, see, e.g., [11, 15, 17, 18, 19, 30]. It has been shown that important classes of these problems can be mathematically converted to be part of the DC (difference of convex maps) optimization class (see, e.g., the books [19, 30]).

During the last decades, important studies on exact or approximate solutions to vector, multiobjective or multicriteria optimization problems (MOP) have aroused great interest both in theory and computation and in practice; see, for instance, [2, 6, 13, 21, 27, 28, 32]. It is well known that the relevant solutions to these problems are those which are efficient in the sense of Pareto. Fractional multiobjective optimization problems (FMOP) particularly have found many important applications in decision making problems, especially, in the fields of economics, management sciences and engineering design [28]. These problems have been widely studied by a large number of authors suggesting several important results on certain properties of the different types of exact or approximate efficient points, including efficiency conditions, duality, etc. One can speak here about an important development that research in multicriteria optimization has had in this direction, and the literature is vast on this subject; see, for instance, [1, 3, 4, 14, 20, 25, 26, 29, 31], to name just a few.

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The most used approach to treat a given FMOP consists in transforming the FMOP by a parametric approach into a nonfractional MOP, which in turn, is converted into a single-objective program using another parametric approach such as the scalarization technique. The scalarized problem can then be solved using standard scalar optimization tools. The main parametric approach that converts a fractional problem to a nonfractional one was introduced in [5]. This technique has been later extended in [29] to multiobjective programs dealing with both efficiency and proper efficiency, and more recently, in [14] and [22] respectively dealing with ϵ-efficiency and weak ϵ-efficiency.

The main objective of our paper is twofold: 1° to establish a complete characterization of ϵ-efficiency (necessary and sufficient Pareto ϵ-optimality condition) for FMOPs; 2° to give the characterization in terms of vector (Pareto) ϵ-subdifferentials. To our knowledge, the only situation where the first point has been addressed in the literature corresponds to FMOPs which minimize the ratios of convex functions by concave functions; otherwise, the question remained open even in the case of the exact ($\epsilon = 0$) efficiency. For the second point, among all the papers cited above and others (not mentioned here), one can find several types of exact or approximate efficiency conditions given only in scalar forms, generally, in terms of exact or approximate scalarized subdifferentials, particularly, of the Fenchel or Clarke type.

In fact, our investigations in this work are conducted in three steps. In the first step, we consider a general unconstrained FMOP where no convexity is assumed. Using a parametric approach transforming the FMOP into a nonfractional MOP, we derive, via a recent result by El Maghri [7], a condition completely characterizing the weakly ϵ-efficient solutions to the FMOP. The condition is given in terms of strong (Fenchel) and weak (Pareto) vector ϵ-subdifferentials associated respectively with the denominator and numerator objective functions. We also show in this step that the previous result is still applicable for convex constrained FMOPs whose numerator objective functions are convex. In a second step, the class of convex constrained FMOPs minimizing ratios of convex functions by concave functions is considered. First, we extend the parametric approach that converts a fractional problem into a nonfractional problem [29] to be able to handle approximate properly efficient solutions as well. In fact, this approach allows to transform the nonconvex FMOP into a convex MOP, and then, to apply El Maghri's vector characterization [8] of the weakly and properly ϵ-efficient solutions given in terms of vector ϵ-subdifferentials. To our knowledge, such a result established for this particular class of FMOP is still new even for obtaining exact ($\epsilon = 0$) weakly or properly Pareto minima. In the third and last step, we end with an application to linear fractional multiobjective programs for which we obtain an explicit vectorial condition, given only in terms of the data, completely characterizing the weakly and properly ϵ-efficient solutions.

2 General preliminary background

Let us first consider the following general vector optimization problem:

$$\text{VOP:} \quad \epsilon\text{--}\operatorname*{Min}_{x \in S} \; F(x)$$

where $F : X \supseteq S \to Y \sqcup \{+\infty\}$, X and Y are real topological vector spaces such that the space Y is separated and equipped with a convex cone Y_+ nontrivial, that is, Y_+ does not coincide with its lineality $l(Y_+) := Y_+ \sqcap -Y_+$. When this latter is null, the cone is said to be pointed. The topological interior $\operatorname{int} Y_+$ of Y_+ sometimes will be required to be nonempty.

The polar cone Y_+^* of Y_+ is the set of $\lambda \in Y^*$ (topological dual) such that $\lambda(Y_+) \subseteq \mathbb{R}_+$, while the strict polar cone $(Y_+^*)^\circ$ is the set of $\lambda \in Y^*$ such that $\lambda(Y_+ \backslash l(Y_+)) \subseteq \mathbb{R}_+ \backslash \{0\}$. The cone Y_+ induces in Y preorder relations: $y \leq_{Y_+} y' \Leftrightarrow y' - y \in Y_+$, $y <_{Y_+} y' \Leftrightarrow y' - y \in \operatorname{int} Y_+$, and, $y \lneq_{Y_+} y' \Leftrightarrow y' - y \in Y_+ \setminus l(Y_+)$. Opposite relations are noted like $\nleq_{Y_+}$, $\nless_{Y_+}$ and $\not\lneq_{Y_+}$. It is adjoined to Y the maximal element $+\infty$ so that $y \leq_{Y_+} +\infty$ for all $y \in Y \sqcup \{+\infty\}$ and obeying (by convention) to the operations $y \pm (+\infty) = +\infty$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and $0 \cdot (+\infty) = 0$.

There are four kinds of Pareto ϵ-solutions to VOP, for a given vector parameter $\epsilon \in Y$, namely, strongly ϵ-efficient solutions, ϵ-efficient solutions, weakly ϵ-efficient solutions and properly ϵ-efficient solution. The sets of such solutions are respectively defined by

$$E_\epsilon^s(F,\ S,\ Y_+) = \{\bar{x} \in S \sqcap \operatorname{dom} F :\ \forall x \in S,\ F(x) \geq_{Y_+} F(\bar{x}) - \epsilon\},$$
$$E_\epsilon^e(F,\ S,\ Y_+) = \{\bar{x} \in S \sqcap \operatorname{dom} F :\ \forall x \in S,\ F(x) \not\lneq_{Y_+} F(\bar{x}) - \epsilon\},$$
$$E_\epsilon^w(F,\ S,\ Y_+) = E_\epsilon^e(F,\ S,\ \operatorname{int} Y_+ \sqcup \{0\})$$
$$E_\epsilon^p(F,\ S,\ Y_+) = \bigsqcup_{\widehat{Y}_+ \in \mathcal{C}(Y_+)} E_\epsilon^e(F,\ S,\ \widehat{Y}_+),$$

where $\mathcal{C}(Y_+) := \{\widehat{Y}_+ \subsetneq Y$ convex cone : $Y_+ \setminus l(Y_+) \subseteq \operatorname{int} \widehat{Y}_+\}$ and $\operatorname{dom} F := \{x \in X : F(x) \in Y\}$ (the effective domain). Note that we are working with Henig's [16] definition as being the most general concept of proper efficiency. The reader can consult [9] for more details on these sets. To simplify the presentation, we will denote by $E_\epsilon^\sigma(F, S, Y_+)$ the set of ϵ-σ-efficient points with respect to the choice of $\sigma \in \{s, e, w, p\}$. Let us notice that strong (ϵ-)efficiency is nothing more than classical (ϵ-)optimality, 0-σ-efficiency reduces to σ-efficiency: $E^\sigma(F, S, Y_+) = E_0^\sigma(F, S, Y_+)$, and, it is easily seen that

$$E_\epsilon^\sigma(F, S, Y_+) \neq \emptyset \quad \Longrightarrow \quad \epsilon \nless_{Y_+}^\sigma 0,$$

where the order symbol $<_{Y_+}^\sigma$ is a unified notation depending of the choice of $\sigma \in \{s, e, w, p\}$:

$$y <_{Y_+}^\sigma y' \quad \Leftrightarrow \quad \begin{cases} y \not\geq_{Y_+} y' & \text{if } \sigma = s, \\ y <_{Y_+} y' & \text{if } \sigma = w, \\ y \lneq_{Y_+} y' & \text{if } \sigma \in \{e, p\}. \end{cases}$$

In order to derive ϵ-efficiency conditions in terms of vector subdifferentials, the VOP may be penalized by

$$\epsilon\text{--}\operatorname*{Min}_{x \in X} F(x) + \delta_S^v(x)$$

in the sense that

$$E_\epsilon^\sigma(F, S, Y_+) = E_\epsilon^\sigma(F + \delta_S^v, X, Y_+), \tag{1}$$

where δ_S^v is the vectorial indicator function of the feasible set S defined by

$$\begin{array}{rccl} \delta_S^v : & X & \to & Y \sqcup \{+\infty\} \\ & x & \mapsto & \delta_S^v(x) = \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{else.} \end{cases} \end{array}$$

This vector penalty function has further adequate properties for this purpose (see [9]).

The vector ϵ-σ-subdifferential is defined in the Pareto sense depending on the choice of $\sigma \in \{s, e, w, p\}$:

$$\partial_\epsilon^\sigma F(\bar{x}) = \{A \in L(X,Y) : \ \bar{x} \in E_\epsilon^\sigma(F - A, X, Y_+)\},$$

where $L(X,Y)$ is the space of linear continuous operators from X to Y. It is taken by convention that $\partial_\epsilon^\sigma F(\bar{x}) = \emptyset$ if $\bar{x} \notin \operatorname{dom} F$. From (1), it is immediate that

$$\bar{x} \in E_\epsilon^\sigma(F, S, Y_+) \quad \Longleftrightarrow \quad 0 \in \partial_\epsilon^\sigma(F + \delta_S^v)(\bar{x}). \tag{2}$$

So, some Pareto ϵ-subdifferential calculus rules ([8, 9]) will be recalled below. Let us first point out that $\partial_0^\sigma F$ is the exact (Pareto) σ-subdifferential ($\partial^\sigma F$) studied in [10], that $\partial_\epsilon^s F(\bar{x})$ is nothing more than the ordinary (Fenchel) ϵ-subdifferential extended to the vector case, and that, in scalar case ($Y = \mathbb{R}$, $Y_+ = \mathbb{R}_+$), all these sets coincide with the classical ϵ-subdifferential ($\partial_\epsilon^s F$) usually denoted by $\partial_\epsilon F$. It is also easily seen that in the finite-dimensional space ($Y = \mathbb{R}^r$, $Y_+ = \mathbb{R}^r_+$), the strong ϵ-subdifferential of $F = (f_1, \dots, f_r)$, for $\epsilon = (\varepsilon_1, \dots, \varepsilon_r) \geq_{Y_+} 0$, reduces to

$$\partial_\epsilon^s F(\bar{x}) = \prod_{i=1}^r \partial_{\varepsilon_i} f_i(\bar{x}). \tag{3}$$

It was shown in [9] that, for $\sigma \in \{s, w, p\}$, the ϵ-σ-subdifferential of any linear operator $A \in L(X,Y)$ is constant:

$$\partial_\epsilon^\sigma A(\bar{x}) = \partial^\sigma A(\bar{x}) \qquad \forall \epsilon \not<_{Y_+}^\sigma 0, \tag{4}$$

where it is assumed that Y_+ must be closed in case of $\sigma = s$ and pointed in case of $\sigma = p$.

Recall first the following definitions.

Definition 1 The vector mapping $F : X \to Y \sqcup \{+\infty\}$ is said to be

(i) proper, if $\operatorname{dom} F \neq \emptyset$;

(ii) Y_+-convex, if $\forall x, x' \in X$, $\forall \alpha \in [0,1]$,
$$F(\alpha x + (1-\alpha)x') \leq_{Y_+} \alpha F(x) + (1-\alpha)F(x');$$

(iii) star Y_+-l.s.c, if $\forall \lambda \in Y_+^*$, $\lambda \circ F$ is lower semicontinuous (in short, l.s.c).

In what follows, $\Gamma(X, Y_+)$ will denote the set of mappings $F : X \to Y \sqcup \{+\infty\}$ proper and Y_+-convex, and, $\Gamma_0(X, Y_+)$ the set of mappings F in $\Gamma(X, Y_+)$ that are star Y_+-l.s.c. Note that $\Gamma_{(0)}(X, \mathbb{R}^r_+) = \prod_{i=1}^r \Gamma_{(0)}(X)$ the set of (l.s.c) proper convex functionals.

Definition 2 *[9]* The vector mapping $F : X \to Y \sqcup \{+\infty\}$ is said to be regular (resp. w-regular, p-regular) ε-subdifferentiable at $\bar{x} \in \operatorname{dom} F$ with $\varepsilon \geq 0$, if

$$\partial_\varepsilon(\lambda \circ F)(\bar{x}) = \bigsqcup_{\substack{\epsilon \in Y_+^\varepsilon, \\ \langle \lambda, \epsilon \rangle = \varepsilon}} \lambda \circ \partial_\epsilon^s F(\bar{x}) \qquad \forall \lambda \in Y_+^* \quad \big(\text{resp. } \forall \lambda \in Y_+^* \setminus \{0\}, \ \forall \lambda \in (Y_+^*)^\circ\big),$$

where $Y_+^0 = \{0\}$ and $Y_+^\varepsilon = Y_+$ if $\varepsilon > 0$, while, $\lambda \circ \partial_\epsilon^s F(\bar{x}) := \{\lambda \circ A \ : \ A \in \partial_\epsilon^s F(\bar{x})\}$.

Remark 1 When $Y = \mathbb{R}^r$, $Y_+ = \mathbb{R}^r_+$, we have $Y_+^* = \mathbb{R}^r_+$, $(Y_+^*)^\circ = \operatorname{int} \mathbb{R}^r_+$. So, by (3), ε-subdifferential regularity of $F = (f_1, \ldots, f_r)$ becomes exactly a well-known chain rule for (scalar) ε-subdifferentials which holds under Moreau–Rockafellar's conditions (see [9]). In scalar case, the concept is always fulfilled: $\partial_\varepsilon(\lambda f)(x) = \lambda \partial_{\frac{\varepsilon}{\lambda}} f(x)$ for all $\lambda \in]0, +\infty[$.

The following formula has been established in [8, see proof of Theorem 4.1, pp. 17]. Let $F : X \to Y \sqcup \{+\infty\}$, $H : X \to Z \sqcup \{+\infty\}$, where Z (as Y) is equipped with a convex cone Z_+, and, let $C \subseteq X$ be a convex set. Assume that one of the following two qualification conditions of the Moreau–Rockafellar (MR) and Attouch-Brézis (AB) type is satisfied:

$$(MR)_1 \quad \begin{cases} F \in \Gamma(X, Y_+),\ H \in \Gamma(X, Z_+),\ X \text{ and } Z \text{ separated locally convex,} \\ \exists\, x_0 \in C \sqcap \operatorname{dom} F \sqcap \operatorname{dom} H \ \text{ s.t.} \\ \delta^v_{-Z_+} \text{ is finite and continuous at } H(x_0). \end{cases}$$

$$(AB)_1 \quad \begin{cases} F \in \Gamma_0(X, Y_+),\ H \in \Gamma_0(X, Z_+),\ C \text{ and } Z_+ \text{ closed,} \\ X \text{ and } Z \text{ Fréchet spaces,} \\ \mathbb{R}_+[Z_+ + H(C \sqcap \operatorname{dom} F \sqcap \operatorname{dom} H)] \text{ is a closed vector subspace of } Z. \end{cases}$$

Then, for $\sigma \in \{p, w\}$, with Y_+ pointed as $\sigma = p$, $\forall \bar{x} \in \operatorname{dom} F \sqcap H^{-1}(-Z_+) \sqcap C$, $\forall \epsilon \not<^\sigma_{Y_+} 0$,

$$\partial^\sigma_\epsilon (F + \delta^v_{-Z_+} \circ H + \delta^v_C)(\bar{x}) = \bigsqcup_{\substack{\epsilon^1 \not<^\sigma_{Y_+} 0,\ \epsilon^2 \geq_{Y_+^\epsilon} 0, \\ \epsilon^1 + \epsilon^2 = \epsilon}} \ \bigsqcup_{\substack{\Lambda \in L_+(Z,Y), \\ -\epsilon^2 \leq_{Y_+} \Lambda(H(\bar{x})) \leq_{Y_+} 0}} \partial^\sigma_{\epsilon^1} (F + \Lambda \circ H + \delta^v_C)(\bar{x}) \quad (5)$$

where $L_+(Z, Y) := \{A \in L(Z, Y) : A(Z_+) \subseteq Y_+\}$ is the set of "positive" linear operators, while, $Y_+^0 := \{0\}$ and $Y_+^\epsilon := Y_+$ if $\epsilon \neq 0$.

It was also established in [8, see proof of Corollary 4.1, pp. 17–18] that if, in addition to the $(MR)_1$ or $(AB)_1$ condition, F and H are finite and continuous at some point of C, F (and resp. H) is σ-regular (resp. regular) ε-subdifferentiable at $\bar{x}$ ($\forall \varepsilon \geq 0$), then the formula (5) becomes more explicit:

$$\partial^\sigma_\epsilon (F + \delta^v_{-Z_+} \circ H + \delta^v_C)(\bar{x}) =$$

$$\bigsqcup_{\substack{\epsilon^i \geq_{Y_+^\epsilon} 0\ (i=1,3,5), \\ \epsilon^2 \geq_{Z_+^\epsilon} 0,\ \epsilon^4 \not<^\sigma_{Y_+} 0, \\ \sum_{i \neq 2} \epsilon^i = \epsilon}} \ \bigsqcup_{\substack{\Lambda \in L_+(Z,Y), \\ \epsilon^4 \not<^\sigma_{Y_+} \Lambda(\epsilon^2) \not<^\sigma_{Y_+} \epsilon^4, \\ -\epsilon^5 \leq_{Y_+} \Lambda(H(\bar{x})) \leq_{Y_+} 0}} \partial^s_{\epsilon^1} F(\bar{x}) + \Lambda \circ \partial^s_{\epsilon^2} H(\bar{x}) + N^v_{\epsilon^3}(\bar{x}, C) + Z_\sigma(X, Y) \qquad (6)$$

where $N^v_\epsilon(\bar{x}, C) = \{A \in L(X, Y) :\ \forall x \in C,\ A(x - \bar{x}) \leq_{Y_+} \epsilon\}$ is the vectorial ϵ-normal cone to C at $\bar{x}$, $Z_w(X, Y) := \{\Theta \in L(X, Y) : \exists \lambda \in Y_+^* \setminus \{0\},\ \lambda \circ \Theta = 0\}$ is the set of weakly "zerolike" operators, $Z_p(X, Y) := \{\Theta \in L(X, Y) : \exists \lambda \in (Y_+^*)^\circ,\ \lambda \circ \Theta = 0\}$ is the set of properly "zerolike" operators ($\Theta \sim_\sigma 0$).

The formula bellow established in [9, Theorem 4.2] expresses the gap between Pareto and Fenchel ϵ-subdifferentials of $F \in \Gamma(X, Y_+)$ σ-regular ε-subdifferentiable at $\bar{x}$ $(\forall \varepsilon \geq 0)$ for $\sigma \in \{p, w\}$:

$$\partial_\epsilon^\sigma F(\bar{x}) = \bigsqcup_{\epsilon \not<_{Y_+}^\sigma \epsilon' \geq_{Y_+^\epsilon} 0} \partial_{\epsilon'}^s F(\bar{x}) + Z_\sigma(X, Y) \qquad \forall \epsilon \not<_{Y_+}^\sigma 0. \tag{7}$$

Also we are going to apply the following characterization of exact or approximate weak efficiency in unconstrained difference vector optimization (non necessarily convex). Note that a similar characterization for proper efficiency remains a challenging open problem.

Theorem 1 *[7] Let F, $G : X \to Y \sqcup \{+\infty\}$ be such that G satisfies the hypothesis:*

$$(\mathcal{H})_1 \qquad \forall \epsilon >_{Y_+} 0,\ \forall x \in \operatorname{dom} G,\ \partial_\epsilon^s G(x) \neq \emptyset.^1$$

Then $\forall \epsilon \not<_{Y_+} 0$,

$$\bar{x} \in E_\epsilon^w(F - G, X, Y_+) \quad \Leftrightarrow \quad \partial_{\epsilon'}^s G(\bar{x}) \subseteq \partial_{\epsilon'+\epsilon}^w F(\bar{x}) \qquad \forall \epsilon' \geq_{Y_+} 0.$$

3 General FMOP

We are concerned in this paper with the following fractional multicriteria optimization problem:

$$\text{FMOP:} \quad \epsilon\text{-}\operatorname*{Min}_{x \in S} \frac{F(x)}{G(x)} = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_r(x)}{g_r(x)} \right)$$

where $F = (f_1, \ldots, f_r) : X \longrightarrow \mathbb{R}^r \sqcup \{+\infty\}$ and $G = (g_1, \ldots, g_r) : X \longrightarrow \mathbb{R}^r \sqcup \{+\infty\}$ is such that $g_i(x) > 0$ for all $x \in S \subseteq X$ and all $i \in I = \{1, ..., r\}$. The space $Y = \mathbb{R}^r$ will be endowed with its natural (partial) order induced by the nonnegative orthant $Y_+ = \mathbb{R}_+^r$.

We aim in this section to characterize weakly (ϵ-)efficient solutions to a general FMOP. To this end, we will use a parametric approach that transforms FMOP to a nonfractional one:

$$\text{VOP}_\mu\text{:} \quad \epsilon\text{-}\operatorname*{Min}_{x \in S} F(x) - \mu G(x) = \Big(f_1(x) - \mu_1 g_1(x), \ldots, f_r(x) - \mu_r g_r(x) \Big)$$

for some vector parameter $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r$.

Lemma 1 *[14, 22] Let $\epsilon = (\varepsilon_1, \ldots, \varepsilon_r) \geq_{Y_+} 0$. Then, for $\sigma \in \{e, w\}$,*

$$\bar{x} \in E_\epsilon^\sigma\big(F/G, S, Y_+\big) \quad \Leftrightarrow \quad \bar{x} \in E_{\bar{\epsilon}}^\sigma\big(F - \bar{\mu} G, S, Y_+\big),$$

where $\bar{\epsilon} = \big(\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_r g_r(\bar{x})\big)$ and $\bar{\mu} = \Big(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \varepsilon_1, \ldots, \frac{f_r(\bar{x})}{g_r(\bar{x})} - \varepsilon_r\Big)$.

[1] This assumption is only required for sufficiency.

The lemma above, first established in [14] for ϵ-efficiency and later in [22] for weak ϵ-efficiency, extends the results obtained in [29] for exact efficiency and proper efficiency in linear fractional multicriteria optimization (the principle being the same for nonlinear FMOPs). In the sequel, a similar extension to proper ϵ-efficiency will be proved.

Remark 2 As $G(\bar{x}) >_{Y_+} 0$, it is clear that the condition $\epsilon \not<_{Y_+} 0$ is equivalent to $\bar{\epsilon} \not<_{Y_+} 0$. So, in Lemma 1, the weaker condition $\epsilon \not<_{Y_+} 0$ can be considered rather than $\epsilon \geq_{Y_+} 0$.

3.1 *Unconstrained FMOP*

We give below a vectorial characterization of weak (ϵ-)efficiency for a general unconstrained ($S = X$) FMOP:

$$\operatorname*{Min}_{x \in X} \frac{F(x)}{G(x)}$$

Theorem 2 *Let $F = (f_1, \ldots, f_r) : X \to \mathbb{R}^r \sqcup \{+\infty\}$, $G = (g_1, \ldots, g_r) : X \to \mathbb{R}^r \sqcup \{+\infty\}$. Let $\epsilon = (\varepsilon_1, \ldots, \varepsilon_r) \not<_{Y_+} 0$, $\bar{\epsilon} = \big(\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_r g_r(\bar{x})\big)$, $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$ $(\forall i \in I)$, $\bar{x} \in X$. Assume that G satisfies the following hypothesis:*

$$(\mathcal{H})_2 \qquad \forall \varepsilon > 0,\ \forall i \in I,\ \forall x \in \operatorname{dom} g_i,\ \ \partial_\varepsilon(\bar{\mu}_i g_i)(x) \neq \emptyset.$$

Then,

$$\bar{x} \in E^w_\epsilon(F/G, X, Y_+) \quad \Leftrightarrow \quad \prod_{i=1}^{r} \partial_{\varepsilon'_i}(\bar{\mu}_i g_i)(\bar{x}) \subseteq \partial^w_{\epsilon' + \bar{\epsilon}}(f_1, \ldots, f_r)(\bar{x})$$

for all $\epsilon' = (\varepsilon'_1, \ldots, \varepsilon'_r) \geq_{Y_+} 0$.

Proof By applying Lemma 1 with $S = X$ and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_r)$, we have

$$\bar{x} \in E^w_\epsilon(F/G, X, Y_+) \quad \Leftrightarrow \quad \bar{x} \in E^w_{\bar{\epsilon}}(F - \bar{\mu}G, X, Y_+).$$

Theorem 1 can be applied with the vector mappings F and $\bar{\mu}G := (\bar{\mu}_1 g_1, \ldots, \bar{\mu}_r g_r)$, since by hypothesis $(\mathcal{H})_2$ and by virtue of (3), the assumption $(\mathcal{H})_1$ of Theorem 1 is fulfilled:

$$\begin{aligned} \bar{x} \in E^w_{\bar{\epsilon}}(F - \bar{\mu}G, X, Y_+) \ &\Leftrightarrow\ \partial^s_{\epsilon'}(\bar{\mu}G)(\bar{x}) \subseteq \partial^w_{\epsilon' + \bar{\epsilon}} F(\bar{x}) \qquad \forall \epsilon' \geq_{Y_+} 0 \\ &\Leftrightarrow\ \prod_{i=1}^{r} \partial_{\varepsilon'_i}(\bar{\mu}_i g_i)(\bar{x}) \subseteq \partial^w_{\epsilon' + \bar{\epsilon}}(f_1, \ldots, f_r)(\bar{x}) \qquad \forall \epsilon' \geq_{Y_+} 0. \quad \square \end{aligned}$$

Remark 3

(a) For the particular case of (exact) 0-efficiency, it suffices to see that $\epsilon = 0$ iff $\bar{\epsilon} = 0$, since $g_i(\bar{x}) \neq 0$ $(\forall i)$.

(b) The hypothesis $(\mathcal{H})_2$ is particularly fulfilled, if $g_i \in \Gamma_0(X)$ and $\bar{\mu}_i \geq 0$ $(\forall i \in I)$ (i.e., $\varepsilon_i \leq \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $\forall i \in I$), because in this case, $\bar{\mu}_i g_i \in \Gamma_0(X)$.

By applying the formulas (7)-(3), we easily deduce the following characterization of exact or approximate weak efficiency expressed only in terms of (scalar) subdifferentials of the data provided that the numerator in FMOP is convex and subdifferentially regular.

Corollary 1 *If, in addition to the assumptions of Theorem 2, $F \in \Gamma(X, Y_+)$ and F is w-regular ε-subdifferentiable at $\bar{x}$ ($\forall \varepsilon \geq 0$), then*

$$\bar{x} \in E^w_\epsilon(F/G, X, Y_+) \quad \Leftrightarrow \quad \prod_{i=1}^r \partial_{\varepsilon'_i}(\bar{\mu}_i g_i)(\bar{x}) \subseteq \bigsqcup_{\substack{(\varepsilon''_1,\ldots,\varepsilon''_r) \geq_{Y_+} 0, \\ (\varepsilon''_1,\ldots,\varepsilon''_r)=0 \ \ if \ \ \epsilon'+\bar{\epsilon}=0, \\ \exists i:\, \epsilon''_i \leq \epsilon'_i + \bar{\epsilon}_i}} \prod_{i=1}^r \partial_{\varepsilon''_i} f_i(\bar{x}) + Z_w(X, \mathbb{R}^r)$$

for all $\epsilon' = (\varepsilon'_1, \ldots, \varepsilon'_r) \geq_{Y_+} 0$.

3.2 *Convex constrained FMOP*

Thanks to the Pareto ϵ-subdifferential calculus presented previously, Theorem 1 allows us to derive vectorial characterizations of weak (ϵ-)efficiency for a FMOP subject to convex inequality constraints:

$$\operatorname*{Min}_{H(x) \leq_{Z_+} 0} \frac{F(x)}{G(x)}$$

where the constraint mapping $H = (h_1, \ldots, h_p) : X \to \mathbb{R}^p \sqcup \{+\infty\}$ and $Z_+ = \mathbb{R}^p_+$. Then, the feasible set $S = \{x \in X : \ h_j(x) \leq 0, \ \forall j \in J\}$, where $J = \{1, \ldots, p\}$, will be assumed to be convex.

Theorem 3 *Let $F = (f_1, \ldots, f_r) : X \to \mathbb{R}^r \sqcup \{+\infty\}$, $G = (g_1, \ldots, g_r) : X \to \mathbb{R}^r \sqcup \{+\infty\}$, $H = (h_1, \ldots, h_p) : X \to \mathbb{R}^p \sqcup \{+\infty\}$. Let $\epsilon = (\varepsilon_1, \ldots, \varepsilon_r) \not\leq_{Y_+} 0$, $\bar{\epsilon} = \big(\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_r g_r(\bar{x})\big)$, $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$ ($\forall i \in I$), $\bar{x} \in X$. Assume that G satisfies the hypothesis $(\mathcal{H})_2$ and (F, H) satisfy one of the following two qualification conditions:*

$$(MR)_2 \quad \begin{cases} f_i \in \Gamma(X), \ h_j \in \Gamma(X) \ (\forall i, j), \ X \textit{ separated locally convex}, \\ \exists\, x_0 \in \operatorname{dom} F \sqcap \operatorname{dom} H \quad \textit{s.t.} \\ \delta^v_{-Z_+} \textit{ is finite and continuous at } H(x_0). \end{cases}$$

$$(AB)_2 \quad \begin{cases} f_i \in \Gamma_0(X), \ h_j \in \Gamma_0(X) \ (\forall i, j), \ X \textit{ Fréchet space}, \\ \mathbb{R}_+[Z_+ + H(\operatorname{dom} F \sqcap \operatorname{dom} H)] \textit{ is a closed vector subspace of } \mathbb{R}^p. \end{cases}$$

Then,

$$\bar{x} \in E^w_\epsilon(F/G, S, Y_+) \quad \Longleftrightarrow \quad \begin{cases} \forall \epsilon' = (\varepsilon'_1, \ldots, \varepsilon'_r) \geq_{Y_+} 0, \ \ \forall A \in \prod_{i=1}^r \partial_{\varepsilon'_i}(\bar{\mu}_i g_i)(\bar{x}), \\ \exists \epsilon^1 \not\leq_{Y_+} 0, \ \ \exists \epsilon^2 \geq_{Y_+^{\epsilon'+\bar{\epsilon}}} 0, \ \ \epsilon^1 + \epsilon^2 = \epsilon' + \bar{\epsilon}, \\ \exists \Lambda = (\lambda_{ij})_{(i,j) \in I \times J} \in L_+(\mathbb{R}^p, \mathbb{R}^r), \\ A \in \partial^w_{\epsilon^1}\Big(f_1 + \sum_{j=1}^p \lambda_{1j} h_j, \ldots, f_r + \sum_{j=1}^p \lambda_{rj} h_j\Big)(\bar{x}), \\ -\epsilon^2 \leq_{Y_+} \Lambda(H(\bar{x})) \leq_{Y_+} 0, \ \ H(\bar{x}) \leq_{Z_+} 0. \end{cases}$$

Proof Taking into account Lemma 1, relation (1) and the assumption $(\mathcal{H})_2$, Theorem 1 applies with the vector mappings $F + \delta^v_S$ and $\bar{\mu} G := (\bar{\mu}_1 g_1, \ldots, \bar{\mu}_r g_r)$:

$$\begin{aligned} \bar{x} \in E^w_\epsilon\big(F/G, S, Y_+\big) \ &\Leftrightarrow \ \bar{x} \in E^w_{\bar{\epsilon}}\big(F - \bar{\mu} G, S, Y_+\big) \\ &\Leftrightarrow \ \bar{x} \in E^w_{\bar{\epsilon}}\big(F + \delta^v_S - \bar{\mu} G, X, Y_+\big) \\ &\Leftrightarrow \ \partial^s_{\epsilon'}(\bar{\mu} G)(\bar{x}) \subseteq \partial^w_{\epsilon'+\bar{\epsilon}}(F + \delta^v_S)(\bar{x}) \qquad \forall \epsilon' \geq_{Y_+} 0. \end{aligned}$$

As $\delta^v_S = \delta^v_{-Z_+} \circ H$, according to $(MR)_2$ or $(AB)_2$, we easily verify that all the hypotheses for applying formula (5) are satisfied, for $\sigma = w$, with $Y = \mathbb{R}^r$, $Z = \mathbb{R}^p$ and $C = X$. Thus,

$$\partial^w_{\epsilon'+\bar\epsilon}(F + \delta^v_{-Z_+} \circ H)(\bar x) = \bigsqcup_{\substack{\epsilon^1 \not<_{Y_+} 0,\ \epsilon^2 \geq_{Y_+^{\epsilon'+\bar\epsilon}} 0, \\ \epsilon^1+\epsilon^2 = \epsilon'+\bar\epsilon}} \ \bigsqcup_{\substack{\Lambda\in L_+(\mathbb{R}^p,\mathbb{R}^r), \\ -\epsilon^2 \leq_{Y_+} \Lambda(H(\bar x)) \leq_{Y_+} 0}} \partial^w_{\epsilon^1}(F + \Lambda \circ H)(\bar x).$$

On the one hand, one has from (3) that $\partial^s_{\epsilon'}(\bar\mu G)(\bar x) = \prod_{i=1}^r \partial_{\varepsilon'_i}(\bar\mu_i g_i)(\bar x)$, on the other hand, taking $\Lambda = (\lambda_{ij})_{(i,j)\in I\times J}$, one can also express the vector mapping $F + \Lambda \circ H$ in terms of its components: $F + \Lambda \circ H = \Big(f_1 + \sum_{j=1}^p \lambda_{1j} h_j, \ldots, f_r + \sum_{j=1}^p \lambda_{rj} h_j\Big)$, which completes the proof of the theorem. □

The following corollary gives a characterization only in terms of ε-subdifferentials of the data whose objective numerators and constraints are more regular.

Corollary 2 *If, in addition to the assumptions of Theorem 3, F and H are finite and continuous at some point, F (and resp. H) is w-regular (resp. regular) ε-subdifferentiable at $\bar x$ ($\forall \varepsilon \geq 0$), then*

$$\bar x \in E^w_\epsilon(F/G, S, Y_+) \iff \begin{cases} \forall \epsilon' = (\varepsilon'_1, \ldots, \varepsilon'_r) \geq_{Y_+} 0,\ \forall A \in \prod\limits_{i=1}^r \partial_{\varepsilon'_i}(\bar\mu_i g_i)(\bar x), \\ \exists \epsilon^1 = (\varepsilon^1_1, \ldots, \varepsilon^1_r) \geq_{Y_+^{\epsilon'+\bar\epsilon}} 0,\ \exists \epsilon^2 = (\varepsilon^2_1, \ldots, \varepsilon^2_p) \geq_{Z_+^{\epsilon'+\bar\epsilon}} 0, \\ \exists \epsilon^3 \not<_{Y_+} 0,\ \exists \epsilon^4 \geq_{Y_+^{\epsilon'+\bar\epsilon}} 0,\ \ \epsilon^1 + \epsilon^3 + \epsilon^4 = \epsilon' + \bar\epsilon, \\ \exists \Lambda = (\lambda_{ij})_{(i,j)\in I\times J} \in L_+(\mathbb{R}^p, \mathbb{R}^r),\ \ \epsilon^3 \not<_{Y_+} \Lambda(\epsilon^2) \not<_{Y_+} \epsilon^3, \\ A \in \prod\limits_{i=1}^r \Big(\partial_{\varepsilon^1_i} f_i(\bar x) + \sum\limits_{j=1}^p \lambda_{ij} \partial_{\varepsilon^2_j} h_j(\bar x)\Big) + Z_w(X, \mathbb{R}^r), \\ -\epsilon^4 \leq_{Y_+} \Lambda(H(\bar x)) \leq_{Y_+} 0,\ \ H(\bar x) \leq_{Z_+} 0. \end{cases}$$

Proof Following the proof of Theorem 3, we have

$$\bar x \in E^w_\epsilon\big(F/G, S, Y_+\big) \Leftrightarrow \prod_{i=1}^r \partial_{\varepsilon'_i}(\bar\mu_i g_i)(\bar x) \subseteq \partial^w_{\epsilon'+\bar\epsilon}(F + \delta^v_{-Z_+} \circ H)(\bar x) \quad \forall \epsilon' \geq_{Y_+} 0.$$

According to the assumptions of Theorem 3 and the corollary, we easily verify that all the hypotheses for applying formula (6) are satisfied with $Y = \mathbb{R}^r$, $Z = \mathbb{R}^p$ and $C = X$. Thus,

$$\partial^w_{\epsilon'+\bar\epsilon}(F + \delta^v_{-Z_+} \circ H)(\bar x) = \bigsqcup_{\substack{\epsilon^i \geq_{Y_+^{\epsilon'+\bar\epsilon}} 0\ (i=1,4), \\ \epsilon^2 \geq_{Z_+^{\epsilon'+\bar\epsilon}} 0,\ \epsilon^3 \not<_{Y_+} 0, \\ \sum_{i\neq 2} \epsilon^i = \epsilon'+\bar\epsilon}} \ \bigsqcup_{\substack{\Lambda\in L_+(\mathbb{R}^p,\mathbb{R}^r), \\ \epsilon^3 \not<_{Y_+} \Lambda(\epsilon^2) \not<_{Y_+} \epsilon^3, \\ -\epsilon^4 \leq_{Y_+} \Lambda(H(\bar x)) \leq_{Y_+} 0}} \partial^s_{\epsilon^1} F(\bar x) + \Lambda \circ \partial^s_{\epsilon^2} H(\bar x) + Z_w(X, \mathbb{R}^r).$$

As $\partial^s_{\epsilon^1} F(\bar x) = \prod_{i=1}^r \partial_{\varepsilon^1_i} f_i(\bar x)$ and $\partial^s_{\epsilon^2} H(\bar x) = \prod_{j=1}^p \partial_{\varepsilon^2_j} h_j(\bar x)$, by putting $\Lambda = (\lambda_{ij})_{(i,j)\in I\times J}$, we get $\partial^s_{\epsilon^1} F(\bar x) + \Lambda \circ \partial^s_{\epsilon^2} H(\bar x) = \prod_{i=1}^r \partial_{\varepsilon^1_i} f_i(\bar x) + \prod_{i=1}^r \sum_{j=1}^p \lambda_{ij} \partial_{\varepsilon^2_j} h_j(\bar x)$ which shows the desired result. □

4 Constrained convex FMOP

By constrained convex FMOP, we intend the problem defined in Section 3.2:

$$\operatorname*{Min}_{H(x)\leq_{Z_+}0} \frac{F(x)}{G(x)}$$

whose objective numerators (f_is) as well as the constraints (h_js) are both convex while the denominators (g_is) are concave ($-g_i$s are convex). For this type of problem, we seek to derive approximate proper or weak efficiency criteria of the Kuhn-Tucker type given in vector forms.

To be able to handle approximate properly efficient solutions, we need to extend the parametric approach [29] that converts a fractional problem into a nonfractional problem dealing with Geoffrion's exact proper efficiency [12].

In $Y = \mathbb{R}^r$ with $Y_+ = \mathbb{R}^r_+$ where $F = (f_1, \ldots, f_r)$, we denote by $E^{Ge}_\epsilon(F, S, Y_+)$, the set of properly ϵ-efficient solutions to VOP in the Geoffrion sense, first defined in [23], as being the set of points $\bar{x} \in E^e_\epsilon(F, S, Y_+)$ such that $\exists M > 0$, $\forall i \in I$, $\forall x \in S$ satisfying $f_i(x) < f_i(\bar{x}) - \varepsilon_i$, $\exists j \in I$ with $f_j(x) > f_j(\bar{x}) - \varepsilon_j$ such that

$$\frac{f_i(\bar{x}) - f_i(x) - \varepsilon_i}{f_j(x) - f_j(\bar{x}) + \varepsilon_j} \leq M.$$

Following the scalarization results due respectively to El Maghri [9, Theorem 3.1] and Liu [24, Theorem 3], Henig's proper ϵ-efficiency and Geoffrion's proper ϵ-efficiency are equivalent in the convex context:

$$E^p_\epsilon(F, S, Y_+) = E^{Ge}_\epsilon(F, S, Y_+) = \bigsqcup_{\lambda >_{Y_+} 0} \langle \lambda, \epsilon\rangle\text{–}\operatorname*{argmin}_{x\in S}\langle \lambda, F(x)\rangle, \tag{8}$$

whenever F is Y_+-convex on S convex.

Lemma 2 *Assume that G satisfies the following hypothesis:*

$$(\mathcal{H})_3 \qquad \exists(\alpha, \beta) \in \mathbb{R}^2,\ \forall i \in I,\ \forall x \in S,\ \ 0 < \alpha \leq g_i(x) \leq \beta.$$

Let $\epsilon = (\varepsilon_1, \ldots, \varepsilon_r) \not\leq_{Y_+} 0$. Then,

$$\bar{x} \in E^{Ge}_\epsilon\big(F/G, S, Y_+\big) \quad \Leftrightarrow \quad \bar{x} \in E^{Ge}_{\bar{\epsilon}}\big(F - \bar{\mu}G, S, Y_+\big),$$

where $\bar{\epsilon} = \big(\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_r g_r(\bar{x})\big)$ and $\bar{\mu} = \Big(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \varepsilon_1, \ldots, \frac{f_r(\bar{x})}{g_r(\bar{x})} - \varepsilon_r\Big)$.

Proof Let $\bar{x} \in E^{Ge}_\epsilon\big(F/G, S, Y_+\big)$. Then, by the very definition, $\bar{x} \in E^e_\epsilon\big(F/G, S, Y_+\big)$, which by virtue of Lemma 1, is equivalent to $\bar{x} \in E^e_{\bar{\epsilon}}\big(F - \bar{\mu}\, G, S, Y_+\big)$. Let $i \in I$ and $x \in S$ such that $f_i(x) - \bar{\mu}_i g_i(x) < f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x}) - \bar{\varepsilon}_i$. Since $f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x}) - \bar{\varepsilon}_i = 0$ and $g_i(x) > 0$, dividing the last inequality by $g_i(x)$, it follows that $\frac{f_i(x)}{g_i(x)} < \bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$.

Hence, by definition, there exists $j \in I$ with $\frac{f_j(x)}{g_j(x)} > \frac{f_j(\bar{x})}{g_j(\bar{x})} - \varepsilon_j$ $(= \bar{\mu}_j)$, which in turn is equivalent to $f_j(x) - \bar{\mu}_j g_j(x) > 0 = f_j(\bar{x}) - \bar{\mu}_j g_j(\bar{x}) - \bar{\varepsilon}_j$, such that

$$\frac{\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(x)}{g_i(x)} - \varepsilon_i}{\frac{f_j(x)}{g_j(x)} - \frac{f_j(\bar{x})}{g_j(\bar{x})} + \varepsilon_j} = \frac{\bar{\mu}_i - \frac{f_i(x)}{g_i(x)}}{\frac{f_j(x)}{g_j(x)} - \bar{\mu}_j} \leq M,$$

for some $M > 0$. Multiplying the numerator of the last inequality by $g_i(x)$ and its denominator by $g_j(x)$ and assuming $(\mathcal{H})_3$, we get

$$\frac{\bar{\mu}_i g_i(x) - f_i(x)}{f_j(x) - \bar{\mu}_j g_j(x)} \leq \frac{\beta M}{\alpha},$$

which shows that $\bar{x} \in E^{Ge}_{\bar{\epsilon}}\big(F - \bar{\mu}G, S, Y_+\big)$. Conversely, let $\bar{x} \in E^{Ge}_{\bar{\epsilon}}\big(F - \bar{\mu}G, S, Y_+\big)$ and let $i \in I$ and $x \in S$ such that $\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$. Hence, $f_i(x) - \bar{\mu}_i g_i(x) < 0 = f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x}) - \bar{\varepsilon}_i$. Thus, by definition, there exists $j \in I$ with $f_j(x) - \bar{\mu}_j g_j(x) > f_j(\bar{x}) - \bar{\mu}_j g_j(\bar{x}) - \bar{\varepsilon}_j = 0$, which in turn is equivalent to $\frac{f_j(x)}{g_j(x)} > \bar{\mu}_j = \frac{f_j(\bar{x})}{g_j(\bar{x})} - \varepsilon_j$, such that

$$\frac{-f_i(x) + \bar{\mu}_i g_i(x)}{f_j(x) - \bar{\mu}_j g_j(x)} \leq M,$$

for some $M > 0$. Dividing the numerator of this inequality by $g_i(x)$ and its denominator by $g_j(x)$ and assuming $(\mathcal{H})_3$, we get

$$\frac{\bar{\mu}_i - \frac{f_i(x)}{g_i(x)}}{\frac{f_j(x)}{g_j(x)} - \bar{\mu}_j} = \frac{\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(x)}{g_i(x)} - \varepsilon_i}{\frac{f_j(x)}{g_j(x)} - \frac{f_j(\bar{x})}{g_j(\bar{x})} + \varepsilon_j} \leq \frac{\beta M}{\alpha},$$

which shows that $\bar{x} \in E^{Ge}_{\epsilon}(F/G, S, Y_+)$. This completes the proof of the Lemma. $\square$

We are now ready to state our ϵ-efficiency conditions.

Theorem 4 *Let $F = (f_1, \ldots, f_r) : X \to \mathbb{R}^r \sqcup \{+\infty\}$, $G = (g_1, \ldots, g_r) : X \to \mathbb{R}^r \sqcup \{+\infty\}$, $H = (h_1, \ldots, h_p) : X \to \mathbb{R}^p \sqcup \{+\infty\}$. Let $\epsilon = (\varepsilon_1, \ldots, \varepsilon_r) \not<^{\sigma}_{Y_+} 0$ with $\sigma \in \{p, w\}$, $\bar{\epsilon} = \big(\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_r g_r(\bar{x})\big)$, $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$ $(\forall i \in I)$, $\bar{x} \in X$. Assume that G satisfies the hypothesis $(\mathcal{H})_3$ and (F, G, H) satisfy one of the following two qualification conditions:*

$$(MR)_3 \quad \begin{cases} f_i - \bar{\mu}_i g_i \in \Gamma(X), \ h_j \in \Gamma(X) \ (\forall i, j), \ X \text{ separated locally convex}, \\ \exists\, x_0 \in \operatorname{dom} F \sqcap \operatorname{dom} G \sqcap \operatorname{dom} H \quad \text{s.t.} \\ \delta^v_{-Z_+} \text{ is finite and continuous at } H(x_0). \end{cases}$$

$$(AB)_3 \quad \begin{cases} f_i - \bar{\mu}_i g_i \in \Gamma_0(X), \ h_j \in \Gamma_0(X) \ (\forall i, j), \ X \text{ Fréchet space}, \\ \mathbb{R}_+[Z_+ + H(\operatorname{dom} F \sqcap \operatorname{dom} G \sqcap \operatorname{dom} H)] \text{ is a closed vector subspace of } \mathbb{R}^p. \end{cases}$$

Then,

$$\bar{x} \in E^{\sigma}_{\epsilon}\,(F/G, S, Y_+) \Longleftrightarrow \begin{cases} \exists \epsilon^1 \not<^{\sigma}_{Y_+} 0, \ \ \exists \epsilon^2 \geq_{Y_+^{\bar{\epsilon}}} 0, \ \ \epsilon^1 + \epsilon^2 = \bar{\epsilon}, \\ \exists \Lambda = (\lambda_{ij})_{(i,j) \in I \times J} \in L_+(\mathbb{R}^p, \mathbb{R}^r), \\ 0 \in \partial^{\sigma}_{\epsilon^1}\Big(f_1 - \bar{\mu}_1 g_1 + \sum_{j=1}^{p} \lambda_{1j} h_j, \ldots, f_r - \bar{\mu}_r g_r + \sum_{j=1}^{p} \lambda_{rj} h_j\Big)(\bar{x}), \\ -\epsilon^2 \leq_{Y_+} \Lambda(H(\bar{x})) \leq_{Y_+} 0, \ \ H(\bar{x}) \leq_{Z_+} 0. \end{cases}$$

Proof According to Lemma 1 and Lemma 2 by taking into account the assumption $(\mathcal{H})_3$, $(MR)_3$ or $(AB)_3$, and (8), it is immediate from (2) that

$$\begin{aligned}\bar{x} \in E^{\sigma}_{\epsilon}\big(F/G, S, Y_+\big) \;&\Leftrightarrow\; \bar{x} \in E^{\sigma}_{\bar{\epsilon}}\big(F - \bar{\mu}G, S, Y_+\big)\\ &\Leftrightarrow\; 0 \in \partial^{\sigma}_{\bar{\epsilon}}\,(F - \bar{\mu}G + \delta^v_S)(\bar{x}),\end{aligned}$$

where $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_r)$. As $\delta^v_S = \delta^v_{-Z_+} \circ H$, according to $(MR)_3$ or $(AB)_3$, we easily verify that all the hypotheses for applying formula (5) are satisfied with $Y = \mathbb{R}^r$, $Z = \mathbb{R}^p$ and $C = X$. Thus,

$$\partial^{\sigma}_{\bar{\epsilon}}(F - \bar{\mu}G + \delta^v_{-Z_+} \circ H)(\bar{x}) = \bigsqcup_{\substack{\epsilon^1 \not<^{\sigma}_{Y_+} 0,\ \epsilon^2 \geq_{Y^{\bar{\epsilon}}_+} 0,\\ \epsilon^1+\epsilon^2 = \bar{\epsilon}}} \;\; \bigsqcup_{\substack{\Lambda\in L_+(\mathbb{R}^p,\mathbb{R}^r),\\ -\epsilon^2\leq_{Y_+}\Lambda(H(\bar{x}))\leq_{Y_+} 0}} \partial^{\sigma}_{\epsilon^1}(F - \bar{\mu}G + \Lambda \circ H)(\bar{x}).$$

The desired result then follows by expressing the mapping $F - \bar{\mu}G + \Lambda \circ H$ in terms of its components: $F - \bar{\mu}G + \Lambda\circ H = \Big(f_1 - \bar{\mu}_1 g_1 + \sum_{j=1}^p \lambda_{1j} h_j, \ldots, f_r - \bar{\mu}_i g_i + \sum_{j=1}^p \lambda_{rj} h_j\Big)$. $\square$

The corollary bellow is obtained by the same arguments as in Corollary 2. Let us recall the notation of σ-“zerolike” matrices Θ for $\sigma \in \{p, w\}$: $\Theta \sim_{\sigma} 0 \Leftrightarrow \Theta \in Z_{\sigma}(X, \mathbb{R}^r)$.

Corollary 3 *If, in addition to the assumptions of Theorem 4, the maps $F - \bar{\mu}G$ and H are finite and continuous at some point, $F - \bar{\mu}G$ (and resp. H) is σ-regular (resp. regular) ε-subdifferentiable at $\bar{x}$ ($\forall\varepsilon \geq 0$), then*

$$\bar{x} \in E^{\sigma}_{\epsilon}\,(F/G, S, Y_+) \Longleftrightarrow \begin{cases} \exists\epsilon^1 = (\varepsilon^1_1, \ldots, \varepsilon^1_r) \geq_{Y^{\bar{\epsilon}}_+} 0,\ \exists\epsilon^2 = (\varepsilon^2_1, \ldots, \varepsilon^2_p) \geq_{Z^{\bar{\epsilon}}_+} 0,\\ \exists\epsilon^3 \not<^{\sigma}_{Y_+} 0,\ \exists\epsilon^4 \geq_{Y^{\bar{\epsilon}}_+} 0,\ \ \epsilon^1 + \epsilon^3 + \epsilon^4 = \bar{\epsilon},\ \exists\Theta \sim_{\sigma} 0,\\ \exists\Lambda = (\lambda_{ij})_{(i,j)\in I\times J} \in L_+(\mathbb{R}^p, \mathbb{R}^r),\ \ \epsilon^3 \not<^{\sigma}_{Y_+} \Lambda(\epsilon^2) \not<^{\sigma}_{Y_+} \epsilon^3,\\ \Theta \in \prod\limits_{i=1}^r \Big(\partial_{\varepsilon^1_i}(f_i - \bar{\mu}_i g_i)(\bar{x}) + \sum\limits_{j=1}^p \lambda_{ij}\partial_{\varepsilon^2_j} h_j(\bar{x})\Big),\\ -\epsilon^4 \leq_{Y_+} \Lambda(H(\bar{x})) \leq_{Y_+} 0,\ \ H(\bar{x}) \leq_{Z_+} 0. \end{cases}$$

In the case of exact ($\epsilon = 0$) σ-efficiency, by using the fact that $\epsilon = 0$ iff $\bar{\epsilon} = 0$ and the fact that $Y^0_+ = \{0\}$, we easily derive the following new simple vectorial characterizations.

Corollary 4 *Under the assumptions of Theorem 4,*

$$\bar{x} \in E^{\sigma}\,(F/G, S, Y_+) \Longleftrightarrow \begin{cases} \exists\Lambda = (\lambda_{ij})_{(i,j)\in I\times J} \in L_+(\mathbb{R}^p, \mathbb{R}^r),\\ 0 \in \partial^{\sigma}\Big(f_1 - \bar{\mu}_1 g_1 + \sum\limits_{j=1}^p \lambda_{1j} h_j, \ldots, f_r - \bar{\mu}_r g_r + \sum\limits_{j=1}^p \lambda_{rj} h_j\Big)(\bar{x}),\\ \Lambda(H(\bar{x})) = 0,\ H(\bar{x}) \leq_{Z_+} 0. \end{cases}$$

Under the assumptions of Corollary 3,

$$\bar{x} \in E^{\sigma}\,(F/G, S, Y_+) \Longleftrightarrow \begin{cases} \exists\Lambda = (\lambda_{ij})_{(i,j)\in I\times J} \in L_+(\mathbb{R}^p, \mathbb{R}^r),\ \ \exists\Theta \sim_{\sigma} 0,\\ \Theta \in \prod\limits_{i=1}^r \Big(\partial(f_i - \bar{\mu}_i g_i)(\bar{x}) + \sum\limits_{j=1}^p \lambda_{ij}\partial h_j(\bar{x})\Big),\\ \Lambda(H(\bar{x})) = 0,\ \ H(\bar{x}) \leq_{Z_+} 0. \end{cases}$$

5 Linear FMOP

Let $c_i,\ d_i \in \mathbb{R}^n,\ \gamma_i,\ \delta_i \in \mathbb{R}\ (i \in I),\ A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$. Then, the FMOP defined in Section 3.2 can be stated in linear form as bellow:

$$\operatorname*{Min}_{Ax\,\leq_{Z_+} b}\ \frac{F(x)}{G(x)} = \left(\frac{c_1^T x + \gamma_1}{d_1^T x + \delta_1}, \dots, \frac{c_r^T x + \gamma_r}{d_r^T x + \delta_r}\right)$$

where it is understood here that $f_i = c_i^T x + \gamma_i$, $g_i = d_i^T x + \delta_i$ $(i \in I)$ and $H(x) = Ax - b$.

The results obtained in the previous section for what we called convex FMOPs apply of course to linear FMOPs. But due the linear structure, we get in this case completely explicit vectorial ϵ-efficiency criteria.

Theorem 5 *Let $\epsilon = (\varepsilon_1, \dots, \varepsilon_r) \not<^{\sigma}_{Y_+} 0$ with $\sigma \in \{p, w\}$, $\bar{\epsilon} = \big(\varepsilon_1 g_1(\bar{x}), \dots, \varepsilon_r g_r(\bar{x})\big)$, $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$ $(\forall i \in I)$, $\bar{x} \in X$. Assume that G satisfies the hypothesis $(\mathcal{H})_3$. Then,*

$$\bar{x} \in E^{\sigma}_{\epsilon}\left(F/G, S, Y_+\right) \Longleftrightarrow \begin{cases} \exists \epsilon' \geq_{Y_+^{\bar{\epsilon}}} 0,\ \ \bar{\epsilon} \not<^{\sigma}_{Y_+} \epsilon',\ \ \exists \Lambda \in L_+(\mathbb{R}^p, \mathbb{R}^r), \\ C - \Lambda \times A \sim_{\sigma} 0, \\ -\epsilon' \leq_{Y_+} \Lambda(A\bar{x} - b) \leq_{Y_+} 0,\ \ A\bar{x} \leq_{Z_+} b, \end{cases}$$

where $C \in \mathbb{R}^{r\times n}$ stands for the $(r \times n)$ matrix whose i^{th} row is the vector $c_i - \bar{\mu}_i d_i \in \mathbb{R}^n$. In particular,

$$\bar{x} \in E^{\sigma}\left(F/G, S, Y_+\right) \Longleftrightarrow \begin{cases} \exists \Lambda \in L_+(\mathbb{R}^p, \mathbb{R}^r), \\ C - \Lambda \times A \sim_{\sigma} 0, \\ \Lambda(A\bar{x} - b) = 0,\ \ A\bar{x} \leq_{Z_+} b. \end{cases}$$

Proof It is clear that all the hypotheses of Theorem 4 are fulfilled. Together with (4), since the vector mapping $F - \bar{\mu}G + \Lambda \circ H = \Big(f_1 - \bar{\mu}_1 g_1 + \sum_{j=1}^{p} \lambda_{1j} h_j, \dots, f_r - \bar{\mu}_i g_i + \sum_{j=1}^{p} \lambda_{rj} h_j\Big)$ is linear, where $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_r)$ and $\Lambda = (\lambda_{ij})_{(i,j)\in I\times J}$, we then get that

$$\bar{x} \in E^{\sigma}_{\epsilon}\left(F/G, S, Y_+\right) \Longleftrightarrow \begin{cases} \exists \epsilon^1 \not<^{\sigma}_{Y_+} 0,\ \ \exists \epsilon^2 \geq_{Y_+^{\bar{\epsilon}}} 0,\ \ \epsilon^1 + \epsilon^2 = \bar{\epsilon}, \\ \exists \Lambda \in L_+(\mathbb{R}^p, \mathbb{R}^r), \\ 0 \in \partial^{\sigma}\big(F - \bar{\mu}G + \Lambda \circ H\big)(\bar{x}), \\ -\epsilon^2 \leq_{Y_+} \Lambda(H(\bar{x})) \leq_{Y_+} 0,\ \ H(\bar{x}) \leq_{Z_+} 0. \end{cases}$$

It is also, by the very definition, that a linear map is always σ-regular subdifferentiable, which according to (7) applied with $\epsilon = 0$, gives us

$$\partial^{\sigma}\big(F - \bar{\mu}G + \Lambda \circ H\big)(\bar{x}) = \partial^{s}\big(F - \bar{\mu}G + \Lambda \circ H\big)(\bar{x}) + Z_{\sigma}(X, \mathbb{R}^r).$$

As the strong subdifferentiable reduces to the derivative (here the Jacobian matrix):

$$\partial^{s}\big(F - \bar{\mu}G + \Lambda \circ H\big)(\bar{x}) = C - \Lambda \times A,$$

where the matrix $C = (c_i - \bar{\mu}_i d_i)_{i\in I} \in \mathbb{R}^{r\times n}$, by putting $\epsilon' = \epsilon^2 = \bar{\epsilon} - \epsilon^1$, the result of the theorem follows straightforwardly. The case of exact $(\epsilon = 0)$ σ-efficiency is derived by the same arguments as in Corollary 4. $\square$

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