STABLE SET POLYTOPES WITH HIGH LIFT-AND-PROJECT RANKS FOR THE LOVÁSZ–SCHRIJVER SDP OPERATOR

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Abstract. We study the lift-and-project rank of the stable set polytopes of graphs with respect to the Lovász–Schrijver SDP operator LS_+, with a particular focus on a search for relatively small graphs with high LS_+-rank (the least number of iterations of the LS_+ operator on the fractional stable set polytope to compute the stable set polytope). In particular, we provide families of graphs whose LS_+-rank is asymptotically a linear function of its number of vertices, which is the least possible up to improvements in the constant factor (previous best result in this direction, from 1999, yielded graphs whose LS_+-rank only grew with the square root of the number of vertices). We also provide several new LS_+-minimal graphs, most notably a 12-vertex graph with LS_+-rank 4, and study the properties of a vertex-stretching operation that appears to be promising in generating LS_+-minimal graphs.

1. Introduction

In combinatorial optimization, a standard approach for tackling a given problem is to encode its set of feasible solutions geometrically (e.g., via an integer program). While the exact solution set is often difficult to analyze, we can focus on relaxations of this set that have certain desirable properties (e.g., combinatorially simple to describe, approximates the underlying set of solutions well, and/or is computationally efficient to optimize over). In that regard, the *lift-and-project* approach provides a systematic procedure which generates progressively tighter convex relaxations of any given $0,1$ optimization problem. In the last three decades, many procedures that fall under the lift-and-project approach have been devised (see, among others, [SA90, LS91, BCC93, Las01, BZ04, AT16]), and there is an extensive body of work on their general properties and performance on a wide range of discrete optimization problems (see, for instance, [Au14] and the references therein).

Herein, we focus on LS_+, the SDP lift-and-project operator due to Lovász and Schrijver [LS91], and its performance on the stable set problem of graphs. A remarkable property of LS_+ is that applying one iteration of the operator to the fractional stable set polytope of a graph already yields a tractable relaxation of the stable set polytope that is stronger than the Lovász theta body relaxation. Thus, LS_+ has been shown to perform well on the stable set problem for graphs that are perfect or "close" to being perfect [BENT13, BENT17, Wag22, BENW23]. For more analyses of various lift-and-project relaxations of the stable set problem, see (among others) [LS91, Lau03, LT03, BO04, EMN06, GL07, GLRS09, GRS13, AT16, ALT22].

The worst-case behaviours of LS_+ have also been studied extensively. Generally, the LS_+-*rank* of a set is defined to be the number of iterations it takes LS_+ to return its integer hull. It is known that the LS_+-rank of a set $P \subseteq [0,1]^n$ is at most n, and a number of elementary

Date: March 15, 2023.

Key words and phrases. stable set problem, lift and project, combinatorial optimization, semidefinite programming, integer programming.

Levent Tunçel: Research of this author was supported in part by an NSERC Discovery Grant.

polytopes in $\mathbb{R}^n$ have been shown to have LS_+-rank $\Theta(n)$ (see, among others, [Goe98, CD01, GT01, STT07, AT18]).

On the other hand, while the stable set problem is known to be strongly $\mathcal{NP}$-hard, hardness results for LS_+ (or any other lift-and-project operator utilizing semidefinite programming) on the stable set problem have been relatively scarce. Prior to this manuscript, the worst (in terms of performance by LS_+) family of examples was given by the line graphs of odd cliques. Using the fact that the LS_+-rank of the fractional matching polytope of the $(2k+1)$-clique is k [ST99], the natural correspondence between the matchings in a given graph and the stable sets in the corresponding line graph, as well as the properties of the LS_+ operator, it follows that the fractional stable set polytope of the line graph of the $(2k+1)$-clique (which contains $\binom{2k+1}{2} = k(2k+1)$ vertices) has LS_+-rank k, giving a family of graphs G with LS_+-rank $\Theta(\sqrt{|V(G)|})$. This lower bound on the LS_+-rank of the stable set polytopes has not been improved since 1999.

In this manuscript, we present what we believe is the first known family of graphs whose LS_+-rank is asymptotically a linear function of the number of vertices. For a positive integer k, let $[k] := \{1, 2, \ldots, k\}$. Given an integer $k \geq 2$, let H_k be the graph where $V(H_k) := \{i_p : i \in [k], p \in \{0, 1, 2\}\}$, and the edges of H_k are

- $\{i_0, i_1\}$ and $\{i_1, i_2\}$ for every $i \in [k]$;
- $\{i_0, j_2\}$ for all $i, j \in [k]$ where $i \neq j$.

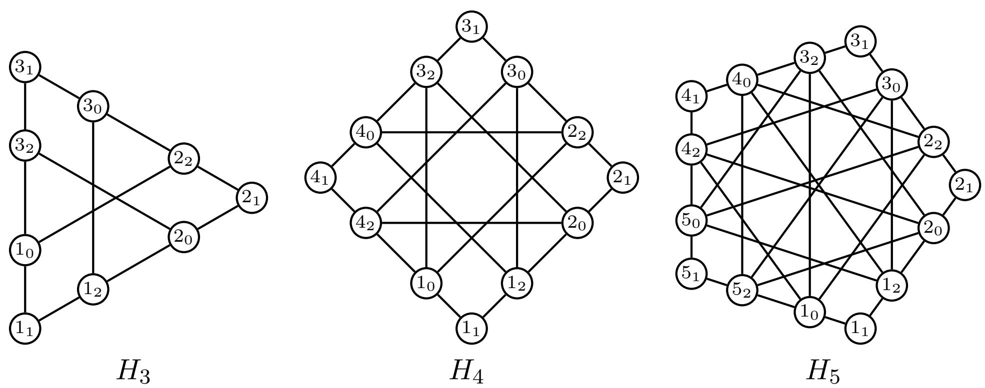

FIGURE 1. Several graphs in the family H_k

Figure 1 illustrates the graphs H_k for several small values of k. Then we have the following.

Theorem 1. *For every $k \geq 3$, the LS_+-rank of the fractional stable set polytope of H_k is at least $\frac{1}{16}|V(H_k)|$.*

We remark that, given $p \in \mathbb{N}$ and polytope $P \subseteq [0,1]^n$ with $O(n^c)$ facets for some constant c, the straightforward formulation of $\mathrm{LS}_+^p(P)$ is an SDP with size $n^{\Omega(p)}$. Since the fractional stable set polytope of the graph H_k has dimension $n = 3k$ with $\Omega(n^2)$ facets, Theorem 1 implies that the SDP described by LS_+ that fails to exactly represent the stable set polytope of H_k has size $n^{\Omega(n)}$. Thus, the consequence of Theorem 1 on the size of the SDPs generated from the LS_+ operator is incomparable with the extension complexity bound due to Lee, Raghavendra, and Steurer [LRS15], who showed that it takes an SDP of size $2^{\Omega(n^{1/13})}$ to exactly represent the stable set polytope of a general n-vertex graph as a projection of a spectrahedron.

Here is how the remainder of our manuscript is organized. In Section 2, we introduce the LS_+ operator and the stable set problem, and establish some notation and basic facts that will aid our subsequent discussion. In Section 3, we study the family of graphs H_k and prove Theorem 1. This result readily leads to the discovery of families of vertex-transitive graphs (detailed in Section 4) whose LS$_+$-rank also exhibits asymptotically linear growth.

In Section 5, we revisit a conjecture by Lipták and the second author [LT03, Conjecture 40], which would imply that there is an ℓ-minimal graph (defined as a graph on 3ℓ vertices with LS$_+$-rank ℓ) for every $\ell \in \mathbb{N}$. We focus on a vertex-stretching graph operation which appears to be promising in generating these ℓ-minimal graphs. We also provide new examples of 3-minimal graphs, as well as what we believe is the first known 4-minimal graph. Finally, in Section 6, we close by mentioning some natural future research directions inspired by our work.

2. Preliminaries

In this section, we establish the necessary definitions and notation for our subsequent analysis.

2.1. The lift-and-project operator LS$_+$.

Here, we define the lift-and-project operator LS$_+$ due to Lovász and Schrijver [LS91] and mention some of its basic properties. Given a convex set $P \subseteq [0,1]^n$, we define the cone

$$\operatorname{cone}(P) := \left\{ \begin{bmatrix} \lambda \\ \lambda x \end{bmatrix} : \lambda \geq 0, x \in P \right\},$$

and index the new coordinate by 0. Given a vector x and an index i, we may refer to the i-entry in x by x_i or $[x]_i$. All vectors are column vectors, so here the transpose of x, $x^\top$, is a row vector. Next, let $\mathbb{S}_+^n$ denote the set of n-by-n symmetric positive semidefinite matrices, and $\operatorname{diag}(Y)$ be the vector formed by the diagonal entries of a square matrix Y. We also let e_i be the i^{th} unit vector.

Given $P \subseteq [0,1]^n$, the operator LS$_+$ first *lifts* P to the following set of matrices:

$$\widehat{\operatorname{LS}}_+(P) := \left\{ Y \in \mathbb{S}_+^{n+1} : Ye_0 = \operatorname{diag}(Y), Ye_i, Y(e_0 - e_i) \in \operatorname{cone}(P)\ \forall i \in [n] \right\}.$$

It then *projects* the set back down to the following set in $\mathbb{R}^n$:

$$\operatorname{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\operatorname{LS}}_+(P), Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix} \right\}.$$

It is easy to see that $\operatorname{LS}_+(P) \subseteq P$ — given $x \in \operatorname{LS}_+(P)$, let $Y \in \widehat{\operatorname{LS}}_+(P)$ be the corresponding *certificate matrix* (and so $Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix}$). Since $Ye_0 = Ye_i + Y(e_0 - e_i)$ for any index $i \in [n]$ and that $\widehat{\operatorname{LS}}_+$ imposes that $Ye_i, Y(e_0 - e_i) \in \operatorname{cone}(P)$, it follows that $Ye_0 \in \operatorname{cone}(P)$, and thus $x \in P$. On the other hand, given any integral vector $x \in P \cap \{0,1\}^n$, observe that $Y := \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^\top \in \widehat{\operatorname{LS}}_+(P)$, and so $x \in \operatorname{LS}_+(P)$. Thus, if we define $P_I := \operatorname{conv}\left(P \cap \{0,1\}^n\right)$ be the *integer hull* of P, then

$$P_I \subseteq \operatorname{LS}_+(P) \subseteq P$$

holds in general. Therefore, $\operatorname{LS}_+(P)$ contains the same set of integral solutions as P. Moreover, if P is a tractable set (i.e., one can optimize a linear function over P in polynomial time), then so is $\operatorname{LS}_+(P)$, and the inclusion is strict unless $P = P_I$. Thus, while it is generally $\mathcal{NP}$-hard to optimize over the integer hull P_I, $\operatorname{LS}_+(P)$ offers a tractable relaxation of P_I that is tighter than the initial relaxation P.

Moreover, we can apply LS$_+$ multiple times to obtain yet tighter relaxations. Given $k \in \mathbb{N}$, let $\operatorname{LS}_+^k(P)$ be the set obtained from applying k successive LS$_+$ operations to P. Then it is well known that

$$P_I = \operatorname{LS}_+^n(P) \subseteq \operatorname{LS}_+^{n-1}(P) \subseteq \cdots \subseteq \operatorname{LS}_+(P) \subseteq P.$$

Thus, LS_+ generates a hierarchy of progressively tighter convex relaxations which converge to P_I in no more than n iterations. The reader may refer to Lovász and Schrijver [LS91] for a proof of this fact and some other properties of the LS_+ operator.

2.2. **The stable set polytope and the LS_+-rank of graphs.** Given a graph $G := (V(G), E(G))$, we define its *fractional stable set polytope* to be

$$\mathrm{FRAC}(G) := \left\{ x \in [0,1]^{V(G)} : x_i + x_j \leq 1, \forall \{i,j\} \in E(G) \right\}.$$

We also define

$$\mathrm{STAB}(G) := \mathrm{FRAC}(G)_I = \mathrm{conv}\left(\mathrm{FRAC}(G) \cap \{0,1\}^{V(G)} \right)$$

to be the *stable set polytope* of G. Notice that $\mathrm{STAB}(G)$ is exactly the convex hull of the incidence vectors of stable sets in G. Also, to reduce cluttering, we will write $\mathrm{LS}_+^k(G)$ instead of $\mathrm{LS}_+^k(\mathrm{FRAC}(G))$.

Given a graph G, we let $r_+(G)$ denote the LS_+-*rank* of G, which is defined to be the smallest integer k where $\mathrm{LS}_+^k(G) = \mathrm{STAB}(G)$. It is well known that $r_+(G) = 0$ (i.e., $\mathrm{STAB}(G) = \mathrm{FRAC}(G)$) if and only if G is bipartite. There has also been recent interest and progress in classifying graphs with $r_+(G) = 1$, which are commonly called LS_+-*perfect graphs* [BENT13, BENT17, Wag22, BENW23]. While the set of LS_+-perfect graphs contains all perfect graphs (whose stable set polytopes are defined by only clique and nonnegativity inequalities), it also contains other well-studied families of graphs such as odd holes, odd antiholes, and wheels. It remains an open problem to find a simple combinatorial characterization of LS_+-perfect graphs.

Next, we mention two simple graph operations that have been critical to the analyses of the LS_+-ranks of graphs. Given a graph G and $S \subseteq V(G)$, we let $G - S$ denote the subgraph of G induced by the vertices $V(G) \setminus S$, and call $G - S$ the graph obtained by the *deletion* of S. (When $S = \{i\}$ for some vertex i, we simply write $G - i$ instead of $G - \{i\}$.) Next, given $i \in V(G)$, let $\Gamma(i) := \{j \in V(G) : \{i,j\} \in E(G)\}$ (i.e., $\Gamma(i)$ is the set of vertices that are adjacent to i). Then the graph obtained from the *destruction* of i in G is defined as

$$G \ominus i := G - (\{i\} \cup \Gamma(i)).$$

Then we have the following.

Theorem 2. *For every graph G,*

(i) [LS91, Corollary 2.16] $r_+(G) \leq \max\{r_+(G \ominus i) : i \in V(G)\} + 1$*;*
(ii) [LT03, Theorem 36] $r_+(G) \leq \min\{r_+(G - i) : i \in V(G)\} + 1$*.*

We are interested in studying relatively small graphs with high LS_+-rank — that is, graphs whose stable set polytope is difficult to obtain for LS_+. First, Lipták and the second author [LT03, Theorem 39] proved the following general upper bound:

Theorem 3. *For every graph G, $r_+(G) \leq \left\lfloor \frac{|V(G)|}{3} \right\rfloor$.*

In Section 3, we prove that the family of graphs H_k satisfies $r_+(H_k) = \Theta(|V(H_k)|)$. This shows that Theorem 3 is asymptotically tight, and rules out the possibility of a sublinear upper bound on the LS_+-rank of a general graph.

Theorem 3 also raises the natural question: Are there graphs on 3ℓ vertices that have LS_+-rank exactly ℓ? Such ℓ-minimal graphs have been found for $\ell = 2$ [LT03] and for $\ell = 3$ [EMN06]. In Section 5, we prove that there are also such examples for $\ell = 4$, and discuss some structural properties of ℓ-minimal graphs in general.

3. A family of graphs G with LS$_+$-rank $\Theta(|V(G)|)$

3.1. **The graphs H_k and their basic properties.** Recall the family of graphs H_k defined in Section 1. For convenience, we let $[k]_p := \{i_p : i \in [k]\}$ for each $p \in \{0, 1, 2\}$. Then notice that one can also construct H_k by starting with a complete bipartite graph with bipartitions $[k]_0$ and $[k]_2$, and then for every $i \in [k]$ subdividing the edge $\{i_0, i_2\}$ into a path of length 2 and labelling the new vertex i_1. Figure 2 illustrates alternative drawings for H_k which highlight this aspect of the family of graphs.

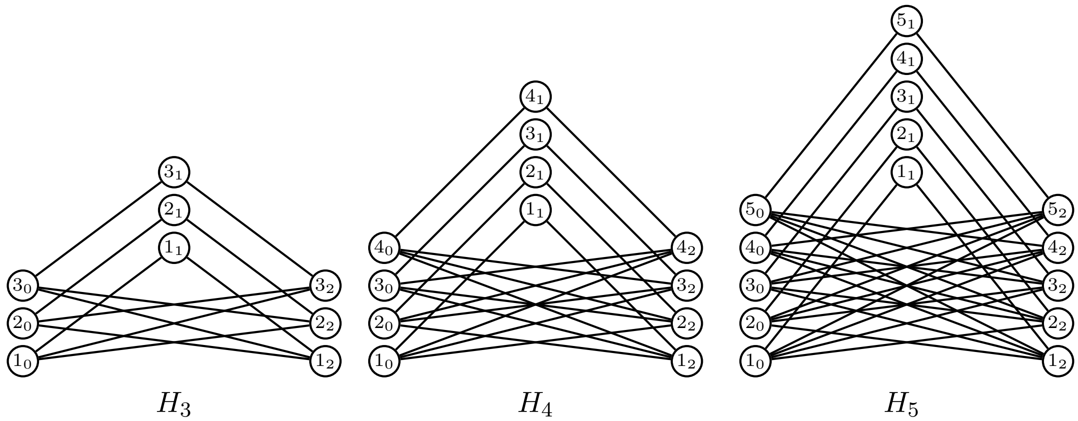

Figure 2. Alternative drawings of the graphs H_k

Notice that the graphs H_k have very rich symmetries, and we mention two particular automorphisms of H_k that are of particular interest. Define $\sigma_1 : V(H_k) \to V(H_k)$ where

$$\sigma_1(i_p) := \begin{cases} (i+1)_p & \text{if } 1 \leq i \leq k-1; \\ 1_p & \text{if } i = k. \end{cases} \tag{1}$$

Also define $\sigma_2 : V(H_k) \to V(H_k)$ where

$$\sigma_2(i_p) := i_{(2-p)} \text{ for every } i \in [k] \text{ and } p \in \{0, 1, 2\}. \tag{2}$$

Visually, σ_1 corresponds to rotating the drawings of H_k in Figure 1 counterclockwise by $\frac{2\pi}{k}$, and σ_2 corresponds to reflecting the drawings of H_k in Figure 2 along the centre vertical line. Also, notice that $H_k \ominus i$ is either isomorphic to H_{k-1} (if $i \in [k]_1$), or is bipartite (if $i \in [k]_0 \cup [k]_2$). As we shall see, these properties are very desirable in our subsequent analysis of H_k.

Next, we describe a family of facets of STAB(H_k) that are of particular interest. Given distinct $i, j \in [k]$, define

$$B_{i,j} := [3k] \setminus \{i_1, i_2, j_0, j_1\}.$$

Then we have the following.

Lemma 4. *For every $k \geq 2$ and for every $i, j \in [k]$, the linear inequality*

$$\sum_{\ell \in B_{i,j}} x_\ell \leq k - 1 \tag{3}$$

is a facet of STAB(H_k).

Proof. We prove our claim by induction on k. When $k = 2$ (3) gives an edge inequality, which is indeed a facet of STAB(H_2) since H_2 is the 6-cycle.

Next, assume $k \geq 3$. By the symmetry of H_k, it suffices to prove the claim for the case $i = 1$ and $j = 2$. First, since $B_{1,2}$ does not contain a stable set of size k, (3) is valid for $\text{STAB}(H_k)$. Next, by the inductive hypothesis,

$$\left(\sum_{\ell \in B_{1,2}} x_\ell \right) - (x_{k_0} + x_{k_1} + x_{k_2}) \leq k - 2 \tag{4}$$

is a facet of $\text{STAB}(H_{k-1})$, and so there exist stable sets $S_1, \ldots, S_{3k-3} \subseteq V(H_{k-1})$ whose incidence vectors are affinely independent and all satisfy (4) with equality. We then define $S_i' := S_i \cup \{k_1\}$ for all $i \in [3k-3]$, $S_{3k-2}' := [k]_0, S_{3k-1}' := [k]_2$, and $S_{3k}' := [k-1]_1 \cup \{k_0, k_2\}$. Then we see that the incidence vectors of $S_1', \ldots, S_{3k}'$ are affinely independent, and they all satisfy (3) with equality. This finishes the proof. □

3.2. **Working from the shadows to prove lower bounds on LS_+-rank.** Next, we aim to exploit the symmetries of H_k to help simplify our analysis of its LS_+-relaxations. Before we do that, we describe the broader framework of this reduction that shall also be useful in analyzing lift-and-project relaxations in other settings. Given a graph G, let $\text{Aut}(G)$ denote the automorphism group of G, and for every $\sigma \in \text{Aut}(G)$ we let $M_\sigma \in \mathbb{R}^{V(G) \times V(G)}$ be the permutation matrix corresponding to σ. We also let χ_S denote the incidence vector of a set S.

Then, given a graph G and $\mathcal{A} := \{A_1, \ldots, A_\ell\}$ a partition of $V(G)$, we say that a set of automorphisms $\mathcal{S} \subseteq \text{Aut}(G)$ is $\mathcal{A}$-*balancing* if, for every $i \in [\ell]$ and for every $j \in A_i$,

$$\sum_{\sigma \in \mathcal{S}} M_\sigma e_j = \frac{|\mathcal{S}|}{|A_i|} \chi_{A_i}.$$

In other words, the automorphisms in $\mathcal{S}$ only maps vertices in A_i to vertices in A_i for every $i \in [\ell]$. Moreover, for every $j \in A_i$, the $|\mathcal{S}|$ images of j under automorphisms in $\mathcal{S}$ spread over A_i evenly, with $|\{\sigma \in \mathcal{S} : \sigma(j) = q\}| = \frac{|\mathcal{S}|}{|A_i|}$ for every $q \in A_i$.

For example, for the graph H_k, consider the vertex partition $\mathcal{A}_1 := \{[k]_0, [k]_1, [k]_2\}$ and $\mathcal{S}_1 := \left\{\sigma_1^i : i \in [k]\right\}$ (where σ_1 is defined in (1)). Then observe that for every $p \in \{0, 1, 2\}$ and $j \in [k]_p$,

$$\sum_{\sigma \in \mathcal{S}_1} M_\sigma e_j = \chi_{[k]_p},$$

and so $\mathcal{S}_1$ is $\mathcal{A}_1$-balancing. Furthermore, if we define $\mathcal{A}_2 := \{[k]_0 \cup [k]_2, [k]_1\}$ and

$$\mathcal{S}_2 := \left\{ \sigma_1^i \circ \sigma_2^j : i \in [k], j \in [2] \right\} \tag{5}$$

(where σ_2 is defined in (2)), one can similarly show that $\mathcal{S}_2$ is $\mathcal{A}_2$-balancing. Then we have the following.

Lemma 5. *Suppose G is a graph, $\mathcal{A} := \{A_1, \ldots, A_\ell\}$ is a partition of $V(G)$, and $\mathcal{S} \subseteq \text{Aut}(G)$ is $\mathcal{A}$-balancing. If $x \in \text{LS}_+^p(G)$, then*

$$x' := \sum_{i=1}^{\ell} \left(\frac{1}{|A_i|} \sum_{j \in A_i} x_j \right) \chi_{A_i}$$

also belongs to $\text{LS}_+^p(G)$.

Proof. First, notice that for every $p \geq 0$, if $x \in \text{LS}_+^p(G)$ and $\sigma \in \text{Aut}(G)$, then $M_\sigma x \in \text{LS}_+^p(G)$. To see this, notice that the property clearly holds when $p = 0$ (and $\text{LS}_+^0(G) = \text{FRAC}(G)$), and that the property is preserved after each application LS_+ because of the invariance of LS_+ under the automorphisms of G. Thus, by the convexity of $\mathrm{LS}_+^p(G)$ and the invariance of LS_+ under the automorphisms of G, given any $\mathcal{S} \subseteq \mathrm{Aut}(G)$, the convex combination

$$x' := \frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} M_\sigma x$$

also belongs to $\mathrm{LS}_+^p(G)$. Now since $\mathcal{S}$ is $\mathcal{A}$-balancing, it follows that $x'_q = \frac{1}{|A_i|} \sum_{j \in A_i} x_j$ for every $i \in [\ell]$ and $q \in A_i$. Then our claim follows. $\square$

Thus, the presence of $\mathcal{A}$-balancing automorphisms allow us to focus on points in $\mathrm{LS}_+^p(G)$ with fewer distinct entries. That is, instead of fully analyzing a family of SDPs in $\mathbb{S}_+^{\Omega(n^p)}$ or its projections $\mathrm{LS}_+^p(G)$, we can work with a spectrahedral shadow in $[0,1]^\ell$ for a part of the analysis. For instance, in the extreme case when G is vertex-transitive, we see that the entire automorphism group $\mathrm{Aut}(G)$ is $\{V(G)\}$-balancing, and so for every $x \in \mathrm{LS}_+^p(G)$ we can use the argument above to deduce that $\frac{1}{|V(G)|} \sum_{j \in V(G)} x_j \bar{e} \in \mathrm{LS}_+^p(G)$.

Now we turn our focus back to the graphs H_k. The presence of an $\mathcal{A}_2$-balancing set of automorphisms (as described in (5)) motivates the study of points in $\mathrm{LS}_+^p(H_k)$ of the following form. Given $a, b \in \mathbb{R}$, we define the vector $w_k(a,b) \in \mathbb{R}^{3k}$ such that

$$[w_k(a,b)]_i := \begin{cases} a & \text{if } i \in [k]_0 \cup [k]_2; \\ b & \text{if } i \in [k]_1. \end{cases}$$

We next describe another valid inequality of $\mathrm{STAB}(H_k)$ that is not a facet, but has the same LS_+-rank as the facets described in (3).

Lemma 6. *The linear inequality*

$$w_k(k-1, k-2)^\top x \leq k(k-1) \tag{6}$$

is valid for $\mathrm{STAB}(H_k)$. *Moreover, the inequalities* (3) *and* (6) *have the same* LS_+*-rank.*

Proof. First, from Lemma 4, we know that the following inequality is valid for $\mathrm{STAB}(H_k)$:

$$\sum_{(i,j) \in [k]^2, i \neq j} \left(\sum_{\ell \in B_{i,j}} x_\ell \right) \leq \sum_{(i,j) \in [k]^2, i \neq j} (k-1). \tag{7}$$

Now, the right hand side of (7) is $k(k-1)(k-1)$. On the other hand, since $|B_{i,j} \cap [k]_0| = |B_{i,j} \cap [k]_2| = k-1$ for all i,j, we see that if $\ell \in [k]_0 \cup [k]_2$, then x_ℓ has coefficient $(k-1)(k-1)$ in the left hand side of (7). A similar argument shows that x_ℓ has coefficient $(k-1)(k-2)$ for all $\ell \in [k]_1$. Thus, (6) is indeed $\frac{1}{k-1}$ times (7), and so the former is valid for $\mathrm{STAB}(H_k)$.

Next, suppose there exists $x \in \mathrm{LS}_+^p(H_k)$ which violates (6). Due to the presence of the $\mathcal{A}_2$-balancing automorphisms $\mathcal{S}_2$, as well as Lemma 5, the point

$$x' := \left(\frac{1}{2k} \sum_{\ell \in [k]_0 \cup [k]_2} x_\ell \right) \chi_{[k]_0 \cup [k]_2} + \left(\frac{1}{k} \sum_{\ell \in [k]_1} x_\ell \right) \chi_{[k]_1}$$

also belongs to $\mathrm{LS}_+^p(H_k)$. For convenience, let $a := \frac{1}{2k} \sum_{\ell \in [k]_0 \cup [k]_2} x_\ell$ and $b := \frac{1}{k} \sum_{\ell \in [k]_1} x_\ell$ (so $x' = w_k(a,b)$). Then the fact that x violates (6) implies that

$$(k-1)(2ka) + (k-2)(kb) > k(k-1),$$

which implies that $2(k-1)a + (k-2)b > k-1$. Then it follows that x' would also violate (3), and the claim follows. $\square$

Thus, we now know that if the inequality (6) has LS_+-rank greater than p, then it must be violated by a point in $\text{LS}_+^p(H_k)$ of the form $w_k(a,b)$. This critical insight enables us to capture important properties of $\text{LS}_+^p(H_k)$ by analyzing a corresponding "shadow" of the set in $\mathbb{R}^2$. More explicitly, given $P \subseteq \mathbb{R}^{3k}$, we define

$$\Phi(P) \coloneqq \left\{(a,b) \in \mathbb{R}^2 : w_k(a,b) \in P\right\}.$$

For example, it is not hard to see that

$$\Phi(\text{FRAC}(H_k)) = \text{conv}\left(\left\{(0,0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0,1)\right\}\right).$$

Likewise, one can check that $w_k\left(\frac{1}{2}, 0\right), w_k\left(\frac{1}{k}, \frac{k-1}{k}\right) \in \text{STAB}(H_k)$ for every k. This and Lemma 6 imply that

$$\Phi(\text{STAB}(H_k)) = \text{conv}\left(\left\{(0,0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{k}, \frac{k-1}{k}\right), (0,1)\right\}\right).$$

Even though $\text{STAB}(H_k)$ is an integral polytope, notice that $\Phi(\text{STAB}(H_k))$ is not integral. Nonetheless, it is clear that

$$\text{LS}_+^p(H_k) = \text{STAB}(H_k) \Rightarrow \Phi(\text{LS}_+^p(H_k)) = \Phi(\text{STAB}(H_k)).$$

Thus, to show that $r_+(H_k) > p$, it suffices to find a point $(a,b) \in \Phi(\text{LS}_+^p(H_k)) \setminus \Phi(\text{STAB}(H_k)$. More generally, given a graph G with a set of $\mathcal{A}$-balancing automorphisms where $\mathcal{A}$ partitions $V(G)$ into ℓ sets, one can adapt our approach and study the LS_+-relaxations of G via analyzing ℓ-dimensional shadows of these sets.

3.3. $\Phi(\text{LS}_+(H_k))$ — **the shadow of the first relaxation.** Next, we aim to study the set $\Phi(\text{LS}_+(H_k))$. To do that, we first look into potential certificate matrices for $w_k(a,b)$ that have plenty of symmetries. Given $k \in \mathbb{N}$ and $a,b,c,d \in \mathbb{R}$, we define the matrix $W_k(a,b,c,d) \in \mathbb{R}^{(3k+1)\times(3k+1)}$ such that $W_k(a,b,c,d) \coloneqq \begin{bmatrix} 1 & w_k(a,b)^\top \\ w_k(a,b) & \overline{W} \end{bmatrix}$, where

$$\overline{W} \coloneqq \begin{bmatrix} a & 0 & a-c \\ 0 & b & 0 \\ a-c & 0 & a \end{bmatrix} \otimes I_k + \begin{bmatrix} c & a-c & 0 \\ a-c & d & a-c \\ 0 & a-c & c \end{bmatrix} \otimes (J_k - I_k).$$

Note that $\otimes$ denotes the Kronecker product, I_k is the k-by-k identity matrix, and J_k is the k-by-k matrix of all ones. Also, the columns of $\overline{W}$ are indexed by the vertices $1_0, 1_1, 1_2, 2_0, 2_1, 2_2, \ldots$ from left to right, with the rows following the same ordering. Then we have the following.

Lemma 7. *Let $k \in \mathbb{N}$ and $a,b,c,d \in \mathbb{R}$. Then $W_k(a,b,c,d) \succeq 0$ if and only if all of the following holds:*

(S1) $c \geq 0$;
(S2) $a - c \geq 0$;
(S3) $(b-d) - (a-c) \geq 0$;
(S4) $2a + (k-2)c - 2ka^2 \geq 0$;
(S5) $(2a + (k-2)c - 2ka^2)(2b + 2(k-1)d - 2kb^2) - (2(k-1)(a-c) - 2kab)^2 \geq 0$.

Proof. Define matrices $\overline{W}_1, \overline{W}_2, \overline{W}_3 \in \mathbb{R}^{3k \times 3k}$ where

$$\overline{W}_1 \coloneqq \frac{1}{2} \cdot J_k \otimes \begin{bmatrix} c & 0 & -c \\ 0 & 0 & 0 \\ -c & 0 & c \end{bmatrix},$$

$$\overline{W}_2 \coloneqq \frac{1}{k} \cdot (kI_k - J_k) \otimes \begin{bmatrix} a-c & c-a & a-c \\ c-a & b-d & c-a \\ a-c & c-a & a-c \end{bmatrix},$$

$$\overline{W}_3 \coloneqq \frac{1}{2k} \cdot J_k \otimes \begin{bmatrix} 2a+(k-2)c-2ka^2 & 2(k-1)(a-c)-2kab & 2a+(k-2)c-2ka^2 \\ 2(k-1)(a-c)-2kab & 2b+2(k-1)d-2kb^2 & 2(k-1)(a-c)-2kab \\ 2a+(k-2)c-2ka^2 & 2(k-1)(a-c)-2kab & 2a+(k-2)c-2ka^2 \end{bmatrix}.$$

Then

$$\begin{aligned} &\overline{W}_1 + \overline{W}_2 + \overline{W}_3 \\ &= \begin{bmatrix} a-a^2 & -ab & a-c-a^2 \\ -ab & b-b^2 & -ab \\ a-c-a^2 & -ab & a-ab^2 \end{bmatrix} \otimes I_k + \begin{bmatrix} c-a^2 & a-c-ab & -a^2 \\ a-c-ab & d-b^2 & a-c-ab \\ -a^2 & a-c-ab & c-a^2 \end{bmatrix} \otimes (J_k - I_k) \\ &= \overline{W} - w_k(a,b)(w_k(a,b))^\top, \end{aligned}$$

which is a Schur complement of $W_k(a,b,c,d)$. Moreover, observe that the columns of $\overline{W}_i$ and $\overline{W}_j$ are orthogonal whenever $i \neq j$. Thus, we see that $W_k(a,b,c,d) \succeq 0$ if and only if $\overline{W}_1, \overline{W}_2$, and $\overline{W}_3$ are all positive semidefinite. Now observe that

$$\begin{aligned} \overline{W}_1 \succeq 0 &\iff \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \succeq 0 \iff \text{(S1)}, \\ \overline{W}_2 \succeq 0 &\iff \begin{bmatrix} a-c & c-a \\ c-a & b-d \end{bmatrix} \succeq 0 \iff \text{(S2) and (S3)}, \\ \overline{W}_3 \succeq 0 &\iff \begin{bmatrix} 2a+(k-2)c-2ka^2 & 2(k-1)(a-c)-2kab \\ 2(k-1)(a-c)-2kab & 2b+2(k-1)d-2kb^2 \end{bmatrix} \succeq 0 \iff \text{(S4) and (S5)}. \end{aligned}$$

Thus, the claim follows. □

Next, for convenience, define $q_k \coloneqq 1 - \sqrt{\frac{k}{2k-2}}$, and

$$p_k(x,y) \coloneqq (2x^2 - x) + 2q_k^2(y^2 - y) + 4q_k xy.$$

Notice that the curve $p_k(x,y) = 0$ is a parabola for all $k \geq 3$. Then, using Lemma 7, we have the following.

Proposition 8. *For every $k \geq 4$,*

$$\Phi(\text{LS}_+(H_k)) \supseteq \left\{(x,y) \in \mathbb{R}^2 : p_k(x,y) \leq 0, x+y \leq 1, x \geq 0, y \geq 0\right\}. \tag{8}$$

Proof. For convenience, let C denote the set on the right hand side of (8). Notice that the boundary points of the triangle $\{(x,y) : x+y \leq 1, x \geq 0, y \geq 0\}$ which lie in C are also boundary points of $\Phi(\text{STAB}(H_k))$. Thus, let us define the set of points

$$C_0 \coloneqq \left\{(x,y) \in \mathbb{R}^2 : p_k(x,y) = 0, \frac{1}{k} < x < \frac{1}{2}\right\}.$$

To prove our claim, it suffices to prove that for all $(a,b) \in C_0$, there exist $c,d \in \mathbb{R}$ such that $W_k(a,b,c,d)$ certifies $w_k(a,b) \in \text{LS}_+(H_k)$.

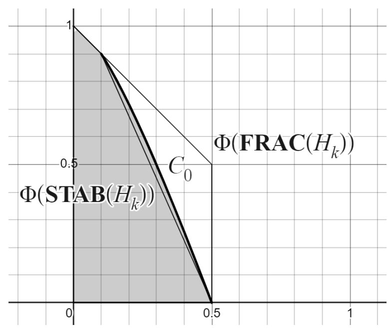

Figure 3. Visualizing the set C for the case $k = 10$

To help visualize our argument, Figure 3 illustrates the set C for the case of $k = 10$. The figure is produced in Desmos. A dynamic version of the demonstration is available at the following link:

`https://www.desmos.com/calculator/r63dsy4nax`

Now given $(a, b) \in \mathbb{R}^2$ (not necessarily in C_0), consider the conditions (S3) and (S5) from Lemma 7:

$$b - a + c \geq d, \tag{9}$$

$$(2a + (k-2)c - 2ka^2)(2b + 2(k-1)d - 2kb^2) - (2(k-1)(a-c) - 2kab)^2 \geq 0. \tag{10}$$

If we substitute $d = b - a + c$ into (10) and solve for c that would make both sides equal, we would obtain the quadratic equation $p_2c^2 + p_1c + p_0 = 0$ where

$$\begin{aligned} p_2 &:= (k-2)(2(k-1)) - (-2(k-1))^2, \\ p_1 &:= (k-2)(2b + 2(k-1)(b-a) - 2kb^2) + (2a - 2ka^2)(2(k-1)) \\ &\quad - 2(-2(k-1))(2(k-1)a - 2kab), \\ p_0 &:= (2a - 2ka^2)(2b + 2(k-1)(b-a) - 2kb^2) - (2(k-1)a - 2kab)^2. \end{aligned}$$

We then define

$$c := \frac{-p_1}{2p_2} = -a^2 - 2ab - \frac{b^2}{2} + \frac{3a}{2} + \frac{b}{2} + \frac{b\,(b-1)}{2\,(k-1)},$$

and $d := b - a + c$. We claim that, for all $(a, b) \in C_0$, $W_k(a, b, c, d)$ would certify $w_k(a, b) \in \text{LS}_+(H_k)$. First, we provide some intuition for the choice of c. Let $\overline{q}_k := 1 + \sqrt{\frac{k}{2k-2}}$ and

$$\overline{p}_k(x, y) := (2x^2 - x) + 2\overline{q}_k^2(y^2 - y) + 4\overline{q}_kxy.$$

Then, if we consider the discriminant $\Delta p := p_1^2 - 4p_0p_2$, one can check that

$$\Delta p = 4(k-1)^2 p_k(a, b)\overline{p}_k(a, b).$$

Thus, when $\Delta p > 0$, there would be two solutions to the quadratic equation $p_2x^2 + p_1x + p_0 = 0$, and c would be defined as the midpoint of these solutions. In particular, when $(a, b) \in C_0$, $p_k(a, b) = \Delta p = 0$, and so $c = \frac{-p_1}{2p_2}$ would indeed be the unique solution that satisfies both (9) and (10) with equality.

Now we verify that $Y := W_k(a, b, c, d)$, as defined, satisfies all the conditions imposed by LS_+. We first show that $W_k(a, b, c, d) \succeq 0$ by verifying the conditions from Lemma 7. Notice that

(S3) and (S5) must hold by the choice of c and d. Next, we check (S1), namely $c \geq 0$. Define the region

$$T := \left\{ (x, y) \in \mathbb{R}^2 : \frac{1}{k} \leq x \leq \frac{1}{2}, \frac{(1-2x)(k-1)}{k-2} \leq y \leq 1 - x \right\}.$$

In other words, T is the triangular region with vertices $\left(\frac{1}{k}, \frac{k-1}{k}\right), \left(\frac{1}{2}, 0\right)$, and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus, T contains C_0. and it suffices to show that $c \geq 0$ over T. Fixing k and viewing c as a function of a and b, we obtain

$$\frac{\partial c}{\partial a} = -2a - 2b + \frac{3}{2}, \quad \frac{\partial c}{\partial b} = \frac{(-4a - 2b + 1)k + 4a + 4b - 2}{2k - 2}.$$

Solving $\frac{\partial c}{\partial a} = \frac{\partial c}{\partial b} = 0$, we obtain the unique solution $(a, b) = \left(\frac{-k+2}{4k}, \frac{4k-2}{4k}\right)$, which is outside of T. Next, one can check that c is non-negative over the three edges of T, and we conclude that $c \geq 0$ over T, and thus (S1) holds. The same approach also shows that both $a - c$ and $2a + (k-2)c - 2ka^2$ are non-negative over T, and thus (S2) and (S4) hold as well, and we conclude that $Y \succeq 0$.

Next, we verify that $Ye_i, Y(e_0 - e_i) \in \text{cone}(\text{FRAC}(H_k))$. By the symmetry of H_k, it suffices to verify these conditions for the vertices $i = 1_0$ and $i = 1_1$.

- Ye_{1_0}: Define $S_1 := \{1_0, 1_2\} \cup \{i_1 : 2 \leq i \leq k\}$ and $S_2 := [k]_0$. Observe that both S_1, S_2 are stable sets of H_k, and that

(11) $$Ye_{1_0} = (a - c) \begin{bmatrix} 1 \\ \chi_{S_1} \end{bmatrix} + c \begin{bmatrix} 1 \\ \chi_{S_2} \end{bmatrix},$$

  Since we verified above that $c \geq 0$ and $a - c \geq 0$, $Ye_{1_0} \in \text{cone}(\text{STAB}(H_k))$, which is contained in $\text{cone}(\text{FRAC}(H_k))$.
- Ye_{1_1}: The non-trivial edge inequalities imposed by $Ye_{1_1} \in \text{cone}(\text{FRAC}(H_k))$ are

(12) $$[Ye_{1_1}]_{2_2} + Y[e_{1_1}]_{3_0} \leq [Ye_{1_1}]_0 \Rightarrow 2(a - c) \leq b,$$

(13) $$[Ye_{1_1}]_{2_0} + [Ye_{1_1}]_{2_1} \leq [Ye_{1_1}]_0 \Rightarrow a - c + d \leq b.$$

  Note that (13) is identical to (S3), which we have already established. Next, we know from (S4) that $c \geq \frac{2ka^2 - 2a}{k-2}$. That together with the fact that $2(k-1)a + (k-2)b \geq k - 1$ for all $(a, b) \in C_0$ and $k \geq 4$ implies (12).
- $Y(e_0 - e_{1_0})$: The non-trivial edge inequalities imposed by $Y(e_0 - e_{1_0}) \in \text{cone}(\text{FRAC}(H_k))$ are

(14) $$[Y(e_0 - e_{1_0})]_{2_2} + [Y(e_0 - e_{1_0})]_{3_0} \leq [Y(e_0 - e_{1_0})]_0 \Rightarrow a + (a - c) \leq 1 - a,$$

(15) $$[Y(e_0 - e_{1_0})]_{2_1} + [Y(e_0 - e_{1_0})]_{2_2} \leq [Y(e_0 - e_{1_0})]_0 \Rightarrow (b - a + c) + a \leq 1 - a.$$

  (14) follows from (12) and the fact that $a - c \geq 0$. For (15), we aim to show that $a + b + c \leq 1$. Define the quantity

$$g(x, y) := 1 - \frac{5}{2}x - \frac{3}{2}y + x^2 + \frac{1}{2}y^2 + 2xy - \frac{y\,(y-1)}{2\,(k-1)}.$$

  Then $g(a, b) = 1 - a - b - c$. Notice that, for all k, the curve $g(x, y) = 0$ intersects with C at exactly three points: $(0, 1), \left(\frac{1}{k}, \frac{k-1}{k}\right)$, and $\left(\frac{1}{2}, 0\right)$. In particular, the curve does not intersect the interior of C. Therefore, $g(x, y)$ is either non-negative or non-positive over C. Since $g(0, 0) = 1$, it is the former. Hence, $g(x, y) \geq 0$ over C (and hence C_0), and (15) holds.

- $Y(e_0 - e_{1_1})$: The non-trivial edge inequalities imposed by $Y(e_0 - e_{1_1}) \in \operatorname{cone}(\text{FRAC}(H_k))$ are

$$[Y(e_0 - e_{1_1})]_{1_0} + [Y(e_0 - e_{1_1})]_{2_2} \leq [Y(e_0 - e_{1_1})]_0 \Rightarrow a + c \leq 1 - b, \tag{16}$$

$$[Y(e_0 - e_{1_1})]_{2_0} + [Y(e_0 - e_{1_1})]_{2_1} \leq [Y(e_0 - e_{1_1})]_0 \Rightarrow c + (b - d) \leq 1 - b. \tag{17}$$

(16) is identical to (14), which we have verified above. Finally, (17) follows from (S3) and the fact that $a + b \leq 1$ for all $(a, b) \in C$.

This completes the proof. □

An immediate consequence of Proposition 8 is the following.

Corollary 9. *For all $k \geq 4$, $r_+(H_k) \geq 2$.*

Proof. For every $k \geq 4$, the set described in (8) is not equal to $\Phi(\text{STAB}(H_k))$. Thus, there exists $w_k(a, b) \in \text{LS}_+(H_k) \setminus \text{STAB}(H_k)$ for all $k \geq 4$, and the claim follows. □

Corollary 9 is sharp — notice that destroying any vertex in H_3 yields a bipartite graph, so it follows from Theorem 2(i) that $r_+(H_3) = 1$. Also, since destroying a vertex in H_4 either results in H_3 or a bipartite graph, we see that $r_+(H_4) = 2$.

3.4. **Showing $r_+(H_k) = \Theta(k)$.** We now develop a few more tools that we need to establish the main result of this section. Again, to conclude that $r_+(H_k) > p$, it suffices to show that $\Phi(\text{LS}_+^p(H_k)) \supset \Phi(\text{STAB}(H_k))$. In particular, we will do so by finding a point in $\Phi(\text{LS}_+^p(H_k)) \setminus \Phi(\text{STAB}(H_k))$ that is very close to the point $\left(\frac{1}{k}, \frac{k-1}{k}\right)$. Given $(a, b) \in \mathbb{R}^2$, let

$$s_k(a, b) := \frac{\frac{k-1}{k} - b}{\frac{1}{k} - a}.$$

That is, $s_k(a, b)$ is the slope of the line that contains the points (a, b) and $\left(\frac{1}{k}, \frac{k-1}{k}\right)$. Next, define

$$f(k, p) := \sup \left\{ s_k(a, b) : (a, b) \in \Phi(\text{LS}_+^p(H_k)), a > \frac{1}{k} \right\}.$$

In other words, $f(k, p)$ is the slope of the tangent line to $\Phi(\text{LS}_+^p(H_k))$ at the point $\left(\frac{1}{k}, \frac{k-1}{k}\right)$ towards the right hand side. Thus, for all $\ell < f(k, p)$, there exists $\epsilon > 0$ where the point $\left(\frac{1}{k} + \epsilon, \frac{k-1}{k} + \ell\epsilon\right)$ belongs to $\Phi(\text{LS}_+^p(H_k))$. For $p = 0$ (and so $\text{LS}_+^p(H_k) = \text{FRAC}(H_k)$), observe that $f(k, 0) = -1$ for all $k \geq 2$ (attained by the point $\left(\frac{1}{2}, \frac{1}{2}\right)$). Next, for $p = 1$, consider the polynomial $p_k(x, y)$ defined before Proposition 8. Then any point (x, y) on the curve $p_k(x, y) = 0$ has slope

$$\frac{\partial}{\partial x} p_k(x, y) = \frac{1 - 4x - 4q_k y}{4q_k^2 y - q_k + 4q_k x}.$$

Thus, by Proposition 8,

$$f(k, 1) \geq \left. \frac{\partial}{\partial x} p_k(x, y) \right|_{(x,y) = \left(\frac{1}{k}, \frac{k-1}{k}\right)} = -1 - \frac{k}{3k^2 - 2(k-1)^2 \sqrt{\frac{2k}{k-1}} - 4k} \tag{18}$$

for all $k \geq 4$. Finally, if $p \geq r_+(H_k)$, then $f(k, p) = -\frac{2(k-1)}{k-2}$ (attained by the point $\left(\frac{1}{2}, 0\right) \in \Phi(\text{STAB}(H_k))$).

We will prove our LS_+-rank lower bound on H_k by showing that $f(k, p) > -\frac{2(k-1)}{k-2}$ for some $p = \Theta(k)$. To do so, we first show that the recursive structure of H_k allows us to establish $(a, b) \in \Phi(\text{LS}_+^p(H_k))$ by verifying (among other conditions) the membership of two particular points in $\Phi(\text{LS}_+^{p-1}(H_{k-1}))$, which will help us relate the quantities $f(k-1, p-1)$ and $f(k, p)$.

Next, we bound the difference $f(k-1,p-1) - f(k,p)$ from above, which implies that it takes LS_+ many iterations to knock the slopes $f(k,p)$ from that of $\Phi(\text{FRAC}(H_k))$ down to that of $\Phi(\text{STAB}(H_k))$.

First, here is a tool that will help us verify certificate matrices recursively.

Lemma 10. *Suppose $a,b,c,d \in \mathbb{R}$ satisfy all of the following:*

(i) $W_k(a,b,c,d) \succeq 0$,
(ii) $2b+2c-d \leq 1$,
(iii) $w_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right), w_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right) \in \text{LS}_+^{p-1}(H_{k-1})$.

Then $W_k(a,b,c,d)$ certifies $w_k(a,b) \in \text{LS}_+^{p}(H_k)$.

Proof. For convenience, let $Y := W_k(a,b,c,d)$. First, $Y \succeq 0$ from (i). Next, we focus on the following column vectors:

- Ye_{1_0}: $Y \succeq 0$ implies that $c \geq 0$ and $a-c \geq 0$ by Lemma 7. Then it follows from (11) that $Ye_{1_0} \in \text{cone}(\text{STAB}(H_k)) \subseteq \text{cone}(\text{LS}_+^{p-1}(H_k))$.
- Ye_{1_1}: (iii) implies $\begin{bmatrix} b \\ w_{k-1}(a-c,d) \end{bmatrix} \in \text{cone}(\text{LS}_+^{p-1}(H_{k-1}))$. Thus,
$$Ye_{1_1} = \begin{bmatrix} b \\ 0 \\ b \\ 0 \\ w_{k-1}(a-c,d) \end{bmatrix} \in \text{cone}(\text{LS}_+^{p-1}(H_k)).$$
- $Y(e_0 - e_{1_0})$: Let $S_1 := [k]_2$, which is a stable set in H_k. Then observe that
$$Y(e_0 - e_{1_0}) = c\begin{bmatrix} 1 \\ \chi_{S_1} \end{bmatrix} + \begin{bmatrix} 1-a-c \\ 0 \\ b \\ 0 \\ w_{k-1}(a-c, b-a+c) \end{bmatrix}.$$
By (iii) and the fact that $\text{cone}(\text{LS}_+^{p-1}(H_k))$ is a convex cone, it follows that $Y(e_0 - e_{1_0}) \in \text{cone}(\text{LS}_+^{p-1}(H_k))$.
- $Y(e_0 - e_{1_1})$: Define $S_2 := [k]_0$, $S_3 := \{1_0, 1_2\} \cup \{i_1 : 2 \leq i \leq k\}$, and $S_4 := \{i_1 : 2 \leq i \leq k\}$, which are all stable sets in H_k. Now observe that
$$\begin{aligned} Y(e_0 - e_{1_1}) =& c\begin{bmatrix} 1 \\ \chi_{S_1} \end{bmatrix} + c\begin{bmatrix} 1 \\ \chi_{S_2} \end{bmatrix} + (a-c)\begin{bmatrix} 1 \\ \chi_{S_3} \end{bmatrix} + (b-d-a+c)\begin{bmatrix} 1 \\ \chi_{S_4} \end{bmatrix} \\ &+ (1-2b-2c+d)\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$
Since $Y \succeq 0$, $b-d-a+c \geq 0$ from (S3). Also, $1-2b-2c+d \geq 0$ by (ii). Thus, $Y(e_0 - e_{1_1})$ is a sum of vectors in $\text{cone}(\text{STAB}(H_k))$, and thus belongs to $\text{cone}(\text{LS}_+^{p-1}(H_k))$.

By the symmetry of H_k and $W_k(a,b,c,d)$, it suffices to verify the membership conditions for the above columns. Thus, it follows that $W_k(a,b,c,d)$ indeed certifies $w_k(a,b) \in \text{LS}_+^{p}(H_k)$. ☐

Example 11. We illustrate Lemma 10 by using it to show that $r_+(H_7) \geq 3$. Let $k = 7, a = 0.1553, b = 0.8278, c = 0.005428$, and $d = 0.6665$. Then one can check (via Lemma 7) that $W_k(a,b,c,d) \succeq 0$, and $2b+2c-d \leq 1$. Also, one can check that $w_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right)$ and $w_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right)$ both belong to $\text{LS}_+(H_{k-1})$ using Proposition 8. Thus, Lemma 10 applies, and $w_k(a,b) \in \text{LS}_+^2(H_k)$. Now observe that $2(k-1)a + (k-2)b = 6.0026 > k-1$, and so $w_k(a,b) \notin \text{STAB}(H_k)$, and we conclude that $r_+(H_7) \geq 3$.

Next, we apply Lemma 10 iteratively to find a lower bound for the LS_+-rank of H_k as a function of k. The following is an updated version of Lemma 10 that gets us a step closer to directly relating $f(k,p)$ and $f(k-1,p-1)$.

Lemma 12. *Suppose $a,b,c,d \in \mathbb{R}$ satisfy all of the following:*
- (i) $W_k(a,b,c,d) \succeq 0$,
- (ii) $2b + 2c - d \leq 1$,
- (iii) $\max\left\{ s_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right), s_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right)\right\} \leq f(k-1,p-1)$.

Then $f(k,p) \geq s_k(a,b)$.

Proof. Given $a,b,c,d \in \mathbb{R}$ that satisfy the given assumptions, define

$$a(\lambda) := \frac{\lambda}{k} + (1-\lambda)a, \qquad b(\lambda) := \frac{\lambda(k-1)}{k} + (1-\lambda)b,$$

$$c(\lambda) := (1-\lambda)c, \qquad d(\lambda) := \frac{\lambda(k-2)}{k} + (1-\lambda)d.$$

Then notice that

$$W_k(a(\lambda), b(\lambda), c(\lambda), d(\lambda)) = \lambda W_k\left(\frac{1}{k}, \frac{k-1}{k}, 0, \frac{k-2}{k}\right) + (1-\lambda)W_k(a,b,c,d). \tag{19}$$

Once can check (e.g., via Lemma 7) that $W_k\left(\frac{1}{k}, \frac{k-1}{k}, 0, \frac{k-2}{k}\right) \succeq 0$ for all $k \geq 2$. Since $W_k(a,b,c,d) \succeq 0$ from (i), it follows from (19) and the convexity of the positive semidefinite cone that $W_k(a(\lambda), b(\lambda), c(\lambda), d(\lambda)) \succeq 0$ for all $\lambda \in [0,1]$.

Now observe that for all $\lambda > 0$, $s_k(a(\lambda), b(\lambda)) = s_k(a,b)$, $s_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{b(\lambda)}, \frac{d(\lambda)}{b(\lambda)}\right) = s_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right)$, and $s_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{1-a(\lambda)-c(\lambda)}, \frac{b(\lambda)-a(\lambda)+c(\lambda)}{1-a(\lambda)-c(\lambda)}\right) = s_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right)$. By assumption (iii), there must be a sufficiently small $\lambda > 0$ where $w_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{b(\lambda)}, \frac{d(\lambda)}{b(\lambda)}\right)$ and $w_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{1-a(\lambda)-c(\lambda)}, \frac{b(\lambda)-a(\lambda)+c(\lambda)}{1-a(\lambda)-c(\lambda)}\right)$ are both contained in $\text{LS}_+^{p-1}(H_k)$. Then Lemma 10 implies that $w_k(a(\lambda), b(\lambda)) \in \text{LS}_+^p(H_k)$, and the claim follows. $\square$

Next, we define four values corresponding to each k that will be important in our subsequent argument:

$$u_1(k) := -\frac{2(k-1)}{k-2}, \qquad u_2(k) := \frac{k-4-\sqrt{17k^2-48k+32}}{2(k-2)},$$

$$u_3(k) := \frac{4(k-1)(-3k+4-2\sqrt{k-1})}{(k-2)(9k-10)}, \qquad u_4(k) := -1 - \frac{k}{3k^2 - 2(k-1)^2\sqrt{\frac{2k}{k-1}} - 4k}.$$

Notice that $u_1(k) = f(k,p)$ for all $p \geq r_+(H_k)$, and $u_4(k)$ is the expression given in (18), the lower bound for $f(k,1)$ that follows from Proposition 8. Then we have the following.

Lemma 13. *For every $k \geq 5$,*

$$u_1(k) < u_2(k) < u_3(k) < u_4(k).$$

Proof. First, one can check that the chain of inequalities holds when $5 \leq k \leq 26$, and that

$$-2 < u_2(k) < \frac{1-\sqrt{17}}{2} < u_3(k) < -\frac{4}{3} < u_4(k) \tag{20}$$

holds for $k = 27$. Next, notice that

$$\lim_{k\to\infty} u_1(k) = 2, \quad \lim_{k\to\infty} u_2(k) = \frac{1-\sqrt{17}}{2}, \quad \lim_{k\to\infty} u_3(k) = -\frac{4}{3},$$

and that $u_i(k)$ is an increasing function of k for all $i \in [4]$. Thus, (20) in fact holds for all $k \geq 27$, and our claim follows. $\square$

Now we are ready to prove the following key lemma which bounds the difference between $f(k-1, p-1)$ and $f(k, p)$.

Lemma 14. *Given $k \geq 5$ and $\ell \in (u_1(k), u_3(k))$, let*

$$\gamma \coloneqq (k-2)(9k-10)\ell^2 + 8(k-1)(3k-4)\ell + 16(k-1)^2,$$

and

$$h(k,\ell) \coloneqq \frac{4(k-2)\ell + 8(k-1)}{\sqrt{\gamma} + 3(k-2)\ell + 8(k-1)} - 2 - \ell.$$

If $f(k-1,p-1) \leq \ell + h(k,\ell)$, then $f(k,p) \leq \ell$.

Proof. Given $\epsilon > 0$, define $a \coloneqq \frac{1}{k} + \epsilon$ and $b \coloneqq \frac{k-1}{k} + \ell\epsilon$. We solve for c, d so that they satisfy condition (ii) in Lemma 12 and (S5) in Lemma 7 with equality. That is,

$$d - 2b - 2c = 1, \tag{21}$$

$$(2a + (k-2)c - 2ka^2)(2b + 2(k-1)d - 2kb^2) - (2(k-1)(a-c) - 2kab)^2 = 0. \tag{22}$$

To do so, we substitute $d = 2b + 2c - 1$ into (22), and obtain the quadratic equation

$$p_2c^2 + p_1c + p_0 = 0$$

where

$$\begin{aligned}
p_2 &\coloneqq (k-2)(4(k-1)) - (-2(k-1))^2,\\
p_1 &\coloneqq (k-2)(2b + 2(k-1)(2b-1) - 2kb^2) + (2a - 2ka^2)(4(k-1))\\
&\quad - 2(-2(k-1))(2(k-1)a - 2kab),\\
p_0 &\coloneqq (2a - 2ka^2)(2b + 2(k-1)(2b-1) - 2kb^2) - (2(k-1)a - 2kab)^2.
\end{aligned}$$

We then define $c \coloneqq \frac{-p_1 + \sqrt{p_1^2 - 4p_0p_2}}{2p_2}$ (this would be the smaller of the two solutions, as $p_2 < 0$), and $d \coloneqq 2b + 2c - 1$. First, we assure that c is well defined. If we consider the discriminant $\Delta p \coloneqq p_1^2 - 4p_0p_2$ as a function of ϵ, then $\Delta p(0) = 0$, and that $\frac{d^2}{d\epsilon^2}\Delta p(0) > 0$ for all $\ell \in (u_1(k), u_3(k))$. Thus, there must exist $\epsilon > 0$ where $\Delta p \geq 0$, and so c, d are well defined.

Next, we verify that $W_k(a,b,c,d) \succeq 0$ for some $\epsilon > 0$ by checking the conditions from Lemma 7. First, by the choice of c, d, (S5) must hold. Next, define the quantities

$$\theta_1 \coloneqq c, \qquad \theta_2 \coloneqq a - c,$$

$$\theta_3 \coloneqq b - d - a + c, \qquad \theta_4 \coloneqq 2a + (k-2)c - 2ka^2.$$

Notice that at $\epsilon = 0$, $\theta_i = 0$ for all $i \in [4]$. Next, given a quantity q that depends on ϵ, we use the notation $q'(0)$ denote the one-sided derivative $\lim_{\epsilon\to 0^+} \frac{q}{\epsilon}$. Then it suffices to show that $\theta_i'(0) \geq 0$ for all $i \in [4]$. Observe that

$$\theta_1'(0) \geq 0 \iff c'(0) \geq 0, \qquad \theta_2'(0) \geq 0 \iff c'(0) \leq 1,$$

$$\theta_3'(0) \geq 0 \iff c'(0) \leq -1 - \ell, \qquad \theta_4'(0) \geq 0 \iff c'(0) \geq \frac{2}{k-2}.$$

Now one can check that
$$c'(0) = \frac{-3k\ell - \sqrt{\gamma} - 4k + 2\ell + 4}{4k-4}.$$
As a function of ℓ, $c'(0)$ is increasing over $(u_1(k), u_3(k))$, with
$$c'(0)\big|_{\ell=u_1(k)} = \frac{2}{k-2}, \qquad c'(0)\big|_{\ell=u_3(k)} = \frac{(6k-4)\sqrt{k-1}+10k-12}{(k-2)(9k-10)}.$$
Thus, for all $k \geq 5$, we see that $\frac{2}{k-2} \leq c'(0) \leq \min\{1, -1-\ell\}$ for all $\ell \in (u_1(k), u_3(k))$, and so there exists $\epsilon > 0$ where $W_k(a,b,c,d) \succeq 0$.

Next, for convenience, let
$$s_1 \coloneqq s_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right), \qquad s_2 \coloneqq s_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right).$$
Notice that both s_1, s_2 are undefined at $\epsilon = 0$, as $\left(\frac{a-c}{b}, \frac{d}{b}\right) = \left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right) = \left(\frac{1}{k-1}, \frac{k-2}{k-1}\right)$ in this case. Now one can check that
$$\lim_{\epsilon \to 0^+} s_1 = \frac{-2\sqrt{\gamma} - 2(k-2)\ell - 8(k-1)}{\sqrt{\gamma} + 3(k-2)\ell + 8(k-1)},$$
$$\lim_{\epsilon \to 0^+} s_2 = \frac{(-2k+3)\sqrt{\gamma} - (2k-1)(k-2)\ell - 8(k-1)^2}{(k-2)\sqrt{\gamma} + (3k-2)(k-2)\ell + 8(k-1)^2}.$$
Observe that for $k \geq 5$ and for all $\ell \in (u_1(k), 0)$, we have
$$0 > \frac{-2\sqrt{\gamma}}{\sqrt{\gamma}} > \frac{(-2k+3)\sqrt{\gamma}}{(k-2)\sqrt{\gamma}},$$
$$0 > \frac{-2(k-2)\ell - 8(k-1)}{3(k-2)\ell + 8(k-1)} > \frac{-(2k-1)(k-2)\ell - 8(k-1)^2}{(3k-2)(k-2)\ell + 8(k-1)^2}.$$
Thus, we conclude that for all k, ℓ under our consideration, $s_1 \geq s_2$ for arbitrarily small $\epsilon > 0$.

Now, notice that $h(k, \ell) = \lim_{\epsilon \to 0^+} s_1 - \ell$. Thus, if $\ell \in (u_1(k), u_3(k))$, then there exists $\epsilon > 0$ where the matrix $W_k(a,b,c,d)$ as constructed is positive semidefinite, satisfies $d \geq 2b + 2c - 1$ (by the choice of c, d), with $s_2 \leq s_1 \leq h(k,\ell) + \ell$. Hence, if $f(k-1, p-1) \geq \ell + h(k, \ell)$, then Lemma 12 applies, and we obtain that $f(k,p) \geq \ell$. ☐

Applying Lemma 14 iteratively, we obtain the following.

Lemma 15. *Given $k \geq 5$, suppose there exists $\ell_1, \ldots, \ell_p \in \mathbb{R}$ where*

(i) *$\ell_p > u_1(k)$, $\ell_2 < u_3(k-p+2)$, and $\ell_1 < u_4(k-p+1)$;*

(ii) *$\ell_i + h(k-p+i, \ell_i) \leq \ell_{i-1}$ for all $i \in \{2, \ldots, p\}$.*

Then $r_+(H_k) \geq p+1$.

Proof. First, notice that $\ell_1 < u_4(k-p+1) \leq f(k-p+1, 1)$ by Proposition 8. Then since $\ell_2 < u_3(k-p+2)$ and $\ell_2 + h(k-p+2, \ell_2) \leq \ell_1$, Lemma 14 implies that $\ell_2 \leq f(k-p+2, 2)$. Iterating this argument results in $\ell_i \leq f(k-p+i, i)$ for every $i \in [p]$. In particular, we have $\ell_p \leq f(k,p)$. Since $\ell_p > u_1(k)$, it follows that $r_+(H_k) > p$, and the claim follows. ☐

Lemmas 14 and 15 provide a simple procedure of establishing LS$_+$-rank lower bounds for H_k.

Example 16. Let $k = 7$. Then $\ell_2 = -2.39$ and $\ell_1 = \ell_2 + h(7, \ell_2)$ certify that $r_+(H_7) \geq 3$. Similarly, for $k = 10$, one can let $\ell_3 = -2.24, \ell_2 = \ell_3 + h(10, \ell_3)$, and $\ell_1 = \ell_2 + h(9, \ell_2)$ and use Lemma 15 to verify that $r_+(H_{10}) \geq 4$.

Next, we prove a lemma that will help us obtain a lower bound for $r_+(H_k)$ analytically.

Lemma 17. *For all $k \geq 5$ and $\ell \in (u_1(k), u_2(k))$, $h(k,\ell) \leq \frac{2}{k-2}$.*

Proof. One can check that the equation $h(k,\ell) = \frac{2}{k-2}$ has three solutions: $\ell = u_1(k), u_2(k)$, and $\frac{k-4-\sqrt{17k^2-48k+32}}{2(k-2)}$ (which is greater than $u_2(k)$). Also, notice that $\frac{\partial}{\partial \ell} h(k,\ell)\big|_{\ell = u_1(k)} = -\frac{1}{k-1} < 0$. Since $h(k,\ell)$ is a continuous function of ℓ over $(u_1(k), u_2(k))$, it follows that $h(k,\ell) \leq \frac{2}{k-2}$ for all ℓ in this range. $\square$

We are finally ready to prove the main result of this section.

Theorem 18. *The* LS$_+$*-rank of H_k is*

- *at least* 2 *for* $4 \leq k \leq 6$*;*
- *at least* 3 *for* $7 \leq k \leq 9$*;*
- *at least* $\lfloor 0.19(k-2) \rfloor + 3$ *for all* $k \geq 10$*.*

Proof. First, $r_+(H_4) \geq 2$ follows from Corollary 9, and $r_+(H_7) \geq 3$ was shown in Example 11 and again in Example 16. Moreover, one can use the approach illustrated in Example 16 to verify that $r_+(H_k) \geq \lfloor 0.19(k-2) \rfloor + 3$ for all k where $10 \leq k \leq 49$. Thus, we shall assume that $k \geq 50$ for the remainder of the proof.

Let $q \coloneqq \lfloor 0.19(k-2) \rfloor$, let $\epsilon > 0$ that we set to be sufficiently small, and define

$$\ell_i \coloneqq \epsilon + u_1(k) + \sum_{j=1}^{q+2-i} \frac{2}{k-1-j}.$$

for every $i \in [q+2]$. (We aim to subsequently apply Lemma 15 with $p = q+2$.) Now notice that

$$\sum_{j=1}^{q} \frac{2}{k-1-i} \leq \int_{k-2-q}^{k-2} \frac{2}{t}\, dt = 2\ln\left(\frac{k-2}{k-2-q}\right),$$

Also, notice that

$$u_2(k-q) - u_1(k) \geq u_2\left(\frac{4}{5}k\right) - u_1(k),$$

as u_2 is an increasing function in k and $q \leq \frac{k}{5}$. Also, one can check that $\overline{w}(k) \coloneqq u_2\left(\frac{4}{5}k\right) - u_1(k)$ is also an increasing function for all $k \geq 5$. Next, we see that

$$2\ln\left(\frac{k-2}{k-2-q}\right) \leq \overline{w}(50) \iff q \leq \left(1 - \frac{1}{\exp(\overline{w}(50)/2)}\right)(k-2)$$

Since $1 - \frac{1}{\exp(\overline{w}(50)/2)} > 0.19$, the first inequality does hold by the choice of q. Hence,

$$\ell_2 - \epsilon = u_1(k) + \sum_{j=1}^{q} \frac{2}{k-1-j} < u_2(k-q).$$

Thus, we can choose ϵ sufficiently small so that $\ell_2 < u_2(k-q)$. Then Lemma 17 implies that $\ell_i + h(k-q-2+i, \ell_i) \leq \ell_{i-1}$ for all $i \in \{2, \ldots, q+2\}$. Also, for all $k \geq 50$, $u_2(k-q) + \frac{1}{k-q-1} < u_4(k-q-1)$. Thus, we obtain that $\ell_1 < u_4(k-q-1)$, and it follows from Lemma 15 that $r_+(H_k) \geq q+3$. $\square$

Since H_k has $3k$ vertices, Theorem 18 (and the fact that $r_+(H_3) = 1$) readily implies Theorem 1. In other words, we now know that for every $\ell \in \mathbb{N}$, there exists a graph on no more than 16ℓ vertices that has LS$_+$-rank ℓ.

Next, we mention that the LS_+-rank lower bound in Theorem 18 also applies to a particular subgraph of H_k. For every $k \geq 2$, let H'_k denote the subgraph of H_k induced by the set $B_{1,2}$ (where $B_{i,j}$ is as defined before Lemma 4). Notice that H'_k contains four fewer vertices than H_k.

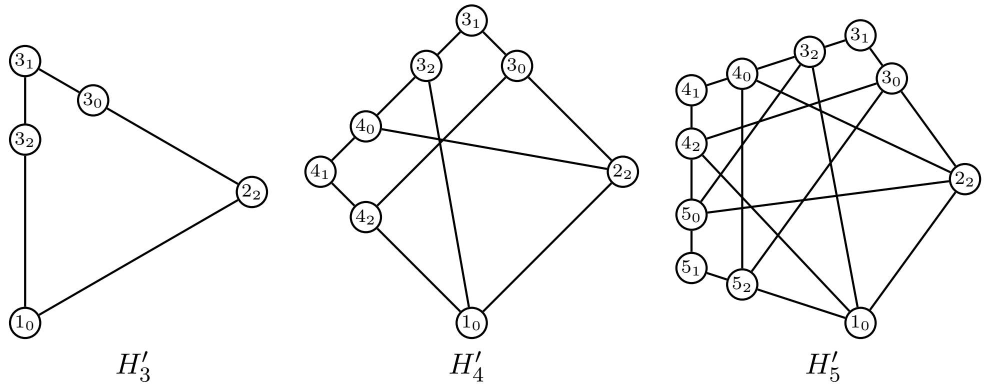

FIGURE 4. Several graphs in the family H'_k

The following is a well-known property of LS_+ that will help us relate points in $\mathrm{LS}_+^p(H_k)$ and $\mathrm{LS}_+^p(H'_k)$.

Lemma 19. *Let $P \subseteq [0,1]^n$ be a polyhedron, and F be any facet of $[0,1]^n$. Then*

$$\mathrm{LS}_+(P \cap F) = \mathrm{LS}_+(P) \cap F.$$

More generally, it follows from Lemma 19 that if $\bar{x} \in \mathrm{LS}_+^p(G)$ and G' is an induced subgraph of G, then the vector obtained from $\bar{x}$ by removing entries not in $V(G')$ is in $\mathrm{LS}_+^p(G')$. Then we have the following.

Proposition 20. *For every $k \geq 4$, the lower bound for $r_+(H_k)$ given in Theorem 18 also applies for $r_+(H'_k)$.*

Proof. For convenience, let $p(k)$ be the lower bound for $r_+(H_k)$ given in Theorem 18. Recall that we proved the lower bound by finding a vector of the form $w_k(a,b) \in \mathrm{LS}_+^{p(k)-1}(H_k)$ which violates (6). Then it follows from Lemma 6 that the same point would also violate (3). Thus, from Lemma 19, the same vector $w_k(a,b)$ (with entries corresponding to vertices outside of $B_{1,2}$ removed) is contained in $\mathrm{LS}_+^{p(k)-1}(H'_k) \setminus \mathrm{STAB}(H'_k)$, and our claim follows. □

3.5. **Chvátal–Gomory rank of** $\mathrm{STAB}(H_k)$**.** Finally, we conclude this section by determining the degree of hardness of $\mathrm{FRAC}(H_k)$ relative to another well-studied cutting plane procedure that is due to Chvátal [Chv73] with earlier ideas from Gomory [Gom58]. Given a set $P \subseteq [0,1]^n$, if $a^\top x \leq \beta$ is a valid inequality of P and $a \in \mathbb{Z}^n$, we say that $a^\top x \leq \lfloor \beta \rfloor$ is a *Chvátal–Gomory cut* for P. Then we define $\mathrm{CG}(P)$, the *Chvátal–Gomory closure* of P, to be the set of points that satisfy all Chvátal–Gomory cuts for P. Note that $\mathrm{CG}(P)$ is a convex set that contains all integral points in P.

Furthermore, given an integer $k \geq 2$, we can recursively define $\mathrm{CG}^k(P) := \mathrm{CG}(\mathrm{CG}^{k-1}(P))$. Then given any valid inequality of P_I, we can define its CG-*rank* (relative to P) to be the smallest integer k where it is valid for $\mathrm{CG}^k(P)$. Then we have the following.

Theorem 21. *Let d be the* CG*-rank of the facet* (3) *of* $\mathrm{STAB}(H_k)$ *relative to* $\mathrm{FRAC}(H_k)$*. Then*

$$\log_4\left(\frac{3k-7}{2}\right) < d \leq \log_2(k-1).$$

The proof of Theorem 21 is provided in Appendix A. Thus, while we showed that the facet (3) has LS_+-rank $\Theta(|V(H_k)|)$, the facet has CG-rank $\Theta(\log(|V(H_k)|)$. We remark that the two results are incomparable in terms of computational complexity since it is generally $\mathcal{NP}$-hard to optimize over $\mathrm{CG}^k(P)$ even for $k = O(1)$. These rank bounds for H_k also provides an interesting contrast with the aforementioned example involving line graphs of odd cliques from [ST99], which have LS_+-rank $\Theta(\sqrt{|V(G)|})$ and CG-rank $\Theta(\log(|V(G)|)$. In the context of the matching problem, odd cliques have CG-rank one with respect to the fractional matching polytope.

4. Symmetric graphs with high LS_+-ranks

So far we have established that there exists a family of graphs (e.g., $H_k, k \geq 2$) which have LS_+-rank $\Theta(|V(G)|)$. However, the previous best result in this context $\Theta(\sqrt{|V(G)|})$ was achieved by a vertex-transitive family of graphs (line graphs of odd cliques). In this section, we show that there also exists a family of vertex-transitive graphs which have LS_+-rank $\Theta(|V(G)|)$.

4.1. **The L_k construction.** In this section, we look into a procedure that is capable of constructing highly symmetric graphs with high LS_+-rank by virtue of containing H_k as an induced subgraph. Given a graph G and an integer $k \geq 2$, we define the graph $L_k(G)$ such that $V(L_k(G)) := \{i_p : i \in [k], p \in V(G)\}$, and vertices i_p, j_q are adjacent in $L_k(G)$ if

- $i = j$ and $\{p, q\} \in E(G)$, or
- $i \neq j$, $p \neq q$, and $\{p, q\} \notin E(G)$.

For an example, let C_4 be the 4-cycle with $V(C_4) := \{0, 1, 2, 3\}$ and

$$E(C_4) := \{\{0,1\}, \{1,2\}, \{2,3\}, \{3,0\}\}.$$

Figure 5 illustrates the graphs $L_2(C_4)$ and $L_3(C_4)$.

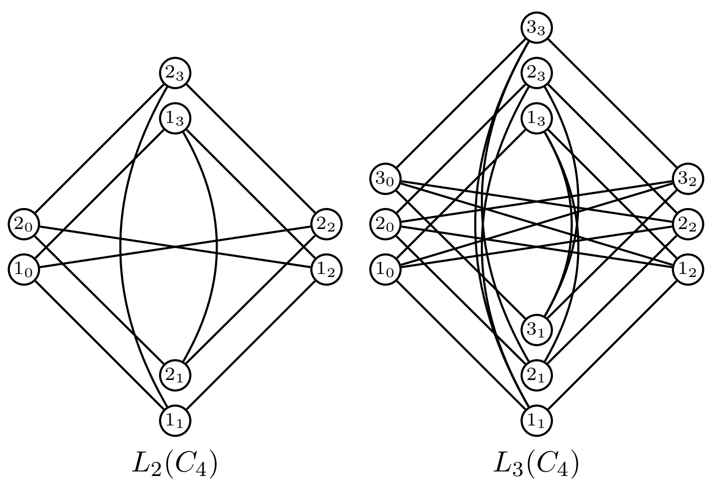

Figure 5. Illustrating the $L_k(G)$ construction on the 4-cycle

Moreover, notice that if we define P_2 to be the graph which is a path of length 2, with $V(P_2) := \{0, 1, 2\}$ and $E(P_2) := \{\{0,1\}, \{1,2\}\}$, then $L_k(P_2) = H_k$ for every $k \geq 2$. Thus, we obtain the following.

Proposition 22. *Let G be a graph that contains P_2 as an induced subgraph. Then the* LS_+*-rank lower bound in Theorem 18 for H_k also applies for $L_k(G)$.*

Proof. Since G contains P_2 as an induced subgraph, there must exist vertices $a, b, c \in V(G)$ where $\{a, b\}, \{b, c\} \in E(G)$, and $\{a, c\} \notin E(G)$. Then the subgraph of $L_k(G)$ induced by the vertices $\{i_p : i \in [k], p \in \{a, b, c\}\}$ is exactly $L_k(P_2) = H_k$. Thus, it follows from Lemma 19 that $r_+(L_k(G)) \geq r_+(H_k)$. □

Since $L_k(C_4)$ has $4k$ vertices, Theorem 18 and Proposition 22 immediately imply the following.

Theorem 23. *Let $k \geq 3$ and $G \coloneqq L_k(C_4)$. Then $r_+(G) \geq \frac{1}{22}|V(G)|$.*

Since $\{L_k(C_4) : k \geq 3\}$ is a family of vertex-transitive graphs, Theorem 23 can also be proved directly by utilizing versions of the techniques in Section 3. The graphs $L_k(C_4)$ are particularly noteworthy because C_4 is the smallest vertex-transitive graph that contains P_2 as an induced subgraph. In general, observe that if G is vertex-transitive, then so is $L_k(G)$. Thus, we now know that there exists a family of vertex-transitive graphs G with $r_+(G) = \Theta(|V(G)|)$.

4.2. **Generalizing the L_k construction.** Next, we study one possible generalization of the aforementioned L_k construction, and mention some interesting graphs it produces. Given graphs G_1, G_2 on the same vertex set V, we define $L_k(G_1, G_2)$ to be the graph with vertex set $\{i_p : i \in [k], p \in V\}$. Vertices i_p, j_q are adjacent in $L_k(G_1, G_2)$ if

- $i = j$ and $\{p, q\} \in E(G_1)$, or
- $i \neq j$ and $\{p, q\} \in E(G_2)$.

Thus, when $G_2 = \overline{G_1}$ (the complement of G_1), then $L_k(G_1, G_2)$ specializes to $L_k(G_1)$. Next, given $\ell \in \mathbb{N}$ and $S \subseteq [\ell]$, let $Q_{\ell,S}$ denote the graph whose vertices are the 2^ℓ binary strings of length ℓ, and two strings are joined by an edge if the number of positions they differ by is contained in S. For example, $Q_{\ell,\{1\}}$ gives the ℓ-cube. Then we have the following.

Proposition 24. *For every $\ell \geq 2$,*

$$L_4(Q_{\ell,\{1\}}, Q_{\ell,\{\ell\}}) = Q_{\ell+2,\{1,\ell+2\}}.$$

Proof. Let $G \coloneqq L_4(Q_{\ell,\{1\}}, Q_{\ell,\{\ell\}})$. Given $i_p \in V(G)$ (where $i \in [4]$ and $p \in \{0,1\}^\ell$), we define the function

$$f(i_p) \coloneqq \begin{cases} 00p & \text{if } i = 1; \\ 01\overline{p} & \text{if } i = 2; \\ 10\overline{p} & \text{if } i = 3; \\ 11p & \text{if } i = 4. \end{cases}$$

Note that $\overline{p}$ denotes the binary string obtained from p by flipping all ℓ bits. Now we see that $\{i_p, j_q\} \in E(G)$ if and only if $f(i_p)$ and $f(j_q)$ differ by either 1 bit or all $\ell + 2$ bits, and the claim follows. □

The graph $Q_{k,\{1,k\}}$ is known as the folded-cube graph, and Proposition 24 implies the following.

Corollary 25. *Let $G \coloneqq Q_{k,\{1,k\}}$ where $k \geq 3$. Then $r_+(G) \geq 2$ if k is even, and $r_+(G) = 0$ if k is odd.*

Proof. First, observe that when k is odd, $Q_{k,\{1,k\}}$ is bipartite, and hence has LS$_+$-rank 0. Next, assume $k \geq 4$ is even. Notice that $Q_{k-2,\{1\}}$ contains a path of length $k-2$ from the all-zeros vertex to the all-ones vertex, while $Q_{k-2,\{k-2\}}$ joins those two vertices by an edge. Thus, $Q_{k,\{1,k\}} = L_4(Q_{k-2,\{1\}}, Q_{k-2,\{k\}})$ contains the induced subgraph $L_4(P_{k-2})$ (where P_{k-2} denotes the graph that is a path of length $k-2$). Since $k-2$ is even, we see that $L_4(P_{k-2})$ can be obtained from $L_4(P_2) = H_4$ by odd subdivision of edges (i.e., replacing edges by paths of odd lengths). Thus, it follows from [LT03, Theorem 16] that $r_+(L_4(P_{k-2})) \geq 2$, and consequently $r_+(Q_{k,\{1,k\}}) \geq 2$. □

Example 26. The case $k = 4$ in Corollary 25 is especially noteworthy. In this case $G \coloneqq Q_{4,\{1,4\}}$ is the (5-regular) Clebsch graph. Observe that $G \ominus i$ is isomorphic to the Petersen graph (which has LS$_+$-rank 1) for every $i \in V(G)$. Thus, together with Corollary 25 we obtain that the Clebsch graph has LS$_+$-rank 2.

Alternatively, one can show that $r_+(G) \geq 2$ by using the fact that the second largest eigenvalue of G is 1. Then it follows from [ALT22, Proposition 4] that $\max\left\{\bar{e}^\top x : x \in \text{LS}_+(G)\right\} \geq 6$, which shows that $r_+(G) \geq 2$ since the largest stable set in G has size 5.

We remark that the Clebsch graph is also special in the following aspect. Given a vertex-transitive graph G, we say that G is *transitive under destruction* if $G \ominus i$ is also vertex-transitive for every $i \in V(G)$. As mentioned above, destroying any vertex in the Clebsch graph results in the Petersen graph, and so the Clebsch graph is indeed transitive under destruction. On the other hand, even though $L(G_1, G_2)$ is vertex-transitive whenever G_1, G_2 are vertex-transitive, the Clebsch graph is the only example which is transitive under destruction we could find using the L_k construction. For instance, one can check that $Q_{k,\{1,k\}} \ominus i$ is not a regular graph for any $k \geq 5$. Also, observe that the Clebsch graph can indeed be obtained from the "regular" L_k construction defined in Section 4.1, as

$$Q_{4,\{1,4\}} = L_4(Q_{2,\{1\}}, Q_{2,\{2\}}) = L_4(C_4, \overline{C_4}) = L_4(C_4).$$

However, one can check that $L_k(C_\ell)$ is transitive under destruction if and only if $(k, \ell) = (4, 4)$ (i.e., the Clebsch graph example), and that $L_k(K_{\ell,\ell})$ is transitive under destruction if and only if $(k, \ell) = (4, 2)$ (i.e., the Clebsch graph example again). It would be fascinating to see what other interesting graphs can result from the L_k construction.

5. Vertex stretching and ℓ-minimal graphs

5.1. **The vertex-stretching operation and subgraphs of H'_k.** In this section, we study a graph operation that has shown promise in producing relatively small graphs with high LS$_+$-ranks, and discuss its implications for the LS$_+$-rank of some subgraphs of H'_k. Given a graph G, vertex $v \in V(G)$, and sets $A_0, A_2 \subset \Gamma(v)$ where $A_0 \cup A_2 = \Gamma(v)$, we define the *stretching* of v in G by applying the following sequence of transformations to G:

- Replace v by three vertices: v_0, v_1, and v_2;
- Join v_0 to all vertices in A_0, and v_2 to all vertices in v_2;
- Add edges $\{v_0, v_1\}$ and $\{v_1, v_2\}$.

For example, Figure 6 shows the graph obtained from stretching vertex 5 in K_5 (with $A_0 = \{2, 3, 4\}$ and $A_2 = \{1, 2, 3\}$).

We remark that our vertex-stretching operation is a slight generalization of the type (i) stretching operation described in [LT03] and later studied in [AEF14] (which further requires that $A_0 \cap A_2 = \emptyset$), as well as the k-stretching operation described in [BENT17] (which further requires that the vertices $A_0 \cap A_2$ induce a clique of size k in G).

Given a graph G, we define $\mathcal{S}(G)$ to be the set of graphs that can be obtained from G by stretching one vertex. Due to its similarity with the aforementioned type (i) vertex-stretching operation studied in [LT03], our vertex-stretching operation shares many of the same structural properties, which we point out below.

Proposition 27. *Let $H \in \mathcal{S}(G)$ be a graph obtained from G by stretching vertex $v \in V(G)$. Then we have the following.*

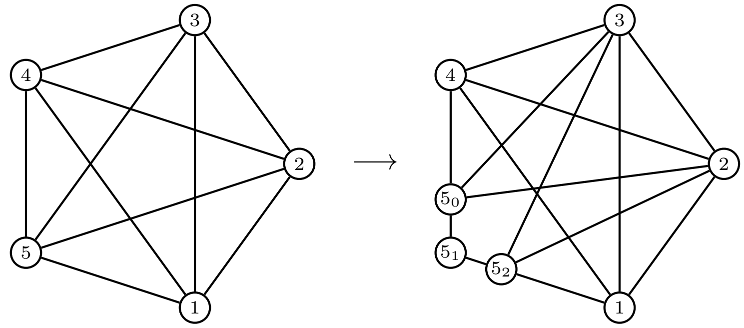

FIGURE 6. Demonstrating the vertex-stretching operation

(i) *If* $a^\top x \leq \beta$ *is valid for* STAB(G)*, then*

$$\left(\sum_{\ell \in V(G)\setminus\{v\}} x_\ell\right) + a_v(x_{v_0} + x_{v_1} + x_{v_2}) \leq \beta + a_v \tag{23}$$

is valid for STAB(H)*.*

(ii) *Given* $x \in \mathbb{R}^{V(G)}$*, define* $x' \in \mathbb{R}^{V(H)}$ *where*

$$x'_i := \begin{cases} x_i & \text{if } i \in V(H) \setminus \{v_0, v_1, v_2\}; \\ x_v & \text{if } i \in \{v_0, v_2\}; \\ 1 - x_v & \text{if } i = v_1. \end{cases}$$

If $x \notin$ STAB(G)*, then* $x' \notin$ STAB(H)*. And if* $x \in \mathrm{LS}_+^p(G)$*, then* $x' \in \mathrm{LS}_+^p(H)$*.*

(iii) $r_+(H) \geq r_+(G)$*.*

Proof. As shown in [LT03], all three properties hold for the more restrictive case when $A_0 \cap A_2 = \emptyset$ in the stretching process. Thus, let $H' \in \mathcal{S}(G)$ such that H' is a subgraph of H, and that H' can be obtained from G by stretching v with $A_0 \cap A_2 = \emptyset$. Then (i) holds for H', and (23) holds for STAB(H'). Thus, (23) is valid for STAB(H) as well since STAB$(H) \subseteq$ STAB(H').

For (ii), the proof for the same property for type (i) stretching [LT03, Lemma 26] applies for this more general case, since the assumption $A_0 \cap A_2 = \emptyset$ was never invoked in that argument. We do remark that in the proof for [LT03, Lemma 26], the entries $X'_{v_i u}$ and X'_{uv_i} should have been defined to be $X_{v_i u}$ and X_{uv_i} respectively, instead of x_u. This error was corrected when Bianchi et al. referred to and paraphrased the argument to prove the same property for their k-stretching operation [BENT17, Theorem 22].

(iii) then follows from (i) and (ii), using the same argument from [LT03, Theorem 25]. □

We also prove a result somewhat similar to Proposition 27(i) that derives some facets of the stable set polytope of the stretched graph.

Proposition 28. *Let* $H \in \mathcal{S}(G)$ *be a graph obtained from* G *by stretching vertex* $v \in V(G)$*, and suppose* $a^\top x \leq \beta$ *is a facet of* STAB(G) *where* $a \geq 0$*. Let* $A_0 := \Gamma(v_0) \cap V(G), A_2 :=$

$\Gamma(v_2) \cap V(G)$), and define the quantities

$$c_0 := \max\left\{ \sum_{i \in S} a_i : S \subseteq V(G) \setminus (\{v\} \cup A_0) \text{ is a stable set} \right\},$$

$$c_2 := \max\left\{ \sum_{i \in S} a_i : S \subseteq V(G) \setminus (\{v\} \cup A_2) \text{ is a stable set} \right\}.$$

Also define $\tilde{a} \in \mathbb{R}^{V(H)}$ where

$$\tilde{a}_i := \begin{cases} a_i & \text{if } i \in V(G) \setminus \{v\}; \\ a_v - (\beta - c_2) & \text{if } i = v_0; \\ a_v - (\beta - c_0) - (\beta - c_2) & \text{if } i = v_1; \\ a_v - (\beta - c_0) & \text{if } i = v_2. \end{cases}$$

If $\tilde{a}_{v_1} \geq 0$, then $\tilde{a}^\top x \leq \beta + \tilde{a}_{v_1}$ is a facet of STAB(H).

Proof. For convenience, let $n := |V(G)|$. Since $a^\top x \leq \beta$ is a facet of STAB(G), there exist stable sets $S_1, \ldots, S_n \subseteq V(G)$ whose incidence vectors are affinely independent and all satisfy $a^\top x \leq \beta$ with equality. Also, let C_0 and C_2 be stable sets that attain the maximum defined in c_0 and c_2, respectively. We then define $S'_1, \ldots, S'_{n+2}$ such that

$$S'_i := \begin{cases} S_i \cup \{v_1\} & \text{if } i \in [n] \text{ and } v \notin S_i; \\ (S_i \setminus \{v\}) \cup \{v_0, v_2\} & \text{if } i \in [n] \text{ and } v \in S_i; \\ C_0 \cup \{v_0\} & \text{if } i = n+1; \\ C_2 \cup \{v_2\} & \text{if } i = n+2. \end{cases}$$

Observe that $S'_1, \ldots, S'_{n+2}$ must all be stable sets in H. Also, using the fact that incidence vectors of $S_1, \ldots, S_n$ are affinely independent and satisfy $a^\top x \leq \beta$ with equality, we see that the incidence vectors of $S'_1, \ldots, S'_{n+2}$ are affinely independent and all satisfy $\tilde{a}^\top x \leq \beta + \tilde{a}_{v_1}$ with equality. Moreover, with the assumption that $\tilde{a}_{v_1} \geq 0$, we obtain that $\tilde{a} \geq 0$, and so if $\tilde{a}^\top x \leq \beta + \tilde{a}_{v_1}$ were not valid for STAB(H) it would have to be violated by the incidence vector of a maximal stable set, which (by the definitions of c_0, c_2 and the stretching operation) would imply that $a^\top x \leq \beta$ is not valid for STAB(G), a contradiction. □

Next, given $k \geq 2$, we recursively define $\mathcal{S}^k(G)$ to be the set of graphs that can be obtained from stretching one vertex of a graph in $\mathcal{S}^{k-1}(G)$. We also let $\mathcal{S}^0(G) := \{G\}$. For instance, notice that $H'_k \in \mathcal{S}^{k-2}(K_k)$ for all $k \geq 2$ — this is apparent if one takes the drawings of H'_k from Figure 4 and relabels the vertices 1_0 and 2_2 by 1 and 2 respectively. We also let $\alpha(G)$ denote the size of the largest stable set in a given graph G. Next, we mention some basic properties of the graphs in $\mathcal{S}^k(G)$.

Lemma 29. *Let G be a graph, and let $H \in \mathcal{S}^k(G)$. Then $\alpha(H) = \alpha(G) + k$.*

Proof. Let $H \in \mathcal{S}(G)$ be a graph obtained from G by stretching vertex $v \in V(G)$. To prove our claim, it suffices to show that $\alpha(H) = \alpha(G) + 1$. Consider a set of vertices $S \subseteq V(G)$. If $v \in S$, then S is a stable set in G if and only if $(S \setminus \{v\}) \cup \{v_0, v_2\}$ is a stable set in H. If $v \notin S$, then S is a stable set in G if and only if $S \cup \{v_1\}$ is a stable set in H. Thus, we see that $\alpha(H) = \alpha(G) + 1$. □

The following is an immediate consequence of the fact that LS$_+$ preserves containment.

Lemma 30. *Given graphs G, H where $V(H) = V(G)$ and $E(H) \subseteq E(G)$,*

(i) *If $a^\top x \leq \beta$ is valid for $\mathrm{LS}_+^p(H)$, then $a^\top x \leq \beta$ is valid for $\mathrm{LS}_+^p(G)$;*

(ii) *If $a^\top x \leq \beta$ is not valid for $\mathrm{LS}_+^p(G)$, then $a^\top x \leq \beta$ is not valid for $\mathrm{LS}_+^p(H)$.*

Given $p \in \mathbb{N}$, define

$$\alpha_{\mathrm{LS}_+^p}(G) := \max\left\{ \bar{e}^\top x : x \in \mathrm{LS}_+^p(G) \right\}.$$

Thus, if $\alpha_{\mathrm{LS}_+^p}(G) > \alpha(G)$, then $r_+(G) \geq p+1$. Moreover, it follows from Lemma 30 that if H is a subgraph of G where $\alpha(H) = \alpha(G)$, then we can conclude that $r_+(H) \geq p+1$ as well. Using this line of reasoning, we see that the LS_+-rank lower bound given in Theorem 18 applies not only to H_k' (as shown in Proposition 20), but also many subgraphs of H_k'.

Proposition 31. *Let $G \in \mathcal{S}^{k-2}(K_k)$ be a subgraph of H_k'. Then the LS_+-rank lower bound for H_k given in Theorem 18 also applies for G.*

Proof. Let $p(k)$ be the lower bound for $r_+(H_k)$ given in Theorem 18. From Proposition 20, we know that $\alpha_{\mathrm{LS}_+^{p(k)-1}}(H_k') > k-1$. But then Lemma 29 implies that $\alpha(H_k') = \alpha(G) = k-1$. Since G is a subgraph of H_k', it follows from Lemma 30 that $\alpha_{\mathrm{LS}_+^{p(k)-1}}(G) > k-1$, and our claim follows. □

Given a graph G, define the *edge density* of G to be $d(G) := \frac{|E(G)|}{\binom{|V(G)|}{2}}$. For instance, $d(G) = 1$ for complete graphs, and $d(G) = 0$ for empty graphs. An interesting contrast that has emerged in the study of lift-and-project relaxations of the stable set polytope of graphs is that dense graphs tend to have high lift-and-project ranks with respect to operators that produce polyhedral relaxations, whereas graphs from both ends of the density spectrum tend to be of small lift-and-project ranks with respect to semidefinite operators. Thus, it is interesting to note that

$$d(H_k') = \frac{k^2 - k - 1}{\binom{3k-4}{2}} = \frac{2}{9} + o(k).$$

Moreover, one can find $G \in \mathcal{S}^{k-2}(K_k)$ that is a subgraph of H_k' with as few as $\frac{k^2+3k-8}{2}$ edges by choosing disjoint A_0 and A_2 during every stretching operation (see Figure 7 for such an example for $k = 5$), and one can check that such graphs have $d(G) \approx \frac{1}{9}$. Thus, we see that there are many graphs with edge densities between $\frac{1}{9}$ and $\frac{2}{9}$ for which the rank lower bound in Theorem 18 applies.

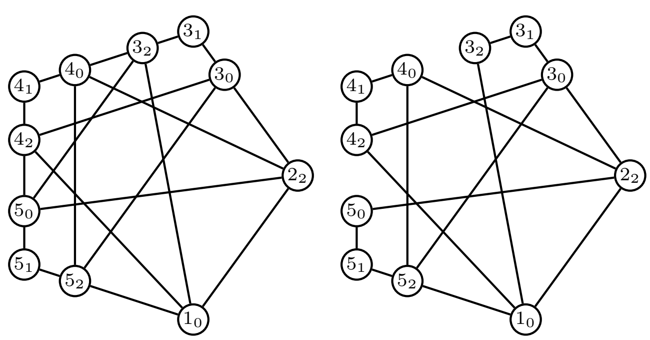

FIGURE 7. H_5' (left) and a subgraph of H_5' in $\mathcal{S}^3(K_5)$ with the fewest possible edges (right)

5.2. **LS_+-minimal graphs via vertex stretching.** In this section, we are interested in studying graphs with the fewest number of vertices with a given LS_+-rank. Given $\ell \in \mathbb{N}$, define $n_+(\ell)$ to be the minimum number of vertices on which there exists a graph G with $r_+(G) = \ell$. It follows immediately from Theorem 3 that $n_+(\ell) \geq 3\ell$ for every $\ell \in \mathbb{N}$. On the other hand, Theorem 1 implies that $n_+(\ell) \leq 16\ell$. Thus, we now know that $n_+(\ell) = \Theta(\ell)$ asymptotically.

Recall that a graph G is ℓ-minimal if $r_+(G) = \ell$ and $|V(G)| = 3\ell$. The following result establishes the close connection between ℓ-minimal graphs and the vertex-stretching operation.

Theorem 32. *Let H be an ℓ-minimal graph where $\ell \geq 2$. Then there exists G where $H \in \mathcal{S}(G)$.*

Proof. First, since $r_+(H) = \ell$, there exists a vertex v_1 where $r_+(H \ominus v_1) = \ell - 1$. This implies that $|V(H \ominus v_1)| \geq 3\ell - 3$, and thus $\deg(v_1) \leq 2$. Since ℓ-minimal graphs cannot contain cut vertices (in which case $\text{STAB}(H)$ would not have a facet of full support), we obtain that $\deg(v_1) = 2$. Let v_0, v_2 denote the two neighbours of v_1. We then define G to be the graph obtained from H by removing $\{v_0, v_1, v_2\}$, and then adding a vertex v that is joined to vertices in $(\Gamma(v_0) \cup \Gamma(v_2)) \setminus \{v_1\}$. We claim that $H \in \mathcal{S}(G)$, and to prove that it only remains to show that both $\Gamma(v_0) \setminus \Gamma(v_2)$ and $\Gamma(v_2) \setminus \Gamma(v_0)$ are non-empty.

Let $a^\top x \leq \beta$ be a facet of $\text{STAB}(H)$ of LS_+-rank ℓ. Then there must be stable sets $S_1, \ldots, S_{3\ell} \subseteq V(H)$ whose incidence vectors are affinely independent and all satisfy $a^\top x \leq \beta$ with equality. Also, since H is ℓ-minimal, a must have full support. Therefore, S_i must be maximal for all $i \in [3\ell]$, and hence belongs to one of the following cases:

(1) $S_i \cap \{v_0, v_1, v_2\} = \{v_1\}$;
(2) $S_i \cap \{v_0, v_1, v_2\} = \{v_0\}$;
(3) $S_i \cap \{v_0, v_1, v_2\} = \{v_2\}$;
(4) $S_i \cap \{v_0, v_1, v_2\} = \{v_0, v_2\}$.

Since $\chi_{S_1}, \ldots, \chi_{S_{3\ell}}$ are affinely independent, one of these stable sets contains v_1 and belongs to (1), so assume without loss of generality that $v_1 \in S_1$. Now consider the matrix A formed by the row vectors $(\chi_{S_2} - \chi_{S_1})^\top, (\chi_{S_3} - \chi_{S_1})^\top, \ldots, (\chi_{S_n} - \chi_{S_1})^\top$. Since $S_1, \ldots, S_{3\ell}$ are affinely independent, A must have linearly independent rows. This means that, if we focus on the submatrix A' of A which consists of just the three columns corresponding to v_0, v_1, and v_2, A' must have rank 3. Then there must exist at least one S_i belonging to each of cases (2), (3), and (4).

Now consider a stable set S_i that belongs to Case (2). Since S_i is maximal, it must contain a vertex that is adjacent to v_2 and not v_0, and so we have found a vertex that belongs to $\Gamma(v_2) \setminus \Gamma(v_0)$. Likewise, analyzing an S_i from Case (3) gives a vertex that belongs to $\Gamma(v_0) \setminus \Gamma(v_2)$. This finishes the proof. □

It is known that ℓ-minimal graphs exist for $\ell \in \{1, 2, 3\}$. For $\ell = 1$, it is easy to see that the 3-cycle is the only 1-minimal graph. The first 2-minimal graph ($G_{2,1}$ in Figure 8) was found by Lipták and the second author [LT03], who also conjectured that ℓ-minimal graphs exist for all $\ell \in \mathbb{N}$. Subsequently, Escalante, Montelar, and Nasini [EMN06] showed that there is only one other 2-minimal graph ($G_{2,2}$ in Figure 8), while providing the first example of a 3-minimal graph ($G_{3,1}$ in Figure 8).

We labelled the vertices of the graphs $G_{2,1}, G_{2,2}$, and $G_{3,1}$ to highlight the fact that all three graphs can be obtained from applying a number of vertex-stretching operations to a complete graph. In particular, every known ℓ-minimal graph — the 3-cycle and the three graphs in Figure 8 — belongs to $\mathcal{S}^{\ell-1}(K_{\ell+2})$. Thus, for the remainder of this subsection, we focus on graphs obtained from stretching vertices of a complete graph, and prove some results about the LS_+-ranks of these graphs. Some of our subsequent arguments rely on the positive semidefiniteness

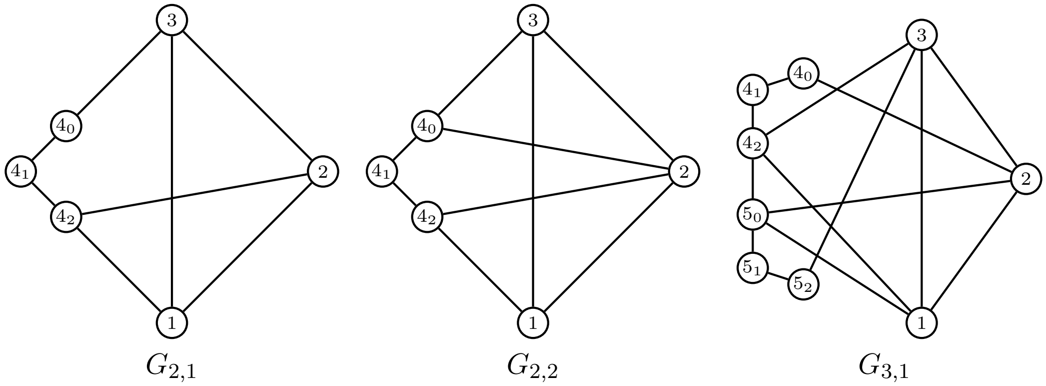

Figure 8. Known 2- and 3-minimal graphs due to [LT03] and [EMN06]

of some specific matrices, and so we first provide a framework for easily and reliably verifying such claims. Given a symmetric matrix $Y \in \mathbb{Z}^{n \times n}$, we say that $U, V \in \mathbb{Z}^{n \times n}$ is a *UV-certificate* of Y if

- $kY = U^\top U + V$ for some $k \in \mathbb{N}$, and,
- V is diagonally dominant.

Observe that the existence of a UV-certificate implies that $Y \succeq 0$ (these certificates are sum-of-squares certificates, and every $Y \in \mathbb{S}_+^n \cap \mathbb{Z}^{n \times n}$ admits such certificates). In all our UV-certificates (U, V), every entry is an integer. Given a UV-certificate to verify that Y is PSD, it suffices to

(i) form $U^\top U + V$ and check that it is equal to kY for some integer k
(ii) check that $V_{ii} \geq \sum_{j \neq i} |V_{ij}|$ for every $i \in [n]$.

Next, we show that if we stretch a vertex in the complete graph, the result is always a graph with LS_+-rank 2.

Proposition 33. *Let $n \geq 4$. Then $r_+(H) = 2$ for all $H \in \mathcal{S}(K_n)$.*

Proof. Let $H \in \mathcal{S}(K_n)$, then $\alpha(H) = 2$ by Lemma 29. Without loss of generality, assume that H is obtained from K_n by stretching vertex n. Also, let G_n be the graph obtained from stretching vertex n in K_n with $A_0 = \{2, 3, \ldots, n\}$ and $A_2 = [n-1]$. (For example, G_4 is the graph $G_{2,2}$ from Figure 8 and G_5 is shown in Figure 6.) Then H must be isomorphic to a subgraph of G_n.

Next, we show that $\alpha_{\mathrm{LS}_+^2}(G_4) > 2$. Consider the certificate matrix

$$Y := \begin{array}{c} \\ \end{array}\begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4_0 & 4_1 & 4_2 \end{matrix} \\ \begin{bmatrix} 200 & 78 & 12 & 78 & 78 & 78 & 78 \\ 78 & 78 & 0 & 0 & 39 & 39 & 0 \\ 12 & 0 & 12 & 0 & 0 & 12 & 0 \\ 78 & 0 & 0 & 78 & 0 & 39 & 39 \\ 78 & 39 & 0 & 0 & 78 & 0 & 39 \\ 78 & 39 & 12 & 39 & 0 & 78 & 0 \\ 78 & 0 & 0 & 39 & 39 & 0 & 78 \end{bmatrix} \end{array}$$

Note that the columns of Y are labelled by the vertices in G_4 they correspond to (the rows of Y follow the same order of indexing). Observe that $Y \succeq 0$ — a UV-certificate for Y is

$$U := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -73 & -26 & -141 & -26 & 60 & 132 & 61 \\ 0 & 124 & 0 & -124 & -200 & 0 & 200 \\ 27 & -181 & 247 & -181 & 51 & 159 & 51 \\ 0 & -527 & 0 & 527 & -326 & 0 & 326 \\ 1 & -166 & -73 & -166 & 449 & -556 & 449 \\ 1224 & 482 & 60 & 482 & 482 & 485 & 482 \end{bmatrix}, \quad V := \begin{bmatrix} 2765 & 917 & 91 & 917 & 316 & -11 & 389 \\ 917 & 1308 & 3 & -212 & -136 & 10 & 29 \\ 91 & 3 & 601 & 3 & -280 & 71 & -139 \\ 917 & -212 & 3 & 1308 & 3 & 10 & -110 \\ 316 & -136 & -280 & 3 & 1328 & -155 & -45 \\ -11 & 10 & 71 & 10 & -155 & 664 & -287 \\ 389 & 29 & -139 & -110 & -45 & -287 & 1207 \end{bmatrix},$$

which gives $7535Y = U^\top U + V$. One can also check that $Ye_i, Y(e_0 - e_i) \in \text{FRAC}(G_4)$ for every $i \in V(G_4)$. This shows that $\bar{x} := \frac{1}{200}(78, 12, 78, 78, 78, 78)^\top \in \text{LS}_+(G_4)$. Now since $\bar{e}^\top \bar{x} = 2.01 > \alpha(G_4)$, we see that $r_+(G_4) \geq 2$. Since G_n contains G_4 as a subgraph for all $n \geq 4$, we conclude that $\alpha_{\text{LS}_+}(G_n) > 2$. Then Lemma 30(ii) implies that $\alpha_{\text{LS}_+}(H) > 2$, and so $r_+(H) \geq 2$.

Finally, notice that $H - v_1$ must be a perfect graph, and so $r_+(H - v_1) \leq 1$. Thus, we conclude that $r_+(H) = 2$. □

We remark that in the proof for $r_+(G_{2,2}) \geq 2$ in [EMN06], the following certificate matrix was given:

$$\frac{1}{2688} \begin{array}{c} \begin{array}{ccccccc} & 4_2 & 1 & 3 & 4_0 & 4_1 & 2 \end{array} \\ \begin{bmatrix} 2688 & 769 & 769 & 769 & 769 & 769 & 1538 \\ 769 & 769 & 0 & 336 & 413\frac{7}{13} & 0 & 0 \\ 769 & 0 & 769 & 0 & 336 & 384 & 0 \\ 769 & 336 & 0 & 769 & 0 & 384 & 0 \\ 769 & 413\frac{7}{13} & 336 & 0 & 769 & 0 & 896 \\ 769 & 0 & 384 & 384 & 0 & 769 & 0 \\ 1538 & 0 & 0 & 0 & 896 & 0 & 1538 \end{bmatrix} \end{array}$$

However, the certificate is incorrect: $Y[2, 4_0] = \frac{896}{2688} = \frac{1}{3} > Y[0, 4_0]$, and thus violates $Ye_{4_0} \in \text{cone}(\text{FRAC}(G_{2,2}))$. In fact, since the vector $\frac{1}{2688}(769, 769, 769, 769, 769, 1538)^\top$ contains only one entry greater than $\frac{1}{3}$, any certificate matrix for this vector cannot contain the entry $\frac{1}{3}$ (which would have to appear in at least 2 columns in the certificate). Still, the claim that $r_+(G_{2,2}) = 2$ is correct, as shown in the proof of Proposition 33.

Next, while all graphs in $\mathcal{S}(K_n)$ have LS$_+$-rank 2, we show that not all graphs in $\mathcal{S}^2(K_n)$ have LS$_+$-rank 3. Given a graph G, we say that a path in G is *sparse* if at most one of the vertices in the path has degree greater than 2 in G. For example, in Figure 9, the graph on the left contains a sparse path $4_1, 4_0, 5_0, 5_1, 5_2$ of length 4, while the graph on the right also contains a sparse path $4_1, 4_2, 3, 5_2, 5_1$ of length 4. Then we have the following.

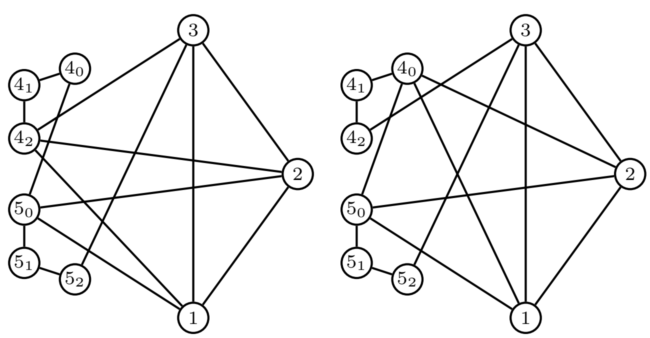

FIGURE 9. Two graphs in $\mathcal{S}^2(K_5)$ with sparse paths

Proposition 34. *Let $n \geq 4$. If $G \in \mathcal{S}^{n-3}(K_n)$ contains a sparse path of length at least 3, then G is not ℓ-minimal.*

Proof. First, we will need the general fact [LT03, Lemma 5] that if H_1, H_2 are subgraphs of H such that $H_1 \cup H_2 = H$ and $H_1 \cap H_2$ is a clique, then $r_+(H) = \max\{r_+(H_1), r_+(H_2)\}$. Now suppose $u_1, u_2, \ldots, u_m$ form a sparse path in G where $m \geq 4$, and let i be the index such that $\deg(u_j) = 2$ for all $j \in [m], j \neq i$. Then notice that $r_+(G - u_i) = r_+(G - U)$, where $U \coloneqq \{u_1, \ldots, u_m\}$. Since $m \geq 4$, $G - U$ has $3(n-2) - |U| < 3(n-3)$ vertices, and so $r_+(G - u_i) = r_+(G - U) \leq n - 4$. As a result, $r_+(G) \leq n - 3$, and G is not ℓ-minimal. □

Thus, both graphs in Figure 9 have LS_+-rank at most 2. (In fact, they have rank 2 since they both contain $G_{2,1}$ as an induced subgraph.) Next, we show that if we stretch a vertex in K_n, and then stretch one of the three new vertices in the stretched graph, the resulting graph cannot have LS_+-rank 3.

Proposition 35. *Let $n \geq 4$. Suppose $G_1 \in \mathcal{S}(K_n)$ is obtained by stretching vertex n in K_n, and $G_2 \in \mathcal{S}^2(K_n)$ is obtained by stretching vertex n_0, n_1, or n_2 in G_1. Then $r_+(G_2) = 2$.*

Proof. First, if G_2 is obtained from G_1 by stretching n_1, then n_{10}, n_{11}, n_{12} all have degree 2, and the vertices $n_{10}, n_{11}, n_{12}, n_2$ induce a sparse path of length 3. Thus, $r_+(G_2) \leq 2$ in this case.

Otherwise, assume without loss of generality that G_2 is obtained from G_1 by stretching n_0, and that $\{n_{02}, n_1\} \in E(G_2)$. (Note that $\{n_{00}, n_1\}$ may or may not be an edge.) Now notice that $G_2 - n_{02}$ is a perfect graph and thus has LS_+-rank at most 1. Thus, $r_+(G_2) \leq 2$ in this case as well. Finally, since $r_+(G_1) = 2$ (from Proposition 33) and $G_2 \in \mathcal{S}(G_1)$, Proposition 27(iii) implies that $r_+(G_2) \geq 2$. □

Thus, to obtain a graph with LS_+-rank 3 in $\mathcal{S}^2(K_n)$, it is necessary that we stretch two of the original vertices of K_n. (That is not sufficient though, as shown for the graphs in Figure 9.)

Next, observe that if G is an ℓ-minimal graph, then it is necessary that $\text{STAB}(G)$ has a facet with full support (or G would have a proper subgraph with the same LS_+-rank). We provide more circumstantial evidence that stretching a number of original vertices of a clique is a promising approach for generating ℓ-minimal graphs by showing that the stable set polytope of these graphs all have a full-support facet.

Proposition 36. *Let k, ℓ be integers where $\ell \geq 3$ and $\ell \geq k \geq 0$. Suppose $H \in \mathcal{S}^k(K_\ell)$ is obtained from K_ℓ by stretching k vertices in K_ℓ. Then $\sum_{i \in V(H)} x_i \leq k+1$ is a facet of* $\text{STAB}(H)$.

Proof. We prove our claim by induction on k. When $k = 0$, $H = K_\ell$, and the claim obviously holds. Next, assume $1 \leq k \leq \ell$. Let $T \subseteq [n]$ be the vertices in K_ℓ that were stretched to obtain H, and let $G \in \mathcal{S}^{k-1}(K_\ell)$ be a graph such that $H \in \mathcal{S}(G)$. (So there exists $v \in T$ where H is obtained from G by stretching v.)

By the inductive hypothesis, $\sum_{i \in V(G)} x_i \leq k$ is a facet of $\text{STAB}(G)$. We make use of Proposition 28, and prove that $c_0 = c_2 = k$, which would imply the result. First, it is obvious that $c_0, c_2 \leq k$ since $\alpha(G) = k$. Next, consider $\Gamma(v_0) \subseteq V(H)$. By the definition of the vertex stretching operation, one of the following must hold:

- There exists an index $j \in [n], j \neq v$ where $j \notin T$ (so $j \in V(H)$) and $j \notin \Gamma(v_0)$. Then

$$S \coloneqq \{j\} \cup \{p_1 : p \in T, p \neq v\}$$

  is a stable set that gives $c_0 = k$.

- There exists an index $j \in [n], j \neq v$ where $j \in T$ (so $j_0, j_1.j_2 \in V(H)$) and $j_0, j_1, j_2 \notin \Gamma(v_0)$. Then
$$S \coloneqq \{j_0, j_2\} \cup \{p_1 : p \in T, p \neq v, j\}$$
is a stable set that gives $c_0 = k$.

The same argument shows that $c_2 = k$, and this finishes the proof. □

We remark that the assumption of stretching only the original vertices of K_ℓ in Proposition 36 is necessary, as shown in the following example.

Example 37. Recall the graph $G_{2,2}$ from Figure 8. Observe that $G_{2,2} \in \mathcal{S}(K_4)$, and that $\bar{e}^\top x \leq 2$ is a facet of $\text{STAB}(G_{2,2})$. Now, if we stretch the vertex $4_2 \in V(G_{2,2})$ to obtain $H \in \mathcal{S}^2(K_4)$ as shown in Figure 10, then (using notation defined in Proposition 28) $c_0 = 1$ and $c_2 = 2$, which implies that

$$x_1 + x_2 + x_3 + x_{4_0} + x_{4_1} + x_{4_{20}} \leq 2 \tag{24}$$

is a facet of $\text{STAB}(H)$. (Observe that the subgraph induced by vertices $1, 2, 3, 4_0, 4_1, 4_{20}$ is isomorphic to $G_{2,1}$ from Figure 8.) This implies that $\sum_{i \in V(H)} x_i \leq 3$, which is the sum of (24) and the edge inequality $x_{4_{21}} + x_{4_{22}} \leq 1$, is not a facet of $\text{STAB}(H)$.

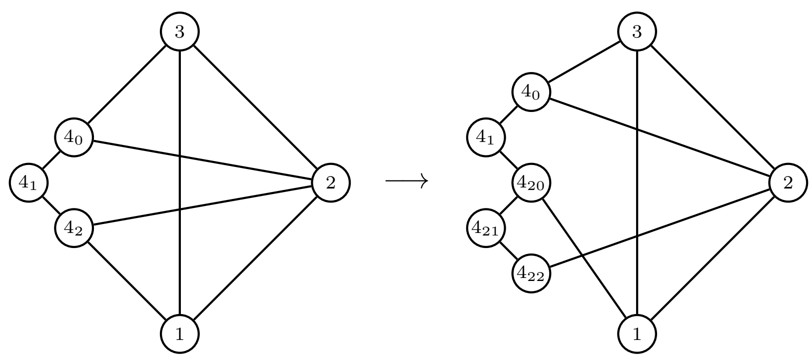

FIGURE 10. A graph in $H \in \mathcal{S}^2(K_4)$ (right) where $\bar{e}^\top x \leq \alpha(H)$ is not a facet of $\text{STAB}(H)$

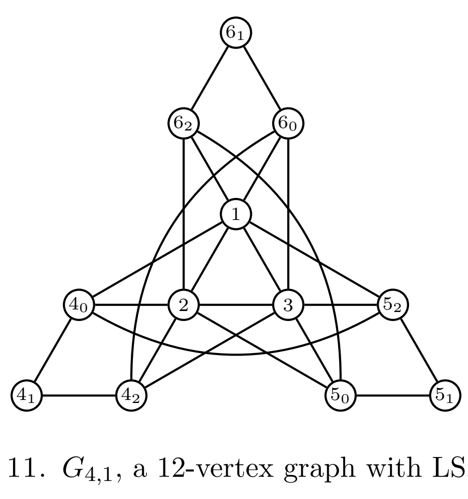

FIGURE 11. $G_{4,1}$, a 12-vertex graph with LS$_+$-rank 4

5.3. **Existence of 4-minimal graphs.** Consider the graph $G_{4,1}$ in Figure 11. We shall show that $r_+(G_{4,1}) = 4$, providing what we believe to be the first known example of a 4-minimal graph (and the first advance in this direction since 2006 [EMN06]). Observe that $G_{4,1} \in \mathcal{S}^3(K_6)$, and is obtained from stretching three of the original vertices in K_6. We also point out two important automorphisms of $G_{4,1}$ that will be useful in simplifying our analysis. Define the functions $f_1, f_2 : V(G_{4,1}) \to V(G_{4,1})$ as follows:

i	1	2	3	4_0	4_1	4_2	5_0	5_1	5_2	6_0	6_1	6_2
$f_1(i)$	2	3	1	5_0	5_1	5_2	6_0	6_1	6_2	4_0	4_1	4_2
$f_2(i)$	1	3	2	5_2	5_1	5_0	4_2	4_1	4_0	6_2	6_1	6_0

Visually, f_1 corresponds to rotating the graph $G_{4,1}$ in Figure 11 counterclockwise by $\frac{2\pi}{3}$, and f_2 corresponds to reflecting the figure along the centre vertical line. Then we have the following.

Theorem 38. *The* LS_+*-rank of* $G_{4,1}$ *is* 4.

Proof. For convenience, let $G := G_{4,1}$ throughout this proof. Since G has 12 vertices, by Theorem 3 it suffices to show that $r_+(G) \geq 4$. Consider the matrix Y_0 defined as follows:

$$
Y_0 := \begin{array}{c} \\ \end{array}
\begin{array}{ccccccccccccc}
 & 1 & 2 & 3 & 4_0 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 & 6_0 & 6_1 & 6_2
\end{array}
$$

$$
Y_0 := \begin{bmatrix}
100000 & 25340 & 25340 & 25340 & 16500 & 75020 & 16500 & 16500 & 75020 & 16500 & 16500 & 75020 & 16500 \\
25340 & 25340 & 0 & 0 & 0 & 17502 & 7838 & 7838 & 17502 & 0 & 0 & 25340 & 0 \\
25340 & 0 & 25340 & 0 & 0 & 25340 & 0 & 0 & 17502 & 7838 & 7838 & 17502 & 0 \\
25340 & 0 & 0 & 25340 & 7838 & 17502 & 0 & 0 & 25340 & 0 & 0 & 17502 & 7838 \\
16500 & 0 & 0 & 7838 & 16500 & 0 & 8073 & 589 & 15911 & 0 & 589 & 15419 & 1081 \\
75020 & 17502 & 25340 & 17502 & 0 & 75020 & 0 & 15419 & 51150 & 15911 & 15911 & 51150 & 15419 \\
16500 & 7838 & 0 & 0 & 8073 & 0 & 16500 & 1081 & 15419 & 589 & 0 & 15911 & 589 \\
16500 & 7838 & 0 & 0 & 589 & 15419 & 1081 & 16500 & 0 & 8073 & 589 & 15911 & 0 \\
75020 & 17502 & 17502 & 25340 & 15911 & 51150 & 15419 & 0 & 75020 & 0 & 15419 & 51150 & 15911 \\
16500 & 0 & 7838 & 0 & 0 & 15911 & 589 & 8073 & 0 & 16500 & 1081 & 15419 & 589 \\
16500 & 0 & 7838 & 0 & 589 & 15911 & 0 & 589 & 15419 & 1081 & 16500 & 0 & 8073 \\
75020 & 25340 & 17502 & 17502 & 15419 & 51150 & 15911 & 15911 & 51150 & 15419 & 0 & 75020 & 0 \\
16500 & 0 & 0 & 7838 & 1081 & 15419 & 589 & 0 & 15911 & 589 & 8073 & 0 & 16500
\end{bmatrix}.
$$

Again, the columns of Y_0 are labelled by the vertices in G they correspond to. with the rows of Y_0 following the same order of indexing.

We prove our claim by showing that $Y_0 \in \widehat{\mathrm{LS}}_+^3(G)$. First, one can check that $Y_0 \succeq 0$ (a UV-certificate is provided in Table 1). Moreover, observe that for all $i, j \in V(G)$,

$$Y_0[i,j] = Y_0[f_1(i), f_1(j)] = Y_0[f_2(i), f_2(j)],$$

and thus the entries of Y_0 exhibit the same symmetries of the graph that are exposed by the automorphisms f_1 and f_2. Hence, to show that $Y_0 \in \widehat{\mathrm{LS}}_+^3(G)$, it suffices to verify the conditions $Y_0 e_i, Y_0(e_0 - e_i) \in \mathrm{cone}(\mathrm{LS}_+^2(G))$ for $i \in \{1, 4_0, 6_1\}$, since for every other vertex j there is an automorphism of G that would map j to one of these three vertices.

Next, notice that

$$
\begin{aligned}
Y_0 e_1 &\leq 17502 \begin{bmatrix} 1 \\ \chi_{\{1,4_1,5_1,6_1\}} \end{bmatrix} + 7838 \begin{bmatrix} 1 \\ \chi_{\{1,4_2,5_0,6_1\}} \end{bmatrix}, \\
Y_0 e_{4_0} &\leq 7838 \begin{bmatrix} 1 \\ \chi_{\{3,4_0,5_1,6_1\}} \end{bmatrix} + 589 \begin{bmatrix} 1 \\ \chi_{\{4_0,4_2,5_0,6_1\}} \end{bmatrix} + 6992 \begin{bmatrix} 1 \\ \chi_{\{4_0,4_2,5_1,6_1\}} \end{bmatrix} \\
&\quad + 492 \begin{bmatrix} 1 \\ \chi_{\{4_0,4_2,5_1,6_2\}} \end{bmatrix} + 589 \begin{bmatrix} 1 \\ \chi_{\{4_0,5_1,6_0,6_2\}} \end{bmatrix}, \\
Y_0(e_0 - e_{6_1}) &\leq 7366 \begin{bmatrix} 1 \\ \chi_{\{2,4_1,5_1,6_0\}} \end{bmatrix} + 476 \begin{bmatrix} 1 \\ \chi_{\{2,4_1,5_2,6_0\}} \end{bmatrix} + 476 \begin{bmatrix} 1 \\ \chi_{\{3,4_0,5_1,6_2\}} \end{bmatrix}
\end{aligned}
$$

$$+ 7366 \begin{bmatrix} 1 \\ \chi_{\{3,4_1,5_1,6_2\}} \end{bmatrix} + 605 \begin{bmatrix} 1 \\ \chi_{\{4_0,4_2,5_1,6_2\}} \end{bmatrix} + 605 \begin{bmatrix} 1 \\ \chi_{\{4_1,5_0,5_2,6_0\}} \end{bmatrix}$$
$$+ 8058 \begin{bmatrix} 1 \\ \chi_{\{4_1,5_1,6_0,6_2\}} \end{bmatrix} + 28 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since all incidence vectors above correspond to stable sets in G, we obtain that $Y_0e_1, Y_0e_{4_0}, Y_0(e_0 - e_{6_1}) \in \operatorname{cone}(\operatorname{STAB}(G)) \subseteq \operatorname{cone}(\operatorname{LS}_+^2(G))$. The details for $Y_0e_{6_1}, Y_0(e_0 - e_1), Y_0(e_0 - e_{4_0}) \in \operatorname{cone}(\operatorname{LS}_+^2(G))$ are provided, respectively, in the proofs of Lemmas 48, 47, and 49 in Appendix B.

Finally, let $\bar{x}$ be the vector such that $Y_0e_0 = 100000 \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix}$. Since $Y_0 \in \widehat{\operatorname{LS}}_+^3(G)$, we have $\bar{x} \in \operatorname{LS}_+^3(G)$. Thus, we see that

$$\alpha_{\operatorname{LS}_+^3}(G) \geq \bar{e}^\top \bar{x} = 4.0008 > 4 = \alpha(G).$$

Thus, $r_+(G) \geq 4$. □

Notice that $G_{4,1}$ contains 24 edges, and by Lemma 30(ii) it follows that every graph in $\mathcal{S}^3(K_6)$ that is a subgraph of $G_{4,1}$ (which can have as few as 21 edges) also has LS$_+$-rank 4, giving more examples of 4-minimal graphs. This implies that the densest 4-minimal graph has density at least $d(G_{4,1}) = \frac{24}{\binom{12}{2}} = \frac{4}{11}$, while the sparsest 4-minimal graph has density at most $\frac{21}{\binom{12}{2}} = \frac{7}{22}$.

Moreover, the fact that $G_{4,1}$ is 4-minimal also provides some new examples of 3-minimal graphs.

Corollary 39. *Let $G_{3,2} := G_{4,1} \ominus 6_1$. Then $G_{3,2}$ is a 3-minimal graph.*

Proof. Since $r_+(G_{4,1}) = 4$, there exists vertex $i \in V(G_{4,1})$ where $G \ominus i$ has LS$_+$-rank 3, which implies that $\deg(i) = 2$, and so $i \in \{4_1, 5_1, 6_1\}$. Now observe that $G_{4,1} \ominus 4_1, G_{4,1} \ominus 5_1, G_{4,1} \ominus 6_1$ are all isomorphic to each other. Thus, $G_{3,2}$ is 3-minimal. □

By Lemma 30(ii) again, every graph in $\mathcal{S}^2(K_5)$ that is a subgraph of $G_{3,2}$ (which includes $G_{3,1}$) is 3-minimal. This implies that 3-minimal graph can have density as high as $d(G_{3,2}) = \frac{16}{\binom{9}{2}} = \frac{4}{9}$ and as low as $d(G_{3,1}) = \frac{14}{\binom{9}{2}} = \frac{7}{18}$.

We close the section by showing that there are no 3-minimal graphs with fewer edges than $G_{3,1}$.

Proposition 40. *Suppose G is a 3-minimal graph. Then $|E(G)| \geq 14$.*

Proof. For a contradiction, suppose $|V(G)| = 9, r_+(G) = 3$, and $|E(G)| \leq 13$. Given $A, B \subseteq V(G)$, we use $\delta(A)$ to denote the number of edges with both endpoints in A, and $\delta(A, B)$ denote the number of edges that have one endpoint in A and one endpoint in B.

Since G is 3-minimal, there must exist vertex v_1 where $r_+(G \ominus v_1) = 2$. This implies that $|V(G \ominus v_1)| \geq 6$, and thus $\deg(v_1) \leq 2$. Since ℓ-minimal graphs cannot have cut vertices, we see that $\deg(v_1) = 2$ and $|V(G \ominus v)| = 6$, and so $G \ominus v$ is isomorphic to either $G_{2,1}$ (8 edges) or $G_{2,2}$ (9 edges).

Let v_0, v_2 be the two neighbours of v_1, and let $A := \{v_0, v_1, v_2\}$ and $B := V(G) \setminus A$. Observe that

$$|E(G)| = \delta(A) + \delta(B) + \delta(A, B). \tag{25}$$

Since $|E(G)| \leq 13, \delta(A) = 2, \delta(B) \geq 8$, we obtain $\delta(A, B) \leq 3$. Again, G being 3-minimal implies that $\deg(v_0), \deg(v_2) \geq 2$, and so we obtain that $2 \leq \delta(\{v_0\}, B) + \delta(\{v_2\}, B) \leq 3$. Thus, we may assume without loss of generality that $\delta(\{v_0\}, B) = 1$, and let u be the only neighbour of v_0 in B.

If $\delta(\{v_2\}, B) = 1$, then u, v_0, v_1, v_2 form a sparse path of length 3 (with $\deg(v_0) = \deg(v_1) = \deg(v_2) = 2$), and Proposition 34 implies that G is not 3-minimal. Now suppose $\delta(\{v_2\}, B) = 2$. This means that $\delta(A, B) = 3$, and so from (25) we know that $|E(G)| = 13, \delta(B) = 8$, and $G - A$ is indeed isomorphic to $G_{2,1}$ and not $G_{2,2}$.

Next, since $r_+(G) = 3$, we obtain that $r_+(G - u) \geq 2$. However, notice that v_1 is a cut vertex in $G - u$. Thus, if we let $A' := \{u, v_0, v_1\}$ and $B' := V(G) \setminus A'$, then we see that $r_+(G - A') \geq 2$. Since $G - A'$ has 6 vertices, it must be isomorphic to $G_{2,1}$ or $G_{2,2}$. Thus, we see that $\delta(B') \geq 8$. Also, $\delta(A') = 2$ and

$$\delta(A', B') = \delta(\{v_1\}, B') + \delta(\{u\}, B') = 1 + (\deg(u) - 1) = \deg(u).$$

Since $13 = |E(G)| = \delta(A') + \delta(A', B') + \delta(B')$, we obtain that $\deg(u) = \delta(A', B') = 3$, and $\delta(B') = 8$. Thus, $G - A'$ is also isomorphic to $G_{2,1}$ and not $G_{2,2}$. For both $G - A$ and $G - A'$ to be isomorphic to $G_{2,1}$, v_2 must be adjacent to the two neighbours of u in $G \ominus v_1$. Thus, G is isomorphic to the graph shown in Figure 12.

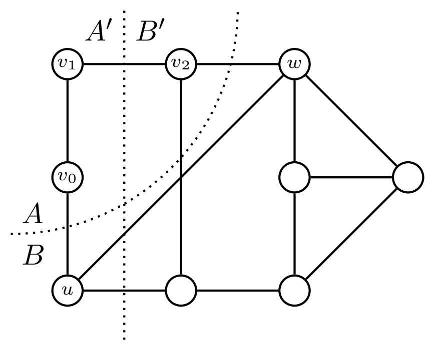

FIGURE 12. Illustrating the proof of Proposition 40

However, notice that $G - w$ has LS_+-rank 1, which contradicts $r_+(G) = 3$. This completes the proof. □

6. Some Future Research Directions

In this section, we mention some follow-up questions to our work in this manuscript that could lead to interesting future research.

Problem 41. What is the exact LS_+-rank of H_k?

While we showed that $r_+(H_k) \geq 0.19k$ asymptotically in Section 3, there is likely room for improvement for this bound. First, Lemma 10 is not sharp. In particular, the assumptions needed for $Y(e_0 - e_{1_0}), Y(e_0 - e_{1_1}) \in \text{LS}_+^{p-1}(H_k)$ are sufficient but not necessary. Using CVX, a package for specifying and solving convex programs [GB14, GB08] with SeDuMi [Stu99], we obtained that $r_+(H_6) \geq 3$. However, there do not exist a, b, c, d that would satisfy the assumptions of Lemma 10 for $k = 6$.

Even so, using Lemma 15 and the approach demonstrated in Example 16, we found computationally that $r_+(H_k) > 0.25k$ for all $k \leq 10000$. One reason for the gap between this computational bound and the analytical bound given in Theorem 18 is that the analytical bound only takes advantage of squeezing ℓ_i's over the interval $(u_1(k), u_2(k))$. Since we were able to show that $h(k, \ell) = \Theta(\frac{1}{k})$ over this interval (Lemma 17), this enabled us to establish a $\Theta(k)$ rank lower bound. Computationally, we see that we could get more ℓ_i's in over the interval $(u_2(k), u_3(k))$. However, over this interval, $h(k, \ell)$ is an increasing function that goes from $\frac{2}{k-2}$ at $u_2(k)$ to $\Theta\left(\frac{1}{\sqrt{k}}\right)$ at $u_3(k)$. This means that simply bounding $h(k, \ell)$ from above by $h(k, u_3(k))$ would only add an additional factor of $\Theta(\sqrt{k})$ in the rank lower bound. Thus, improving the constant factor in Theorem 18 would seem to require additional insights.

As for an upper bound on $r_+(H_k)$, we know that $r_+(H_4) = 2$, and $r_+(H_{k+1}) \leq r_+(H_k) + 1$ for all k. This gives the obvious upper bound of $r_+(H_k) \leq k - 2$. It would be interesting to obtain sharper bounds or even nail down the exact LS_+-rank of H_k.

Problem 42. Is there an ℓ-minimal graph G in $\mathcal{S}^{\ell-1}(K_{\ell+2})$ for all $\ell \in \mathbb{N}$?

Results from [LT03, EMN06] show that the answer is "yes" for $\ell = 1, 2, 3$. Our first 4-minimal graph shows that this is also true for $\ell = 4$. Does the pattern continue for larger ℓ? And more importantly, how can we verify the LS_+-rank of these graphs analytically, as opposed to primarily relying on specific numerical certificates?

Problem 43. Given $\ell \in \mathbb{N}$, what are the maximum and minimum possible edge densities of ℓ-minimal graphs?

Given $\ell \in \mathbb{N}$, let $d^+(\ell)$ (resp. $d^-(\ell)$) be the maximum (resp. minimum) possible edge density of an ℓ-minimal graph. It was previously known that $d^+(1) = d^-(1) = 1$ (attained by the 3-cycle), $d^+(2) = \frac{3}{5}$ $(G_{2,2})$, $d^-(2) = \frac{8}{15}$ $(G_{2,1})$, and $d^-(3) \leq \frac{7}{18}$ $(G_{3,1})$. We proved in Proposition 40 that $d^-(3) = \frac{7}{18}$, and the new 3- and 4-minimal graphs we presented in Section 5 show that $d^+(3) \geq \frac{4}{9}, d^+(4) \geq \frac{4}{11}$, and $d^-(4) \leq \frac{7}{22}$. Can we prove tight bounds for $d^+(\ell)$ and/or $d^-(\ell)$ in general?

Problem 44. What can we say about the lift-and-project ranks of graphs for other positive semidefinite lift-and-project operators?

After LS_+, many stronger semidefinite lift-and-project operators (such as Las [Las01], BZ_+ [BZ04], Θ_k [GPT10], and SA_+ [AT16]) have been proposed. While these stronger operators are capable of producing tighter relaxations than LS_+, these SDP relaxations can also be more computationally challenging to solve. For instance, while the LS_+^k-relaxation of a set $P \subseteq [0,1]^n$ involves $O(n^k)$ PSD constraints of order $O(n)$, the operators $\text{Las}^k, \text{BZ}_+^k$ and SA_+^k all impose one (or more) PSD constraint of order $\Omega(n^k)$ in their formulations. It would be interesting to determine the corresponding properties of graphs which are minimal with respect to these stronger lift-and-project operators.

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Appendix A. On the Chvátal–Gomory rank of H_k

We provide the proof of Theorem 21 herein. First, we need a lemma about the valid inequalities of $\text{STAB}(H_k)$.

Lemma 45. *Suppose* $a^\top x \leq \beta$ *is valid for* $\text{STAB}(H_k)$ *where* $a \in \mathbb{Z}_+^{3k} \setminus \{0\}$*. Then* $\frac{\beta}{a^\top \bar{e}} > \frac{1}{3}$.

Proof. We consider two cases. First, suppose that $a_{i_1} = 0$ for all $i \in [k]$. Since $[k]_p$ is a stable set in H_k for $p \in \{0, 1, 2\}$, observe that

$$a^\top \bar{e} = a^\top (\chi_{[k]_0} + \chi_{[k]_1} + \chi_{[k]_2}) \leq \beta + 0 + \beta = 2\beta.$$

Thus, we obtain that $\frac{\beta}{a^\top \bar{e}} \geq \frac{1}{2} > \frac{1}{3}$ in this case. Otherwise, we may choose $i \in [k]$ where $a_{i_1} > 0$. Consider the stable sets

$$S_0 := ([k]_0 \setminus \{i_0\}) \cup \{i_1\}, S_1 := ([k]_1 \setminus \{i_1\}) \cup \{i_0, i_2\}, S_2 := ([k]_2 \setminus \{i_2\}) \cup \{i_1\}.$$

Now $\chi_{S_0} + \chi_{S_1} + \chi_{S_2} = \bar{e} + e_{i_1}$. Since $a_{i_1} > 0$, this implies that $a^\top \bar{e} < 3\beta$, and so $\frac{\beta}{a^\top \bar{e}} > \frac{1}{3}$ in this case as well. □

We will also need the following result.

Lemma 46. [CCH89, Lemma 2.1] *Let* $P \subseteq \mathbb{R}^n$ *be a rational polyhedron. Given* $u, v \in \mathbb{R}^n$ *and positive real numbers* $m_1, \ldots, m_d \in \mathbb{R}$*, define*

$$x^{(i)} := u - \left(\sum_{i=1}^{d} \frac{1}{m_i} \right) v$$

for all $i \in [d]$*. Suppose*

(i) $u \in P$*, and*
(ii) *for all* $i \in [d]$*,* $a^\top x^{(i)} \leq \beta$*, for every inequality* $a^\top x \leq \beta$ *that is valid for* P_I *and satisfies* $a \in \mathbb{Z}^n$ *and* $a^\top v < m_i$*.*

Then $x^{(i)} \in \text{CG}^i(P)$ *for all* $i \in [d]$*.*

We are now ready to prove Theorem 21.

Proof of Theorem 21. We first prove the rank lower bound. Given $d \geq 0$, let $k := \frac{1}{3}(2^{2d+1} + 7)$ (then $d = \log_4\left(\frac{3k-7}{2}\right)$). We show that the CG-rank of the inequality $\sum_{\ell \in B_{i,j}} x_\ell \leq k - 1$ is at least $d + 1$ using Lemma 46.

Let $u := \frac{1}{2}\bar{e}, v := \bar{e}$, and $m_i := 2^{2i+1}$ for all $i \in [d]$. Then notice that $x^{(i)} = \frac{2^{2i+1}+1}{3 \cdot 2^{2i+1}}\bar{e}$ for all $i \in [d]$. Now suppose $a^\top x \leq \beta$ is valid for $\text{STAB}(H_k)$ where a is an integral vector and $a^\top v < m_i$ (which translates to $a^\top \bar{e} < 2^{2i+1}$). Now Lemma 45 implies that $\frac{\beta}{a^\top \bar{e}} > \frac{1}{3}$. Furthermore, using the fact that $\beta, a^\top \bar{e}$ are both integers, $a^\top \bar{e} < 2^{2i+1}$, and $2^{2i+1} \equiv 2 \pmod 3$, we obtain that $\frac{\beta}{a^\top \bar{e}} \geq \frac{2^{2i+1}+1}{3 \cdot 2^{2i+1}}$, which implies that $a^\top x^{(i)} \leq \beta$. Thus, it follows from Lemma 46 that $x^{(i)} \in \text{CG}^i(H_k)$ for every $i \in [d]$.

In particular, we obtain that $x^{(d)} = \frac{2^{2d+1}+1}{3 \cdot 2^{2d+1}} \bar{e} \in \mathrm{CG}^d(H_k)$. However, notice that $x^{(d)}$ violates the inequality $\sum_{\ell \in B_{i,j}} x_\ell \leq k-1$ for $\mathrm{STAB}(H_k)$, as

$$\frac{k-1}{|B_{i,j}|} = \frac{k-1}{3k-4} = \frac{2^{2d+1}+4}{3 \cdot 2^{2d+1}+3} > \frac{2^{2d+1}+1}{3 \cdot 2^{2d+1}}.$$

Next, we turn to proving the rank upper bound. Given $d \in \mathbb{N}$, let $k := 2^d + 1$ (then $d = \log_2(k-1)$). We prove that $\sum_{\ell \in B_{i,j}} x_\ell \leq k-1$ is valid for $\mathrm{CG}^d(H_k)$ by induction on d. When $d = 1$, we see that $k = 3$ and $B_{i,j}$ induces a 5-cycle, so the claim holds.

Now assume $d \geq 2$, and $k = 2^d + 1$. Let i, j be distinct, fixed indices in $[k]$. By the inductive hypothesis, if we let $T \subseteq [k] \setminus \{i,j\}$ where $|T| = 2^{d-1} - 1$, then the inequality

$$x_{i_0} + x_{j_2} + \sum_{\ell \in T} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1}$$

is valid for $\mathrm{CG}^{d-1}(H_k)$ (since $\{\ell_0, \ell_1, \ell_2 : \ell \in T\} \cup \{i_0, j_2\}$ induce a copy of H'_{k-1}, which is a subgraph of H_k). Averaging the above inequality over all possible choices of T, we obtain that

$$x_{i_0} + x_{j_2} + \frac{2^{d-1}-1}{k-2} \sum_{\ell \in [k] \setminus \{i,j\}} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} \tag{26}$$

is valid for $\mathrm{CG}^{d-1}(H_k)$. Next, using an $B_{i,j}$ inequality on an H'_{k-1} subgraph plus two edge inequalities, we obtain that for all $T \subseteq [k] \setminus \{i,j\}$ where $|T| = 2^{d-1} + 1$, the inequality

$$\sum_{\ell \in T} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} + 2$$

is valid for $\mathrm{CG}^{d-1}(H_k)$. Averaging the above inequality over all choices of T, we obtain

$$\frac{2^{d-1}+1}{k-2} \sum_{\ell \in [k] \setminus \{i,j\}} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} + 2. \tag{27}$$

Taking the sum of (26) and $\frac{k-2^{d-1}-1}{2^{d-1}+1}$ times (27), we obtain that

$$x_{i_0} + x_{j_2} + \sum_{\ell \in [k] \setminus \{i,j\}} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq \frac{k-2^{d-1}-1}{2^{d-1}+1}(2^{d-1}+2) + 2^{d-1} \tag{28}$$

is valid for $\mathrm{CG}^{d-1}(H_k)$. Now observe that the left hand side of (28) is simply $\sum_{\ell \in B_{i,j}} x_\ell$. On the other hand, the right hand side simplifies to $k - 2 + \frac{k}{2^{d-1}+1}$. Since $k = 2^d + 1, 1 < \frac{k}{2^{d-1}+1} < 2$, and so the floor of the right hand side of (28) is $k-1$. This shows that the inequality $\sum_{\ell \in B_{i,j}} x_\ell \leq k-1$ has CG-rank at most d. $\square$

Appendix B. Proofs of Lemmas 47, 48, and 49

The following lemmas provide the deferred technical details from the proof of Theorem 38. To reduce cluttering, given $S \subseteq [n]$ we will let $\hat{\chi}_S$ denote the vector $\begin{bmatrix} 1 \\ \chi_S \end{bmatrix} \in \mathbb{R}^{n+1}$.

Lemma 47. *Let Y_0 be as defined in the proof of Theorem 38. Then $Y_0(e_0 - e_1) \in \mathrm{cone}(\mathrm{LS}_+^2(G_{4,1}))$.*

Proof. First, notice that $[Y_0(e_0 - e_1)]_1 = 0$. Thus, let $G' := G_{4,1} - 1$ and v be the restriction of $Y_0(e_0 - e_1)$ to the coordinates indexed by $\operatorname{cone}(\mathrm{LS}_+^2(G'))$. Then, by Lemma 19, it suffices to show that $v \in \operatorname{cone}(\mathrm{LS}_+^2(G'))$. Consider the matrix

$$Y_2 := \begin{array}{c}\begin{matrix} & 2 & 3 & 4_0 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 & 6_0 & 6_1 & 6_2 \end{matrix} \\ \begin{bmatrix} 74660 & 25340 & 25340 & 16500 & 57518 & 8662 & 8662 & 57518 & 16500 & 16500 & 49680 & 16500 \\ 25340 & 25340 & 0 & 0 & 25340 & 0 & 0 & 17166 & 8174 & 8363 & 16977 & 0 \\ 25340 & 0 & 25340 & 8174 & 17166 & 0 & 0 & 25340 & 0 & 0 & 16977 & 8363 \\ 16500 & 0 & 8174 & 16500 & 0 & 8320 & 342 & 16158 & 0 & 0 & 14067 & 2433 \\ 57518 & 25340 & 17166 & 0 & 57518 & 0 & 7678 & 41360 & 16158 & 16494 & 34971 & 14067 \\ 8662 & 0 & 0 & 8320 & 0 & 8662 & 984 & 7678 & 342 & 0 & 8662 & 0 \\ 8662 & 0 & 0 & 342 & 7678 & 984 & 8662 & 0 & 8320 & 0 & 8662 & 0 \\ 57518 & 17166 & 25340 & 16158 & 41360 & 7678 & 0 & 57518 & 0 & 14067 & 34971 & 16494 \\ 16500 & 8174 & 0 & 0 & 16158 & 342 & 8320 & 0 & 16500 & 2433 & 14067 & 0 \\ 16500 & 8363 & 0 & 0 & 16494 & 0 & 0 & 14067 & 2433 & 16500 & 0 & 8137 \\ 49680 & 16977 & 16977 & 14067 & 34971 & 8662 & 8662 & 34971 & 14067 & 0 & 49680 & 0 \\ 16500 & 0 & 8363 & 2433 & 14067 & 0 & 0 & 16494 & 0 & 8137 & 0 & 16500 \end{bmatrix}\end{array}.$$

We claim that $Y_2 \in \widehat{\mathrm{LS}}_+^2(G')$. First, one can verify that $Y_2 \succeq 0$ (a UV-certificate is provided in Table 1). Also, notice that the function f_2 (restricted to $V(G')$) is an automorphism of G'. Moreover, observe that for all $i, j \in V(G')$, $Y_2[i,j] = Y_2[f_2(i), f_2(j)]$. Thus, by symmetry, it only remains to prove the conditions $Y_2 e_i, Y_2(e_0 - e_i) \in \operatorname{cone}(\mathrm{LS}_+(G'))$ for $i \in \{2, 4_0, 4_1, 4_2, 6_0, 6_1\}$.

First, notice that

- $[Y_2 e_{4_1}]_0 = [Y_2 e_{4_1}]_{4_1}$, $[Y_2 e_{4_1}]_{4_0} = [Y_2 e_{4_1}]_{4_2} = 0$, and that the following matrix certifies that $Y_2 e_{4_1}$ (with the entries corresponding to vertices $4_0, 4_1, 4_2$ removed) belongs to $\operatorname{cone}(\mathrm{LS}_+(G' \ominus 4_1))$.

$$Y_{21} := \begin{array}{c}\begin{matrix} & 2 & 3 & 5_0 & 5_1 & 5_2 & 6_0 & 6_1 & 6_2 \end{matrix} \\ \begin{bmatrix} 57518 & 25340 & 17164 & 7678 & 41360 & 16158 & 16496 & 34970 & 14068 \\ 25340 & 25340 & 0 & 0 & 19057 & 6283 & 5860 & 19444 & 0 \\ 17164 & 0 & 17164 & 0 & 17164 & 0 & 0 & 12010 & 5117 \\ 7678 & 0 & 0 & 7678 & 0 & 7678 & 3125 & 4516 & 0 \\ 41360 & 19057 & 17164 & 0 & 41360 & 0 & 10718 & 26585 & 10400 \\ 16158 & 6283 & 0 & 7678 & 0 & 16158 & 5778 & 8385 & 3668 \\ 16496 & 5860 & 0 & 3125 & 10718 & 5778 & 16496 & 0 & 8910 \\ 34970 & 19444 & 12010 & 4516 & 26585 & 8385 & 0 & 34970 & 0 \\ 14068 & 0 & 5117 & 0 & 10400 & 3668 & 8910 & 0 & 14068 \end{bmatrix}\end{array}$$

- $[Y_2 e_{6_1}]_0 = [Y_2 e_{6_1}]_{6_1}$, $[Y_2 e_{6_1}]_{6_0} = [Y_2 e_{6_1}]_{6_2} = 0$, and that the following matrix certifies that $Y_2 e_{6_1}$ (with the entries corresponding to vertices $6_0, 6_1, 6_2$ removed) belongs to $\operatorname{cone}(\mathrm{LS}_+(G' \ominus 6_1))$.

$$Y_{22} := \begin{array}{c}\begin{matrix} & 2 & 3 & 4_0 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 \end{matrix} \\ \begin{bmatrix} 49680 & 16977 & 16977 & 14068 & 34970 & 8662 & 8662 & 34970 & 14068 \\ 16977 & 16977 & 0 & 0 & 16977 & 0 & 0 & 11129 & 5848 \\ 16977 & 0 & 16977 & 5848 & 11129 & 0 & 0 & 16977 & 0 \\ 14068 & 0 & 5848 & 14068 & 0 & 8220 & 442 & 13626 & 0 \\ 34970 & 16977 & 11129 & 0 & 34970 & 0 & 7578 & 21344 & 13626 \\ 8662 & 0 & 0 & 8220 & 0 & 8662 & 1084 & 7578 & 442 \\ 8662 & 0 & 0 & 442 & 7578 & 1084 & 8662 & 0 & 8220 \\ 34970 & 11129 & 16977 & 13626 & 21344 & 7578 & 0 & 34970 & 0 \\ 14068 & 5848 & 0 & 0 & 13626 & 442 & 8220 & 0 & 14068 \end{bmatrix}\end{array}$$

- $[Y_2(e_0 - e_2)]_2 = 0$, and that the following matrix certifies that $Y_2(e_0 - e_2)$ (with the entry corresponding to vertex 2 removed) belongs to $\operatorname{cone}(\mathrm{LS}_+(G' - 2))$.

$$\begin{array}{ccccccccccc} & 3 & 4_0 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 & 6_0 & 6_1 & 6_2 \end{array}$$

$$Y_{23} := \begin{bmatrix} 49320 & 25340 & 16500 & 32178 & 8662 & 8662 & 40354 & 8324 & 8137 & 32703 & 16500 \\ 25340 & 25340 & 6118 & 19222 & 0 & 0 & 25340 & 0 & 0 & 19368 & 5972 \\ 16500 & 6118 & 16500 & 0 & 8107 & 595 & 15905 & 0 & 2465 & 10494 & 6006 \\ 32178 & 19222 & 0 & 32178 & 0 & 7688 & 24409 & 7769 & 5672 & 21928 & 10250 \\ 8662 & 0 & 8107 & 0 & 8662 & 974 & 7688 & 555 & 0 & 5933 & 2729 \\ 8662 & 0 & 595 & 7688 & 974 & 8662 & 0 & 8067 & 70 & 8592 & 0 \\ 40354 & 25340 & 15905 & 24409 & 7688 & 0 & 40354 & 0 & 7763 & 24111 & 16243 \\ 8324 & 0 & 0 & 7769 & 555 & 8067 & 0 & 8324 & 374 & 7950 & 257 \\ 8137 & 0 & 2465 & 5672 & 0 & 70 & 7763 & 374 & 8137 & 0 & 8067 \\ 32703 & 19368 & 10494 & 21928 & 5933 & 8592 & 24111 & 7950 & 0 & 32703 & 0 \\ 16500 & 5972 & 6006 & 10250 & 2729 & 0 & 16243 & 257 & 8067 & 0 & 16500 \end{bmatrix}$$

- $[Y_2(e_0 - e_{4_0})]_{4_0} = 0$, and that the following matrix certifies that $Y_2(e_0 - e_{4_0})$ (with the entry corresponding to vertex 4_0 removed) belongs to $\text{cone}(\text{LS}_+(G' - 4_0))$.

$$\begin{array}{ccccccccccc} & 2 & 3 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 & 6_0 & 6_1 & 6_2 \end{array}$$

$$Y_{24} := \begin{bmatrix} 58160 & 25340 & 17164 & 57518 & 342 & 8320 & 41360 & 16500 & 16496 & 35612 & 14068 \\ 25340 & 25340 & 0 & 25228 & 0 & 0 & 19229 & 6068 & 5788 & 19552 & 0 \\ 17164 & 0 & 17164 & 17063 & 0 & 0 & 17055 & 0 & 0 & 12187 & 4977 \\ 57518 & 25228 & 17063 & 57518 & 0 & 7946 & 41199 & 16198 & 16378 & 35219 & 13979 \\ 342 & 0 & 0 & 0 & 342 & 340 & 1 & 190 & 0 & 340 & 1 \\ 8320 & 0 & 0 & 7946 & 340 & 8320 & 0 & 8190 & 3046 & 5274 & 0 \\ 41360 & 19229 & 17055 & 41199 & 1 & 0 & 41360 & 0 & 10763 & 26612 & 10417 \\ 16500 & 6068 & 0 & 16198 & 190 & 8190 & 0 & 16500 & 5653 & 8919 & 3601 \\ 16496 & 5788 & 0 & 16378 & 0 & 3046 & 10763 & 5653 & 16496 & 0 & 9091 \\ 35612 & 19552 & 12187 & 35219 & 340 & 5274 & 26612 & 8919 & 0 & 35612 & 0 \\ 14068 & 0 & 4977 & 13979 & 1 & 0 & 10417 & 3601 & 9091 & 0 & 14068 \end{bmatrix}$$

Also, notice that $Y_{21}e_0 = Y_{21}(e_{5_1} + e_{5_2})$. Thus, if we let Y'_{21} be the matrix obtained from Y_{21} by removing the 0^{th} row and column, then we see that $Y'_{21} \succeq 0 \Rightarrow Y_{21} \succeq 0$. The UV-certificates of Y'_{21}, Y_{22}, Y_{23}, and Y_{24} are provided in Table 1.

Next, observe that

$$\begin{aligned}
Y_2 e_2 &\leq 8291\hat{\chi}_{\{2,4_1,5_1,6_0\}} + 8873\hat{\chi}_{\{2,4_1,5_1,6_1\}} + 72\hat{\chi}_{\{2,4_1,5_2,6_0\}} + 8104\hat{\chi}_{\{2,4_1,5_2,6_1\}}, \\
Y_2 e_{4_0} &\leq 6365\hat{\chi}_{\{3,4_0,5_1,6_1\}} + 1811\hat{\chi}_{\{3,4_0,5_1,6_2\}} + 342\hat{\chi}_{\{4_0,4_2,5_0,6_1\}} + 7361\hat{\chi}_{\{4_0,4_2,5_1,6_1\}} \\
&\quad + 617\hat{\chi}_{\{4_0,4_2,5_1,6_2\}} + 4\hat{\chi}_{\{4_0,5_1,6_0,6_2\}}, \\
Y_2 e_{4_2} &\leq 642\hat{\chi}_{\{4_0,4_2,5_0,6_1\}} + 7678\hat{\chi}_{\{4_0,4_2,5_1,6_1\}} + 342\hat{\chi}_{\{4_2,5_0,5_2,6_1\}}, \\
Y_2 e_{6_0} &\leq 6764\hat{\chi}_{\{2,4_1,5_1,6_0\}} + 1599\hat{\chi}_{\{2,4_1,5_2,6_0\}} + 4\hat{\chi}_{\{4_0,5_1,6_0,6_2\}} + 7300\hat{\chi}_{\{4_1,5_1,6_0,6_2\}} \\
&\quad + 833\hat{\chi}_{\{4_1,5_2,6_0,6_2\}}, \\
Y_2(e_0 - e_{4_1}) &\leq 7254\hat{\chi}_{\{3,4_0,5_1,6_1\}} + 1004\hat{\chi}_{\{3,4_0,5_1,6_2\}} + 489\hat{\chi}_{\{4_0,4_2,5_0,6_1\}} + 6472\hat{\chi}_{\{4_0,4_2,5_1,6_1\}} \\
&\quad + 1291\hat{\chi}_{\{4_0,4_2,5_1,6_2\}} + 137\hat{\chi}_{\{4_0,5_1,6_0,6_2\}} + 495\hat{\chi}_{\{4_2,5_0,5_2,6_1\}}, \\
Y_2(e_0 - e_{4_2}) &\leq 832\hat{\chi}_{\{2,4_1,5_1,6_0\}} + 3414\hat{\chi}_{\{2,4_1,5_1,6_1\}} + 496\hat{\chi}_{\{2,4_1,5_2,6_0\}} + 919\hat{\chi}_{\{2,4_1,5_2,6_1\}} \\
&\quad + 5480\hat{\chi}_{\{3,4_0,5_1,6_1\}} + 2094\hat{\chi}_{\{3,4_0,5_1,6_2\}} + 2827\hat{\chi}_{\{3,4_1,5_1,6_1\}} + 1634\hat{\chi}_{\{3,4_1,5_1,6_2\}} \\
&\quad + 700\hat{\chi}_{\{4_0,5_1,6_0,6_2\}} + 526\hat{\chi}_{\{4_1,5_0,5_2,6_0\}} + 1070\hat{\chi}_{\{4_1,5_05_2,6_1\}} + 690\hat{\chi}_{\{4_1,5_1,6_0,6_2\}} \\
&\quad + 455\hat{\chi}_{\{4_1,5_2,6_0,6_2\}} + 126\hat{\chi}_{\{4_2,5_0,5_2,6_1\}} + \frac{44735}{57518} Y_2 e_{4_1}, \\
Y_2(e_0 - e_{6_0}) &\leq 275\hat{\chi}_{\{2,4_1,5_1,6_1\}} + 186\hat{\chi}_{\{3,4_0,5_1,6_1\}} + 2333\hat{\chi}_{\{3,4_0,5_1,6_2\}} + 186\hat{\chi}_{\{3,4_1,5_1,6_1\}} \\
&\quad + 5933\hat{\chi}_{\{3,4_1,5_1,6_2\}} + 140\hat{\chi}_{\{4_0,4_2,5_1,6_2\}} + 227\hat{\chi}_{\{4_1,5_0,5_2,6_1\}} + \frac{48880}{49680} Y_2 e_{6_1},
\end{aligned}$$

$$Y_2(e_0 - e_{6_1}) \leq 7474\hat{\chi}_{\{2,4_1,5_1,6_0\}} + 978\hat{\chi}_{\{2,4_1,5_2,6_0\}} + 978\hat{\chi}_{\{3,4_0,5_1,6_2\}} + 7474\hat{\chi}_{\{3,4_1,5_1,6_2\}} + 1454\hat{\chi}_{\{4_0,5_1,6_0,6_2\}} + 5168\hat{\chi}_{\{4_1,5_1,6_0,6_2\}} + 1454\hat{\chi}_{\{4_1,5_2,6_0,6_2\}}.$$

Since all incidence vectors above correspond to stable sets in G', and we already showed earlier that $Y_2e_{4_1}, Y_2e_{6_1} \in \text{cone}(\text{LS}_+(G'))$, we obtain that all the vectors above belong to $\text{cone}(\text{LS}_+(G'))$. Thus, we conclude that $Y_0(e_0 - e_1) \in \text{cone}(\text{LS}_+^2(G_{4,1}))$. □

Lemma 48. *Let Y_0 be as defined in the proof of Theorem 38. Then $Y_0e_{6_1} \in \text{cone}(\text{LS}_+^2(G_{4,1}))$.*

Proof. First, notice that $[Y_0e_{6_1}]_0 = [Y_0e_{6_1}]_{6_1}$, and $[Y_0e_{6_1}]_{6_0} = [Y_0e_{6_1}]_{6_2} = 0$. Thus, let $G' := G_{4,1} \ominus 6_1$ and v be the restriction of $Y_0e_{6_1}$ to the coordinates indexed by $\text{cone}(\text{LS}_+^2(G'))$. Then, by Lemma 19, it suffices to show that $v \in \text{cone}(\text{LS}_+^2(G'))$. Consider the matrix

$$Y_1 := \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4_0 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 \end{array} \\ \begin{bmatrix} 75020 & 25340 & 17502 & 17502 & 15419 & 51150 & 15911 & 15911 & 51150 & 15419 \\ 25340 & 25340 & 0 & 0 & 0 & 17400 & 7940 & 7940 & 17400 & 0 \\ 17502 & 0 & 17502 & 0 & 0 & 17502 & 0 & 0 & 9571 & 7931 \\ 17502 & 0 & 0 & 17502 & 7931 & 9571 & 0 & 0 & 17502 & 0 \\ 15419 & 0 & 0 & 7931 & 15419 & 0 & 7488 & 396 & 14993 & 0 \\ 51150 & 17400 & 17502 & 9571 & 0 & 51150 & 0 & 15485 & 27920 & 14993 \\ 15911 & 7940 & 0 & 0 & 7488 & 0 & 15911 & 396 & 15485 & 396 \\ 15911 & 7940 & 0 & 0 & 396 & 15485 & 396 & 15911 & 0 & 7488 \\ 51150 & 17400 & 9571 & 17502 & 14993 & 27920 & 15485 & 0 & 51150 & 0 \\ 15419 & 0 & 7931 & 0 & 0 & 14993 & 396 & 7488 & 0 & 15419 \end{bmatrix} \end{array}.$$

We claim that $Y_1 \in \widehat{\text{LS}}_+^2(G')$. First, one can verify that $Y_1 \succeq 0$ (a UV-certificate is provided in Table 1). Also, notice that the function f_2 (restricted to $V(G')$) is an automorphism of G'. Moreover, observe that for all $i, j \in V(G')$, $Y_1[i,j] = Y_1[f_2(i), f_2(j)]$. Thus, by symmetry, it only remains to prove the conditions $Y_1e_i, Y_1(e_0 - e_i) \in \text{cone}(\text{LS}_+(G'))$ for $i \in \{1, 2, 4_0, 4_1, 4_2\}$.

First, notice that $[Y_1e_{4_1}]_0 = [Y_1e_{4_1}]_{4_1}$, $[Y_1e_{4_1}]_{4_0} = [Y_1e_{4_1}]_{4_2} = 0$, and that the following matrix certifies that $Y_1e_{4_1}$ (with the entries corresponding to vertices $4_0, 4_1, 4_2$ removed) belongs to $\text{cone}(\text{LS}_+(G' \ominus 4_1))$. (See Table 1 for a UV-certificate.)

$$Y_{11} := \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 5_0 & 5_1 & 5_2 \end{array} \\ \begin{bmatrix} 51150 & 17400 & 17502 & 9571 & 15485 & 27920 & 14993 \\ 17400 & 17400 & 0 & 0 & 7544 & 9856 & 0 \\ 17502 & 0 & 17502 & 0 & 0 & 10450 & 7052 \\ 9571 & 0 & 0 & 9571 & 0 & 9571 & 0 \\ 15485 & 7544 & 0 & 0 & 15485 & 0 & 7941 \\ 27920 & 9856 & 10450 & 9571 & 0 & 27920 & 0 \\ 14993 & 0 & 7052 & 0 & 7941 & 0 & 14993 \end{bmatrix} \end{array}$$

Now consider the following vectors:

$$\begin{array}{lcccccccccl} & 1 & 2 & 3 & 4_0 & 4_1 & 4_2 & 5_0 & 5_1 & 5_2 & \\ z^{(1)} := [& 51150 & 17400 & 17502 & 9571 & 0 & 51150 & 0 & 5485 & 27920 & 14933 &]^\top \\ z^{(2)} := [& 51150 & 17502 & 17400 & 9571 & 0 & 51150 & 0 & 14933 & 27920 & 15485 &]^\top \\ z^{(3)} := [& 57518 & 25340 & 0 & 17164 & 14068 & 34970 & 16496 & 16158 & 41360 & 7678 &]^\top \\ z^{(4)} := [& 49680 & 0 & 16977 & 169771 & 40683 & 4970 & 8662 & 8662 & 34970 & 14068 &]^\top \end{array}$$

Notice that $z^{(1)} \in \text{cone}(\text{LS}_+(G'))$ follows from $Y_1e_{4_1} \in \text{cone}(\text{LS}_+(G'))$ as shown above. Then it follows from the symmetry of G' that $z^{(2)} \in \text{cone}(\text{LS}_+(G'))$ as well. $z^{(3)}, z^{(4)} \in \text{cone}(\text{LS}_+(G'))$ follows respectively from $Y_2e_{4_1}, Y_2e_{6_1} \in \text{cone}(\text{LS}_+(G_{4,1} - 1))$, as shown in Lemma 47. Next, observe that

$$
\begin{aligned}
Y_1 e_1 &\leq 17400\hat{\chi}_{\{1,4_1,5_1\}} + 7940\hat{\chi}_{\{1,4_2,5_0\}},\\
Y_1 e_2 &\leq 9571\hat{\chi}_{\{2,4_1,5_1\}} + 7931\hat{\chi}_{\{2,4_1,5_2\}},\\
Y_1 e_{4_0} &\leq 7931\hat{\chi}_{\{3,4_0,5_1\}} + 396\hat{\chi}_{\{4_0,4_2,5_0\}} + 7092\hat{\chi}_{\{4_0,4_2,5_1\}},\\
Y_1 e_{4_2} &\leq 414\hat{\chi}_{\{1,4_1,5_0\}} + 7523\hat{\chi}_{\{2,4_1,5_1\}} + 7974\hat{\chi}_{\{3,4_1,5_1\}},\\
Y_1(e_0 - e_1) &\leq 874\hat{\chi}_{\{2,4_1,5_1\}} + 3160\hat{\chi}_{\{2,4_1,5_2\}} + 3160\hat{\chi}_{\{3,4_0,5_1\}} + 874\hat{\chi}_{\{3,4_1,5_1\}} + 1100\hat{\chi}_{\{4_0,4_2,5_1\}}\\
&\quad + 1100\hat{\chi}_{\{4_1,5_0,5_2\}} + \frac{39412}{49680}z^{(4)},\\
Y_1(e_0 - e_2) &\leq 1729\hat{\chi}_{\{1,4_1,5_0\}} + 6165\hat{\chi}_{\{1,4_1,5_1\}} + 626\hat{\chi}_{\{1,4_2,5_0\}} + 1749\hat{\chi}_{\{1,4_2,5_1\}}\\
&\quad + 4298\hat{\chi}_{\{3,4_0,5_1\}} + 2999\hat{\chi}_{\{3,4_1,5_1\}} + 1009\hat{\chi}_{\{4_0,4_2,5_0\}} + 1751\hat{\chi}_{\{4_0,4_2,5_1\}}\\
&\quad + 1959\hat{\chi}_{\{4_1,5_0,5_2\}} + 971\hat{\chi}_{\{4_2,5_0,5_2\}} + \frac{34235}{57518}z^{(3)} + 27\hat{\chi}_{\emptyset},\\
Y_1(e_0 - e_{4_0}) &\leq 498\hat{\chi}_{\{1,4_1,5_0\}} + 2500\hat{\chi}_{\{1,4_1,5_1\}} + 639\hat{\chi}_{\{1,4_2,5_0\}} + 6832\hat{\chi}_{\{1,4_2,5_1\}}\\
&\quad + 1613\hat{\chi}_{\{2,4_1,5_1\}} + 1022\hat{\chi}_{\{2,4_1,5_2\}} + 1421\hat{\chi}_{\{3,4_1,5_1\}} + 478\hat{\chi}_{\{4_1,5_0,5_2\}}\\
&\quad + 952\hat{\chi}_{\{4_2,5_0,5_2\}} + \frac{20799}{51150}z^{(1)} + \frac{22819}{51150}z^{(2)} + 28\hat{\chi}_{\emptyset},\\
Y_1(e_0 - e_{4_1}) &\leq 452\hat{\chi}_{\{1,4_1,5_0\}} + 7504\hat{\chi}_{\{2,4_1,5_1\}} + 7931\hat{\chi}_{\{3,4_0,5_1\}} + 7955\hat{\chi}_{\{3,4_1,5_1\}} + 28\hat{\chi}_{\emptyset},\\
Y_1(e_0 - e_{4_2}) &\leq 234\hat{\chi}_{\{1,4_1,5_0\}} + 195\hat{\chi}_{\{2,4_1,5_2\}} + 7935\hat{\chi}_{\{3,4_0,5_1\}} + 93\hat{\chi}_{\{3,4_1,5_1\}}\\
&\quad + \frac{46278}{51150}z^{(1)} + \frac{4354}{51150}z^{(2)} + 20\hat{\chi}_{\emptyset}.
\end{aligned}
$$

Since all incidence vectors above correspond to stable sets in G', we obtain that all the vectors above belong to $\operatorname{cone}(\mathrm{LS}_+(G'))$. Thus, we conclude that $Y_0 e_{6_1} \in \operatorname{cone}(\mathrm{LS}_+^2(G_{4,1}))$. □

Lemma 49. *Let Y_0 be as defined in the proof of Theorem 38. Then $Y_0(e_0 - e_{4_0}) \in \operatorname{cone}(\mathrm{LS}_+^2(G_{4,1}))$.*

Proof. For convenience, let $G := G_{4,1}$ throughout this proof. Using $Y_0 e_{6_1} \in \operatorname{cone}(\mathrm{LS}_+^2(G))$ from Lemma 48 and the symmetry of G, we know that the vector

$$
z := \begin{bmatrix} \overset{1}{75020} & \overset{2}{17502} & \overset{3}{25340} & \overset{4_0}{17502} & \overset{4_1}{0} & \overset{4_2}{75020} & \overset{5_0}{0} & \overset{5_1}{15419} & \overset{5_2}{51150} & \overset{6_0}{15911} & \overset{6_1}{15911} & \overset{6_2}{51150} & 15419 \end{bmatrix}^\top
$$

belongs to $\operatorname{cone}(\mathrm{LS}_+^2(G))$. Now observe that

$$
\begin{aligned}
Y_0(e_0 - e_{4_0}) \leq \frac{2}{3}z + \frac{1}{3}\Big(&7726\hat{\chi}_{\{1,4_1,5_0,6_1\}} + 17105\hat{\chi}_{\{1,4_1,5_1,6_1\}} + 16187\hat{\chi}_{\{1,4_2,5_1,6_1\}}\\
&+ 8509\hat{\chi}_{\{2,4_1,5_1,6_0\}} + 8324\hat{\chi}_{\{2,4_1,5_1,6_1\}} + 8509\hat{\chi}_{\{2,4_1,5_2,6_1\}} + 9486\hat{\chi}_{\{3,4_1,5_1,6_1\}}\\
&+ 8017\hat{\chi}_{\{3,4_1,5_1,6_2\}} + 7403\hat{\chi}_{\{4_1,5_1,6_0,6_2\}} + 9170\hat{\chi}_{\{4_2,5_0,5_2,6_1\}} + 24\hat{\chi}_{\emptyset}\Big).
\end{aligned}
$$

Notice that all incidence vectors above correspond to stable sets in G. Since $\operatorname{cone}(\mathrm{LS}_+^2(G))$ is a lower-comprehensive convex cone, it follows that $Y_0(e_0 - e_{4_0}) \in \operatorname{cone}(\mathrm{LS}_+^2(G))$. □

Finally, we provide in Table 1 the UV-certificates of all PSD matrices used in Theorem 38 and Lemmas 47, 48, and 49.

Y_0

U

−2	1	1	1	1	1	1	1	1	1	1	1	1
0	114	−8	−107	152	27	−114	202	411	348	−353	−440	−234
0	−57	128	−71	−320	−492	−335	292	270	69	29	221	268
−361	−710	−711	−711	−88	434	−88	−88	434	−88	−89	434	−87
0	−135	521	−387	477	230	31	395	−171	−771	−872	−60	741
0	525	−145	−379	731	−65	−873	−779	−167	409	48	231	462
0	0	0	0	−984	0	984	−984	0	984	−984	0	984
820	−787	−788	−787	1222	−569	1222	1221	−569	1221	1222	−569	1222
0	1693	−2745	1052	−901	857	−628	1238	−329	−652	−337	−529	1280
0	−2192	−370	2562	909	116	−1116	326	−801	1102	−1235	685	14
0	1845	−595	−1250	308	−990	1042	1707	−2079	1128	−2014	3069	−2170
0	−378	1787	−1409	−2148	2972	−1904	1341	−2343	1854	808	−629	50
8215	2154	2154	2154	1246	6313	1246	1246	6313	1246	1246	6313	1246

V

11050	1142	1601	781	−196	621	−196	624	621	624	−557	621	165
1142	15917	−509	−499	−1882	3390	−156	−27	432	−915	−1711	−214	1694
1601	−509	12805	−677	958	−1473	1373	−587	−70	59	−379	235	227
781	−499	−677	14384	1269	58	20	75	1698	−1891	−868	901	703
−196	−1882	958	1269	13455	−739	−38	−706	1544	−297	−936	409	−415
621	3390	−1473	58	−739	11866	−186	540	−364	186	−530	381	544
−196	−156	1373	20	−38	−186	8568	−1358	220	469	156	−1144	−163
624	−27	−587	75	−706	540	−1358	10158	1776	−61	476	−1550	172
621	432	−70	1698	1544	−364	220	1776	11890	−1026	1209	198	23
624	−915	59	−1891	−297	186	469	−61	−1026	13347	−1927	−193	1281
−557	−1711	−379	−868	−936	−530	156	476	1209	−1927	11530	−292	43
621	−214	235	901	409	381	−1144	−1550	198	−193	−292	9703	1337
165	1694	227	703	−415	544	−163	172	23	1281	43	1337	8773

Y_2

U

−2	1	1	0	0	1	1	0	0	1	1	0
0	1	−2	72	70	−5	5	−70	−72	0	0	0
−67	−50	−51	82	87	−35	−35	88	82	−57	−58	−56
−56	−129	−129	35	−44	−250	−250	−44	35	142	278	143
0	−110	110	−110	141	384	−384	−141	111	52	0	−52
0	216	−216	−18	25	125	−125	−25	18	−567	0	567
−144	−423	−423	−424	255	253	252	255	−424	−48	263	−48
362	−556	−556	538	−221	531	531	−221	538	546	−248	546
0	928	−928	442	−429	492	−492	429	−442	429	0	−429
−62	82	82	−588	568	−470	−470	568	−588	1034	−1489	1034
0	−1020	1021	1031	−1020	432	−432	1020	−1031	−371	0	371
3377	1203	1203	673	2686	338	338	2686	673	717	2294	717

V

4918	41	−26	−35	−29	−89	−233	38	−35	−130	576	−9
41	6219	172	148	1281	−511	448	−1143	908	−675	173	−80
−26	172	4074	−79	−16	399	−532	329	1079	54	115	−922
−35	148	−79	5725	−112	974	−28	−20	201	−548	953	−344
−29	1281	−16	−112	4852	−1238	−471	−31	−79	−27	−22	−1183
−89	−511	399	974	−1238	5976	−273	−691	12	−533	26	−469
−233	448	−532	−28	−471	−273	6481	−948	934	−803	289	−295
38	−1143	329	−20	−31	−691	−948	4677	−53	−1083	36	−14
−35	908	1079	201	−79	12	934	−53	5504	−279	953	−613
−130	−675	54	−548	−27	−533	−803	−1083	−279	4546	77	−53
576	173	115	953	−22	26	289	36	953	77	9521	−142
−9	−80	−922	−344	−1183	−469	−295	−14	−613	−53	−142	4375

Y'_{21}

U

−44	−1	0	−20	−1	0	0	0
233	22	465	229	20	−410	−389	86
385	793	−241	−149	305	244	−460	−545
439	137	−877	−360	468	−700	−18	959
1565	−1333	−1075	528	−1287	508	−788	−327
655	−1159	930	−530	1923	1995	−641	1022
−1375	1105	−499	1255	−949	1180	−1505	1780
2466	1620	307	4002	775	986	3444	835

V

3758	−873	−519	−1	636	9	−429	−479
−873	7118	−18	2774	−91	−423	−1334	−135
−519	−18	1822	17	282	−78	70	−402
−1	2774	17	8485	−2504	−708	22	687
636	−91	282	−2504	6178	−107	586	225
9	−423	−78	−708	−107	3903	−635	−904
−429	−1334	70	22	586	−635	5449	602
−479	−135	−402	687	225	−904	602	5052

Y_{22}

U

5	−3	−3	−1	−1	−3	−3	−1	−1
−1	−12	9	8011	8038	−189	190	−8035	−8009
−7048	−7604	−7605	6746	7345	−5664	−5664	7345	6752
−1	8338	−8339	10389	−11032	−29236	29235	11032	−10389
5767	23052	23052	28635	−19951	−26290	−26290	−19952	28635
0	82739	−82739	16909	−15913	35612	−35612	15913	−16909
32757	−40385	−40385	61612	−35034	61878	61878	−35034	61613
0	55744	−55744	−88585	87206	−48485	48485	−87206	88585
239799	88332	88332	60363	177671	34664	34664	177671	60363

V

245290	67346	43642	33983	37	8790	67640	11795	6716
67346	262681	−47235	−60196	1386	−15331	−35125	−4652	21255
43642	−47235	230858	−23610	−7313	−78942	460	18674	−1875
33983	−60196	−23610	293133	−56794	−5587	28035	3646	3783
37	1386	−7313	−56794	116143	891	−8115	−5728	−26068
8790	−15331	−78942	−5587	891	159853	−29478	−14768	−1859
67640	−35125	460	28035	−8115	−29478	217945	−6899	−36239
11795	−4652	18674	3646	−5728	−14768	−6899	124459	2908
6716	21255	−1875	3783	−26068	−1859	−36239	2908	120960

Y_{23}

U

−1	−1	0	0	−1	0	0	0	−1	0	0
3	0	−10	−5	−6	−1	11	−3	−6	23	24
−1	7	26	29	7	−50	−61	−4	4	18	20
−56	−71	75	74	−61	−45	63	55	−78	−15	−13
−20	−29	71	77	−26	125	−34	−186	−29	4	12
3	−6	219	−154	−525	−3	−17	16	490	160	−165
71	395	245	−200	−264	256	−195	248	−296	−219	283
−382	1053	−281	−46	−387	−664	309	−650	−488	157	−530
−61	135	−946	906	−666	−101	25	−28	493	−776	714
25	−59	764	−724	359	−748	769	−732	442	−855	883
2625	1525	824	1782	403	373	2243	358	385	1800	822

V

3268	−126	−24	−112	−237	198	−265	583	192	−200	631
−126	5487	−1306	−32	22	−488	501	−233	−286	8	753
−24	−1306	3523	−177	4	802	29	9	−228	−99	75
−112	−32	−177	2815	−3	−4	−57	928	47	597	−258
−237	22	4	−3	4247	119	−1336	82	−822	−708	8
198	−488	802	−4	119	3240	189	165	337	−120	68
−265	501	29	−57	−1336	189	4825	96	−29	796	459
583	−233	9	928	82	165	96	3654	262	374	−122
192	−286	−228	47	−822	337	−29	262	4115	−118	199
−200	8	−99	597	−708	−120	796	374	−118	4024	−139
631	753	75	−258	8	68	459	−122	199	−139	4828

Y_{24}

U

−23	−1	−1	0	0	−1	−1	−1	0	0	−1
20	7	6	−12	−25	0	−15	−19	0	5	10
−4	−1	2	44	43	−2	−42	−42	−2	−2	0
72	31	31	−119	136	21	3	6	13	7	20
−41	−70	16	−41	−1	−164	−61	26	192	188	−6
−19	−131	−304	−36	25	107	83	−101	−95	203	256
−13	−170	−62	−28	16	369	166	−186	294	26	−360
82	−767	605	68	15	358	−253	330	−375	269	44
33	510	−537	20	15	373	−620	652	−252	506	−562
−222	245	337	−215	−3	−333	402	−620	−773	659	−567
2058	955	634	2045	7	242	1551	501	533	1338	451

V

3039	256	−4	11	33	616	259	212	195	517	213
256	2748	−376	−1	−168	−556	192	−60	147	−215	−197
−4	−376	2699	−164	9	−114	−33	−861	7	513	−52
11	−1	−164	3056	−981	−238	125	−117	651	−6	−116
33	−168	9	−981	2948	−11	−251	5	−464	102	−130
616	−556	−114	−238	−11	3982	71	516	145	253	−36
259	192	−33	125	−251	71	4861	−1304	80	1182	218
212	−60	−861	−117	5	516	−1304	3960	18	−14	8
195	147	7	651	−464	145	80	18	2459	90	−12
517	−215	513	−6	102	253	1182	−14	90	5059	1081
213	−197	−52	−116	−130	−36	218	8	−12	1081	2689

Y_1

U

−3	1	1	1	1	1	1	1	1	1
0	0	312	−312	−663	−1314	−1080	1080	1313	665
639	262	1834	1834	−1199	−1101	1007	1007	−1100	−1201
−573	−2813	−954	−953	−1543	1304	1872	1872	1305	−1544
0	0	1938	−1939	3269	304	−1822	1822	−303	−3269
0	0	6254	−6254	−1331	−1182	4069	−4069	1182	1331
−2501	2818	1055	1055	−5112	3150	−4138	−4138	3151	−5112
−123	−9594	4954	4954	2208	1240	−3751	−3751	1240	2208
0	0	3904	−3904	−6195	8605	−5528	5528	−8605	6195
22183	7784	5318	5318	4055	15768	4304	4304	15768	4055

V

15782	3040	944	1517	−961	810	−1074	−1074	3245	−256
3040	28790	−3737	−924	−1274	561	−1625	−1625	294	−3563
944	−3737	23948	−126	−1931	2745	203	−3361	−4007	3326
1517	−924	−126	21978	6048	−1446	−7055	153	828	−318
−961	−1274	−1931	6048	37807	−12613	−459	−3449	188	3617
810	561	2745	−1446	−12613	36247	852	−2975	6408	−2004
−1074	−1625	203	−7055	−459	852	18359	322	−974	2597
−1074	−1625	−3361	153	−3449	−2975	322	18359	1369	1267
3245	294	−4007	828	188	6408	−974	1369	32772	−4348
−256	−3563	3326	−318	3617	−2004	2597	1267	−4348	27264

Y_{11}

U

−2	1	1	1	1	1	1
1218	1284	1319	2378	−1339	−3175	−1273
11	2338	−2256	−81	−4392	40	4398
−981	5772	5824	−8802	−2137	−362	−2372
169	−10840	10788	−537	−4810	−823	6490
2335	1031	−1134	−5111	10842	−11453	9705
26748	9331	9439	5079	7256	15899	7008

V

32150	−109	12752	4226	506	−460	13236
−109	21193	568	−2619	−6911	−795	−2712
12752	568	20891	130	−600	863	−5224
4226	−2619	130	19072	−4217	463	−1160
506	−6911	−600	−4217	20400	2412	−2691
−460	−795	863	463	2412	32551	−3917
13236	−2712	−5224	−1160	−2691	−3917	37492

TABLE 1. UV-certificates for matrices in the proofs of Theorem 38 and Lemmas 47, 48, and 49