

Chapter 18

A PDE-Constrained Generalized Nash Equilibrium Approach for Modeling Gas Transport with Strategic Suppliers

Veronika Grimm^[0000-0002-9025-9731],
Michael Hintermüller^[0000-0001-9471-2479],
Olivier Huber^[0000-0002-4447-8650],
Lars Schewe^[0000-0002-3778-262X],
Martin Schmidt^[0000-0001-6208-5677], and
Gregor Zöttl^[0000-0002-2729-1173]

Abstract We investigate a class of generalized Nash equilibrium problems (GNEPs) in which the objectives of the individuals are interdependent and the shared constraint consists of a system of partial differential equations. This setup is motivated by the modeling of strategic interactions of competing firms, which explicitly take into account the dynamics of transporting a commodity such as natural gas or hydrogen. We establish the existence of a variational equilibrium of the GNEP. In the case of symmetric firms, we identify an equivalent optimization problem. We use this model to numerically explore the impact of linepacking, that is the use of the network as a temporary storage device. In particular, we study the firms' decisions under various linepacking abilities and analyze which market participants benefit from it.

Veronika Grimm
Energy Systems & Market Design, University of Technology Nuremberg (UTN), Germany,
e-mail: veronika.grimm@utn.de

Michael Hintermüller
Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany,
e-mail: michael.hintermueller@wias-berlin.de

Olivier Huber
Wisconsin Institute for Discovery, University of Wisconsin-Madison, USA, e-mail: ohuber2@wisc.edu

Lars Schewe
School of Mathematics and Maxwell Institute for Mathematical Sciences, The University of Edinburgh,
UK, e-mail: lars.schewe@ed.ac.uk

Martin Schmidt
Department of Mathematics, Trier University, Germany, e-mail: martin.schmidt@uni-trier.de

Gregor Zöttl
Industrial Organization and Energy Markets, FAU Erlangen-Nürnberg, Germany, e-mail: gregor.zoettl@fau.de

18.1 Introduction

The framework of equilibrium problems considerably contributes to the understanding of a wide array of real-world problems, ranging from management and microeconomics applications to a variety of design problems in engineering. Specifically, non-cooperative game theory has turned out to be a useful tool in analyzing the actions of agents (also denoted by players) on markets. In such a context, one seeks solutions in which each player's individual objective is maximized subject to individual (also denoted private) constraints, and no player has an incentive to take decisions deviating from a *Nash equilibrium* [30]. For classic Nash equilibrium problems (NEPs), the coupling between individuals occurs in the objective functionals only, whereas the feasible sets of player's decisions are independent. If the latter are also interdependent, one faces a so-called generalized Nash equilibrium problem (GNEP).

In this work, we investigate a particular GNEP in which the dynamics of transporting a given good are described by a system of partial differential equations (PDEs), which constitutes a shared constraint that is taken into account by all players. This is highly relevant, e.g., in the case of liberalized energy markets, where conversion and trading of energy carriers is organized by competing private companies that seek to maximize their profits in a non-cooperative fashion. Within a short time horizon, as in an intraday market, it is important to consider a high-fidelity model of the dynamics when transporting goods with slow transients such as natural gas or hydrogen. Moreover, in our setup, the objective functionals are interdependent, so that the actions of a player's rivals directly affect the value of that player's objective. This is thus closely related to classic oligopolistic situations of strategically interacting firms in a market context; see, e.g., [10] for a seminal article in this area. The focus here is on the solution concept called *variational equilibrium* (VE) for which the multipliers associated with the shared constraints have a clear economic interpretation.

In view of the aforementioned application regarding liberalized gas markets, in this work we first provide conditions for the existence of a solution to the governing system of PDEs with enough regularity to ensure the existence of multipliers to the shared constraints. Second, we show the existence of a VE that solves the underlying jointly convex GNEP. For symmetric setups, we identify a reformulation of the original equilibrium problem as a single optimization problem, which allows us to derive a solution algorithm for finding such a VE. Finally, we apply our algorithm to a suitable test instance, which is motivated by the trading and transporting of natural gas. Our numerical results highlight crucial dynamical aspects in the context of natural gas such as the impact and value of *linepacking*. The latter refers to the action of changing the amount of gas stored in the pipe, the so-called linepack. When such a linepacking occurs, the pipe acts as a temporary storage device. This allows to take advantage of price variations or to overcome fluctuations in supply and demand.

Our work contributes to several strands of the literature. Indeed, equilibrium problems that are subject to network constraints have received considerable attention. In the context of energy markets, these settings have been studied mainly for the analysis of electricity and gas markets. There, we find many contributions that consider strategic interactions of firms in the presence of shared and static, i.e., stationary, network

constraints; see, e.g., [7, 33, 37, 39] as well as, more recently, [22]. Due to its physical properties, a steady-state approximation of electricity transmission is well accepted for many market applications. Owing to slower speeds of propagation, a stationary approach cannot capture, however, all aspects of the dynamic evolution of gas transported in a single pipe or in an entire network.

Another strand of literature explicitly considers strategic interaction of firms in a dynamic context—but in the absence of shared network constraints. In several contributions, firms experience adjustment costs when changing their output; see [6, 24, 40]. Other contributions have analyzed situations in which strategic firms exploit a private resource stock [28] or accumulate private productive capacities over time [27]. Similar to our approach, in these contributions, the interactive nature of the situation analyzed also arises due to firms' profits being influenced by all rivals' actions. However, there are no shared constraints that restrict the choices of the firms.

Several works on dynamic oligopoly models do consider shared constraints; [3, 4] study differential games in which firms jointly exploit a common renewable production asset. More recent articles further elaborate on this; see, e.g., [8, 9, 42], who consider different types of strategic behavior among firms. Note that for all these problems, the shared constraint can be represented by a single state variable governed by an ordinary differential equation (ODE). For a proper modeling of the physics underlying the transportation aspect of our setup, we have to consider a spatially distributed state variable governed by a system of PDEs.

Several recent works analyze GNEPs involving shared constraints governed by PDEs [13, 16, 19, 20, 25]. While the structural assumptions vary among these contributions, the results and methods are mostly applied to instances in which objective functionals are of tracking type and the shared constraint is a scalar PDE. In our setup, we consider objectives that model market interaction of strategic firms. In addition, the shared constraint comprises a system of PDEs, which allows to appropriately capture the physical phenomena under consideration.

The remainder of this paper is structured as follows. After this introduction, we present the stylized GNEP model coupling trading and transport phenomena in Sect. 18.2. Section 18.3 provides general well-posedness results for the system of PDEs under investigation. In Sect. 18.4, we establish the existence of a VE for the jointly convex GNEP under consideration. A reformulation of the GNEP is given that allows for a computational approach to determine a VE. We illustrate the important role of linepacking in Sect. 18.5 and finally conclude in Sect. 18.6.

Notations For a given Banach space X , its dual is given by X^* . If $A: X \rightarrow Y$ is a linear and continuous mapping between Banach spaces, its adjoint is denoted by $A^*: Y^* \rightarrow X^*$. The continuous and dense embedding of a Banach space X into Y is represented by $X \hookrightarrow Y$. With Ω being an open interval of \mathbb{R} , for any integer p (and even for $p = +\infty$), the function space $L^p(\Omega)$ is the standard Lebesgue space. For any real s , $H^s(\Omega)$ is the $L^2(\Omega)$ -based Sobolev space of order s . Let $f: \mathbb{R} \rightarrow X$ and $T > 0$. If $t \mapsto \|f(t)\|_X$ lies in $L^p(0, T)$ and is Bochner measurable, then $f \in L^p(0, T; X)$. The space $C([0, T]; X)$ consists of all X -valued continuous functions on $[0, T]$ equipped with the norm of uniform convergence. For more background on these spaces, see, e.g., [1]. As we have only one spatial dimension, denoted by x , the Laplacian operator Δ has

the same action as taking second partial derivatives $\partial^2/\partial x^2$. When applied to a vector-valued function, its action is to take second partial derivatives of each component. As a shorthand notation, for any function f depending on x and t , f_t , f_x , and f_{xx} denote the first-order derivative with respect to t , x , and the second-order derivative with respect to x , respectively. For a given set S , its closure is denoted by \bar{S} , its interior by $\text{int } S$, and its indicator function by δ_S . Given a convex set S , its normal cone is denoted by $N_S(\cdot): X \rightrightarrows X^*$, where “ \rightrightarrows ” indicates that it is a set-valued map. The subdifferential of a convex function $g: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is represented by $\partial g: X \rightrightarrows X^*$.

18.2 A GNEP Model for Gas Transport with Strategic Suppliers

We investigate a stylized market situation where a commodity, e.g., natural gas or hydrogen, which is supplied by several competing firms has to be transported through a single pipe e . The pipe’s diameter $D^e > 0$ is typically much smaller than its length $L^e > 0$. Hence, a one-dimensional model for the flow of gas is appropriate. Let $A^e := (D^e)^2\pi/4$ be the constant cross-sectional area of the pipe. We consider the case of an isothermal flow, i.e., the temperature $T_{\text{emp}} > 0$ of gas is constant. We denote the density by ρ^e , the pressure by p^e , and the speed of sound by $c_s = \sqrt{R_s T_{\text{emp}}}$ for a given specific gas constant R_s . The state equation is given by the ideal gas law $p^e = c_s^2 \rho^e$, and the velocity of gas is assumed to be small with respect to c_s . Given the operational conditions in a pipeline system, the evolution of the gas can be modeled by the system

$$\begin{aligned} p_t^e - \varepsilon p_{xx}^e + \frac{c_s^2}{A^e} q_x^e &= 0, \\ q_t^e - \varepsilon q_{xx}^e + A^e p_x^e &= -\sigma^e A^e \frac{\check{q}^e}{\check{p}^e} \left(q^e - \frac{\check{q}^e}{2\check{p}^e} p^e \right) - \frac{A^e g \sin \varphi^e}{c_s^2} p^e, \end{aligned} \quad (18.1)$$

where $\varepsilon > 0$, $\sigma^e := 16c_s^2\lambda^e/(\pi^2(D^e)^5)$, $\lambda^e \geq 0$ is the friction coefficient, g is the gravitational acceleration, and φ^e is the constant angle of the pipe with the horizontal plane. The pair (\check{p}, \check{q}) , with $\check{p} > 0$ and $\check{q} > 0$, is a reference state. System (18.1) comes from the linearization and viscosity regularization of the hyperbolic system known as “ISO2” in Chap. 2, “A Catalog of Gas Network Models: PDEs, Coupling Conditions, and Numerical Schemes”, which presents different models for gas flow in pipelines depending on the operational conditions. The boundary conditions for (18.1) are of Dirichlet type. Among other gases, flows of natural gas and hydrogen can be modeled by (18.1).

Alternatively to the single-pipe scenario, a small pipeline network whose graph is a directed tree, could be considered. In such a setting, at a node at which multiple pipes meet, two relations have to hold: (i) The pressures at the common endpoints of these pipes are identical and (ii) the inflows and outflows of gas are balanced. In our subsequent exposition we will, however, stick to the single-pipe setting for convenience as this allows us to emphasize the GNEP aspect. We consider supply decisions of strategic firms which ship and sell the commodity through a pipe. The two nodes are

labeled in and out, respectively. For the ease of notation, subsequently we drop all superscripts related to the pipe's identifier. Further, we assume that M strategic firms (producers) may inject gas at the in node and sell it at the out node. Concerning the equilibrium setting, we consider the case of Cournot competition, i.e., suppliers choose quantities supplied and sold, then market prices at the output node result in equilibrium. More specifically: With the goal of maximizing their respective profit over a finite time period $[0, T]$, $T > 0$, each firm strategically decides on the injected mass flow q_i^{in} at pressure p_i^{in} and the withdrawn mass flow q_i^{out} at pressure p_i^{out} . Our setup thus considers an open loop situation in which strategic players simultaneously choose their optimal strategies which then determines all transported and consumed quantities for all $t \in [0, T]$.

Note that we use the terms “firm”, “agent”, or “player” as synonyms. We denote by $u_i := (p_i^{\text{in}}, q_i^{\text{in}}, p_i^{\text{out}}, q_i^{\text{out}})$ the decisions of firm i , and let $u := (u_1, \dots, u_M)$ be the entire decision vector. With respect to notation, we use the classic game-theoretic convention that for any i , $u = (u_i, u_{-i})$, where u_{-i} collects the decisions of all the other firms except i . Mathematically, it is supposed that each u_i belongs to $U \subseteq L^2(0, T)^4$. In the pipe, the state variable is $y := (p, q): (0, L) \times (0, T) \rightarrow \mathbb{R}^2$. The aforementioned Dirichlet boundary conditions are given by $y(0, \cdot) = u_\Gamma^0$ and $y(L, \cdot) = u_\Gamma^L$, where the nodal state values $u_\Gamma^0 := (p^{\text{in}}, q^{\text{in}})$ and $u_\Gamma^L := (p^{\text{out}}, q^{\text{out}})$ are related to the firms' actions by

$$u_\Gamma^0 = \left(\sum_{i=1}^M \frac{p_i^{\text{in}}}{M}, \sum_{i=1}^M q_i^{\text{in}} \right), \quad u_\Gamma^L = \left(\sum_{i=1}^M \frac{p_i^{\text{out}}}{M}, \sum_{i=1}^M q_i^{\text{out}} \right). \quad (18.2)$$

The relations on the mass flows fit the above stated conditions at a junction. For the pressures, this is also the case if all firms choose the same pressures, and we focus on this case. Furthermore, we consider lower and upper bound constraints on the pressures and mass flows. For given bounds $\underline{y} := (\underline{p}, \underline{q})$ and $\bar{y} := (\bar{p}, \bar{q})$, these state constraints are represented by

$$y \in K := \{y \in C([0, T]; C([0, L])^2) \mid \underline{y} \leq y(x, t) \leq \bar{y} \forall (x, t) \in [0, L] \times [0, T]\}, \quad (18.3)$$

where the space $C([0, T]; C([0, L])^2)$ is chosen to ensure the existence of the associated Lagrange multipliers for a primal-dual stationarity characterization of optimal solutions or equilibria. Since firms are fully aware of both the system (18.1) and the bound constraints (18.3), they all take into account the same, and thus shared, network constraints. Let S be the so-called control-to-state operator, which gives the solution (state) y of the PDE system for a given (control) u . Then, the inclusion $y = S(u) \in K$ represents the shared constraint of the underlying GNEP.

Now, at any time $t \in [0, T]$, there is a given demand of gas at the consumption node out. The price of gas is given by an inverse demand function $P(t, q^{\text{out}}(t))$, where $q^{\text{out}}(t)$ is the total amount of gas at the exit node. On the other hand, each firm chooses to inject the amount $q_i^{\text{in}}(t)$ at a cost of $c_i(t) q_i^{\text{in}}(t)$. For our subsequent investigation, we make the following assumptions on the data.

Assumption. (i) We have $P(t, q^{\text{out}}(t)) := \beta(t) - \alpha(t) q^{\text{out}}(t)$, and both α and β are measurable functions in t .

- (ii) *There exist finite numbers α_{\min} and α_{\max} such that $0 < \alpha_{\min} \leq \alpha(t) \leq \alpha_{\max}$ holds for almost all $t \in (0, T)$.*
- (iii) *For any firm $i \in \{1, \dots, M\}$, there exists a constant $\bar{c}_i > 0$ such that $|c_i(t)| \leq \bar{c}_i$ for almost all $t \in (0, T)$ and c_i is measurable.*

For any $i \in \{1, \dots, M\}$, the maximization problem for firm $i \in \{1, \dots, M\}$ reads

$$\max_{u_i \in \mathbf{U}} \int_0^T \left(P \left(t, \sum_{k=1}^M q_k^{\text{out}}(t) \right) q_i^{\text{out}}(t) - c_i(t) q_i^{\text{in}}(t) \right) dt - \mathcal{R}_i(u_i) \quad (18.4a)$$

$$\text{s.t. } 0 \leq q_i^{\text{in}}(t) \leq \bar{q}_i^{\text{in}} \quad \text{for almost all } t \in (0, T), \quad (18.4b)$$

$$\int_0^T \left(q_i^{\text{out}}(t) - q_i^{\text{in}}(t) \right) dt \leq 0, \quad (18.4c)$$

$$S(u_i, u_{-i}) \in K, \quad (18.4d)$$

where \bar{q}_i^{in} is a given production-specific capacity bound. The convex function \mathcal{R}_i in the objective function (18.4a) is there to influence, specifically regularize, the state at the end of the time period (i.e., at $t = T$) in the pipe. Indeed, the initial state $y_0 := (p_0, q_0)$ is likely to be rather regular (in space), but without any additional consideration the final state is only required to satisfy the constraint (18.4d). Constraint (18.4c) ensures that over the entire time period each firm has injected at least as much gas as it has withdrawn. This is important since at the beginning of the time period, the pipe does contain some gas as we require the initial state y_0 to fulfill the given upper and lower bounds. Hence, the constraint (18.4c) ensures that there is no free gas for any of the firms. In problem (18.4), the constraints (18.4b) and (18.4c) are private, whereas (18.4d) is shared among producers. The collection of these M -many optimization problems gives rise to a GNEP, which is studied in more detail in Sect. 18.4.

As a benchmark for the GNEP solution, we consider the corresponding *welfare maximization problem*. In fact, the overall welfare is given by the total consumer surplus minus the total production cost:

$$\int_0^T \left(\int_0^{q^{\text{out}}(t) = \sum_{i=1}^M q_i^{\text{out}}(t)} P(t, \tau) d\tau - \sum_{i=1}^M c_i(t) q_i^{\text{in}}(t) \right) dt.$$

After replacing the innermost integral by its expression and noting that we only need to consider the total amount of gas withdrawn, the associated welfare maximization problem reads

$$\begin{aligned} \max_{\mathbf{u} \in \mathbf{U}} \quad & \int_0^T \left(\beta(t) q^{\text{out}}(t) - \frac{1}{2} \alpha(t) q^{\text{out}}(t) q^{\text{out}}(t) - \sum_{i=1}^M c_i(t) q_i^{\text{in}}(t) \right) dt - \mathcal{R}(\mathbf{u}) \\ \text{s.t.} \quad & 0 \leq q_i^{\text{in}}(t) \leq \bar{q}_i^{\text{in}} \quad \text{for almost all } t \in (0, T) \text{ and all } i \in \{1, \dots, M\}, \\ & \int_0^T \left(q^{\text{out}}(t) - \sum_{i=1}^M q_i^{\text{in}}(t) \right) dt \leq 0, \\ & S(\mathbf{u}) \in K, \end{aligned} \quad (18.5)$$

where $\mathbf{u} := (p^{\text{in}}, p^{\text{out}}, (q_1^{\text{in}}, \dots, q_M^{\text{in}}), q^{\text{out}})$ and \mathcal{R} sums up all \mathcal{R}_i , $i = 1, \dots, M$. Note that from any solution u^* of the GNEP with identical pressure variables, i.e., (u_i^*, u_{-i}^*) is a solution to (18.4) for all $i \in \{1, \dots, M\}$ and $(p_i^{\text{in}}, p_i^{\text{out}}) = (p_j^{\text{in}}, p_j^{\text{out}})$ for all $i, j \in \{1, \dots, M\}$, a feasible point for the welfare maximization problem can be constructed. Therefore, from a solution of the GNEP, a lower bound on the optimal function value in (18.5) can be obtained.

18.3 Well-Posedness of the Gas Flow PDE System

Before analyzing our GNEP, we first address the well-posedness of the PDE system (18.1) over a pipe with a solution of sufficient regularity to ensure the existence of multipliers for the shared constraint.

The spatial domain of the pipe is $\Omega := (0, L)$, with $L > 0$, and we set $\Omega_T := \Omega \times (0, T)$ for a time horizon $T > 0$. The system of PDEs modeling the flow of gas is given by

$$p_t - \varepsilon p_{xx} + \frac{c_s^2}{A} q_x = 0 \quad \text{in } \Omega_T, \quad (18.6a)$$

$$q_t - \varepsilon q_{xx} + A p_x = -\sigma A \frac{\check{q}}{\check{p}} \left(q - \frac{\check{q}}{2\check{p}} p \right) - \frac{Ag \sin \varphi}{c_s^2} p \quad \text{in } \Omega_T, \quad (18.6b)$$

$$p(0, \cdot) = p^{\text{in}}, \quad p(L, \cdot) = p^{\text{out}}, \quad q(0, \cdot) = q^{\text{in}}, \quad q(L, \cdot) = q^{\text{out}} \quad \text{in } (0, T), \quad (18.6c)$$

$$p(\cdot, 0) = p_0, \quad q(\cdot, 0) = q_0 \quad \text{in } \Omega, \quad (18.6d)$$

where (p_0, q_0) is the initial state and $(\check{p}, \check{q}) \in C([0, T]; C(\bar{\Omega})^2)$, with \check{p} bounded away from zero, is a given reference state. For our analysis, we transform (18.6) into a system of PDEs with homogeneous Dirichlet boundary conditions. For this purpose, we require the boundary data $(p^{\text{in}}, p^{\text{out}}, q^{\text{in}}, q^{\text{out}})$ to be in the Sobolev space $H^1(0, T)^4$. Then, for the boundary data $u_\Gamma^p := (p^{\text{in}}, p^{\text{out}})$ and $u_\Gamma^q := (q^{\text{in}}, q^{\text{out}})$, we construct extensions u_Ω^p (resp. u_Ω^q) of u_Γ^p (resp. u_Γ^q) to the entire domain Ω . Specifically, for any $(x, t) \in \Omega_T$, we set $u_\Omega^p(x, t)$ and $u_\Omega^q(x, t)$ to be the linear interpolation of the values of u_Γ^p and u_Γ^q at both ends of the pipe. More precisely, we define the operator $\mathcal{E}: H^1(0, T)^2 \rightarrow H^1(0, T; H^s(\Omega))$, with s being an arbitrary positive integer, as

$$\mathcal{E}(v_1, v_2)(t)(x) := \frac{L-x}{L} v_1(t) + \frac{x}{L} v_2(t). \quad (18.7)$$

This yields $u_\Omega^p := \mathcal{E}(u_\Gamma^p)$ as well as $u_\Omega^q := \mathcal{E}(u_\Gamma^q)$. From the expression in (18.7), $(u_\Omega^p)_x$ and $(u_\Omega^q)_x$ are constant with respect to x . Hence, these extensions indeed belong to $H^1(0, T; H^s(\Omega))$. We then consider the shifted state variables $\tilde{p} := p - u_\Omega^p$ and $\tilde{q} := q - u_\Omega^q$, which are solutions to the system

$$\tilde{p}_t - \varepsilon \tilde{p}_{xx} + \frac{c_s^2}{A} \tilde{q}_x = -(u_\Omega^p)_t - \frac{c_s^2}{A} (u_\Omega^q)_x \quad \text{in } \Omega_T, \quad (18.8a)$$

$$\begin{aligned} \tilde{q}_t - \varepsilon \tilde{q}_{xx} + A \tilde{p}_x + \sigma A \frac{\check{q}}{\check{p}} \tilde{q} + \left[\frac{Ag \sin \varphi}{c_s^2} - \sigma A \frac{\check{q}^2}{2\check{p}^2} \right] \tilde{p} \\ = -(u_\Omega^q)_t - A(u_\Omega^p)_x + \sigma A \frac{\check{q}}{\check{p}} u_\Omega^q + \left[\frac{Ag \sin \varphi}{c_s^2} - \sigma A \frac{\check{q}^2}{2\check{p}^2} \right] u_\Omega^p \quad \text{in } \Omega_T, \quad (18.8b) \end{aligned}$$

$$\tilde{p}(0, \cdot) = \tilde{p}(L, \cdot) = 0, \quad \tilde{q}(0, \cdot) = \tilde{q}(L, \cdot) = 0 \quad \text{in } (0, T), \quad (18.8c)$$

$$\tilde{p}(\cdot, 0) = \tilde{p}_0 := p_0 - u_\Omega^p(\cdot, 0), \quad \tilde{q}(\cdot, 0) = \tilde{q}_0 := q_0 - u_\Omega^q(\cdot, 0) \quad \text{in } \Omega. \quad (18.8d)$$

Remember that the feasible set of each firm includes the constraint $y \in K$ with K given in (18.3), and that for analytical but also numerical reasons one wishes to characterize a solution of (18.4) by Lagrange multipliers. This leads us to seek solutions of (18.6) with higher regularity than the classic parabolic one; cf. (18.10) below for the latter. In this vein, our subsequent approach provides us, under appropriate assumptions, with a unique solution of (18.6) in the space

$$Y := \{y \in L^2(0, T; H^2(\Omega)^2) \cap L^\infty(0, T; H^1(\Omega)^2) \text{ and } y_t \in L^2(0, T; L^2(\Omega)^2)\}.$$

Then, by the Aubin–Lions Lemma (see Lemma 18.5 in Appendix A), we have the (even compact) embedding of Y into $C([0, T]; C(\Omega)^2)$. The latter space is instrumental in proving the existence of an interior point for the constraint $y \in K$. As a consequence, Slater’s constraint qualification (CQ) is satisfied, which yields the existence of a (bounded) Lagrange multiplier for this constraint.

For the proof of higher regularity we proceed as follows. First, the existence and uniqueness of a solution with classic parabolic regularity is obtained via classic results on abstract parabolic problems. Then, for the additional regularity, we do not wish to follow the usual path of showing temporal regularity and then spatial regularity as terms involving the time derivative of the control are already present in (18.8a) and (18.8b). Rather, we only show additional spatial regularity, first for (\tilde{p}, \tilde{q}) and then for $(\tilde{p}_t, \tilde{q}_t)$. Indeed, under appropriate regularity of the shifted initial state and the right-hand side, the solution (\tilde{p}, \tilde{q}) satisfies

$$\begin{aligned} (\tilde{p}, \tilde{q}) &\in L^2(0, T; H^2(\Omega)^2) \cap L^\infty(0, T; H_0^1(\Omega)^2), \\ (\tilde{p}_t, \tilde{q}_t) &\in L^2(0, T; L^2(\Omega)^2), \end{aligned}$$

which leads to $(p, q) \in Y$.

Let $\mathcal{V} := H_0^1(\Omega)^2$ and $\mathcal{H} := L^2(\Omega)^2$. We denote any element v of \mathcal{V} , resp. \mathcal{H} , by (v_p, v_q) , where both v_p and v_q belong to $H_0^1(\Omega)$, resp. $L^2(\Omega)$. This specific notation is inspired by the variable names for gas pressure and flow. It is well-known that $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and that the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact. Therefore, we have the Gelfand triple

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*.$$

Let $(\cdot, \cdot)_{\mathcal{H}}$ denote the inner product of \mathcal{H} and let $\|\cdot\|_{\mathcal{H}}$ be the corresponding norm. For any v, w in \mathcal{H} , we have $(v, w)_{\mathcal{H}} = (v_p, w_p)_{L^2(\Omega)} + (v_q, w_q)_{L^2(\Omega)}$ and $\|v\|_{\mathcal{H}}^2 = \|v_p\|_{L^2(\Omega)}^2 + \|v_q\|_{L^2(\Omega)}^2$. The norm of \mathcal{V} is denoted by $\|\cdot\|_{\mathcal{V}}$. In the following, we consider a system of parabolic PDEs with the lower-order terms having generic coefficients. This allows us to streamline the presentation and also invoke these results for an associated adjoint system in future works. Let the bilinear form $a_{\tau}: [0, T] \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ be defined as

$$a_{\tau}(t, v, w) := -\varepsilon (\nabla v, \nabla w)_{\mathcal{H}} + \theta_p (\nabla v_q, w_p)_{L^2(\Omega)} + \theta_q (\nabla v_p, w_q)_{L^2(\Omega)} + \sum_{i,j \in \{p,q\}} (\gamma_{ij}(t) v_i, w_j)_{L^2(\Omega)} + \tau (v, w)_{\mathcal{H}}, \quad (18.9)$$

where $\varepsilon > 0$, both θ_p and θ_q are non-zero constants, and each γ_{ij} belongs to $C([0, T]; C(\bar{\Omega}))$ with magnitude bounded by $\tilde{\gamma}_{ij}$. The parameter τ is fixed later. By applying the Cauchy–Schwarz inequality to each term in (18.9), we see that the bilinear form a_{τ} is bounded, i.e., there exists $C_{a_{\tau}} > 0$ such that $|a_{\tau}(t, v, w)| \leq C_{a_{\tau}} \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}}$ for all $v, w \in \mathcal{V}$ and $t \in [0, T]$. Let $\mathcal{L}_{\tau}(t)$ be an operator in \mathcal{H} with action

$$\mathcal{L}_{\tau}(t)v := -\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \Delta v + \begin{pmatrix} 0 & \theta_p \\ \theta_q & 0 \end{pmatrix} \nabla v + \begin{pmatrix} \gamma_{pp}(t) + \tau & \gamma_{pq}(t) \\ \gamma_{qp}(t) & \gamma_{qq}(t) + \tau \end{pmatrix} v,$$

and domain $D(\mathcal{L}_{\tau}(t)) \equiv (H^2(\Omega) \cap H_0^1(\Omega))^2$. Recall that $v \in D(\mathcal{L}_{\tau}(t))$ is equivalent to

$$v \in \mathcal{V}, \quad \mathcal{L}_{\tau}(t)v \in \mathcal{H}, \quad \text{and} \quad (\mathcal{L}_{\tau}(t)v, w)_{\mathcal{H}} = a_{\tau}(t, v, w) \quad \text{for all } w \in \mathcal{V}.$$

We are going to investigate a problem of the following form: Find $v \in L^2(0, T; \mathcal{V})$ with $v_t \in L^2(0, T; \mathcal{V}^*)$ such that given $v_0 \in \mathcal{H}$ and $\ell \in L^2(0, T; \mathcal{V}^*)$, it satisfies $v(0) = v_0$ and for all $w \in \mathcal{V}$, we have

$$\frac{d}{dt} (v(\cdot), w)_{\mathcal{H}} + a_0(\cdot, v(\cdot), w) = (\ell(\cdot), w)_{\mathcal{H}}$$

in the sense of $\mathcal{D}^*(0, T)$, the space of distribution on $(0, T)$. This is referred to as Problem (P). As it is usual in the parabolic case, rather than requiring the coercivity of the bilinear form a_0 , it is sufficient to check that a_0 satisfies a weaker condition, namely Gårding's inequality (also known as weak coercivity).

Lemma 18.1 (Gårding's inequality for a_0) *Consider the bilinear form defined in (18.9). Then, there exists two constants $\bar{\tau}, \zeta$, with $\zeta > 0$, such that for all $v \in \mathcal{V}$ and $t \in [0, T]$, we have*

$$a_0(t, v, v) + \bar{\tau} \|v\|_{\mathcal{H}}^2 \geq \zeta \|v\|_{\mathcal{V}}^2.$$

Proof. Take $t \in [0, T]$. We use classic inequalities to get the following bound on the second term in the right-hand side of (18.9):

$$(\nabla v_q, v_p)_{L^2(\Omega)} \geq -\|\nabla v_q\|_{L^2(\Omega)} \|v_p\|_{L^2(\Omega)} \geq -\kappa_2 \|\nabla v_q\|_{L^2(\Omega)}^2 - \frac{1}{4\kappa_2} \|v_p\|_{L^2(\Omega)}^2$$

for any $\kappa_2 > 0$. Similarly, for the third term of (18.9) we have

$$(\nabla v_p, v_q)_{L^2(\Omega)} \geq -\kappa_3 \|\nabla v_p\|_{L^2(\Omega)}^2 - \frac{1}{4\kappa_3} \|v_q\|_{L^2(\Omega)}^2,$$

with $\kappa_3 > 0$. Finally, each term in the sum (fourth term of (18.9)) is bounded from below by

$$\begin{aligned} (\gamma_{pp}(t)v_p, v_p)_{L^2(\Omega)} &\geq -\bar{\gamma}_{pp} \|v_p\|_{L^2(\Omega)}^2, \quad (\gamma_{qq}(t)v_q, v_q)_{L^2(\Omega)} \geq -\bar{\gamma}_{qq} \|v_q\|_{L^2(\Omega)}^2, \\ (\gamma_{pq}(t)v_p, v_q)_{L^2(\Omega)} &\geq -\kappa_{pq} \bar{\gamma}_{pq} \|v_p\|_{L^2(\Omega)}^2 - \frac{\bar{\gamma}_{pq}}{4\kappa_{pq}} \|v_q\|_{L^2(\Omega)}^2, \\ (\gamma_{qp}(t)v_q, v_p)_{L^2(\Omega)} &\geq -\kappa_{qp} \bar{\gamma}_{pq} \|v_q\|_{L^2(\Omega)}^2 - \frac{\bar{\gamma}_{pq}}{4\kappa_{qp}} \|v_p\|_{L^2(\Omega)}^2, \end{aligned}$$

where $\kappa_{pq} > 0$ and $\kappa_{qp} > 0$ involve $\bar{\gamma}_{pq}$ and $\bar{\gamma}_{pq}$. Now, we set $\kappa_2 := \frac{\varepsilon}{2|\theta_p|}$ and $\kappa_3 := \frac{\varepsilon}{2|\theta_q|}$ and let $\bar{\tau}_p := \bar{\gamma}_{pp} + \bar{\gamma}_{pq}\kappa_{pq} + \bar{\gamma}_{qp}(4\kappa_{qp})^{-1}$ as well as $\bar{\tau}_q := \bar{\gamma}_{qq} + \bar{\gamma}_{qp}\kappa_{qp} + \bar{\gamma}_{pq}(4\kappa_{pq})^{-1}$. Then, we get

$$\begin{aligned} a_0(t, v, v) &\geq \frac{\varepsilon}{2} \|\nabla v\|_{\mathcal{H}}^2 - \left(\frac{|\theta_p|}{2\varepsilon} + \bar{\tau}_p \right) \|v_p\|_{L^2(\Omega)}^2 - \left(\frac{|\theta_q|}{2\varepsilon} + \bar{\tau}_q \right) \|v_q\|_{L^2(\Omega)}^2 \\ &\geq \frac{\varepsilon}{2} \|v\|_{\mathcal{V}}^2 - \left(\frac{\theta_p^2}{2\varepsilon} + \bar{\tau}_p + \frac{\varepsilon}{2} \right) \|v_p\|_{L^2(\Omega)}^2 - \left(\frac{\theta_q^2}{2\varepsilon} + \bar{\tau}_q + \frac{\varepsilon}{2} \right) \|v_q\|_{L^2(\Omega)}^2. \end{aligned}$$

With $\bar{\tau}$ being the maximum of the coefficients in front of $\|v_p\|_{L^2(\Omega)}^2$ and $\|v_q\|_{L^2(\Omega)}^2$, we obtain

$$a_0(t, v, v) \geq \frac{\varepsilon}{2} \|v\|_{\mathcal{V}}^2 - \bar{\tau} \|v\|_{\mathcal{H}}^2,$$

which completes the proof. \square

Since the time horizon T is finite, we continue our investigation with the coercive bilinear form $a := a_{\bar{\tau}}$. Indeed, if z is the solution of

$$\begin{aligned} \frac{d}{dt} (z, w)_{\mathcal{H}} + a(t, z, w) &= (e^{\bar{\tau}t} \ell, w)_{\mathcal{H}} \quad \text{for all } w \in \mathcal{V}, \\ z(0) &= v_0 \end{aligned}$$

in the sense of $\mathcal{D}^*(0, T)$, then $v = z e^{\bar{\tau}t}$ is a solution of Problem (P). For more details, see [11, Remark 2, p. 512], [31, p. 383] or [35, Appendix A]. In the following, we set $\mathcal{L} := \mathcal{L}_{\bar{\tau}}$ and we continue to use ℓ to denote the constant terms of the system of PDEs.

Proposition 18.1 *Suppose that $v_0 \in \mathcal{H}$ and $\ell \in L^2(0, T; \mathcal{V}^*)$. Then, there exists a unique solution v to Problem (P) with*

$$v \in L^2(0, T; \mathcal{V}) \quad \text{and} \quad v_t \in L^2(0, T; \mathcal{V}^*). \quad (18.10)$$

Moreover, there exists $C > 0$ such that the following estimate holds for all $t \in (0, T]$:

$$\|v(t)\|_{\mathcal{H}}^2 + \|v\|_{L^2(0, t; \mathcal{V})}^2 \leq C(\|v_0\|_{\mathcal{H}}^2 + \|\ell\|_{L^2(0, t; \mathcal{V}^*)}^2). \quad (18.11)$$

Proof. The existence result is classic; see Theorem 1 and 2 in [11] (on Pages 512–513), Theorem 1.37 in [21], or Theorem 11.3 in [31]. The estimate is provided in [11, Page 515] and [21, Theorem 1.35]. \square

For the additional regularity, we follow the procedure outlined in [11, Remark 2, Page 532] for our system. Let us first check that over the domain Ω , the operator \mathcal{L} satisfies the Ladyzhenskaya–Sobolevski inequality.

Lemma 18.2 (Ladyzhenskaya–Sobolevski-type inequality) *There exists $c_1, c_2 > 0$ such that for any fixed $t \in [0, T]$, we have*

$$(\mathcal{L}(t)v, -\Delta v)_{\mathcal{H}} \geq c_1 \|v\|_{H^2(\Omega)^2}^2 - c_2 \|v\|_{\mathcal{H}}^2 \quad \text{for all } v \in D(\mathcal{L}(t)).$$

Proof. We proceed as in [29, Proof of Theorem 1.1]. Let $v \in D(\mathcal{L}(t)) \equiv (H^2(\Omega) \cap H_0^1(\Omega))^2$ and consider the splitting $v =: (v_p, v_q)$. First, thanks to Lemma 18.7, we get that the second-order terms in \mathcal{L} satisfy

$$\varepsilon (\Delta v_p, \Delta v_p)_{L^2(\Omega)} + \varepsilon (\Delta v_q, \Delta v_q)_{L^2(\Omega)} \geq \frac{\varepsilon}{2} \|v\|_{H^2(\Omega)^2}^2 - \varepsilon(C'' + 1) \|v\|_{\mathcal{H}}^2 \quad (18.12)$$

for some constant $C'' > 0$. Then, moving on to the first-order terms in \mathcal{L} , we have

$$\begin{aligned} (\nabla v_q, \Delta v_p)_{L^2(\Omega)} &\geq -\|\nabla v_q\|_{L^2(\Omega)} \|\Delta v_p\|_{L^2(\Omega)} \\ &\geq -\kappa_1 \|v_q\|_{H^2(\Omega)} \|\Delta v_p\|_{L^2(\Omega)} - C_{\kappa_1} \|v_q\|_{L^2(\Omega)} \|\Delta v_p\|_{L^2(\Omega)} \\ &\geq -\frac{\kappa_1}{2} \|v_q\|_{H^2(\Omega)}^2 - C_{\kappa_1} C_{\kappa'_1} \|v_q\|_{L^2(\Omega)}^2 - (\kappa'_1 + \frac{\kappa_1}{2}) \|v_p\|_{H^2(\Omega)}^2, \end{aligned} \quad (18.13)$$

where we used the relation (18.41) in Lemma 18.6 given in Appendix A and Cauchy's inequality. Similarly, we have

$$(\nabla v_p, \Delta v_q)_{L^2(\Omega)} \geq -\frac{\kappa_2}{2} \|v_p\|_{H^2(\Omega)}^2 - C_{\kappa_2} C_{\kappa'_2} \|v_p\|_{L^2(\Omega)}^2 - (\kappa'_2 + \frac{\kappa_2}{2}) \|v_q\|_{H^2(\Omega)}^2. \quad (18.14)$$

Finally, with $\tau_{pp} = \tau_{qq} = \bar{\tau}$ and $\tau_{pq} = \tau_{qp} = 0$, for i, j being p or q we have

$$((\tau_{ij} + \gamma_{ij}(t))v_i, \Delta v_j)_{L^2(\Omega)} \geq (\tau_{ij} + \bar{\gamma}_{ij}) \left(-\kappa_{i,j} \|v_j\|_{H^2(\Omega)}^2 - C_{\kappa_{i,j}} \|v_i\|_{L^2(\Omega)}^2 \right) \quad (18.15)$$

for all $t \in [0, T]$. Summarizing, we derived lower bounds for $(\mathcal{L}(t)v, \Delta v)_{\mathcal{H}}$ in terms of $\|v_p\|_{H^2(\Omega)}^2$, $\|v_q\|_{H^2(\Omega)}^2$, $\|v_p\|_{L^2(\Omega)}^2$, and $\|v_q\|_{L^2(\Omega)}^2$ for all $t \in [0, T]$. Furthermore, the magnitude of the terms $\|v_p\|_{H^2(\Omega)}^2$ and $\|v_q\|_{H^2(\Omega)}^2$ in (18.13)–(18.15) can be controlled by choosing κ_1 , κ'_1 , κ_2 , κ'_2 , and $\kappa_{i,j}$. Hence, for any values of θ_p , θ_q , $\bar{\tau}$, and $\bar{\gamma}_{ij}$, one can select these quantities such that the contribution of $\|v_p\|_{H^2(\Omega)}^2$ (resp. $\|v_q\|_{H^2(\Omega)}^2$) in (18.13)–(18.15) is larger than $-\frac{\varepsilon}{4} \|v_p\|_{H^2(\Omega)}^2$ (resp. $-\frac{\varepsilon}{4} \|v_q\|_{H^2(\Omega)}^2$). Finally, using (18.12), we obtain

$$(\mathcal{L}(t)v, -\Delta v)_{\mathcal{H}} \geq \frac{\varepsilon}{4} \|v\|_{H^2(\Omega)^2}^2 - C_{\text{total}} \|v\|_{\mathcal{H}}^2,$$

with a suitably chosen $C_{\text{total}} > 0$. \square

Now, we are ready to state the desired regularity result.

Proposition 18.2 *Suppose that $v_0 \in \mathcal{V}$ and $\ell \in L^2(0, T; \mathcal{H})$. Then, the solution v to Problem (P) satisfies*

$$v \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\Omega)^2)$$

with the estimate

$$\|v\|_{L^\infty(0, T; \mathcal{V})} + \|v\|_{L^2(0, T; H^2(\Omega)^2)} \leq C(\|v_0\|_{\mathcal{V}} + \|\ell\|_{L^2(0, T; \mathcal{H})})$$

for some $C > 0$.

Proof. We follow [11, Remark 2, Page 532]. To this end, we consider the following approximation

$$\mathcal{V}_m := \text{span} \{\mathcal{W}_j\}_{j=1}^m$$

of \mathcal{V} , where $\mathcal{W}_j = (e_j, e_j)$ is defined by the (normalized) eigenfunctions e_j of the Laplacian in the Dirichlet problem

$$-\Delta e_j = \nu_j e_j \quad \text{in } \Omega$$

and $e_j(0) = e_j(L) = 0$. These functions form an orthonormal basis of $L^2(\Omega)$ and belong to $H_0^1(\Omega) \cap C^\infty(\Omega)$; see [5, Theorem 9.31]. Note that the sequence $\{e_j/\sqrt{\nu_j + 1}\}_{j=1}^\infty$ forms an orthonormal basis of $H_0^1(\Omega)$. For the initial state of the finite-dimensional problem over \mathcal{V}_m , we choose $v^{(m)}(0) \in \mathcal{V}_m$ such that $v^{(m)}(0)$ converges to v_0 as $m \rightarrow \infty$ in \mathcal{V} . With $v^{(m)} = \sum_{j=1}^m \zeta_j \mathcal{W}_j$ being the solution to the Galerkin approximation of order m to Problem (P), for almost all $t \in (0, T)$, we have

$$\left(\frac{d}{dt} v^{(m)}(t), w \right)_{\mathcal{H}} + a(t, v^{(m)}(t), w) = (\ell(t), w)_{\mathcal{H}} \quad (18.16)$$

for all $w \in \mathcal{V}_m \subset \mathcal{H}$. For almost all t in $(0, T)$, we take $w = \nu_j \zeta_j(t) \mathcal{W}_j$ in (18.16) and sum all the equalities. This yields

$$\left(\frac{d}{dt} v^{(m)}(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}} + \left(\mathcal{L}(t) v^{(m)}(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}} = \left(\ell(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}}.$$

After integrating by part the first term and exchanging the partial derivatives, we have

$$\left(\frac{d}{dt} \nabla v^{(m)}(t), \nabla v^{(m)}(t) \right)_{\mathcal{H}} + \left(\mathcal{L}(t) v^{(m)}(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}} = \left(\ell(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}}.$$

Then, we integrate w.r.t. t and get

$$\begin{aligned} & \frac{1}{2} \|\nabla v^{(m)}(t)\|_{\mathcal{H}}^2 + \int_0^t \left(\mathcal{L}(s) v^{(m)}(s), -\Delta v^{(m)}(s) \right)_{\mathcal{H}} ds \\ &= \int_0^t \left(\ell(s), -\Delta v^{(m)}(s) \right)_{\mathcal{H}} ds + \frac{1}{2} \|\nabla v^{(m)}(0)\|_{\mathcal{H}}^2. \end{aligned}$$

Using Lemma 18.2, there exists strictly positive constants c_1 and c_2 such that

$$\begin{aligned} & \frac{1}{2} \|\nabla v^{(m)}(t)\|_{\mathcal{H}}^2 + c_1 \int_0^t \|v^{(m)}(s)\|_{H^2(\Omega)^2}^2 ds \\ & \leq \frac{1}{2} \|\nabla v^{(m)}(0)\|_{\mathcal{H}}^2 + \int_0^t \|\ell(s)\|_{\mathcal{H}} \|\Delta v^{(m)}(s)\|_{\mathcal{H}} ds + c_2 \int_0^t \|v^{(m)}(s)\|_{\mathcal{H}}^2 ds, \end{aligned} \quad (18.17)$$

where we used that $\ell \in L^2(0, T; \mathcal{H})$. Now, there exists $C_{\frac{c_1}{2}} > 0$ such that

$$\int_0^t \|\ell(s)\|_{\mathcal{H}} \|\Delta v^{(m)}(s)\|_{\mathcal{H}} ds \leq C_{\frac{c_1}{2}} \int_0^t \|\ell(s)\|_{\mathcal{H}}^2 ds + \frac{c_1}{2} \int_0^t \|v^{(m)}(s)\|_{H^2(\Omega)^2}^2 ds.$$

Using the estimate in (18.11), we know that there exists a positive constant C_{c_2} such that

$$c_2 \int_0^t \|v^{(m)}(s)\|_{\mathcal{H}}^2 ds \leq C_{c_2} \left[\int_0^t \|\ell(s)\|_{\mathcal{V}}^2 ds \right] + \frac{1}{2} \|v^{(m)}(0)\|_{\mathcal{H}}^2 - \frac{1}{2} \|v^{(m)}(t)\|_{\mathcal{H}}^2$$

holds. To get an upper bound independent of m , we use the last two relations in (18.17) and the boundedness of the sequence $v^{(m)}(0) \in \mathcal{V}_m$ as it converges to v_0 as $m \rightarrow \infty$ in \mathcal{V} . Thus, we get

$$\frac{1}{2} \|v^{(m)}(t)\|_{\mathcal{V}}^2 + \frac{c_1}{2} \int_0^t \|v^{(m)}(s)\|_{H^2(\Omega)^2}^2 ds \leq C_0 \|v_0\|_{\mathcal{V}}^2 + C_{\text{rhs}} \int_0^t \|\ell(s)\|_{\mathcal{H}}^2 ds. \quad (18.18)$$

Hence, $v^{(m)} \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\Omega)^2)$. Then, using classic arguments, at the limit $m \rightarrow \infty$, the solution v also fulfills (18.18). The estimate is then obtained based on this inequality. \square

Let us now turn our attention to the characterization of the control-to-state mapping. We start with the following relation between ℓ and the boundary values.

Lemma 18.3 *Let $u_\Gamma \in H^1(0, T)^4$ and ℓ be the right-hand side of the equations (18.8a)–(18.8b). Then, there exists $C > 0$ such that*

$$\|\ell\|_{L^2(0, T; \mathcal{H})} \leq C \|u_\Gamma\|_{H^1(0, T)^4}.$$

Proof. In this proof, C is a positive constant that may take different values depending on the context. We show that the inequality $\|\ell\|_{L^2(0, T; \mathcal{H})}^2 \leq C \|u_\Gamma\|_{H^1(0, T)^4}^2$ holds. First, note that for any $(x, t) \in \Omega_T$, we have

$$(u_\Omega^p)_t(x, t) = \frac{L-x}{L} (p^{\text{in}})_t(t) + \frac{x}{L} (p^{\text{out}})_t(t), \quad (u_\Omega^p)_x(x, t) = \frac{p^{\text{out}}(t) - p^{\text{in}}(t)}{L},$$

and similar relations hold for $(u_\Omega^q)_t$ and $(u_\Omega^q)_x$. With $\ell =: (\ell^p, \ell^q)$, for all $s \in (0, T)$, we have

$$\|\ell^p(\cdot, s)\|_{L^2(\Omega)}^2 \leq C \left(\|(u_\Omega^p)_t(\cdot, s)\|_{L^2(\Omega)}^2 + \|(u_\Omega^q)_x(\cdot, s)\|_{L^2(\Omega)}^2 \right).$$

Using the expressions above for $(u_\Omega^p)_t$ and $(u_\Omega^p)_x$, we obtain

$$\|\ell^p(\cdot, s)\|_{L^2(\Omega)}^2 \leq C \left((p^{\text{in}}(s))^2 + ((p^{\text{out}})_t(s))^2 + (q^{\text{in}}(s))^2 + (q^{\text{out}}(s))^2 \right).$$

Now for ℓ^q , using similar arguments, we have

$$\begin{aligned} & \|\ell^q(\cdot, s)\|_{L^2(\Omega)}^2 \\ & \leq C \left(\|(u_\Omega^q)_t(\cdot, s)\|_{L^2(\Omega)}^2 + \|(u_\Omega^p)_x(\cdot, s)\|_{L^2(\Omega)}^2 + \|u_\Omega^q(\cdot, s)\|_{L^2(\Omega)}^2 + \|u_\Omega^p(\cdot, s)\|_{L^2(\Omega)}^2 \right) \\ & \leq C \left((q^{\text{in}}(s))^2 + ((q^{\text{out}})_t(s))^2 + (p^{\text{in}}(s))^2 + ((p^{\text{out}})_t(s))^2 \right. \\ & \quad \left. + (p^{\text{in}}(s))^2 + (p^{\text{out}}(s))^2 + (q^{\text{in}}(s))^2 + (q^{\text{out}}(s))^2 \right). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} & \|\ell\|_{L^2(0,T;\mathcal{H})}^2 \\ & = \|\ell^p\|_{L^2(0,T;L^2(\Omega))}^2 + \|\ell^q\|_{L^2(0,T;L^2(\Omega))}^2 \\ & \leq C \left[\|p^{\text{in}}\|_{L^2(0,T)}^2 + \|p^{\text{out}}\|_{L^2(0,T)}^2 + \|q^{\text{in}}\|_{L^2(0,T)}^2 + \|q^{\text{out}}\|_{L^2(0,T)}^2 \right. \\ & \quad \left. + \|(p^{\text{in}})_t\|_{L^2(0,T)}^2 + \|(p^{\text{out}})_t\|_{L^2(0,T)}^2 + \|(q^{\text{in}})_t\|_{L^2(0,T)}^2 + \|(q^{\text{out}})_t\|_{L^2(0,T)}^2 \right]. \end{aligned}$$

This concludes the proof. \square

Finally, the desired regularity result for the solution of the system can be obtained. As the shifted initial state has to be in $H_0^1(\Omega)^2$, there are compatibility conditions between the initial state and the value of the controls at $t = 0$. To capture these, we define

$$Z := \{(y_0, u_\Gamma) \in H^1(\Omega)^2 \times H^1(0, T)^4 \mid p_0(0) = p^{\text{in}}(0), p_0(L) = p^{\text{out}}(0), \\ q_0(0) = q^{\text{in}}(0), q_0(L) = q^{\text{out}}(0)\}.$$

Lemma 18.4 *Given $y_0 \in H^1(\Omega)^2$, assume that the boundary control u_Γ satisfies $(y_0, u_\Gamma) \in Z$. Then, the unique solution $y = (p, q)$ of (18.6) has regularity*

$$y \in L^\infty(0, T; H^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \quad \text{and} \quad y_t \in L^2(0, T; \mathcal{H}),$$

and there is a linear and continuous mapping $S_{\text{PDE}}: Z \rightarrow Y$ such that $y = S_{\text{PDE}}(y_0, u_\Gamma)$.

Proof. The hypothesis allows us to invoke Lemma 18.3 and Proposition 18.2. Given data $(y_0, u_\Gamma) \in Z$, let v be the unique solution of Problem (P) with initial state $v_0 = y_0 - (\mathcal{E}(u_\Gamma^p), \mathcal{E}(u_\Gamma^q))^\top(\cdot, 0)$. From Proposition 18.2, we have

$$v \in L^\infty(0, T; H_0^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \subset L^2(0, T; (H^2(\Omega) \cap H_0^1(\Omega))^2), \quad (18.19)$$

and then $v \in D(\mathcal{L}(t))$ for all $t \in [0, T]$. Moreover, Problem (P) is equivalent to $v_t = \ell - \mathcal{L}v$ in $L^2(0, T; \mathcal{V}^*)$ plus initial and boundary conditions; see [11, Remark 2, p. 512]. As

$\ell - \mathcal{L}v \in L^2(0, T; \mathcal{H}) \subset L^2(0, T; \mathcal{V}^*)$, we have $v_t \in L^2(0, T; \mathcal{H})$. Furthermore, with v satisfying (18.19), the estimate

$$\begin{aligned} \|v_t\|_{L^2(0, T; \mathcal{H})} &\leq \|\ell\|_{L^2(0, T; \mathcal{H})} + \|\mathcal{L}v\|_{L^2(0, T; \mathcal{H})} \\ &\leq \|\ell\|_{L^2(0, T; \mathcal{H})} + C_{\mathcal{L}} \|v\|_{L^2(0, T; H^2(\Omega)^2)} \end{aligned}$$

holds due to the boundedness of \mathcal{L} . In combination with the estimates from Proposition 18.2 and Lemma 18.3, we get

$$\|v\|_{L^\infty(0, T; \mathcal{V})} + \|v_t\|_{L^2(0, T; \mathcal{H})} + \|v\|_{L^2(0, T; H^2(\Omega)^2)} \leq C(\|v_0\|_{\mathcal{V}} + \|u_\Gamma\|_{H^1(0, T)^4}).$$

From v , the unique solution y of (18.6) is given by

$$y = v + \begin{pmatrix} \mathcal{E}(u_\Gamma^p) \\ \mathcal{E}(u_\Gamma^q) \end{pmatrix}.$$

Recall that $\mathcal{E}: H^1(0, T)^2 \rightarrow H^1(0, L; H^2(\Omega))$ is linear and bounded. Therefore, we get the boundedness of S_{PDE} . \square

Before moving on to the analysis of the GNEP, let us define the control-to-state mapping \mathcal{S} . The compatibility condition $(y_0, u_\Gamma) \in Z$ restricts the solution operator to be well-defined on a subset of $H^1(0, T)^4$. More precisely, with $u_\Gamma^{y_0} := (p_0(0), p_0(L), q_0(0), q_0(L))$, the set of boundary controls compatible with y_0 is $U_{y_0} := \{u_\Gamma \in H^1(0, T)^4 \mid u_\Gamma(0) = u_\Gamma^{y_0}\}$. With $\widehat{u}^{y_0} \in U_{y_0}$ the constant function $\widehat{u}^{y_0} \equiv u_\Gamma^{y_0}$, the relation between any boundary control $u_\Gamma \in U_{y_0}$ and the state can be written as

$$y = S_{\text{PDE}}(y_0, u_\Gamma) = S_{\text{PDE}}(y_0, \widehat{u}^{y_0}) + S_{\text{PDE}}(0, u_\Gamma - \widehat{u}^{y_0}),$$

where

$$u_\Gamma - \widehat{u}^{y_0} \in \widehat{U} := \{\widehat{u} \in H^1(0, T)^4 \mid \widehat{u}(0) = 0\} \quad (18.20)$$

is the Banach space (with the norm of $H^1(0, T)^4$) of controls compatible with the initial state being 0. With $\widehat{S}_0: \widehat{U} \rightarrow Y$ given by $\widehat{u} \mapsto S_{\text{PDE}}(0, \widehat{u})$, the control-to-state operator $S: U_{y_0} \rightarrow Y$ is given by

$$S(u_\Gamma) := S_{\text{PDE}}(y_0, \widehat{u}^{y_0}) + \widehat{S}_0(u_\Gamma - \widehat{u}^{y_0}) = y. \quad (18.21)$$

18.4 Analysis of the Jointly Convex GNEP

Given the well-posedness of the PDE governing the gas flow, we now study the underlying GNEP. The space of pressures (resp. mass flows) at the pipe's endpoints is $P := H^1(0, T)^2$ (resp. $Q := H^1(0, T)^2$). Each firm's decision variable $u_i =: (p_i, q_i)$, where p_i and q_i are the pressure and mass flow variables, is an element of $U := P \times Q$, and we define $\mathcal{U} := U^M$ as we consider M -many firms. The set of initial states compatible with the constraint $y \in K$ is given by:

$$Y_0^{\text{ad}} := \{y_0 \in H^1(\Omega)^2 \mid \underline{y} \leq y_0(x) \leq \bar{y} \text{ for all } x \in [0, L]\},$$

where $\underline{y}, \bar{y} \in \mathbb{R}$ with $\underline{y} \leq \bar{y}$ are fixed. The mapping in (18.2) between the players' decisions and the boundary values is $\mathcal{L}: \mathcal{U} \rightarrow U$ with

$$\mathcal{L} := (\mathcal{L}^p, \mathcal{L}^q) \quad \text{with} \quad \mathcal{L}^p(u) := \frac{1}{M} \sum_{i=1}^M p_i \quad \text{and} \quad \mathcal{L}^q(u) := \sum_{i=1}^M q_i.$$

While this does not impose the continuity of the pressure, we shall see that this particular choice of \mathcal{L}^p allows us to show existence of solutions to the game such that the continuity of pressure holds.

In our analysis below we invoke the following notation. As previously discussed, the control-to-state operator \mathcal{S} is only well-defined on a subset of \mathcal{U} that lacks the vector space structure. To facilitate the handling of the compatibility conditions, we perform a change of variables. Given $y_0 = (p_0, q_0)$, consider the constant functions $\widehat{p}^0(t) := (p_0(0), p_0(L))$ and $\widehat{q}^0(t) := (q_0(0)/M, q_0(L)/M)$ for all $t \in [0, T]$. With $\widehat{u}_i^{y_0} := (\widehat{p}^0, \widehat{q}^0)$, the shifted decisions for player i are given by $\widehat{u}_i = u_i - \widehat{u}_i^{y_0}$, and these quantities are collected in \widehat{u} and \widehat{u}^{y_0} . With $\widehat{P} := \{\widehat{p} \in P \mid \widehat{p}(0) = 0\}$ and $\widehat{Q} := \{\widehat{q} \in Q \mid \widehat{q}(0) = 0\}$, the assumptions stated in (18.25a) and (18.25b) below allow us to define the feasible sets for the pressures and mass flows as $\widehat{P}^{\text{ad}} := \{\widehat{p} \in \widehat{P} \mid \widehat{p} + \widehat{p}^0 \in P^{\text{ad}}\}$ and $\widehat{Q}_i^{\text{ad}} := \{\widehat{q} \in \widehat{Q} \mid \widehat{q} + \widehat{q}^0 \in Q_i^{\text{ad}}\}$. In the light of the definition of \widehat{U} in (18.20), we note that $\widehat{U} = \widehat{P} \times \widehat{Q}$. Also, as closed subspaces of $H^1(0, T)^2$, equipped with the norm of $H^1(0, T)^2$, both \widehat{P} and \widehat{Q} are Banach spaces. Furthermore, from the inclusions $H_0^1(0, T)^2 \subset \widehat{P} \subset L^2(0, T)^2$ and $\widehat{P}^* \subset H^{-1}(0, T)^2$, we infer that \widehat{P} is dense in $L^2(0, T)^2$. Coupled with the continuous injection of $H^1(0, T)$ in $L^2(0, T)$, we deduce that we have the Gelfand triples $\widehat{P} \hookrightarrow L^2(0, T)^2 \hookrightarrow \widehat{P}^*$ and $\widehat{Q} \hookrightarrow L^2(0, T)^2 \hookrightarrow \widehat{Q}^*$. The space $\mathcal{Z} := (\widehat{U}^*)^M$ is isomorphic to $\widehat{\mathcal{U}}^*$, denoted by $\mathcal{Z} \simeq \widehat{\mathcal{U}}^*$. We can thus identify any element of \mathcal{Z} with a unique element of $\widehat{\mathcal{U}}^*$ and vice-versa. The duality pairing between $\widehat{\mathcal{U}}$ and \mathcal{Z} is given by $\langle z, u \rangle_{\mathcal{Z}, \widehat{\mathcal{U}}} = \sum_{i=1}^M \langle z_i, u_i \rangle_{\widehat{\mathcal{U}}^*, \widehat{\mathcal{U}}}$. Then, the feasible set for \widehat{u}_i is a subset of the Banach space \widehat{U} . The affine mapping $\widehat{S}: \widehat{\mathcal{U}} \rightarrow Y$, from the players' actions to the state, is given by

$$y = \widehat{S}(\widehat{u}) = S_{\text{PDE}}(y_0, \widehat{u}^{y_0}) + \widehat{S}_0(\mathcal{L}(\widehat{u})).$$

With $\mathcal{L}_i: \widehat{P} \times \widehat{Q} \rightarrow \widehat{P} \times \widehat{Q}$ given by $\mathcal{L}_i(\widehat{u}_i) := (\widehat{p}_i, \widehat{q}_i/M)$ for all $i \in \{1, \dots, M\}$, we define $\widehat{S}_i(\widehat{u}_i) := \widehat{S}_0(\mathcal{L}_i(\widehat{u}_i))$. Then, for a given \widehat{u}_{-i} , the mapping from each player's control \widehat{u}_i to the state y reads

$$\widehat{S}(\widehat{u}_i, \widehat{u}_{-i}) = \widehat{S}_i(\widehat{u}_i) + S_{\text{PDE}}(y_0, \widehat{u}^{y_0}) + \widehat{S}_0(\mathcal{L}(0, \widehat{u}_{-i})).$$

Note that $\widehat{S}_i: \widehat{U} \rightarrow Y$ is well defined and linear. The feasible set for each agent is given by the set-valued map $\widehat{C}_i: \widehat{U}^{M-1} \rightrightarrows \widehat{U}$ defined as

$$\widehat{C}_i(\widehat{u}_{-i}) := \{\widehat{u}_i \in \widehat{U}_i^{\text{ad}} \mid \widehat{S}(\widehat{u}_i, \widehat{u}_{-i}) \in K\}, \quad (18.22)$$

where $\widehat{U}_i^{\text{ad}} := \widehat{P}^{\text{ad}} \times \widehat{Q}_i^{\text{ad}}$. As both $\widehat{U}_i^{\text{ad}}$ and K are non-empty, closed, and convex, the fact that \widehat{S} is affine ensures that the images of $\widehat{C}_i(\widehat{u}_{-i})$ are closed and convex (if non-empty). The part of the objective functional that intertwines the players' decision variables is the revenue functional

$$r_i(q_i^{\text{out}}, q_{-i}^{\text{out}}) := \int_0^T \left(\alpha(t) \sum_{k=1}^M q_k^{\text{out}}(t) - \beta(t) \right) q_i^{\text{out}}(t) dt.$$

With $\widehat{u}_i := (\widehat{p}_i^{\text{in}}, \widehat{p}_i^{\text{out}}, \widehat{q}_i^{\text{in}}, \widehat{q}_i^{\text{out}})$, in the shifted variables, we get

$$\begin{aligned} \widehat{r}_i(\widehat{q}_i^{\text{out}}, \widehat{q}_{-i}^{\text{out}}) &:= r_i(\widehat{q}_i^{\text{out}} + q_0(L)/M, \widehat{q}_{-i}^{\text{out}} + q_0(L)/M) \\ &= \int_0^T \left[\alpha(t) \sum_{k=1}^M \widehat{q}_k^{\text{out}}(t) - (\beta(t) - \alpha(t)q_0(L)) \right] [\widehat{q}_i^{\text{out}}(t) + q_0(L)/M] dt. \end{aligned}$$

Note that for all $\widehat{q}_{-i}^{\text{out}} \in L^2(0, T)^{M-1}$, the mapping $\widehat{q}_i^{\text{out}} \mapsto \widehat{r}_i(\widehat{q}_i^{\text{out}}, \widehat{q}_{-i}^{\text{out}})$ is convex and continuously Fréchet differentiable. The expression of the gradient w.r.t. the first variable reads

$$\nabla_{\widehat{q}_i^{\text{out}}} \widehat{r}_i(\widehat{q}_i^{\text{out}}, \widehat{q}_{-i}^{\text{out}}) = \alpha \sum_{k=1}^M \widehat{q}_k^{\text{out}} + \alpha \widehat{q}_i^{\text{out}} - (\beta - \widehat{\alpha}q_0(L))$$

with $\widehat{\alpha} := \alpha(1 + M)/M$. Let $e \in \widehat{U}^*$, $d_1: \widehat{U} \times \widehat{U}^{M-1} \rightarrow \widehat{U}^*$, and $d_2: \widehat{U} \rightarrow \widehat{U}^*$ such that for any $w_i = (w_i^{p^0}, w_i^{p^L}, w_i^{q^0}, w_i^{q^L}) \in \widehat{U}$ the following holds:

$$\langle d_1(\widehat{u}_i, \widehat{u}_{-i}), w_i \rangle_{\widehat{U}^*, \widehat{U}} = \left(\alpha \sum_{k=1}^M \widehat{q}_k^{\text{out}}, w_i^{q^L} \right)_{L^2(0, T)}, \quad (18.23a)$$

$$\langle d_2(\widehat{u}_i), w_i \rangle_{\widehat{U}^*, \widehat{U}} = \left(\alpha \widehat{q}_i^{\text{out}}, w_i^{q^L} \right)_{L^2(0, T)}, \quad (18.23b)$$

$$\langle e, w_i \rangle_{\widehat{U}^*, \widehat{U}} = \left(\beta - \widehat{\alpha}q_0(L), w_i^{q^L} \right)_{L^2(0, T)}. \quad (18.23c)$$

We are now ready to state our main result.

Theorem 18.1 *Given an initial state $(p_0, q_0) \in Y_0^{\text{ad}}$, consider the following jointly convex GNEP with M players, where the optimization problem associated with player $i \in \{1, \dots, M\}$ is given by*

$$\begin{aligned} \min_{u_i \in \widehat{U}} \quad & f_i(u_i, u_{-i}) := \int_0^T \left(\alpha(t) \sum_{k=1}^M q_k^{\text{out}}(t) - \beta(t) \right) q_i^{\text{out}}(t) dt + g_i(u_i) \\ \text{s.t.} \quad & u_i \in P^{\text{ad}} \times Q_i^{\text{ad}}, \quad S(u_i, u_{-i}) \in K, \end{aligned} \quad (18.24)$$

where $g_i(u_i)$ is any function defined as $g_i(u_i) = g^P(p_i) + g_i^Q(q_i)$ with both g^P and g_i^Q being lower semicontinuous and proper convex functions. Assume that the subdifferentials ∂g^P and ∂g_i , for each player, have full domain and that the sets K , $P^{\text{ad}} \subset P$ as

well as $Q_i^{\text{ad}} \subset Q$ are closed and convex. In addition, P^{ad} and Q_i^{ad} satisfy the property that

$$p \in P^{\text{ad}} \implies p^{\text{in}}(0) = p_0(0) \text{ and } p^{\text{out}}(0) = p_0(L), \quad (18.25a)$$

$$q \in Q_i^{\text{ad}} \implies Mq_i^{\text{in}}(0) = q_0(0) \text{ and } Mq_i^{\text{out}}(0) = q_0(L). \quad (18.25b)$$

Furthermore, suppose that there exists $\bar{u} \in \prod_{i=1}^M (P^{\text{ad}} \times Q_i^{\text{ad}})$ such that Slater's constraint qualification (CQ)

$$S(\bar{u}) \in \text{int } K \quad (18.26)$$

holds. Finally, assume that the subdifferential of the sum $Mg^P(p) + \sum_{i=1}^M g_i^q(q_i)$, when restricted to $P^{\text{ad}} \times \prod_{i=1}^M Q_i^{\text{ad}}$, is coercive. Then, there exists a variational equilibrium to the GNEP with all pressures p_i , $i \in \{1, \dots, M\}$, being identical.

Proof. We proceed in three steps: First, we show that there exists a solution to the GNEP for which the first-order conditions for the players' optimization problems exhibit the same multiplier for the shared constraint $S(u) \in K$. Second, we formulate a reduced generalized equation with only one boundary pressure variable and show that its solution set is non-empty. Lastly, from any solution of the latter problem, we construct a solution to an operator equation associated with the first part and therefore to the jointly convex GNEP. Note that this investigation is performed in the shifted variable spaces introduced before the theorem statement, as it facilitates the handling of the compatibility conditions.

Step 1. We focus on the first-order optimality condition for the minimization problem associated with player i . For this purpose, take any fixed \hat{u}_{-i} such that $\widehat{C}_i(\hat{u}_{-i}) \neq \emptyset$ holds. Since for any \hat{u}_{-i} , the partial map $f_i(\cdot, \hat{u}_{-i})$ is lower semicontinuous and convex, for \hat{u}_i to be optimal, it is necessary and sufficient that the inclusion

$$0 \in \partial(f_i(\cdot, \hat{u}_{-i}) + \delta_{\widehat{C}_i(\hat{u}_{-i})})(\hat{u}_i) \quad (18.27)$$

holds, where $\delta_{\widehat{C}_i(\hat{u}_{-i})}$ is the indicator function of the set $\widehat{C}_i(\hat{u}_{-i})$. Let $\widehat{g}^P(\widehat{p}_i) := g^P(\widehat{p}_i + \widehat{p}^0)$ and $\widehat{g}_i^q(\widehat{q}_i) := g_i^q(\widehat{q}_i + \widehat{q}^0)$. As \widehat{r}_i is C^1 and $\partial\widehat{g}_i^q$ has full domain, by applying the sum rule for convex subdifferentials we obtain

$$\partial(f_i(\cdot, \hat{u}_{-i}) + \delta_{\widehat{C}_i(\hat{u}_{-i})})(\hat{u}_i) = d_1(\hat{u}_i, \hat{u}_{-i}) + d_2(\hat{u}_i) - e + \partial\widehat{g}_i^q(\hat{u}_i) + \partial(\delta_{\widehat{C}_i(\hat{u}_{-i})})(\hat{u}_i),$$

where d_1 , d_2 , and e were defined in (18.23a)–(18.23c). Using the chain rule for convex subdifferentials we get that for any $\hat{u}_i \in \widehat{U}_i^{\text{ad}}$ with $\widehat{S}(\hat{u}_i, \hat{u}_{-i}) \in K$, the inclusion

$$\partial\delta_{\widehat{C}_i(\hat{u}_{-i})}(\hat{u}_i) \supseteq N_{\widehat{U}_i^{\text{ad}}}(\hat{u}_i) + \partial(\delta_K(\widehat{S}(\hat{u}_i, \hat{u}_{-i}))) \quad (18.28)$$

holds. Hence, any solution to the generalized equation

$$0 \in d_1(\hat{u}_i, \hat{u}_{-i}) + d_2(\hat{u}_i) - e + \partial\widehat{g}_i^q(\hat{u}_i) + N_{\widehat{U}_i^{\text{ad}}}(\hat{u}_i) + \partial(\delta_K(\widehat{S}(\hat{u}_i, \hat{u}_{-i}))) \quad (18.29)$$

satisfies (18.27) and therefore is optimal for player i . Recall from (18.21) that $\widehat{u}_i \mapsto \widehat{S}(\widehat{u}_i, \widehat{u}_{-i})$ is an affine mapping. Invoking again the chain rule for convex subdifferentials, we have

$$\partial(\delta_K(\widehat{S}(\widehat{u}_i, \widehat{u}_{-i}))) \supseteq \widehat{S}_i^* N_K(\widehat{S}(\widehat{u}_i, \widehat{u}_{-i})). \quad (18.30)$$

With $\mu_i \in N_K(\widehat{S}(\widehat{u}_i, \widehat{u}_{-i}))$, from (18.29) we get the generalized equation

$$e \in d_1(\widehat{u}_i, \widehat{u}_{-i}) + d_2(\widehat{u}_i) + \partial \widehat{g}_i(\widehat{u}_i) + N_{\widehat{U}_i^{\text{ad}}}(\widehat{u}_i) + \widehat{S}_i^* \mu_i.$$

Any $\widehat{u} \in \widehat{\mathcal{U}}$ that satisfies the above inclusion for all $i \in \{1, \dots, M\}$ is a solution of the GNEP. Note that if an appropriate constraint qualification, like Slater's CQ, were to hold for $\widehat{C}_i(\widehat{u}_{-i})$ as given in (18.22), then both (18.28) and (18.30) would hold with equality. Since the state variable $y = \widehat{S}(\widehat{u}_i, \widehat{u}_{-i})$ is the same for all players, we can restrict all μ_i to be equal. This leads us to the following operator equations:

$$e \in d_1(\widehat{u}_i, \widehat{u}_{-i}) + d_2(\widehat{u}_i) + \partial \widehat{g}_i(\widehat{u}_i) + N_{\widehat{U}_i^{\text{ad}}}(\widehat{u}_i) + \widehat{S}_i^* \mu, \quad i \in \{1, \dots, M\}, \quad (18.31a)$$

$$\mu \in N_K(\widehat{S}(\widehat{u})). \quad (18.31b)$$

Any $\widehat{u} \in \widehat{\mathcal{U}}$ that satisfies the above relations is a variational equilibrium of the GNEP. To conclude the first step and streamline the remainder of the proof, let us provide an equivalent system of inclusions that characterize a solution to the jointly convex GNEP. Let $\tilde{d}_1: \widehat{Q} \times \widehat{Q}^{M-1} \rightarrow \widehat{Q}^*$ and $\tilde{d}_2: \widehat{Q} \rightarrow \widehat{Q}^*$ be such that

$$\begin{aligned} \langle \tilde{d}_1(\widehat{q}_i, \widehat{q}_{-i}), w_i \rangle_{\widehat{Q}^*, \widehat{Q}} &= \left(\alpha \sum_{k=1}^M \widehat{q}_k^{\text{out}}, w_i^{\text{out}} \right)_{L^2(0,T)}, \\ \langle \tilde{d}_2(\widehat{q}_i), w_i \rangle_{\widehat{Q}^*, \widehat{Q}} &= (\alpha \widehat{q}_i^{\text{out}}, w_i^{\text{out}})_{L^2(0,T)}, \\ \langle \tilde{e}, w_i \rangle_{\widehat{Q}^*, \widehat{Q}} &= (\beta - \widehat{\alpha} q_0(L), w_i^{\text{out}})_{L^2(0,T)}, \end{aligned}$$

for any $w_i = (w_i^{\text{in}}, w_i^{\text{out}}) \in \widehat{Q}$. Remembering that $\mathcal{Z} \simeq \widehat{\mathcal{U}}^*$, we conclude that the solution set of the system (18.31a)–(18.31b) coincides with the one of

$$0 \in \partial \widehat{g}^p(\widehat{p}_i) + N_{\widehat{P}^{\text{ad}}}(\widehat{p}_i) + M^{-1} \eta^p, \quad i \in \{1, \dots, M\}, \quad (18.32a)$$

$$\tilde{e} \in \tilde{d}_1(\widehat{q}_i, \widehat{q}_{-i}) + \tilde{d}_2(\widehat{q}_i) + \partial \widehat{g}_i^q(\widehat{q}_i) + N_{\widehat{Q}_i^{\text{ad}}}(\widehat{q}_i) + \eta^q, \quad i \in \{1, \dots, M\}, \quad (18.32b)$$

$$\mu \in N_K(\widehat{S}(\widehat{u})), \quad (18.32c)$$

$$\begin{pmatrix} \eta^p \\ \eta^q \end{pmatrix} = \widehat{S}_0^* \mu. \quad (18.32d)$$

Step 2. We are interested in the existence of a solution for which the endpoint pressure is the same for all players. For this purpose, we consider the operator equation in the

space $\tilde{\mathcal{U}} := \tilde{P} \times \tilde{Q}^M$. Let $\tilde{\mathcal{L}}: \tilde{P} \times \tilde{Q}^M \rightarrow \tilde{P} \times \tilde{Q}$ be given by $\tilde{\mathcal{L}}(\tilde{p}, \tilde{q}) = (\tilde{p}, \sum_{i=1}^M \tilde{q}_i)$. The relation between the state variable y and the controls $(\tilde{p}, \tilde{q}) \in \tilde{\mathcal{U}}$ is now given by $\tilde{S}(\tilde{p}, \tilde{q}) := S_{\text{PDE}}(y_0, \tilde{u}^{y_0}) + \tilde{S}_0(\tilde{\mathcal{L}}(\tilde{p}, \tilde{q}))$. The constraint set is $\tilde{C} := \{(\tilde{p}, \tilde{q}) \in \tilde{P}^{\text{ad}} \times \tilde{Q}^{\text{ad}} \mid \tilde{S}(\tilde{p}, \tilde{q}) \in K\}$ with $\tilde{Q}^{\text{ad}} := \prod_{i=1}^M \tilde{Q}_i^{\text{ad}}$. Thanks to the convexity of \tilde{P}^{ad} , Condition (18.26) ensures the non-emptiness of \tilde{C} . With $\tilde{\mathcal{Z}} := \tilde{P}^* \times (\tilde{Q}^*)^M$, we associate the operators $\tilde{\mathcal{A}}: \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{Z}}$ and $\tilde{\mathcal{B}}: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{Z}}$ with

$$\tilde{\mathcal{A}}(\tilde{p}, \tilde{q}) = \begin{pmatrix} 0 \\ \tilde{d}_1(\tilde{q}_1, \tilde{q}_{-1}) + \tilde{d}_2(\tilde{q}_1) \\ \vdots \\ \tilde{d}_1(\tilde{q}_M, \tilde{q}_{-M}) + \tilde{d}_2(\tilde{q}_M) \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{B}}(\tilde{p}, \tilde{q}) = \begin{pmatrix} \partial M \tilde{g}^p(\tilde{p}) \\ \partial \tilde{g}_1^q(\tilde{q}_1) \\ \vdots \\ \partial \tilde{g}_M^q(\tilde{q}_M) \end{pmatrix}. \quad (18.33)$$

Our goal is to show existence of a solution to the generalized equation

$$\tilde{\mathbf{e}} \in \tilde{\mathcal{A}}'(\tilde{p}, \tilde{q}) + \tilde{\mathcal{B}}'(\tilde{p}, \tilde{q}) + N_{\tilde{C}}(\tilde{p}, \tilde{q}), \quad (18.34)$$

where $\tilde{\mathbf{e}} := (0, \tilde{e}, \dots, \tilde{e})$ and $\tilde{\mathcal{A}}'$ as well as $\tilde{\mathcal{B}}'$ are defined by identifying the sets $\tilde{\mathcal{A}}(\tilde{p}, \tilde{q})$ and $\tilde{\mathcal{B}}(\tilde{p}, \tilde{q})$ with the corresponding subsets of $\tilde{\mathcal{U}}^*$. Conversely, the operator $\tilde{\mathcal{C}}: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{Z}}$ is defined by setting $\tilde{\mathcal{C}}(\tilde{p}, \tilde{q})$ to the corresponding subset of $N_{\tilde{C}}(\tilde{p}, \tilde{q})$. We show that the sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$ is maximally monotone and start by establishing that $\tilde{\mathcal{A}}$ is monotone and demicontinuous. The latter property is true whenever for any sequence $\{v_n\} \subset \tilde{D}\tilde{\mathcal{A}}$ converging strongly to v in $\tilde{\mathcal{U}}$, we have that $\tilde{\mathcal{A}}(v_n)$ converges weakly to $\tilde{\mathcal{A}}(v)$ in $\tilde{\mathcal{Z}}$; see [2, Definition 1.137]. With $z^p \in \tilde{P}$, $z_i^q \in \tilde{Q}$, $z_i^q = (z_i^{\text{in}}, z_i^{\text{out}})$, and z^q collecting all the z_i^q , we have

$$\begin{aligned} \langle \tilde{\mathcal{A}}(\tilde{p}, \tilde{q}), (z^p, z^q) \rangle_{\tilde{\mathcal{Z}}, \tilde{\mathcal{U}}} &= \sum_{i=1}^M \left[\left(\alpha \sum_{k=1}^M \tilde{q}_k^{\text{out}}, z_i^{\text{out}} \right)_{L^2(0,T)} + (\alpha \tilde{q}_i^{\text{out}}, z_i^{\text{out}})_{L^2(0,T)} \right] \\ &= \left(\sum_{k=1}^M \tilde{q}_k^{\text{out}}, \sum_{i=1}^M \alpha z_i^{\text{out}} \right)_{L^2(0,T)} + \sum_{i=1}^M (\tilde{q}_i^{\text{out}}, \alpha z_i^{\text{out}})_{L^2(0,T)}. \end{aligned}$$

As $\alpha \in L^\infty(0, T)$, we have $\alpha z_i^{\text{out}} \in L^2(0, T)$. Then, for any sequence $\{(\tilde{p}_n, \tilde{q}_n)\} \subset \tilde{D}\tilde{\mathcal{A}}$ strongly converging to (\tilde{p}, \tilde{q}) , we have weak convergence of all the terms in the sum above and $\tilde{\mathcal{A}}(\tilde{p}_n, \tilde{q}_n)$ converges weakly to $\tilde{\mathcal{A}}(\tilde{p}, \tilde{q})$. For the monotonicity, as $\tilde{q} \mapsto \tilde{d}_1(\tilde{q}_i, \tilde{q}_{-i})$ is linear, the operator $\tilde{\mathcal{A}}$ is a linear. Taking $z^p = \tilde{p}$ and $z^q = \tilde{q}$ in the previous relation yields

$$\begin{aligned} \langle \tilde{\mathcal{A}}(\tilde{p}, \tilde{q}), (\tilde{p}, \tilde{q}) \rangle_{\tilde{\mathcal{Z}}, \tilde{\mathcal{U}}} &= \sum_{i=1}^M \left[\left(\alpha \sum_{k=1}^M \tilde{q}_k^{\text{out}}, \tilde{q}_i^{\text{out}} \right)_{L^2(0,T)} + (\alpha \tilde{q}_i^{\text{out}}, \tilde{q}_i^{\text{out}})_{L^2(0,T)} \right] \\ &= \left(\alpha \sum_{k=1}^M \tilde{q}_k^{\text{out}}, \sum_{i=1}^M \tilde{q}_i^{\text{out}} \right)_{L^2(0,T)} + \sum_{i=1}^M (\alpha \tilde{q}_i^{\text{out}}, \tilde{q}_i^{\text{out}})_{L^2(0,T)}. \end{aligned}$$

Since $\alpha(t) \geq \alpha_{\min} > 0$ for almost all $t \in (0, T)$, we infer that

$$\langle \tilde{\mathcal{A}}(\tilde{p}, \tilde{q}), (\tilde{p}, \tilde{q}) \rangle_{\tilde{Z}, \tilde{U}} \geq \alpha_{\min} \left\| \sum_{i=1}^M \tilde{q}_i^{\text{out}} \right\|_{L^2(0,T)}^2 + \alpha_{\min} \sum_{i=1}^M \|\tilde{q}_i^{\text{out}}\|_{L^2(0,T)}^2.$$

As $\tilde{\mathcal{A}}$ has full domain, is monotone and demicontinuous, it is maximally monotone; see [2, Corollary 1.142]. The maximal monotonicity of the sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$ follows from the fact that both $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ have full domain.

For the existence of a solution to (18.34), we rely on the coercivity of the maximally monotone sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$; see [2, Definition 1.138]. By the assumption on the subdifferential of $Mg^p(p) + \sum_{i=1}^M g_i^q(q_i)$, namely $\tilde{\mathcal{B}}$, we have the coercivity of the sum $\tilde{\mathcal{B}} + \tilde{\mathcal{C}}$. As $\tilde{\mathcal{A}}$ has full domain and is maximally monotone, the sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$ is coercive. Now under the isomorphism between \tilde{Z} and \tilde{U}^* , the operator $\tilde{\mathcal{A}}' + \tilde{\mathcal{B}}' + N_{\tilde{C}}$ is also coercive and maximally monotone. It is well-known that a coercive and maximally monotone operator from a reflexive Banach space to its dual is surjective; see, e.g., [2, Theorem 1.143]. Therefore, there is a solution of the operator equation

$$\mathbf{e} \in (\tilde{\mathcal{A}}' + \tilde{\mathcal{B}}' + N_{\tilde{C}})(u). \quad (18.35)$$

Step 3. Finally, we construct from any solution (p^*, q^*) of this generalized equation a solution u^* of the system (18.31a)–(18.31b) and, thus, of the GNEP. First, since (18.26) holds at $\tilde{u} = (\tilde{p}, \tilde{q})$, we infer that with $p' := (\frac{1}{M} \sum_{i=1}^M \tilde{p}_i) - \tilde{p}^0$ and $q' \in \tilde{Q}^M$ defined as $q'_i := \tilde{q}_i - \tilde{q}^0$, the inclusion

$$\tilde{S}(p', q') \in \text{int } K$$

holds and $(p', q') \in \tilde{C}$. Invoking the chain rule for convex subdifferentials, we get that for any $(\tilde{p}, \tilde{q}) \in \tilde{C}$, we have

$$N_{\tilde{C}}(\tilde{p}, \tilde{q}) = N_{\tilde{p}^{\text{ad}} \times \tilde{Q}^{\text{ad}}}(\tilde{p}, \tilde{q}) + \tilde{\mathcal{L}}^* \tilde{S}_0^* N_K(\tilde{S}(\tilde{p}, \tilde{q})).$$

Let $\mu^* \in N_K(\tilde{S}(p^*, q^*))$ be such that the inclusion (18.35) holds. With $\eta^* := \tilde{S}_0^* \mu^*$, we get $(\eta^p)^* \in \tilde{P}^*$ and $(\eta^q)^* \in \tilde{Q}^*$ from η^* . Then, from (18.35) we obtain that

$$0 \in \partial M \widehat{g}^p(p^*) + N_{\tilde{p}^{\text{ad}}}(p^*) + (\eta^p)^*$$

holds. Since $\partial M \widehat{g}^p(p^*) = M \partial \widehat{g}^p(p^*)$, the following inclusion is satisfied:

$$0 \in \partial \widehat{g}^p(p^*) + N_{\tilde{p}^{\text{ad}}}(p^*) + M^{-1}(\eta^p)^*. \quad (18.36)$$

We define $u^* \in \widehat{U}^{\text{ad}}$ by setting $u_i^* := (p^*, q_i^*)$ and since $\tilde{S}(p^*, q^*) = S(u^*)$ by construction, we have $\mu^* \in N_K(S(u^*))$. Using the definition of the operators in (18.33) and of $\tilde{\mathcal{L}}$, for all $i \in \{1, \dots, M\}$ we get

$$0 \in \tilde{d}_1(q_i^*, q_{-i}^*) + \tilde{d}_2(q_i^*) + \partial \widehat{g}_i^q(q_i^*) + N_{\tilde{Q}_i^{\text{ad}}}(q_i^*) + (\eta^q)^*. \quad (18.37)$$

Since $\mu^* \in N_K(S(u^*))$, $\eta^* = \tilde{S}_0^* \mu^*$, and (18.36)–(18.37) hold for all u_i^* , we infer that u^* satisfies (18.32). \square

Corollary 18.1 *Given an initial state $(p_0, q_0) \in Y_0^{\text{ad}}$, consider the GNEP formed by the collection of problems of the form (18.4). Suppose that for each firm i , the functional $\mathcal{R}_i(p_i, q_i) = \mathcal{R}_i^q(q_i) + \mathcal{R}^p(p_i)$ with both $\mathcal{R}_i^q: Q \rightarrow \mathbb{R}$ and $\mathcal{R}^p: P \rightarrow \mathbb{R}$ being lower semicontinuous, proper and convex with subdifferentials having full domain. We set*

$$\begin{aligned} Q_i^{\text{ad}} = \{ & (q^{\text{in}}, q^{\text{out}}) \in Q \mid q_0(0) = Mq^{\text{in}}(0), q_0(L) = Mq^{\text{out}}(0), \\ & \int_0^T q^{\text{out}}(t) - q^{\text{in}}(t) dt \leq 0, \\ & 0 \leq q^{\text{in}} \leq \bar{q}_i^{\text{in}}, 0 \leq q^{\text{out}} \leq \bar{q} \text{ a.e. in } (0, T)\} \end{aligned}$$

and

$$P^{\text{ad}} = \{(p^{\text{in}}, p^{\text{out}}) \in P \mid \underline{p} \leq p^{\text{in}}, p^{\text{out}} \leq \bar{p}, p_0(0) = p^{\text{in}}(0) \text{ and } p_0(L) = p^{\text{out}}(0)\}.$$

Assume that there exists $\bar{u} \in \prod_{i=1}^M (P^{\text{ad}} \times Q_i^{\text{ad}})$ such that $S(\bar{u}) \in \text{int } K$ holds. Suppose that the subdifferential of the sum $M\mathcal{R}^p(p) + \sum_{i=1}^M \mathcal{R}_i^q(q_i)$, when restricted to $P^{\text{ad}} \times \prod_{i=1}^M Q_i^{\text{ad}}$, is coercive. Then, there exists a variational equilibrium to the GNEP with all pressures p_i being identical.

Proof. To fit into the format of (18.24), we let

$$g_i(u_i) = \int_0^T c_i(t) q_i^{\text{in}}(t) dt + \mathcal{R}_i(u_i)$$

and note that all the hypotheses are satisfied. \square

Remark 18.1 The existence of feasible controls satisfying (18.26) is automatically given whenever the initial state y_0 is a stationary solution satisfying the bound constraints and there exists feasible controls to satisfy the injection requirements given in (18.25b). The coercivity condition is satisfied whenever the feasible set of each agent is bounded. Otherwise, like in our test instance, this condition is satisfied whenever each feasible set is bounded in the L^2 sense and each \mathcal{R} is the square of the L^2 norm of the weak time derivative; see (18.43) below.

Remark 18.2 For the existence of a variational equilibrium, the mild Slater condition (18.26), involving all players' decisions, needs to hold. If one seeks a stronger characterization of the solution set of the GNEP, so that any variational equilibrium is a solution of System (18.31a)–(18.31b), then we need both (18.28) and (18.30) to hold with equality. For this, a sufficient condition is that for all i , and any feasible \hat{u}_{-i} , there exists $\hat{u}_i \in \hat{U}_i^{\text{ad}}$ such that

$$\hat{S}(\hat{u}_i, \hat{u}_{-i}) \in \text{int } K,$$

which is much more stringent.

Furthermore, if the producers have the same feasible sets and cost functionals, then a solution to the GNEP can be constructed from the solution to an optimization problem. We refer to this case as the *symmetric case*. Note that this is a simplified but not a trivial

setting since this symmetric case does not necessarily lead to symmetric equilibria; see, e.g., [32].

Proposition 18.3 *Consider the GNEP of Theorem 18.1 with a given initial state y_0 and assume that the players' problems are identical. Suppose that all the assumptions of Theorem 18.1 hold, except for the boundedness of the P^{ad} and Q^{ad} . Let $Q_{\text{opt}}^{\text{ad}} := MQ^{\text{ad}}$ and suppose that there exists $\bar{u} \in P^{\text{ad}} \times Q_{\text{opt}}^{\text{ad}}$ such that $S(\bar{u}) \in \text{int} K$ holds. Then, a variational equilibrium of the GNEP can be constructed from any solution of the optimization problem*

$$\begin{aligned} \min_{u \in U} \quad & \int_0^T \left(\frac{M+1}{2M} \alpha(t) q^{\text{out}}(t) - \beta(t) \right) q^{\text{out}}(t) dt + g(u) \\ \text{s.t.} \quad & u \in P^{\text{ad}} \times Q_{\text{opt}}^{\text{ad}}, \quad S(u) \in K, \end{aligned} \quad (18.38)$$

where $g(p, q) = Mg^P(p) + Mg_i^q(M^{-1}q)$.

Proof. We use the same functional analytic setting as in the proof of Theorem 18.1 and we also perform the change of variables $\hat{p} := p - \hat{p}^0$ and $\hat{q} := q - \hat{q}^0$ with $\hat{p}^0(t) := (p_0(0), p_0(L))$ and $\hat{q}^0(t) := (q_0(0), q_0(L))$ for all $t \in [0, T]$. We also use the same functional analytic setting as in the previous proof. The space $\hat{Q}_{\text{opt}}^{\text{ad}}$ is defined as $\{\hat{q} \in \hat{Q} \mid \hat{q} + \hat{q}^0 \in Q_{\text{opt}}^{\text{ad}}\}$. We show that any (\hat{p}, \hat{q}) satisfying the first-order optimality condition of this problem can be used to construct an element $\hat{u} \in \hat{P} \times \hat{Q}$ satisfying the first-order optimality condition (18.34). From basic results of convex analysis, we have $\partial_{\hat{q}} \hat{g}(\hat{p}, \hat{q}) = \partial g_i^q(M^{-1}(\hat{q} + \hat{q}^0))$. Recall that $\hat{\alpha} = (1 + M^{-1})\alpha$. The revenue part $\hat{r}_{\text{opt}}: L^2(0, T) \rightarrow \mathbb{R}$ of the objective functional with the shifted variables reads

$$\hat{r}_{\text{opt}}(\hat{q}^{\text{out}}) = \int_0^T \left[\frac{1}{2} \hat{\alpha}(t) \hat{q}^{\text{out}}(t) - \beta(t) + \frac{1}{2} \hat{\alpha}(t) q_0(L) \right] [\hat{q}^{\text{out}}(t) + q_0(L)] dt.$$

It is Fréchet differentiable and its gradient is given by

$$\nabla \hat{r}_{\text{opt}}(\hat{q}^{\text{out}}) = \hat{\alpha} \hat{q}^{\text{out}} + \hat{\alpha} q_0(L) - \beta.$$

With the assumptions on the cost function and the fact that Slater's CQ holds, any solution of the optimization problem satisfies

$$\begin{aligned} 0 &= M \partial g^P(p^*) + \zeta_p + \eta^P, \\ \tilde{e} &= \begin{pmatrix} 0 \\ \alpha(1 + M^{-1})(q^{\text{out}})^* \end{pmatrix} + \partial g_i^q(M^{-1}(q^* + \hat{q}^0)) + \zeta_q + \eta^q, \end{aligned}$$

where $\zeta_p \in N_{\hat{P}^{\text{ad}}}(p^*)$, $\zeta_q \in N_{\hat{Q}_{\text{opt}}^{\text{ad}}}(q^*)$, $\mu \in N_K(\hat{S}(u^*))$, (η^P, η^q) are obtained from $\hat{S}_0^* \mu$ and \tilde{e} satisfies $\langle \tilde{e}, w_i \rangle_{\hat{Q}^*, \hat{Q}} = (\beta - \hat{\alpha} q_0(L), w_i^{\text{out}})_{L^2(0, T)}$ for any $w_i = (w_i^{\text{in}}, w_i^{\text{out}}) \in \hat{Q}$. Note that $\tilde{\zeta}_p := M^{-1} \zeta_p$ also belongs to $N_{\hat{P}^{\text{ad}}}(p^*)$. Now let $\hat{q}_i = M^{-1} q^*$ and $\hat{\mathbf{q}}$ collects all \hat{q}_i . Then, after some algebraic manipulations, we get that the following holds:

$$0 = \partial g^p(p^\star) + \tilde{\zeta}_p + M^{-1}\eta^p,$$

$$\tilde{e} = \begin{pmatrix} 0 \\ \alpha \sum_k \widehat{q}_k^{\text{out}} + \alpha \widehat{q}_i^{\text{out}} \end{pmatrix} + \partial g_i^q(\widehat{q}_i + M^{-1}\widehat{q}^0) + \zeta_q + \eta^q.$$

Note that $g_i^q(\widehat{q}_i + M^{-1}q_0(L)) = \widehat{g}_i^q(\widehat{q}_i)$ holds by the definition of the latter function. Since $\widehat{S}(u^\star) = \widehat{S}(p^\star, \widehat{\mathbf{q}})$, we have $\mu \in N_K(\widehat{S}(p^\star, \widehat{\mathbf{q}}))$. Then, one can see that for all i , the generalized equation (18.32) holds with $u_i = (p^\star, \widehat{q}_i)$. Therefore, the strategy profile $u = (u_1, \dots, u_M)$ is a variational equilibrium of the GNEP. \square

Existence of a solution of the optimization problem (18.38) is easily obtained using classical arguments. The feasible set is a non-empty, convex, and closed subset of a reflexive Banach space. In addition to the hypotheses of Proposition 18.3, assume that

$$\lim_{\|u\| \rightarrow +\infty} g(u) + \delta_{P^{\text{ad}} \times Q^{\text{opt}}}(u) = +\infty.$$

This ensures that the level sets of the objective functional are bounded over the feasible set. The objective functional is also proper lower semicontinuous and convex with full domain. Then, there exists a minimizer; see, e.g., [2, Theorem 2.11].

18.5 Test Cases: The Influence of Linepacking on Market Outcomes

The dynamics of transport in networks, as formalized in our analysis, capture important intertemporal aspects that are highly relevant in real-world market interactions. One important example arises in gas markets, where transport dynamics enable the network itself to function as a temporal storage device—a phenomenon referred to as linepacking. The importance of linepacking in the operation of real-world gas transport is documented, for instance, in [26, Page 212]. It has been estimated that for, e.g., the Great Britain national gas transmission system, the average hourly linepack stores energy of about 3743 GWh, but that the available linepack flexibility varies widely over the year [38]. In that study, the available within-day flexibility of the transmission system ranged from 13 GWh to 427 GWh.

In what follows, based on a stylized numerical test example, we use our framework to illustrate how the ability of linepacking can influence market outcomes in gas markets. Specifically, we investigate how linepacking is used for storage by market participants and who benefits to which degree.

To this end, we consider M identical and strategically acting firms that trade natural gas, which is transported over a single pipe. The time horizon of 4 h is split into two periods of 2 h each. Purchase cost $c(t)$ and inverse demand $P(t, q(t)) = \beta(t) + \alpha(t)q(t)$ do not change within each period. With $i \in \{1, 2\}$, we use c_i , α_i , and β_i to denote their values, which are given by

$$c(t) = \begin{cases} 388.8, & t \in [0, 2 \text{ h}), \\ 432.0, & t \in [2 \text{ h}, 4 \text{ h}]. \end{cases}$$

$$\alpha(t) = \begin{cases} -16.6, & t \in [0, 2 \text{ h}), \\ -20.7, & t \in [2 \text{ h}, 4 \text{ h}], \end{cases}$$

$$\beta(t) = \begin{cases} 1500.8, & t \in [0, 2 \text{ h}), \\ 2000.3, & t \in [2 \text{ h}, 4 \text{ h}], \end{cases}$$

The purchase cost of gas is 10 % cheaper in the first period and the inverse demand function $P(t, q(t))$ reflects higher demand in the second period. This choice of parameters thus incentivizes producers to use the pipe as a temporal storage for the gas acquired in the first period for its use in the second one to reduce the overall costs. Taking piecewise constant data allows us, moreover, to easily formulate a “static” counterpart of the considered scenario. For this counterpart, we associate a mass flow q_i and boundary pressures $(p_i^{\text{in}}, p_i^{\text{out}})$ for each period. We consider the flow to be stationary and that each triple $(q_i, p_i^{\text{in}}, p_i^{\text{out}})$ satisfies the Weymouth equation as a stationary approximation of gas flow in a pipe. Our choice of parameters ensures that no congestion occurs for the stationary flow. In the stationary case, this implies that the bounds on pressure and mass flow do not influence the market outcomes. As we will see, however, these bounds play an important role in the instationary model.

The physical parameters of the instance, inspired by [34], are as follows: The pipe has length $L = 50$ km, a cross-sectional diameter $D = 0.5$ m, a slope angle $\varphi = 0$, and its wall has a roughness $k = 1 \times 10^{-4}$ m. The gas temperature is $T_{\text{emp}} = 15$ °C, its molar mass is and $m = 0.016\,043$ kg mol $^{-1}$. The latter two values are those for methane, which usually dominates the composition of natural gas. With $R = 8.3145$ J mol $^{-1}$ K $^{-1}$ being the universal gas constant, the specific gas constant $R_s := R/m$ has value 518.26 J kg K $^{-1}$. The friction coefficient is modeled using the formula by Nikuradse, i.e., $\lambda = (2 \log_{10}(\frac{D}{k}) + 1.138)^{-2}$; see Page 25 in [26] for a discussion of this model.

The initial state is assumed to be stationary and is given by $q_0 = 48.289$ kg s $^{-1}$ and the pressure at the entry node being 50 bar. Remember that the pressure and mass flow admit, in a pointwise sense, upper (\bar{p}, \bar{q}) and lower $(\underline{p}, \underline{q})$ bounds in the pipe. The values for the last three bounds are fixed to

$$\underline{p} = 40 \text{ bar}, \quad \underline{q} = 10.9 \text{ kg s}^{-1}, \quad \bar{q} = 163 \text{ kg s}^{-1}.$$

The value of \bar{p} directly influences the maximal linepack capability in our setup and, thus, we will vary it in our test cases. By using the ideal gas law $p = c_s^2 \rho$, the total amount of gas in the pipe linearly depends on the integral of the pressure profile over the pipe. Thus, pressure is the variable we monitor when studying linepack effects. In our setup, we vary \bar{p} in the range of [50.1, 70.0] bar.

With this calibration, we solve the model described in (18.4). More precisely, we are in the setting of Proposition 18.3. The specific discretization of the PDE system (18.1) is described in Appendix C. To keep the presentation focused on the actual results, we moved the detailed problem formulations for this test case to Appendix B. There, we also present the modeling environments and solvers used for the computations.

Let us first present the time evolution of the physical quantities in the case of 3 firms. We start with the evolution of the mass flows at the endpoints of the pipe as shown in Fig. 18.1. For most parts of the time horizon, the output q^{out} is constant but has rather

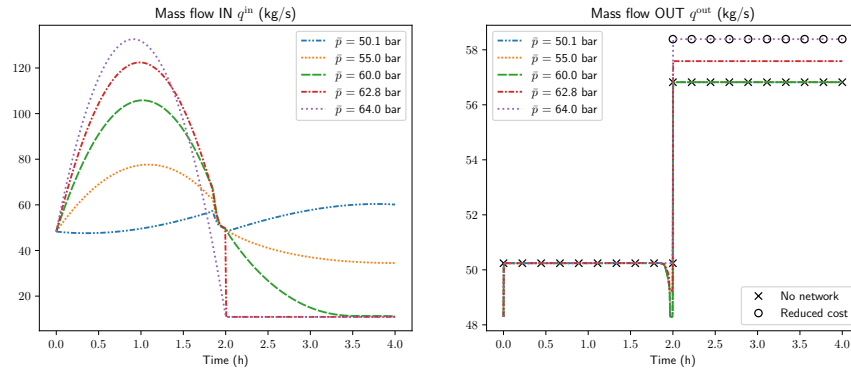


Fig. 18.1: Mass flow boundary controls q^{in} and q^{out} for different values of the upper pressure bound \bar{p} . On the right figure, the crosses indicate the optimal flow given by the static model. The circles indicate the optimal mass flow during the second period with the cost of gas being c_1 rather than c_2 also for the static model.

different values in each period. Each value is very close to the one obtained by the static model, which is indicated by the crosses in the figure. Two transitions are present: (i) at the beginning from the initial state to the desired output value and (ii) at the interface of the two time periods at $t = 2$ h. The latter has a drop in the outflow, which we will discuss later. The evolution of q^{in} is distinct for each value of \bar{p} . The general trend is that when \bar{p} increases, more gas is injected during the first period and less during the second one. This leads to the situation that only the minimum amount q is injected during or towards the end of the second period. The exception to this behavior is for $\bar{p} = 50.1$ bar, where the ability to steer the system from one operational point to another one is difficult due to the small gap between the initial pressure and its upper bound.

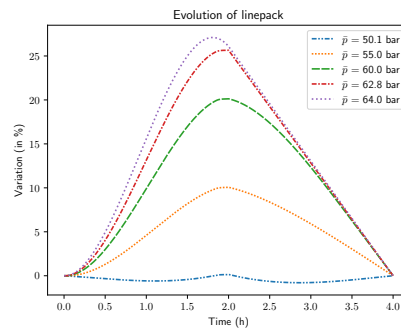


Fig. 18.2: Evolution of the linepack (total mass of gas in the pipe). The instance is the same as in Fig. 18.1.

In Fig. 18.2, the evolution of the linepack relative to its initial value is displayed. Except for $\bar{p} = 50.1$ bar, the curves are similar with an increase in the first period, leading to a maximum around the switching time of 2 h, which is finally followed by an almost linear decrease. A higher value of \bar{p} leads to a higher linepack peak. Any value \bar{p} greater than 64 bar yields the same result for this instance. As mentioned before, the linepack is related to the pressure profile in the pipe. Figure 18.3 displays the values

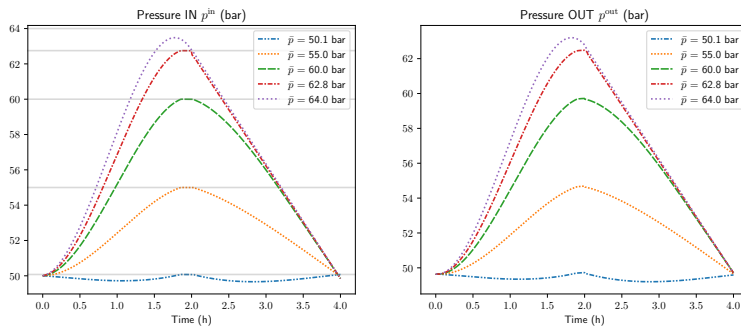


Fig. 18.3: Evolution of the pressure at the endpoints. The instance is the same as in Fig. 18.1.

of p^{in} and p^{out} over the time horizon. For a given \bar{p} , these pressures evolve similarly except for the period just before $t = 2$ h. Except for $\bar{p} = 64$ bar, p^{in} reaches a plateau close to its upper bound. Note that this is not the case for p^{out} or the linepack, which is due to the drop in q^{out} for this short time interval, leading to a higher linepack. This drop disappears when the upper pressure constraint is not active; see, e.g., the plot for $\bar{p} = 64$ bar.

We now move on to the economical interpretation of these results. First, note that the cost and demand structure indeed incentivizes linepacking. The case $\bar{p} = 50.1$ bar is an exception since the physical constraints appear to be too tight—remember that $p^{\text{in}}(0) = 50$ bar—so that linepacking is not observed to a significant degree. There, the transition between the different desired output values drives the decisions. With \bar{p} large enough, the quantity of gas stored during the first period covers the amount sold during the second one with $q^{\text{in}}(t) = \underline{q}$ for $t \in [2 \text{ h}, 4 \text{ h}]$. If more gas can be stored, this leads to an increased value of q^{out} . This lasts until q^{out} reaches the solution of the static problem with purchase cost c_1 rather than c_2 ; see the results for $\bar{p} = 64$ bar. Then, a further increase in the linepack ability does not yield any change.

Next, we look at the price of gas at the output node as shown in Fig. 18.4. There, as a reference, we display the purchase cost $c(t)$ and static strategic prices given by $c(t) + \beta M^{-1} q^{\text{out}}(t)$. These would occur if firms choose standard strategic markups for a given $q^{\text{out}}(t)$. Again, the two transitions induce price variations. Furthermore, the discontinuity of P at $t = 2$ h leads to a jump in prices. For low and moderate linepack capability, i.e., $\bar{p} \in \{50.1, 55.0, 60.0\}$ bar, pricing apparently takes place based on the current purchase cost c . Firms, rather than consumers, benefit from savings due to

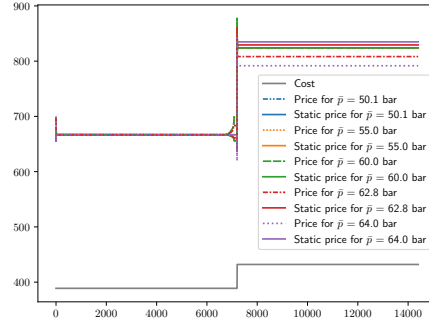


Fig. 18.4: Evolution of the “static” and actual price at the output node. The instance is the same as in Fig. 18.1.

linepacking. When the storage capacity is large, i.e., $\bar{p} \in \{62.8, 64.0\}$ bar, we observe a significant decrease of consumer prices. In these cases, the firms can purchase all the gas that they plan to sell minus the minimum amount induced by \underline{q} . The static price prediction, as the relevant marginal purchase cost, is not only determined by the purchase cost c_2 but also takes into account c_1 . If the upper pressure bound is never active, then c_1 is the only relevant purchase cost for the entire time horizon. Market prices in hours 2 to 4 are then also based on c_1 only.

To further understand how the market participants benefit from additional linepack ability, we look at the evolution of the aggregated firms’ profits, the consumer surplus, and the overall welfare over time if \bar{p} varies in $[51, 70]$ bar. In addition to the case of 3 firms, we solved the same instance in the case of perfect competition. To better see the different occurring effects, we split the overall firms profit into three components:

$$\begin{aligned}
 & \int_0^{4h} \left(P(t, q^{\text{out}}(t)) q^{\text{out}}(t) - c(t) q^{\text{in}}(t) \right) dt \\
 = & \int_0^{4h} \left(q^{\text{out}}(t) - q^{\text{in}}(t) \right) c(t) dt \\
 & + \int_0^{4h} \left(\alpha M^{-1} q^{\text{out}}(t) \right) q^{\text{out}}(t) dt \\
 & + \int_0^{4h} \left(P(t, q^{\text{out}}(t)) - \left(\alpha M^{-1} q^{\text{out}}(t) + c(t) \right) \right) q^{\text{out}}(t) dt \\
 =: & \boxed{\text{linepack savings}} + \boxed{\text{static strategic profits}} + \boxed{\text{network rent}}.
 \end{aligned} \tag{18.39}$$

The first term captures the reduction in total purchase cost thanks to linepacking that allows them to separate purchase and selling time. Next, we have profits obtained by the exercise of static market power. Last, the network rent captures the impact of the network on market prices. These are defined as the earnings obtained from the difference between the observed market price and the static price. In Fig. 18.5, the welfare and its two components are displayed. We can observe that there are two

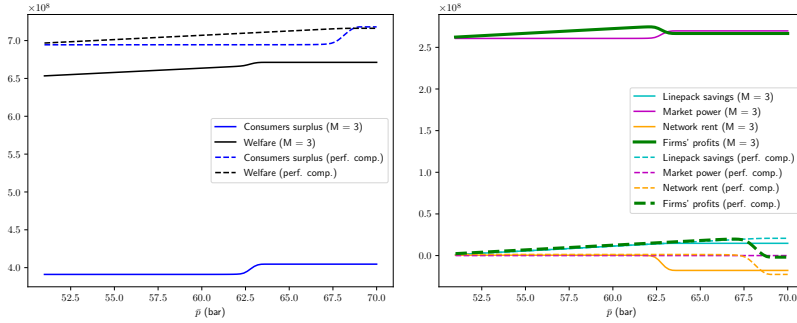


Fig. 18.5: Aggregated consumer surplus and welfare (left) and firms' profits partitioned in components (right) for the test instance in the case of 3 producers and under perfect competition.

different regimes with different solution structures: Low linepacking ability and high linepacking ability. As seen in Fig. 18.5 (left), if the linepacking ability is low, the consumer surplus remains constant, while the welfare increases with \bar{p} . This implies that firms' profits become larger, which is confirmed in Fig. 18.5 (right), where we see that the linepack savings are increasing. This structure of the solution stays the same for linepack abilities below a threshold of around $\bar{p} = 62$ bar for 3 firms and $\bar{p} = 67$ bar for the case of perfect competition. With \bar{p} above these values, the structure of the solution changes. In those cases additional linepack capability increases the availability of cheaper gas. As discussed previously, the market price in the second period is then based on both c_1 and c_2 . From this, the consumers benefit and their surplus increases, leading to higher welfare values. Finally, for very high linepack capability, purchase cost c_2 does no longer influence the market price at all, but it is exclusively determined based on the lower purchase cost c_1 , inducing lower market prices in the second period. This beneficial effect of linepacking, inducing lower market prices is observed for a larger range of linepack capabilities in case of strategically acting firms rather than for the perfectly competitive case. This is due to the fact that strategically behaved firms in general tend to sell less gas and smaller linepack capabilities allow to fully cover all sales by period 1 purchases. As a consequence of the drop in market prices for large linepack capabilities we observe a drop of firms profits in those cases, which is caused by network rents becoming negative in those cases; see Fig. 18.5 (right). By looking at its definition in (18.39), it is clear that as soon as the market price drops during the second period, this quantity becomes negative. As firms are forced to purchase a minimum amount of gas for all points in time t ($q^{\text{in}}(t) \geq \bar{q}$) in the case of perfect competition, the firms' overall profit even becomes negative for large \bar{p} .

We have thus seen that our setup is suitable to analyze dynamic network transport in a market environment and thus contributes to the understanding of phenomena such as linepacking, which is highly important in the context of gas markets. We are able to analyze how the ability of linepacking can influence market outcomes in gas markets and which market participants benefit to which degree. Interestingly, for low linepacking

capabilities mainly firms can increase their profits due to an improved purchase cost structure. This reverts in case of high linepack capabilities where reduced market prices induce a significant increase of consumer surplus.

18.6 Conclusion

We investigated a stylized market situation in which strategic suppliers transport and sell a commodity (e.g., natural gas or hydrogen) to consumers. This has been modeled as a GNEP in which the evolution equations for the gas flow are shared constraints. We showed the well-posedness of this system of PDEs with a solution regular enough to ensure the existence of Lagrange multipliers. Then, we proved the existence of a variational equilibrium of the GNEP, which is consistent with the continuity of pressures at the nodes. When producers are identical, we identified an equivalent optimization problem in the sense that a variational equilibrium can be constructed from any solution to the latter. We illustrated the insights gained via such a PDE model compared to a static one. In particular, we highlighted how linepacking impacts the market outcomes and determined who benefits from additional linepacking ability. For this purpose, we considered a simple instance in which the time horizon is split into two periods and gas is cheaper during the first one. While the quantity of gas sold is most of the time close to the solution of the static problem, the inflow is rather different and changes with the linepacking ability. Indeed, the producers seek to reduce their overall cost by storing cheap gas as much as possible in the pipe. There, linepacking only benefits the producers and this lasts until it is possible to store enough gas to cover the sale of the second period while fulfilling the flow constraints. Then, the amount sold during that time interval increases until it reaches the value given by the solution of the static problem for the second period, but with the cost of the first one. This indicates a decoupling between injection and withdrawal of gas from the pipe and induces an increase of the consumer surplus as well as a decrease of the firms' profit, while the overall welfare is still increasing.

These results encourage further investigation of such models. On the one hand, other models for the evolution of the gas flow can be considered; see, e.g., [12]. In this case, the shared constraint consists of a semilinear system of hyperbolic PDEs. On the other hand, one can depart from the deterministic setting and investigate the market outcomes when uncertainties are present. These could originate either from the physical model, where, e.g., the friction coefficient is unknown, or from the economical data such as demand or supply.

A Fundamental Results

In this section, we review some fundamental results from convex optimization and the theory of PDEs. For our work, we need to prove that our problems have solutions of high regularity (see Sect. 18.3) to ensure the existence of multipliers of the state

constraints. For this, we need to review some results from PDE theory, starting with the Aubin–Lions lemma.

Lemma 18.5 (Aubin–Lions lemma; [36, Corollary 4]) *Let X_0 , X , and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1 . For $1 \leq p, q \leq +\infty$, let*

$$W := \{u \in L^p(0, T; X_0) \mid u_t \in L^q(0, T; X_1)\}.$$

- (i) *If $p < +\infty$, then the embedding of W into $L^p(0, T; X)$ is compact.*
(ii) *If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0, T]; X)$ is compact.*

We use this lemma with $X_0 = H_0^1(\Omega)^2$, $X = C(\bar{\Omega})^2$, and $X_1 = H^{-1}(\Omega)^2$. The proof of additional regularity of the solution to the PDE system uses the Ladyzhenskaya–Sobolevski inequality for the Laplacian. For convenience, we provide a proof of this result and also state its main ingredient: the interpolation inequalities of Gagliardo–Nirenberg.

Lemma 18.6 (Interpolation inequalities of Gagliardo–Nirenberg; [5, p. 234]) *Let I be an open and bounded interval. Then, there exists a constant $C > 0$ such that for any $v \in H^2(I)$, we have*

$$\|\nabla v\|_{L^2(I)} \leq C \|v\|_{H^2(I)}^{1/2} \|v\|_{L^2(I)}^{1/2}, \quad (18.40)$$

and for a given $\kappa > 0$, there exists C_κ such that

$$\|\nabla v\|_{L^2(I)} \leq \|v\|_{H^1(I)} \leq \kappa \|v\|_{H^2(I)} + C_\kappa \|v\|_{L^2(I)}. \quad (18.41)$$

We can now state and prove the Ladyzhenskaya–Sobolevski inequality for the Laplacian.

Lemma 18.7 (Ladyzhenskaya–Sobolevski inequality for Δ) *Let I be an open and bounded interval. Then, there exists a constant $C > 0$ such that for all $v \in H^2(I)$, we have*

$$(\Delta v, \Delta v)_{L^2(I)} \geq \frac{1}{2} \|v\|_{H^2(I)}^2 - C \|v\|_{L^2(I)}^2.$$

Proof. From Inequality (18.40) in Lemma 18.6, we have

$$\|\nabla v\|_{L^2(I)}^2 \leq C' \|v\|_{H^2(I)} \|v\|_{L^2(I)}.$$

Using Cauchy’s inequality, there exists $C'' > 0$ such that

$$\|\nabla v\|_{L^2(I)}^2 \leq \frac{1}{2} \|v\|_{H^2(I)}^2 + C'' \|v\|_{L^2(I)}^2$$

holds and by further using the definition of the H^2 -norm as well as by rearranging terms, we obtain

$$\begin{aligned}\|\Delta v\|_{L^2(I)}^2 &\geq \|\nabla v\|_{L^2(I)}^2 + \|v\|_{L^2(I)}^2 - 2(C'' + 1)\|v\|_{L^2(I)}^2, \\ 2\|\Delta v\|_{L^2(I)}^2 &\geq \|v\|_{H^2(I)}^2 - 2(C'' + 1)\|v\|_{L^2(I)}^2.\end{aligned}$$

Finally, we get

$$(\Delta v, \Delta v)_{L^2(I)} \geq \frac{1}{2}\|v\|_{H^2(I)}^2 - (C'' + 1)\|v\|_{L^2(I)}^2. \quad \square$$

B Optimization Models Used for the Test Instance

In Sect. 18.5, all producers are identical. In this case, we can solve the optimization problem given in Proposition 18.3. Remember that the price functional in the symmetric case is given by

$$\tilde{P}(t, q(t)) = \beta(t) - \frac{M+1}{2M}\alpha(t)q(t).$$

With $u = (q^{\text{in}}, q^{\text{out}}, p^{\text{in}}, p^{\text{out}})$, the optimization problem associated with the symmetric case reads

$$\max_{u \in H^1(0, T)^4} \int_0^T \left(\tilde{P}(t, q^{\text{out}}(t)) q^{\text{out}}(t) - c(t)q^{\text{in}}(t) \right) dt - \mathcal{R}(p, q, u) \quad (18.42a)$$

$$\text{s.t. } p_t - \varepsilon p_{xx} + \frac{c_s^2}{A} q_x = 0 \quad \text{in } (0, L) \times (0, T), \quad (18.42b)$$

$$q_t - \varepsilon q_{xx} + A p_x = -\frac{1}{2}\sigma A \left(\frac{2\check{q}}{\check{p}} q - \frac{\check{q}^2}{\check{p}^2} p \right) \quad \text{in } (0, L) \times (0, T), \quad (18.42c)$$

$$p(0, \cdot) = p^{\text{in}}, \quad p(L, \cdot) = p^{\text{out}} \quad \text{in } (0, T), \quad (18.42d)$$

$$q(0, \cdot) = q^{\text{in}}, \quad q(L, \cdot) = q^{\text{out}} \quad \text{in } (0, T), \quad (18.42e)$$

$$p(\cdot, 0) = p_0, \quad q(\cdot, 0) = q_0 \quad \text{in } (0, L), \quad (18.42f)$$

$$\underline{p} \leq p \leq \bar{p}, \quad \underline{q} \leq q \leq \bar{q} \quad \text{in } [0, L] \times [0, T], \quad (18.42g)$$

$$p_0(0) = p^{\text{in}}(0), p_0(L) = p^{\text{out}}(0), q_0(0) = q^{\text{in}}(0), q_0(L) = q^{\text{out}}(0), \quad (18.42h)$$

$$\int_0^T q^{\text{out}}(t) - q^{\text{in}}(t) dt \leq 0. \quad (18.42i)$$

Thanks to the regularity of the solution to the system in (18.42b)–(18.42f), any feasible (boundary) controls also satisfy the same bounds as in (18.42g) pointwise in time. As in our instance the lower and upper bounds at the nodes are identical to the ones in the pipe, we do not add explicit bounds at the nodes. The compatibility conditions between the initial state and the decision variables are given in (18.42h). Remember that Inequality (18.42i) ensures that the market participants do not get gas for free out of the initial state. The regularization functional \mathcal{R} has the structure

$$\mathcal{R}(p, q, u) = \gamma_1^p \|\nabla p^{\text{in}}\|_{L^2(0, T)^2}^2 + \gamma_1^q \|\nabla q^{\text{in}}\|_{L^2(0, T)^2}^2 + \eta_1 \|\nabla q(\cdot, T)\|_{L^2(\Omega)}^2 \quad (18.43)$$

with non-negative coefficients. By the regularity of the solution, we have $q \in C([0, T]; H^1(\Omega))$, which ensures that $\nabla q(\cdot, T) \in L^2(\Omega)$. The last term aims at avoiding unrealistic final states. For any sequence $u_n \in H^1(0, T)^4$ such that $\|\nabla u_n\|_{L^2(0, T)^4} \rightarrow +\infty$, we have $\lim \mathcal{R}(S(u_n), u_n) = +\infty$. As any feasible control of Problem (18.42) is bounded in $L^2(0, T)^4$, the level sets of the objective functional are bounded. The latter is also lower semicontinuous, proper, and its subdifferential has full domain. Furthermore, the feasible set is non-empty, closed, and convex. Thus, there exists a solution of Problem (18.42). In order to discretize System (18.42), we use a discretization scheme that is motivated by analogous schemes for the semilinear Euler equations; for details, see Appendix C. It better captures the problem structure than a generic discretization scheme for parabolic PDE.

Problem (18.42) is parametric in the linearization point (\check{p}, \check{q}) . Inspired by sequential programming approaches, we seek a tuple (u^*, p^*, q^*) such that $(u^*, p^*, q^*) \in \text{SOL}(p^*, q^*)$, where SOL is the solution mapping of Problem (18.42). Starting with $(\bar{p}^0, \bar{q}^0) = (p_0(0), q_0)$, a sequence of solutions $(\bar{u}^k, \bar{p}^k, \bar{q}^k) \in \text{SOL}(\bar{p}^{k-1}, \bar{q}^{k-1})$ is computed. Whenever the relative difference between $(\bar{p}^{k-1}, \bar{q}^{k-1})$ and (\bar{p}^k, \bar{q}^k) falls below a given threshold, the procedure stops and a solution is returned.

The corresponding stationary model is given as follows. The time interval $[0, T]$ is partitioned into J subintervals $[t_k, t_{k+1})$. The flow q is constant along the pipe and the pressure monotonically changes according to the Weymouth equation

$$p^2(x) = p^2(0) - \sigma q |q| x;$$

see Sect. 2.3.1.2 in [26] for a derivation of this relation. Hence, for each time subinterval, the state consists of the scalar flow q_k and the endpoint pressures p_k^{in} and p_k^{out} . With α_k , β_k , and c_k being the constant data of the inverse demand and cost functions on each subinterval $[t_k, t_{k+1})$, we obtain the problem

$$\begin{aligned} \max_{q, p^{\text{in}}, p^{\text{out}}} \quad & \sum_{k=1}^J (t_{k+1} - t_k) [(\beta_k - (M^{-1} + 1)\alpha_k q_k) q_k - c_k q_k] \\ \text{s.t.} \quad & (p_k^{\text{out}})^2 = (p_k^{\text{in}})^2 - \Lambda q_k^2, \quad 1 \leq k \leq J, \\ & \underline{p} \leq p_k^{\text{in}} \leq \bar{p}, \quad \underline{p} \leq p_k^{\text{out}} \leq \bar{p}, \quad \underline{q} \leq q_k \leq \bar{q}, \quad 1 \leq k \leq J, \end{aligned} \quad (18.44)$$

with $\Lambda := \sigma L$. As q_k is constant on each time subintervals, Constraint (18.42i) is automatically satisfied.

We use two different computational setups for the PDE-constrained and the stationary problem. The discretized version of Problem (18.44) is a convex QP. The latter is solved using the barrier algorithm of Gurobi 9.5.0; see [17]. We use JuMP as the modeling environment; see [14]. The complete instance data as well as the code to perform the numerical computations and create the figures can be found at [23]. The nonconvex stationary problem (18.44) is solved by Ipopt 3.14.1 via the Pyomo modeling environment; see [41] and [18], respectively.

C Discretization Scheme

Coming back to Problem (18.42), as a first step towards solving such instances, we pursue a discretize-then-optimize approach. For the discretization of the system in (18.42b)–(18.42f), rather than using a generic scheme for parabolic PDEs, we retain a numerical scheme that can be related to the linearized version of the ISO2 system. Indeed, the relations in (18.42b)–(18.42c) are obtained from the semilinear Euler equations

$$\begin{aligned} p_t + \frac{c_s^2}{A} q_x &= 0, \\ q_t + A p_x &= -\frac{1}{2} \sigma A \frac{q|q|}{p} - g A \sin \varphi \frac{p}{c_s^2}, \end{aligned}$$

by linearizing the right-hand side and adding a viscosity term. Using the Riemann invariants $R_{\pm} := c_s A^{-1} q \pm p$, we obtain the diagonal system

$$\begin{aligned} (R_+)_t + c_s (R_+)_x &= F(R_+, R_-), \\ (R_-)_t - c_s (R_-)_x &= F(R_+, R_-), \end{aligned} \quad (18.45)$$

in which the coupling between R_+ and R_- is through the right-hand side

$$F(R_+, R_-) := -\frac{\lambda c_s}{4D} (R_+ + R_-) \frac{|R_+ + R_-|}{R_+ - R_-} - g \sin \varphi \frac{R_+ - R_-}{2c}.$$

This diagonal form is useful to investigate theoretical properties of the system as well as for developing numerical schemes; see [15]. For the relations in (18.42b)–(18.42c), the same change of variables results in

$$\begin{aligned} (R_+)_t - \varepsilon (R_+)_{xx} + c_s (R_+)_x &= F_{\text{aff}}(R_+, R_-), \\ (R_-)_t - \varepsilon (R_-)_{xx} - c_s (R_-)_x &= F_{\text{aff}}(R_+, R_-), \end{aligned} \quad (18.46)$$

where F_{aff} is the affine approximation of F around the operational points $\check{p} > 0$ and $\check{q} > 0$.

Let us first discretize the linearized version of the hyperbolic system (18.45), where F has been substituted by F_{aff} . Specifically, we integrate R_{\pm} along the characteristic lines using a triangular grid as illustrated on Fig. 18.6. There, the solid line from node i to node k (resp. from j to k) represents the characteristic line for R_+ (resp. R_-). The characteristics passing through a given pair (\bar{x}, \bar{t}) are the images of the mappings $\xi_+(s, \bar{x}, \bar{t}) := (\bar{x} + c_s(s - \bar{t}), s)$ and $\xi_-(s, \bar{x}, \bar{t}) := (\bar{x} - c_s(s - \bar{t}), s)$, respectively. Along the characteristics, System (18.45) is compactly given by

$$\frac{d}{ds} (R_{\pm} \circ \xi_{\pm}) = (R_{\pm})_t \circ \xi_{\pm} + c_s (R_{\pm})_x \circ \xi_{\pm} = F(R_+, R_-) \circ \xi_{\pm}. \quad (18.47)$$

Let us denote by R_{\pm}^i , R_{\pm}^j , and R_{\pm}^k the values of the Riemann invariants at the nodes i , j , and k . We have $\Delta x = x_l - x_{l-1} = x_{l+1} - x_l$, $\Delta t = t_{n+1} - t_n$, and $\Delta x = c_s \Delta t$ as well as $R_+^k = R_+(\xi_+(t_n + \Delta t, x_{l-1}, t_n))$ and $R_-^k = R_-(\xi_-(t_n + \Delta t, x_{l+1}, t_n))$. Integrating

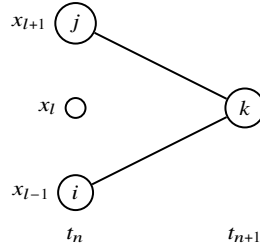


Fig. 18.6: Triangular grid used for integration

System (18.47) along the characteristics starting from node i (node j) for R_+ (R_-) and using the midpoint rule for approximating the right-hand side yields

$$\begin{aligned} R_+^k - R_+^i &= \frac{\Delta t}{2} [F_{\text{aff}}(R_+^i, R_-^i) + F_{\text{aff}}(R_+^k, R_-^k)], \\ R_-^k - R_-^j &= \frac{\Delta t}{2} [F_{\text{aff}}(R_+^j, R_-^j) + F_{\text{aff}}(R_+^k, R_-^k)]. \end{aligned}$$

Next, we relate this discretized system with one obtained by discretizing the viscosity regularized system. We proceed just with R_+ as the reasoning is similar with R_- . We discretize the left-hand side of the first relation in (18.46) using a finite difference scheme with $\Delta x = c_s \Delta t$ and $\varepsilon = c_s \Delta x$. Using a central difference formula for the second-order term, one obtains

$$\begin{aligned} & \frac{R_+^k - R_+(x_l, t_n)}{\Delta t} - c_s \Delta x \frac{R_+^j + R_+^i - 2R_+(x_l, t_n)}{(\Delta x)^2} + c_s \frac{R_+^j - R_+(x_l, t_n)}{\Delta x} \\ &= \frac{R_+^k - R_+(x_l, t_n)}{\Delta t} - \frac{R_+^j + R_+^i - 2R_+(x_l, t_n)}{\Delta t} + \frac{R_+^j - R_+(x_l, t_n)}{\Delta t} \\ &= \frac{R_+^k - R_+^i}{\Delta t}. \end{aligned}$$

With our choice of the right-hand side, the values of the Riemann invariants at (x_l, t_n) do not appear in any relation. This gives the grid its triangular shape.

Acknowledgements This work was supported by the German Research Foundation (DFG) within Subprojects A05, B02, B07, B08, and B09 in the Sonderforschungsbereich/Transregio 154 ‘‘Mathematical Modelling, Simulation and Optimization using the Example of Gas Networks’’ (Project ID 239904186).

This research has been performed as part of the Energie Campus Nurnberg and is supported by funding of the Bavarian State Government.

References

1. R. Adams and J. Fournier. *Sobolev Spaces*. Academic Press, 2 edition, 2003.
2. V. Barbu and T. Precupanu. *Convexity and Optimization in Banach Spaces*. Springer Monographs in Mathematics. Springer Netherlands, 4 edition, 2012. doi:10.1007/978-94-007-2247-7.
3. H. Benckekroun. Unilateral production restrictions in a dynamic duopoly. *Journal of Economic Theory*, 111(2):214–239, 2003. doi:10.1016/S0022-0531(03)00090-5.
4. H. Benckekroun. Comparative dynamics in a productive asset oligopoly. *Journal of Economic Theory*, 138(1):237–261, 2008. doi:10.1016/j.jet.2006.10.012.
5. H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, 2010. doi:10.1007/978-0-387-70914-7.
6. G. Caruana and L. Einav. Production targets. *The RAND Journal of Economics*, 39(4):990–1017, 2008. doi:10.1111/j.1756-2171.2008.00047.x.
7. X. Chen and Z. Wang. Differential variational inequality approach to dynamic games with shared constraints. *Mathematical Programming*, 146(1):379–408, 2014. doi:10.1007/s10107-013-0689-1.
8. L. Colombo and P. Labrecciosa. On the Markovian efficiency of Bertrand and Cournot equilibria. *Journal of Economic Theory*, 155:332–358, 2015. doi:10.1016/j.jet.2014.11.007.
9. L. Colombo and P. Labrecciosa. Stackelberg versus Cournot: A differential game approach. *Journal of Economic Dynamics and Control*, 101:239–261, 2019. doi:10.1016/j.jedc.2018.10.007.
10. A. Cournot. *Recherches sur les principes mathématiques de la théorie des richesses*. Hachette Paris, 1838.
11. R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology: Evolution Equations I*. Springer Berlin Heidelberg, 2000. doi:10.1007/978-3-642-58090-1.
12. P. Domschke, B. Hiller, J. Lang, V. Mehrmann, R. Morandin, and C. Tischendorf. Gas network modeling: An overview. Technical report, Technische Universität Darmstadt, 2021. URL: <https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/411>.
13. A. Dreves and J. Gwinner. Jointly convex generalized Nash equilibria and elliptic multiobjective optimal control. *Journal of Optimization Theory and Applications*, 168(3):1065–1086, 2016. doi:10.1007/s10957-015-0788-7.
14. I. Dunning, J. Huchette, and M. Lubin. JuMP: A modeling language for mathematical optimization. *SIAM Review*, 59(2):295–320, 2017. doi:10.1137/15M1020575.
15. M. Gugat, J. Habermann, M. Hintermüller, and O. Huber. Constrained exact boundary controllability of a semilinear model for pipeline gas flow. *European Journal of Applied Mathematics*, 34(3):532–553, 2023. doi:10.1017/S0956792522000389.
16. M. Gugat and S. Steffensen. Dynamic boundary control games with networks of strings. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(4):1789–1813, 2018. doi:10.1051/cocv/2017082.
17. Gurobi Optimization, LLC. Gurobi Optimizer Reference Manual, 2021. URL: <https://www.gurobi.com>.
18. W. E. Hart, C. D. Laird, J.-P. Watson, D. L. Woodruff, G. A. Hackebeil, B. L. Nicholson, and J. D. Sirola. *Pyomo – Optimization Modeling in Python*, volume 67 of *Springer Optimization and Its Applications*. Springer, 2017. doi:10.1007/978-3-319-58821-6.
19. M. Hintermüller and T. Surowiec. A PDE-constrained generalized Nash equilibrium problem with pointwise control and state constraints. *Pacific Journal on Optimization*, 9(2):251–273, 2013.
20. M. Hintermüller, T. Surowiec, and A. Kämmler. Generalized Nash equilibrium problems in Banach spaces: Theory, Nikaido–Isoda-based path-following methods, and applications. *SIAM Journal on Optimization*, 25(3):1826–1856, 2015. doi:10.1137/14096829X.
21. M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints*, volume 23 of *Mathematical Modelling: Theory and Applications*. Springer Netherlands, 2009. doi:10.1007/978-1-4020-8839-1.

22. P. Holmberg and A. B. Philpott. On supply-function equilibria in radial transmission networks. *European Journal of Operational Research*, 271(3):985–1000, 2018. doi:10.1016/j.ejor.2018.05.070.
23. O. Huber. PDE-Cournot-code, 2023. URL: <https://github.com/xhub/PDE-Cournot-code>, doi:10.5281/zenodo.7702535.
24. B. Jun and X. Vives. Strategic incentives in dynamic duopoly. *Journal of Economic Theory*, 116(2):249–281, 2004. doi:10.1016/j.jet.2003.08.005.
25. C. Kanzow, V. Karl, D. Steck, and D. Wachsmuth. The multiplier-penalty method for generalized Nash equilibrium problems in Banach spaces. *SIAM Journal on Optimization*, 29(1):767–793, 2019. doi:10.1137/17M114114X.
26. T. Koch, B. Hiller, M. E. Pfetsch, and L. Schewe, ed. *Evaluating Gas Network Capacities*. Society for Industrial and Applied Mathematics, 2015. doi:10.1137/1.9781611973693.
27. L. Lambertini and A. Palestini. On the feedback solutions of differential oligopoly games with hyperbolic demand curve and capacity accumulation. *European Journal of Operational Research*, 236(1):272–281, 2014. doi:10.1016/j.ejor.2013.12.008.
28. A. Ledvina and R. Sircar. Dynamic Bertrand oligopoly. *Applied Mathematics & Optimization*, 63(1):11–44, 2011. doi:10.1007/s00245-010-9110-0.
29. P. L. Lions. Une inégalité pour les opérateurs elliptiques du second ordre. *Annali di Matematica Pura ed Applicata*, 127(1):1–11, 1981. doi:10.1007/bf01811715.
30. J. F. Nash. Equilibrium points in n -person games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950. doi:10.1073/pnas.36.1.48.
31. M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*, volume 13. Springer-Verlag, 2004. doi:10.1007/b97427.
32. S. S. Reynolds and B. J. Wilson. Bertrand–Edgeworth competition, demand uncertainty, and asymmetric outcomes. *Journal of Economic Theory*, 92(1):122–141, 2000. doi:10.1006/jeth.1999.2624.
33. D. A. Schiro, J.-S. Pang, and U. V. Shanbhag. On the solution of affine generalized Nash equilibrium problems with shared constraints by Lemke’s method. *Mathematical Programming*, 142(1):1–46, 2013. doi:10.1007/s10107-012-0558-3.
34. M. Schmidt, D. Aßmann, R. Burlacu, J. Humpola, I. Joormann, N. Kanelakis, T. Koch, D. Oucharif, M. Pfetsch, L. Schewe, R. Schwarz, and M. Sirvent. GasLib – A library of gas network instances. *Data*, 2(4):40, 2017. doi:10.3390/data2040040.
35. C. Schwab and R. Stevenson. Space-time adaptive wavelet methods for parabolic evolution problems. *Mathematics of Computation*, 78(267):1293–1318, 2009. doi:10.1090/s0025-5718-08-02205-9.
36. J. Simon. Compact sets in the space $L^p(0, T; B)$. *Annali di Matematica Pura ed Applicata*, 146(1):65–96, 1986. doi:10.1007/bf01762360.
37. J.-Y. Wei and Y. Smeers. Spatial oligopolistic electricity models with Cournot generators and regulated transmission prices. *Operations Research*, 47(1):102–112, 1999. doi:10.1287/opre.47.1.102.
38. G. Wilson and P. Rowley. Flexibility in Great Britain’s gas networks: analysis of linepack and linepack flexibility using hourly data. Briefing note, UK Energy Research Centre, 2019. URL: <https://ukerc.ac.uk/publications/linepack/>.
39. R. Wilson. Supply function equilibrium in a constrained transmission system. *Operations Research*, 56(2):369–382, 2008. doi:10.1287/opre.1070.0421.
40. F. Wirl. Dynamic demand and noncompetitive intertemporal output adjustments. *International Journal of Industrial Organization*, 28(3):220–229, 2010. doi:10.1016/j.ijindorg.2009.08.002.
41. A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106(1):25–57, 4 2005. doi:10.1007/s10107-004-0559-y.
42. B. Xin and M. Sun. A differential oligopoly game for optimal production planning and water savings. *European Journal of Operational Research*, 269(1):206–217, 2018. doi:10.1016/j.ejor.2017.07.016.