The alternating simultaneous Halpern–Lions–Wittmann–Bauschke algorithm for finding the best approximation pair for two disjoint intersections of convex sets

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Abstract: Given two nonempty and disjoint intersections of closed and convex subsets, we look for a best approximation pair relative to them, i.e., a pair of points, one in each intersection, attaining the minimum distance between the disjoint intersections. We propose an iterative process based on projections onto the subsets which generate the intersections. The process is inspired by the Halpern-Lions-Wittmann-Bauschke algorithm and the classical alternating process of Cheney and Goldstein, and its advantage is that there is no need to project onto the intersections themselves, a task which can be rather demanding. We prove that under certain conditions the two interlaced subsequences converge to a best approximation pair. These conditions hold, in particular, when the space is Euclidean and the subsets which generate the intersections are compact and strictly convex. Our result extends the one of Aharoni, Censor and Jiang ["Finding a best approximation pair of points for two polyhedra", *Computational Optimization and Applications* 71 (2018), 509–523] which considered the case of finite-dimensional polyhedra.

Keywords: Alternating algorithm, best approximation pair, Dini's Theorem, disjoint intersections, projection methods, Simultaneous Halpern–Lions–Wittmann–Bauschke (S-HLWB) algorithm.

Contents

1. Introduction

1.1. Background:

We consider the problem of finding a best approximation pair relative to two closed and convex disjoint sets A and B, namely a pair $(a, b) \in A \times B$ that attains the distance $\|a - b\| = \inf \{\|x - y\| \mid x \in A,\ y \in B\}$. This problem goes back to the classical 1959 work of Cheney and Goldstein [15] which employs proximity maps (i.e., metric projections, Euclidean nearest point projections) and is at the heart of what is nowadays called "finding best approximation pairs relative to two sets".

We consider the important situation where each of the two sets is a nonempty intersection of a finite family of compact and convex sets and the space is Euclidean (i.e., a finite-dimensional Hilbert space). The practical importance of this situation stems from its relevance to real-world situations wherein the feasibility-seeking modelling is used and there are two disjoint constraints sets. One set represents "hard" constraints, i.e., constraints that must be met, while the other set represents "soft" constraints which should be observed as much as possible. Under such circumstances, the desire to find a point in the hard constraints intersection set that will be closest to the intersection set of soft constraints leads to the problem of finding a best approximation pair of the two sets which are intersections of constraints sets: see, e.g., [18] and [26], and also the more general approach [16], for applications in signal processing.

One way to approach the best approximation pair problem is to project alternatingly onto each of the two intersections and then to take the limit, as done, for instance, in [4, Section 3], [5, Theorem 4.8], [15, Theorem 4], [20, Theorem 4.1], [21, Theorem 1.4] and [26, Theorem 1], or to apply iteratively other kinds of operators (related to orthogonal projections), as done in [8, Theorem 3.13], [18, p. 656] and [23, Corollary 2.8]; see also the review [13]. While this approach is satisfying from the mathematical point of view, it can be rather problematic from the computational point of view because applying a metric projection operator to the intersection of finitely-many sets is commonly a non-trivial problem on its own which might be solved only approximately. (We note, parenthetically, that there exist, of course, iterative methods which can find asymptotically, and sometimes exactly, a projection of a point onto the intersection of a finite family of closed and convex subsets: see, for instance, [6, 7, 10, 12, 14] and some of the references therein).

We approach the best approximation pair problem from a constructive algorithmic point of view. A major advantage of our approach is that it eliminates the need to project onto each intersection, and replaces these computationally demanding projections by a certain weighted sum of projections onto each of the members which induce the intersection set. The method that we employ is a simultaneous version of the HLWB algorithm of [7, Corollary 30.2], which we apply alternatingly to the two intersections (the letters HLWB are acronym of the names of the authors of the corresponding papers: Halpern in [19], Lions in [22], Wittmann in [25], and Bauschke in [3]; the acronym HLWB was dubbed in [11]).

More precisely, the iterative process that we consider is divided into sweeps, where in the odd numbered sweeps we project successively onto a collection of compact and convex subsets A_i defining $A := \bigcap_{i=1}^{I} A_i \neq \emptyset$, $I \in \mathbb{N}$, and construct from all of these projections certain weighted sums, and in even numbered sweeps we act similarly but with the collection of compact and convex subsets B_j defining $B := \bigcap_{j=1}^{J} B_j \neq \emptyset$, $J \in \mathbb{N}$, where $A \bigcap B = \emptyset$. An important component in the method is that the number of successive weighted sums increases from sweep to sweep. Under the assumption that there exists a unique best approximation pair, we are able to show that our algorithmic scheme converges to this pair. This assumption, of uniqueness of the best approximation pair, is satisfied, in particular, when all the sets A_i, B_j, $i \in \{1, 2, \ldots, I\}$ and $j \in \{1, 2, \ldots, J\}$ are strictly convex, as we show in Proposition 16 below.

Our work is motivated by the work of Aharoni et al. [1], in which the authors present an algorithm which looks for a best approximation pair relative to given two disjoint convex polyhedra. In this work they propose a process based on projections onto the half-spaces defining the two polyhedra, and this process is essentially a sequential (that is, non-simultaneous) alternating version of the HLWB algorithm. While the given subsets in their case are essentially linear (affine), in our case they are essentially nonlinear (non-affine); on the other hand, the subsets in [1] are not necessarily bounded as in our case, and no uniqueness assumption on the set of best approximation pairs is imposed in [1] like is done here; hence our work extends, but does not generalize, their work.

The proof of our main convergence result (Theorem 32 below) is partly inspired by the proof of convergence of the main result in [1], namely [1, Theorem 1], but important differences exist because the settings are different. In particular, along the way we present and use a simple but useful generalization of the celebrated Dini's theorem for uniform convergence (Proposition 24 below).

We note that also [9] is motivated by [1] and extends it to closed and convex subsets which are not just polyhedra. The authors of [9] study the Douglas–Rachford algorithm, a dual-based proximal method, a proximal distance algorithm, and a stochastic subgradient descent approach. The presentation in [9] is based on reformulating the problem into a minimization problem and applying various approaches to it, theoretically and experimentally, without convergence results (see also [17] for a dual reformulation of the best approximation problem as a maximization problem, with some explicit examples), whereas our present work maintains the original problem formulation and employs tools from fixed point theory.

1.2. Paper layout:

The paper is organized as follows. In Section 2 we give definitions and preliminaries. In Section 3 we present a sufficient condition which ensures that there is a unique best approximation pair. In Section 4 we define the Simultaneous-HLWB (S-HLWB) operator and introduce some of its features. In Section 5 we present our new method for finding a best approximation pair in a finite-dimensional Hilbert space. In Section 6 we prove our main result, namely the convergence of the alternating S-HLWB algorithmic sequence.

2. Preliminaries

For the reader's convenience we include in this section some properties of operators in Hilbert space that will be used in the sequel. Although our setting is a Euclidean space, many of the definitions and results mentioned in this section can be generalized to infinite-dimensional spaces. We use the excellent books of Bauschke and Combettes [7] and of Cegielski [10] as our desk-copy in which all the results of this section can be found. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $X \subseteq \mathcal{H}$ be a nonempty, closed and convex subset. Let C be a nonempty closed and convex subset of $\mathcal{H}$, let $x \in \mathcal{H}$, and let $c \in C$. Denote the distance from x to C by $d(x, C) := \inf_{c \in C} \|x - c\|$, it is well-known that the infimum is attained at a unique point called the **projection (or the orthogonal projection) of x onto C,** and it is denoted by $P_C(x)$. Id stands below for the unit operator and riC stands for the relative interior of a set C. We recall that $dist\,(A,B) := \inf\{\|u-v\| \mid u \in A, v \in B\}$ whenever A and B are nonempty subsets of X, and we say that $dist\,(A,B)$ is attained whenever there are $a \in A$ and $b \in B$ such that $\|a-b\| = dist\,(A,B)$; in this case we call (a,b) a **best approximation pair relative to A and B**. It is straightforward to check that in this case, if in addition A and B are closed and convex, then $a = P_A b$ and $b = P_B a$.

Let r be a nonengative number. The closed ball $B\,[x,r]$ is defined by

$$B\,[x,r] := \{y \in \mathcal{H} \mid \|x-y\| \leq r\}\,, \tag{2.1}$$

for each $x \in \mathcal{H}$.

It is well-known that closed balls in a Euclidean space are compact, see, e.g., [7, Fact 2.33]. For each $m \in \mathbb{N}$ we denote by Δ_m the $(m-1)$-dimensional unit simplex, that is $\Delta_m := \{u \in \mathbb{R}^m \mid u \geq 0,\, \sum_{i=1}^m u_i = 1\}$.

Definition 1. *An operator $T : X \to \mathcal{H}$ is* Nonexpansive *(NE) if $\|T(x) - T(y)\| \leq \|x-y\|$ for all $x, y \in X$.*

We have the following.

Lemma 2. *[10, Lemma 2.1.12] Given $m \in \mathbb{N}$, let $S_i : X \to X$, be nonexpansive for all $i \in \{1, 2, \ldots, m\}$. . Then:*

i. A convex combination $S := \sum_{i=1}^m w_i S_i$ is nonexpansive.

ii. A composition $S := S_m S_{m-1} \cdots S_1$ is nonexpansive.

For the proof of the first part of the following proposition see, [10, Theorem 2.2.21 (i) and (iv)]. The proof of the second part (and, actually, of the first part as well), can be found in [15, Theorem 3]

Proposition 3. *i. The orthogonal projection $P_C : \mathcal{H} \to \mathcal{H}$ onto a nonempty, closed and convex subset C of the space is NE, and its fixed point set is C itself.*

ii. If for some pair $(x,y) \in \mathcal{H}^2$ equality holds in the NE inequality related to P_C, that is, if $\|P_C x - P_C y\| = \|x-y\|$, then $\|x - P_C x\| = \|y - P_C y\|$.

Definition 4. *A **steering parameters sequence** $\{\tau_k\}_{k=0}^\infty$ is a real sequence in $(0,1)$ such that $\tau_k \to 0$, $\sum_{k=0}^\infty \tau_k = +\infty$, and $\sum_{k=0}^\infty |\tau_{k+1} - \tau_k| < +\infty$.*

Remark 5. An example of a steering parameter sequence $\{\tau_k\}_{k=1}^\infty$ is $\tau_k := \dfrac{1}{k}$ for all $k \in \mathbb{N}$.

The following theorem is the corollary [7, Corollary 30.2] with only some symbols changed to make it agree with the notations that we use. The iterative process of (2.2) represents the **simultaneous HLWB (S-HLWB) algorithm**.

Theorem 6. *Let X be a nonempty closed convex subset of $\mathcal{H}$, let $\{T_\ell\}_{\ell=1}^{L}$, $L \in \mathbb{N}$ be a finite family of nonexpansive operators from X to X such that $C = \bigcap_{\ell=1}^{L} \mathrm{Fix}T_\ell \neq \emptyset$, let $\{w_\ell\}_{\ell=1}^{L}$ be real numbers in $(0,1]$ such that $\sum_{\ell=1}^{L} w_\ell = 1$, and let $x \in X$ (called an anchor point). Let $\{\tau_k\}_{k=0}^{\infty}$ be a sequence of steering parameters, let $x^0 \in X$, and set*

$$(\forall k \geq 0)\ x^{k+1} = \tau_k x + (1-\tau_k)\sum_{\ell=1}^{L} w_\ell T_\ell x^k. \tag{2.2}$$

Then $\lim_{k\to\infty} x^k = P_C x$.

The theorem below was proved by Cheney and Goldstein in [15, Theorem 2]. See also , [18, Theorem 3] and [26, Theorem 3] for different proofs in the case where one of the sets is bounded ([18, Theorem 3] proves one direction, while [26, Theorem 3] probes both directions and in the case where the Hilbert space is complex).

Theorem 7. *Let K_1 and K_2 be two nonempty, closed and convex sets in a real Hilbert space. Let P_i denote the orthogonal projection onto K_i, $i \in \{1,2\}$. Then x is a fixed point of P_1P_2 if and only if x is a point of K_1 nearest K_2. Moreover, in this case $\|x - P_2 x\| = dist(K_1, K_2)$.*

The following result is also due to Cheney and Goldstein in [15, Theorem 4].

Theorem 8. *Let K_1 and K_2 be two nonempty, closed and convex sets in a real Hilbert space. Let P_i be the orthogonal projection onto K_i, $i \in \{1,2\}$ and let $Q := P_1P_2$. If either one of the sets K_1 and K_2 is compact, or one of these sets is finite-dimensional and the distance between these sets is attained, then for each x in the space the sequence $\left\{Q^k(x)\right\}_{k=1}^{\infty}$ converges to a fixed point of Q.*

Proposition 9. *[10, Theorem 2.1.14] Given $m \in \mathbb{N}$, suppose that $U_i : X \to \mathcal{H}$, $i \in \{1,2,\ldots,m\}$ are nonexpansive operators with a common fixed point and $U := \sum_{i\in I} w_i U_i$, where $w_i > 0$ for all $i \in \{1,2,\ldots,m\}$ and $\sum_{i=1}^{m} w_i = 1$. Then*

$$\mathrm{Fix}U = \bigcap_{i=1}^{m} \mathrm{Fix}U_i. \tag{2.3}$$

The next proposition concerns the fixed point set of compositions of orthogonal projections.

Proposition 10. *[7, Corollary 4.51] Given $m \in \mathbb{N}$, suppose that $T_i : \mathcal{H} \to \mathcal{H}$, $i \in \{1,2,\ldots,m\}$ are averaged nonexpansive operators, namely for each $i \in \{1,2,\ldots,m\}$ there are $\alpha_i \in (0,1)$ and a nonexpansive operator $S_i : \mathcal{H} \to \mathcal{H}$ such that $T_i =$*

$\alpha_i \mathrm{Id} + (1-\alpha_{\mathrm{i}})S_i$. *Suppose further that* $\bigcap_{i=1}^{m} \mathrm{Fix} T_i \neq \emptyset$. *If* $T := T_m T_{m-1} \cdots T_1$, *then*

$$\mathrm{Fix} T = \bigcap_{i=1}^{m} \mathrm{Fix} T_i. \tag{2.4}$$

We now recall two basic notions of convergence.

Definition 11. *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces and let* $\{f_k\}_{k=1}^{\infty}$, $f_k : X \to Y$ *be a sequence of functions. We say that* $\{f_k\}_{k=1}^{\infty}$

(i) converges pointwise to the function $f : X \to Y$ *if for every* $\varepsilon > 0$ *and every* $x \in X$ *there exists a positive integer* N_ε *such that for all* $k > N_\varepsilon$,

$$d_Y(f_k(x), f(x)) < \varepsilon. \tag{2.5}$$

(ii) converges uniformly to the function $f : X \to Y$ *if for every* $\varepsilon > 0$, *there exists a positive integer* N_ε *such that for all* $k > N_\varepsilon$ *and all* $x \in X$,

$$d_Y(f_k(x), f(x)) < \varepsilon. \tag{2.6}$$

3. A sufficient condition for the existence of a unique best approximation pair

In this section, we present a sufficient condition which ensures that there is a unique best approximation pair $(a, b) \in A \times B$. We note that Lemma 13 below (which should be known) holds in any real normed space, and Proposition 16 below holds in any real inner product space. In what follows the underlying space will be denoted by X.

Definition 12. *A subset* C *of* X *is called strictly convex if for all* x *and* y *in* C *and all* $t \in (0, 1)$, *the point* $tx + (1-t)\,y$ *is in the interior of* C.

Lemma 13. *Given* $m \in \mathbb{N}$ *and* m *strictly convex subsets* $C_1, C_2 \ldots, C_m$ *of* X, *their intersection is also strictly convex.*

Proof. Assume that $\bigcap_{i=1}^{m} C_i \neq \emptyset$, otherwise the assertion is obvious (void). Let $x, y \in \bigcap_{i=1}^{m} C_i$ and $t \in (0, 1)$ be arbitrary. Since each C_i is strictly convex $tx + (1-t)\,y$ is in the interior of C_i for all $i \in \{1, 2, \ldots, m\}$, there is some $\rho_i > 0$ such that the ball with center at $tx + (1-t)\,y$ and radius ρ_i is contained in C_i for all $i \in \{1, 2, \ldots, m\}$.

Since $\min\{\rho_i \mid i \in \{1, 2, \ldots, m\}\} > 0$, the ball with center $tx + (1-t)\,y$ and radius $\min\{\rho_i \mid i \in \{1, 2, \ldots, m\}\} > 0$ is contained in C_i for all $i \in \{1, 2, \ldots, m\}$, namely in $\bigcap_{i=1}^{m} C_i$, and so $tx + (1-t)\,y$ is in the interior of $\bigcap_{i=1}^{m} C_i$. Since x and y where arbitrary points in $\bigcap_{i=1}^{m} C_i$ and t was an arbitrary point in $(0, 1)$, we conclude from the definition of strict convexity that $\bigcap_{i=1}^{m} C_i$ is strictly convex. □

Example 14. Examples of bounded strictly convex sets: balls, ellipses, the intersection of a paraboloid with a bounded strictly convex set, the intersection of one arm of a hyperboloid with a bounded strictly convex set, level-sets of coercive and continuous strictly convex functions (namely, sets of the form $\{x \in X \mid f(x) \leq \alpha\}$, where $\alpha \in \mathbb{R}$ is fixed, $f : X \to \mathbb{R}$ is continuous and satisfies $\lim_{\|x\| \to \infty} f(x) = \infty$ (coercivity), and $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ for all $t \in (0,1)$ and $x, y \in X$ (strict convexity)).

The next lemma and proposition ensure that there is a unique best approximation pair.

Lemma 15. *Suppose that A and B are nonempty and convex subsets of a real inner product space X and suppose that $a \in A$ and $b \in B$ satisfy $\|a-b\| = dist(A,B) > 0$. Then A is contained in the closed half-space $\{y \in X \mid \langle y-a, a-b\rangle \geq 0\}$, and B is contained in the closed half-space $\{y \in X \mid \langle y-b, b-a\rangle \geq 0\}$. Moreover, the point a is a boundary point of A, and the point b is a boundary point of B.*

Proof. We prove the assertion regarding A. The proof regarding B is similar. Since $\|a-b\| = dist(A,B)$, we have $a = P_A(b)$. Thus, from the well known characterization of the orthogonal projection [10, Theorem 1.2.4], any $y \in A$ satisfies $\langle y-a, a-b\rangle \geq 0$, namely $A \subseteq \{y \in X \mid \langle y-a, a-b\rangle \geq 0\}$.

To see that a is a boundary point of A, let D be an arbitrary (non-degenerate) ball centered at a, with say, radius $r > 0$. Since $a \in A \cap D$, it remains to show that D contains points outside A. Indeed, let $t \in (0, r/\|b-a\|)$. Then $a + t(b-a) \in D$ but $\langle (a + t(b-a)) - a, a-b\rangle = -t\|a-b\|^2 < 0$, and so $a + t(b-a) \notin \{y \in X \mid \langle y-a, a-b\rangle \geq 0\}$. Since $A \subseteq \{y \in X \mid \langle y-a, a-b\rangle \geq 0\}$, by the previous paragraph, we conclude that $a + t(b-a)$ cannot be in A, as required. Therefore, a is a boundary point of A. □

Proposition 16. *Let X be a real inner product space. (i) If $A \subseteq X$ and $B \subseteq X$ are nonempty and $dist(A,B)$ is attained, then there is at least one best approximation pair $(a,b) \in A \times B$. (ii) If $A \subseteq X$ and $B \subseteq X$ are nonempty, strictly convex, and $dist(A,B) > 0$, then there is at most one best approximation pair $(a,b) \in A \times B$. (iii) If $A \subseteq X$ and $B \subseteq X$ are nonempty, strictly convex, satisfy $dist(A,B) > 0$, and the distance between them is attained, then there is exactly one best approximation pair $(a,b) \in A \times B$. In particular, there is exactly one best approximation pair $(a,b) \in A \times B$ whenever $A \subseteq X$ and $B \subseteq X$ are nonempty, strictly convex, compact and $dist(A,B) > 0$.*

Proof. Part (i) is just a restatement of the assertion that $dist(A,B)$ is attained.

We now turn to Part (ii). Suppose that $(a,b) \in A \times B$ and $\left(\tilde{a}, \tilde{b}\right) \in A \times B$ are two arbitrary best approximation pairs. We will show that $(a,b) = \left(\tilde{a}, \tilde{b}\right)$. Since both (a,b) and $\left(\tilde{a}, \tilde{b}\right)$ are best approximation pairs, we have $\|a-b\| = dist(A,B) = \left\|\tilde{a} - \tilde{b}\right\|$. Let $t \in (0,1)$ be arbitrary and let $a_t := ta + (1-t)\tilde{a}$ and $b_t := tb + (1-t)\tilde{b}$. By the convexity of A and B we have $a_t \in A$ and $b_t \in B$, and therefore, in particular, $dist(A,B) \leq \|a_t - b_t\|$. On the other hand, the triangle inequality and the assumption that (a,b) and $\left(\tilde{a}, \tilde{b}\right)$ are best approximation pairs imply that $\|a_t - b_t\| \leq t\|a-b\| + (1-t)\left\|\tilde{a} - \tilde{b}\right\| = t dist(A,B) + (1-t) dist(A,B) = dist(A,B)$. Hence,

$\|a_t - b_t\| = dist\,(A, B)$ and (a_t, b_t) is a best approximation pair too. Thus, Lemma 15 implies that a_t is a boundary point of A and b_t is a boundary point of B.

Suppose, by way of nagation, that $a \neq \tilde{a}$. Then a_t is strictly inside the line segment $[a, \tilde{a}]$, and hence the strict convexity of A implies that a_t is in the interior of A, contradicting the claim proved in the previous paragraphs that a_t is a boundary point of A. Consequently $a = \tilde{a}$, and similarly $b = \tilde{b}$. Therefore, $(a, b) = \left(\tilde{a}, \tilde{b}\right)$, and hence all the best approximation pairs (if they exist) are equal. Thus, there is at most one best approximation pair.

Now we show that the assumptions of Part (iii) hold. From Part (ii) we know that there is at most one approximation pair, and from Part (i) there is at least one best approximation pair. We conclude therefore that there is exactly one best approximation pair $(a, b) \in A \times B$. This conclusion holds when one considers the final case mentioned in Part (iii) since, as is well-known and easily follows from compactness, $dist\,(A, B)$ is attained whenever A and B are nonempty and compact (one simply takes a cluster point of $\left\{\left(a^k, b^k\right)\right\}_{k=1}^{\infty}$, where $\left(a^k, b^k\right) \in A \times B$ satisfies $\left\|a^k - b^k\right\| < dist\,(A, B) + (1/k)$ for all $k \in \mathbb{N}$). □

Remark 17. The strict convexity assumption in Proposition 16(iii) is sufficient but not necessary, as shown in Figure 3.1.

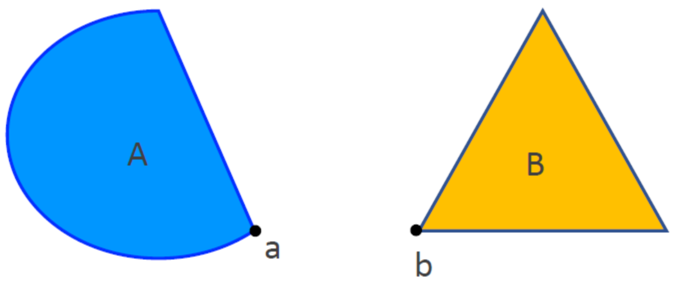

Figure 3.1: A case where there is a unique best approximation pair despite the fact that A and B are not strictly convex

4. The Simultaneous-HLWB projection operator

Inspired by the iterative process of (2.2) which represents the simultaneous HLWB (S-HLWB) algorithm, we define the Simultaneous-HLWB operator and introduce, in the sequel, some of its properties. Given are two families of nonempty, closed and convex sets $\mathcal{A} := \{A_i\}_{i=1}^{I}$ and $\mathcal{B} := \{B_j\}_{j=1}^{J}$ in a Euclidean space $\mathcal{H}$ for some positive integers I and J. Assume that $A := \bigcap_{i=1}^{I} A_i \neq \emptyset$ and $B := \bigcap_{j=1}^{J} B_j \neq \emptyset$ but $A \bigcap B = \emptyset$, where $A_i, B_j \subseteq B\,[0, \rho]$ for some $\rho > 0$. Our overall goal is to find a best approximation pair relative to A and B.

Definition 18. *We denote by*

$$M_{\mathcal{G},\tau}[d] := \tau d + (1-\tau)\sum_{\ell=1}^{L} w_\ell P_{C_\ell},\ L \in \mathbb{N} \tag{4.1}$$

the operators $M_{\mathcal{G},\tau}[d] : \mathcal{H} \to \mathcal{H}$, *with a fixed anchor point* $d \in \mathcal{H}$, *with respect to the family* $\mathcal{G} := \{C_\ell\}_{\ell=1}^{L}$ *of nonempty closed convex sets such that* $\bigcap_{\ell=1}^{L} C_\ell \neq \emptyset$ $\tau \in (0,1]$ *a real parameter and* $w_\ell \in (0,1]$ *are positive weights such that* $\sum_{\ell=1}^{L} w_\ell = 1$. *We call such operators* ***simultaneous-HLWB operators****. I.e., for* $x \in \mathcal{H}$,

$$M_{\mathcal{G},\tau}[d](x) := \tau d + (1-\tau)\sum_{\ell=1}^{L} w_\ell P_{C_\ell}(x). \tag{4.2}$$

5. The alternating simultaneous HLWB algorithm

Motivated by the strategy of the original algorithm in [3, Algorithm in Equation (2)], we present here our new method for finding a best approximation pair in a finite-dimensional Hilbert space, namely the alternating simultaneous HLWB algorithm (A-S-HLWB). But before doing so, at the end of this section, we need a technical preparation.

First we look at products of S-HLWB operators.

Definition 19. *For a family* $\mathcal{G} := \{C_\ell\}_{\ell=1}^{I}$ *of nonempty closed convex sets such that* $\bigcap_{\ell=1}^{L} C_\ell \neq \emptyset$ *and some given anchor point* $d \in \mathcal{H}$ *define, for any* $q = 0, 1, 2, \ldots,$ *the operators* $Q_{\mathcal{G},q}[d]$, *associated with* $\mathcal{G}$, *by*

$$Q_{\mathcal{G},q}[d] := \prod_{t=0}^{q} M_{\mathcal{G},\tau_t^{\mathcal{G}}}[d] = M_{\mathcal{G},\tau_q^{\mathcal{G}}}[d]\left(M_{\mathcal{G},\tau_{q-1}^{\mathcal{G}}}[d]\left(\cdots\left(M_{\mathcal{G},\tau_0^{\mathcal{G}}}[d]\right)\right)\right) \tag{5.1}$$

where $\left\{\tau_p^{\mathcal{G}}\right\}_{p=0}^{\infty}$ *is some given steering parameters sequence.*

When we choose in Definition 19 $\mathcal{C} = \mathcal{A}$, $q = s \in \mathbb{N}$ and $d = u \in \mathcal{H}$, we obtain from the family $\mathcal{A} := \{A_i\}_{i=1}^{I}$ and its associated steering parameters sequence $\left\{\tau_g^{\mathcal{A}}\right\}_{g=0}^{\infty}$, the operators $Q_{\mathcal{A},s}[u]$ as follows,

$$Q_{\mathcal{A},s}[u] := \prod_{t=0}^{s} M_{\mathcal{A},\tau_t^{\mathcal{A}}}[u]. \tag{5.2}$$

When we choose in the Definition 19 $\mathcal{C} = \mathcal{B}$, $q = r \in \mathbb{N}$ and $d = v \in \mathcal{H}$, we obtain from the family $\mathcal{B} := \{B_i\}_{i=1}^{I}$ and its associated steering parameters sequence $\left\{\tau_h^{\mathcal{B}}\right\}_{h=0}^{\infty}$, the operators $Q_{\mathcal{B},r}[v]$ as follows,

$$Q_{\mathcal{B},r}[v] := \prod_{t=0}^{r} M_{\mathcal{B},\tau_t^{\mathcal{B}}}[v]. \tag{5.3}$$

Each of these operators is formed by a finite composition of operators and we will prove the existence of fixed points for them.

In the following lemma we show the relation between the operator (5.1) and the iterative process (2.2).

Lemma 20. *Under the assumptions of Theorem 6 with anchor point $d \in X$ and $T_\ell := P_\ell := P_{C_\ell}$ for all $\ell \in \{1, 2, \ldots, L\}$, the sequnce $\left\{x^k\right\}_{k=0}^{\infty}$, generated by (2.2), is*

$$x^0 \in X,\ x^{k+1} = \tau_k d + (1 - \tau_k) \sum_{\ell=1}^{L} w_\ell P_\ell x^k. \tag{5.4}$$

Consider the sequence

$$y^0 \in X,\ y^{k+1} := Q_{\mathcal{G},k}[d]\left(y^0\right) = \prod_{t=0}^{k} M_{\mathcal{G},\tau_t^{\mathcal{G}}}[d]\left(y^0\right), \tag{5.5}$$

which is defined by using the operator (5.1) of Definition 19. If $y^0 = x^0$ and $\tau_k = \tau_k^{\mathcal{G}}$ for all nonnegative integers k, then *$y^k = x^k$ for every $k \geq 0$, and*

$$\lim_{k\to\infty} x^k = \lim_{k\to\infty} y^k = P_C d. \tag{5.6}$$

Proof. By (5.4) we have

$$x^1 = \tau_0 d + (1 - \tau_0) \sum_{\ell=1}^{L} w_\ell P_\ell x^0, \tag{5.7}$$

and for all $k \in \mathbb{N}$, and we have

$$x^{k+1} = M_{\mathcal{G},\tau_k}[d]\left(x^k\right) = \tau_k d + (1 - \tau_k) \sum_{\ell=1}^{L} w_\ell P_\ell\left(x^k\right). \tag{5.8}$$

By (5.5) we have

$$y^1 = M_{\mathcal{G},\tau_0^{\mathcal{G}}}[d]\left(y^0\right) = \tau_0^{\mathcal{G}} d + \left(1 - \tau_0^{\mathcal{G}}\right) \sum_{\ell=1}^{L} w_\ell P_{C_\ell}(y^0), \tag{5.9}$$

and for all $k \in \mathbb{N}$, and we have

$$y^{k+1} = M_{\mathcal{G},\tau_k^{\mathcal{G}}}[d]\left(y^k\right) = \tau_k^{\mathcal{G}} d + \left(1 - \tau_k^{\mathcal{G}}\right) \sum_{\ell=1}^{L} w_\ell P_\ell\left(y^k\right). \tag{5.10}$$

From (5.7), (5.9) and the assumptions that $x^0 = y^0$ and $\tau_0 = \tau_0^{\mathcal{G}}$, it follows that $x^1 = y^1$. Now we assume that the equality between x^ℓ and y^ℓ holds for all integers ℓ from 0 to k. From (5.8), (5.10) and the assumptions that $x^k = y^k$ and $\tau_k = \tau_k^{\mathcal{G}}$, it follows that $x^{k+1} = y^{k+1}$ as well, and hence we conclude, by mathematical induction, that $x^k = y^k$ for all nonnegative integers k. Finally, (5.6) follows from Theorem

6. □

We shall employ the operators of (4.1) and (5.1), from here onward, always with an anchor point that is identical with the initial point on which the operators act. To this end we define $d := x$ in (4.1) and obtain the new operator

$$\widehat{M}_{\mathcal{G},\tau}(x) := M_{\mathcal{G},\tau}[x](x) = \tau x + (1-\tau)\sum_{\ell=1}^{L} w_\ell P_{C_\ell}(x), \tag{5.11}$$

thus,

$$\widehat{M}_{\mathcal{G},\tau} = \tau \mathrm{Id} + (1-\tau)\sum_{\ell=1}^{L} w_\ell P_{C_\ell}. \tag{5.12}$$

For the choice $d := x$ and $\tau := \tau_t^{\mathcal{G}}$ in (5.1) we obtain the new operator

$$\begin{aligned}\widehat{Q}_{\mathcal{G},q}(x) := Q_{\mathcal{G},q}[x](x) &= \prod_{t=0}^{q} M_{\mathcal{G},\tau_t^{\mathcal{G}}}[x](x) = \prod_{t=0}^{q} \widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}}(x) \\ &= \prod_{t=0}^{q}\left(\tau_t^{\mathcal{G}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{G}}\right)\sum_{\ell=1}^{L} w_\ell P_{C_\ell}\right)(x). \end{aligned} \tag{5.13}$$

This yields, for the choice $\mathcal{G} = \mathcal{A}$ and $q = s \in \mathbb{N}$,

$$\widehat{Q}_{\mathcal{A},s}(x) := Q_{\mathcal{A},s}[x](x), \tag{5.14}$$

and, for the choice $\mathcal{G} = \mathcal{B}$ and $q = r \in \mathbb{N}$,

$$\widehat{Q}_{\mathcal{B},r}(x) := Q_{\mathcal{B},r}[x](x). \tag{5.15}$$

Lemma 21. *The operator of (5.11) and the operator of (5.13) are NE operators.*

Proof. By Lemma 2(i) a convex combination of NE operators is NE and hence $\sum_{\ell=1}^{L} w_\ell P_{C_\ell}$ is NE, and hence, again because of Lemma 2(i) and because the identity operator is NE, also $\widehat{M}_{\mathcal{G},\tau}$ is NE. By Lemma 2(ii) the operator $\widehat{Q}_{\mathcal{G},q}$ is also NE. □

A key point in our development is the property described in the next lemma about the successive application of these two operators.

Lemma 22. *For every $x \in \mathcal{H}$*

$$\lim_{r\to\infty}\left(\lim_{s\to\infty}\left(\widehat{Q}_{\mathcal{B},r}\left(\widehat{Q}_{\mathcal{A},s}(x)\right)\right)\right) = P_B\left(P_A(x)\right). \tag{5.16}$$

Proof. Lemma 20 implies that for all x and y in $\mathcal{H}$

$$\lim_{s\to\infty}\left(\widehat{Q}_{\mathcal{A},s}(x)\right) = P_A(x), \tag{5.17}$$

and

$$\lim_{r\to\infty}\left(\widehat{Q}_{\mathcal{B},r}\left(y\right)\right)=P_{B}\left(y\right). \tag{5.18}$$

Thus, due to the continuity of $\widehat{Q}_{\mathcal{B},r}$ for all $r\in\mathbb{N}$, which follows from the nonexpansivity property of the operators (Lemma 21),

$$\begin{aligned}\lim_{r\to\infty}\left(\lim_{s\to\infty}\left(\widehat{Q}_{\mathcal{B},r}\left(\widehat{Q}_{\mathcal{A},s}\left(x\right)\right)\right)\right)&=\lim_{r\to\infty}\left(\widehat{Q}_{\mathcal{B},r}\left(\lim_{s\to\infty}\left(\widehat{Q}_{\mathcal{A},s}\left(x\right)\right)\right)\right)\\&=\lim_{r\to\infty}\left(\widehat{Q}_{\mathcal{B},r}\left(P_{A}\left(x\right)\right)\right)=P_{B}\left(P_{A}\left(x\right)\right).\end{aligned} \tag{5.19}$$

□

Next is the classical theorem named after Dini, see, e.g., [24, Theorem 7.13, page 150] or [2, Theorem 8.2.6], which tells us when pointwise convergence of a sequence of functionals (i.e., operators into the real line) implies its uniform convergence.

Theorem 23. ***Dini's Theorem***. *Let K be a compact metric space. Let $f:K\to\mathbb{R}$ be a continuous functional and let $\{f_k\}_{k=1}^{\infty}$, $f_k:K\to\mathbb{R}$, be a sequence of continuous functionals. If $\{f_k\}_{k=1}^{\infty}$ is monotonic and converges pointwise to f then $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f.*

We will need the following generalization of Dini's Theorem for general operators (not necessarily functionals), which we derive from Theorem 23. This proposition might be known, but we have not seen it in the literature.

Proposition 24. *Let (K,d_K) be a compact metric space and let (Y,d_Y) be a metric space. For each $k\in\mathbb{N}$ suppose that $T_k:K\to Y$ is a continuous operator and assume that there is a continuous operator $T:K\to Y$ such that $\{T_k\}_{k=1}^{\infty}$ converges pointwise to T. If $d_Y(T_{k+1}(x),T(x))\leq d_Y(T_k(x),T(x))$ for all $k\in\mathbb{N}$ and all $x\in K$, then $\{T_k\}_{k=1}^{\infty}$ converges uniformly to T.*

Proof. Denote $f_k(x):=d_Y(T_k(x),T(x))$ for all $k\in\mathbb{N}$ and $x\in K$. By the continuity of T_k and T and the continuity of the metric, it follows that $f_k:K\to[0,\infty)$ is continuous for every $k\in\mathbb{N}$. Since $\{T_k\}_{k=1}^{\infty}$ converges pointwise to T, it follows that $0=\lim_{k\to\infty}d_Y(T_k(x),T(x))=\lim_{k\to\infty}f_k(x)=\lim_{k\to\infty}|f_k(x)-0|$, namely $\{f_k\}_{k=1}^{\infty}$ converges pointwise to $f\equiv 0$. Since we assume that $d_Y(T_{k+1}(x),T(x))\leq d_Y(T_k(x),T(x))$ for all $x\in K$ and $k\in\mathbb{N}$, we have $f_{k+1}(x)=d_Y(T_{k+1}(x),T(x))\leq d_Y(T_k(x),T(x))=f_k(x)$ for all $x\in K$ and $k\in\mathbb{N}$, namely the sequence $\{f_k\}_{k=1}^{\infty}$ is monotone. Hence, using the fact that K is compact, we conclude from Theorem 23 that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to the zero function. Thus, given $\varepsilon>0$, there exists some $k_\varepsilon\in\mathbb{N}$ such that $d_Y(T_k(x),T(x))=|f_k(x)-0|<\varepsilon$ for all $x\in K$ and $k_\varepsilon\leq k\in\mathbb{N}$, namely $\{T_k\}_{k=1}^{\infty}$ does converge uniformly to T. □

Lemma 25. *The fixed point set of the operator $\widehat{M}_{\mathcal{G},\tau}$ of (5.11) is equal to the, assumed nonempty, intersection $\bigcap_{\ell=1}^{L}C_\ell$.*

Proof. Let us denote $S(x) := \sum_{\ell=1}^{L} w_\ell P_{C_\ell}(x)$. The identity operator is NE and any $x \in \mathcal{H}$ is a fixed point of it. By Proposition 3 $\mathrm{Fix}P_{C_\ell} = C_\ell$, hence $\bigcap_{\ell=1}^{L} \mathrm{Fix}P_{C_\ell} = \bigcap_{\ell=1}^{L} C_\ell \neq \emptyset$. The projections P_{C_ℓ} are NEs by Proposition 3. By Proposition 9 we have $\mathrm{Fix}S = \bigcap_{\ell=1}^{L} \mathrm{Fix}P_{C_\ell}$, therefore, $\mathrm{FixId} \bigcap \mathrm{Fix}S = \bigcap_{\ell=1}^{L} C_\ell \neq \emptyset$. The operator S is a convex combination of the NE operators P_{C_ℓ}, so, it is also NE. For each $\tau \in (0,1)$ the operator $\widehat{M}_{\mathcal{G},\tau}$ is an averaged NE operator since

$$\widehat{M}_{\mathcal{G},\tau} = \tau \mathrm{Id} + (1-\tau) S. \tag{5.20}$$

Hence, by Proposition 9, we have

$$\mathrm{Fix}\widehat{M}_{\mathcal{G},\tau} = \mathrm{FixId} \bigcap \mathrm{Fix}S = \bigcap_{\ell=1}^{L} C_\ell. \tag{5.21}$$

□

In the following lemma we prove that the fixed point set of a composition of operators of the kind defined by (5.20) is nonempty.

Lemma 26. *For each $q \in \mathbb{N}$, we have*

$$\mathrm{Fix}\widehat{Q}_{\mathcal{G},q} = \bigcap_{\ell=1}^{L} C_\ell. \tag{5.22}$$

Proof. Lemma 21 implies that the operators $\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}}$ are NEs for each $\tau_t^{\mathcal{G}} \in (0,1)$. By Lemma 25

$$\mathrm{Fix}\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}} = \bigcap_{\ell=1}^{L} C_\ell \neq \emptyset, \quad \text{for each } \tau_t^{\mathcal{G}}. \tag{5.23}$$

By (5.21) it is clear that

$$\bigcap_{t=0}^{q} \mathrm{Fix}\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}} = \bigcap_{t=0}^{q} \bigcap_{\ell=1}^{L} C_\ell = \bigcap_{\ell=1}^{L} C_\ell \neq \emptyset. \tag{5.24}$$

We observe that (5.20) implies that $\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}}$ is an averaged NE operator for each $t \in \{1, 2, \ldots, q\}$. Since $\mathrm{Fix}\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}} \neq \emptyset$, as follows from (5.23), and since $\widehat{Q}_{\mathcal{G},q}$ is a finite composition of the averaged NE operators $\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}}$, $t \in \{1, 2, \ldots, q\}$, it follows from (5.24) and Proposition 10 that

$$\mathrm{Fix}\widehat{Q}_{\mathcal{G},q} = \mathrm{Fix} \prod_{t=0}^{q} \widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}} = \bigcap_{t=0}^{q} \mathrm{Fix}\widehat{M}_{\mathcal{G},\tau_t^{\mathcal{G}}} = \bigcap_{\ell=1}^{L} C_\ell. \tag{5.25}$$

□

Lemma 27. *For each $k \in \mathbb{N}$ let $T_k := \widehat{Q}_{\mathcal{G},k-1}$. Given $L \in \mathbb{N}$, let $C := \bigcap_{\ell=1}^{L} C_\ell$ and assume that C is nonempty. Then $\{T_k\}_{k=1}^{\infty}$ converges pointwise to P_C and the* *convergence is uniform on every compact set (and, in particular, on every closed ball).*

Proof. Denote $T := P_C$. Fix an arbitrary $x \in \mathcal{H}$ and let $z := T(x)$. Then $z = P_C \in C$ and so $\widehat{M}_{\mathcal{G},\tau_k^{\mathcal{G}}}(z) = z$ for all $k \in \mathbb{N}$ by Lemma 25. Observe that, by the definition of T_{k+1}, we have $T_{k+1} = \widehat{M}_{\mathcal{G},\tau_k^{\mathcal{G}}} T_k$ whenever $k \geq 2$. Since $\widehat{M}_{\mathcal{G},\tau_k^{\mathcal{G}}}$ is NE, according to Lemma 21, it follows from the previous lines that for all $k \geq 2$,

$$\begin{aligned} \|T_{k+1}(x) - T(x)\| &= \|T_{k+1}(x) - z\| \\ &= \left\| \widehat{M}_{\mathcal{G},\tau_k^{\mathcal{G}}}(T_k(x)) - \widehat{M}_{\mathcal{G},\tau_k^{\mathcal{G}}}(z) \right\| \\ &\leq \|T_k(x) - z\| = \|T_k(x) - T(x)\| . \end{aligned} \tag{5.26}$$

Now, if we choose $d := y^0 \in \mathcal{H}$ in (5.5), then we obtain the iterative process $y^{k+1} := \widehat{Q}_{\mathcal{G},k}(y^0) = T_{k+1}(y^0)$, and Lemma 20 then guarantees that $\lim_{k\to\infty} y^k = P_C(y^0)$ for every $y^0 \in \mathcal{H}$. Thus, $\{T_k\}_{k=1}^{\infty}$ converges pointwise to P_C. As a result of (5.26) and the fact that T_k is continuous for all k, as a composition of continuous operators, Proposition 24 yields the uniform convergence of $\{T_k\}_{k=1}^{\infty}$ to $T = P_C$ on every compact set, and in particular on every closed ball. ☐

Theorem 28. *Let $\mathcal{H}$ be a finite-dimensional real Hilbert space and let $\{R_k\}_{k=1}^{\infty}$ and $\{S_k\}_{k=1}^{\infty}$ where $R_k : \mathcal{H} \to \mathcal{H}$ and $S_k : \mathcal{H} \to \mathcal{H}$, for all $k \in \mathbb{N}$, be infinite sequences of operators defined for all $k \in \mathbb{N} \cup \{0\}$ by*

$$R_{k+1} := \widehat{Q}_{\mathcal{A},k} = \prod_{t=0}^{k} \widehat{M}_{\mathcal{A},\tau_t^{\mathcal{A}}} = \prod_{t=0}^{k} \left(\tau_t^{\mathcal{A}} \mathrm{Id} + \left(1 - \tau_t^{\mathcal{A}}\right) \sum_{\ell=1}^{L} w_\ell P_{A_\ell} \right) \tag{5.27}$$

and

$$S_{k+1} := \widehat{Q}_{\mathcal{B},k} = \prod_{t=0}^{k} \widehat{M}_{\mathcal{B},\tau_t^{\mathcal{B}}} = \prod_{t=0}^{k} \left(\tau_t^{\mathcal{B}} \mathrm{Id} + \left(1 - \tau_t^{\mathcal{B}}\right) \sum_{\ell=1}^{L} w_\ell P_{B_\ell} \right). \tag{5.28}$$

Further assume that there is some $\rho > 0$ such that $A_i, B_j \subseteq B[0,\rho]$ for all $i \in \{1, 2, \ldots, I\}$ and $j \in \{1, 2, \ldots, J\}$. Then the sequance defined for all $k \in \mathbb{N} \cup \{0\}$ by

$$T_{k+1} := S_{k+1} R_{k+1} = \widehat{Q}_{\mathcal{B},k} \widehat{Q}_{\mathcal{A},k} = \left(\prod_{t=0}^{k} \widehat{M}_{\mathcal{B},\tau_t^{\mathcal{B}}} \right) \left(\prod_{t=0}^{k} \widehat{M}_{\mathcal{A},\tau_t^{\mathcal{A}}} \right) \tag{5.29}$$

converges uniformly to $P_B P_A$ on $B[0,\rho]$.

Proof. By Lemma 27, with $\mathcal{G} = \mathcal{A}$ and $C = A$, $\{R_k\}_{k=1}^{\infty}$ converges uniformly to P_A on $B[0,\rho]$, i.e., for every $\varepsilon_1 > 0$, there exists an integer N_1 such that

$$\|R_{k+1}(x) - P_A(x)\| = \left\| \widehat{Q}_{\mathcal{A},k}(x) - P_A(x) \right\| < \varepsilon_1, \tag{5.30}$$

for all $k > N_1$ and all $x \in B[0,\rho]$.

By Lemma 27 again, now with $\mathcal{G} = \mathcal{B}$ and $C = B$, $\{S_k\}_{k=1}^{\infty}$ converges uniformly to P_B on $B[0,\rho]$, i.e., for every $\varepsilon_2 > 0$, there exists an integer N_2 such that

$$\|S_{k+1}(y) - P_B(y)\| = \left\|\widehat{Q}_{\mathcal{B},k}(y) - P_B(y)\right\| < \varepsilon_2, \tag{5.31}$$

for all $k > N_2$ and all $y \in B[0,\rho]$.

Now we prove that $y^k := \widehat{Q}_{\mathcal{A},k}(x)$ is in the ball $B[0,\rho]$ for all $k \in \mathbb{N} \cup \{0\}$ and for all $x \in B[0,\rho]$. Since $x \in B[0,\rho]$ and A_i and B_j are contained in $B[0,\rho]$ for all $i \in I$ and $j \in J$, by assumption, we have $P_{A_\ell}(x) \in A_\ell \subseteq B[0,\rho]$ for all $\ell \in \{1,2,\ldots,L\}$ and so, by the convexity of $B[0,\rho]$, also $\sum_{\ell=1}^{L} w_\ell P_{A_\ell}(x) \in B[0,\rho]$. Again by the convexity of $B[0,\rho]$ and the fact that $\tau_0^{\mathcal{A}} \in (0,1)$, also $\widehat{M}_{\mathcal{A},\tau_0^{\mathcal{A}}}(x) = \tau_0^{\mathcal{A}} x + \left(1-\tau_0^{\mathcal{A}}\right)\sum_{\ell=1}^{L} w_\ell P_{A_\ell}(x)$ is in $B[0,\rho]$, namely $y^0 = \widehat{M}_{\mathcal{A},\tau_0^{\mathcal{A}}}(x) \in B[0,\rho]$. By induction on k and the equality $y^k = \widehat{Q}_{\mathcal{A},k}(x) = \widehat{M}_{\mathcal{A},\tau_k^{\mathcal{A}}}(\widehat{Q}_{\mathcal{A},k-1}(x))$ we see that indeed $y^k \in B[0,\rho]$ for every $k \in \mathbb{N} \cup \{0\}$.

Now we combine the results established above, where in the following calculations $k \in \mathbb{N}$ is larger than $\max\{N_1, N_2\}$, the triangle inequality implies (5.35), the fact that P_B is NE by Proposition 3 implies (5.36), and (5.37) follows by choosing $\varepsilon_1 := \varepsilon_2 := \dfrac{\varepsilon}{2}$ for $\varepsilon > 0$ in (5.30) and (5.31), respectively, and $y = \widehat{Q}_{\mathcal{A},k}(x)$ in (5.31) (where $x \in B[0,\rho]$ and we also use the fact that $\widehat{Q}_{\mathcal{A},k}(x) \in B[0,\rho]$ for all $x \in B[0,\rho]$ as we showed earlier):

$$\left\|\widehat{Q}_{\mathcal{B},k}\widehat{Q}_{\mathcal{A},k}(x) - P_B P_A(x)\right\| \tag{5.32}$$

$$= \left\|\widehat{Q}_{\mathcal{B},k}\left(\widehat{Q}_{\mathcal{A},k}(x)\right) - P_B(P_A(x))\right\| \tag{5.33}$$

$$= \left\|\widehat{Q}_{\mathcal{B},k}\left(\widehat{Q}_{\mathcal{A},k}(x)\right) - P_B\left(\widehat{Q}_{\mathcal{A},k}(x)\right) + P_B\left(\widehat{Q}_{\mathcal{A},k}(x)\right) - P_B(P_A(x))\right\| \tag{5.34}$$

$$\le \left\|\widehat{Q}_{\mathcal{B},k}\left(\widehat{Q}_{\mathcal{A},k}(x)\right) - P_B\left(\widehat{Q}_{\mathcal{A},k}(x)\right)\right\| + \left\|P_B\left(\widehat{Q}_{\mathcal{A},k}(x)\right) - P_B(P_A(x))\right\| \tag{5.35}$$

$$\le \left\|\widehat{Q}_{\mathcal{B},k}\left(\widehat{Q}_{\mathcal{A},k}(x)\right) - P_B\left(\widehat{Q}_{\mathcal{A},k}(x)\right)\right\| + \left\|\widehat{Q}_{\mathcal{A},k}(x) - P_A(x)\right\| \tag{5.36}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{5.37}$$

Therefore, by Definition 11(ii), the sequence $\{T_{k+1}\}_{k=0}^{\infty}$ converges uniformly to $P_B P_A$, i.e.,

$$\lim_{k\to\infty} T_{k+1} = \lim_{k\to\infty}\left(\widehat{Q}_{\mathcal{B},k}\widehat{Q}_{\mathcal{A},k}\right) = P_B P_A. \tag{5.38}$$

□

Since the main idea of our algorithm is that the number q of successive projections onto each of the two intersections increases from one sweep to the next, we prove the following theorem, which is a cornerstone of our analysis that will follow, for $q \to \infty$.

Theorem 29. *Given are two families of closed and convex sets $\{A_i\}_{i=1}^{I}$ and $\{B_j\}_{j=1}^{J}$ in a finite-dimensional real Hilbert space $\mathcal{H}$, for some positive integers I and J. Assume that $A_i, B_j \subseteq B[0,\rho]$ for all $i \subseteq \{1,2,\ldots,I\}$ and $j \subseteq \{1,2,\ldots,J\}$ for some $\rho > 0$. Given are also two sequences of parameters $\left\{\tau_k^{\mathcal{A}}\right\}_{k=0}^{\infty}$ and $\left\{\tau_k^{\mathcal{B}}\right\}_{k=0}^{\infty}$. Assume* *that* $A := \cap_{i=1}^{I} A_i \neq \emptyset$ *and* $B := \cap_{j=1}^{J} B_j \neq \emptyset$ *but* $A \cap B = \emptyset$. *Let* $\{\widehat{Q}_{\mathcal{A},k}\}_{k=0}^{\infty}$ *and* $\{\widehat{Q}_{\mathcal{B},k}\}_{k=0}^{\infty}$ *be sequences of operators defined in (5.14) and in (5.15), respectively, and create the sequences* $\{\widehat{Q}_{\mathcal{B},k}\widehat{Q}_{\mathcal{A},k}\}_{k=0}^{\infty}$ *and* $\{\widehat{Q}_{\mathcal{A},k}\widehat{Q}_{\mathcal{B},k}\}_{k=0}^{\infty}$*, of their products. Then the limit operators of the sequences of products exist, and the fixed point sets are:*

$$\operatorname{Fix}\left(\lim_{k\to\infty}\left(\widehat{Q}_{\mathcal{B},k}\widehat{Q}_{\mathcal{A},k}\right)\right) = \{z \in B \mid \|z - P_A(z)\| = dist\,(A,B)\}, \tag{5.39}$$

and

$$\operatorname{Fix}\left(\lim_{k\to\infty}\left(\widehat{Q}_{\mathcal{A},k}\widehat{Q}_{\mathcal{B},k}\right)\right) = \{w \in A \mid \|w - P_B(w)\| = dist\,(A,B)\}. \tag{5.40}$$

Proof. Both A and B are nonempty (by assumption). Moreover, both of them are compact since each of them is an intersection of closed and bounded (hence compact) subsets. As a result, and as is well-known, the distance $dist\,(A,B)$ between them is attained. Hence [15, Theorem 2] ensures that

$$\operatorname{Fix}(P_B P_A) = \{z \in B \mid \|z - P_A(z)\| = dist\,(A,B)\} \tag{5.41}$$

and

$$\operatorname{Fix}(P_A P_B) = \{z \in A \mid \|z - P_B(z)\| = dist\,(A,B)\}. \tag{5.42}$$

Since $\lim_{k\to\infty}\left(\widehat{Q}_{\mathcal{B},k}\widehat{Q}_{\mathcal{A},k}\right) = P_B P_A$ and $\lim_{k\to\infty}\left(\widehat{Q}_{\mathcal{A},k}\widehat{Q}_{\mathcal{B},k}\right) = P_A P_B$, as follows from Theorem 28, the previous lines imply (5.39) and (5.40). ☐

Our formulation and proof of Lemma 30 below are inspired by [1, Lemma 2], but a few slight differences exist between what is written in [1] and what we do here, partly because the proof of [1, Lemma 2] contains a few minor gaps and other issues, and also because it is possible to slightly generalize [1, Lemma 2] as we do.

Lemma 30. *Let* K_1 *and* K_2 *be two nonempty, closed and convex sets in a real Hilbert space* $\mathcal{H}$*, and let* P_i *be the orthogonal projection onto* K_i*,* $i \in \{1,2\}$*. If* S *is a nonempty and compact subset of* $\mathcal{H}$ *such that* $P_2P_1(S) = S$*, and if* $\lim_{k\to\infty}(P_2P_1)^k(z)$ *exists for all* $z \in S$ *and is a fixed point of* P_2P_1*, then* $S \subseteq \operatorname{Fix}(P_2P_1)$*, and any point of* S *is a point of* K_2 *which is nearest to* K_1*. In particular, if either* K_1 *or* K_2 *is also compact, or one of these sets is finite-dimensional and* $dist\,(K_1,K_2)$ *is attained, then* $S \subseteq \operatorname{Fix}(P_2P_1)$*, and any point of* S *is a point of* K_2 *which is nearest to* K_1*.*

Proof. Define

$$S' := \{s \in S \mid P_2P_1(s) = s\}. \tag{5.43}$$

Fix some $z \in S$. By our assumption $\lim_{k\to\infty}(P_2P_1)^k(z)$ exists and is a fixed point of P_2P_1. Since $P_2P_1(S) = S$, in particular $P_2P_1(z) \in S$, and so, by induction, $(P_2P_1)^k(z) \in S$ for all $k \in \mathbb{N}$. Hence, since S is closed (because it is compact), the limit $\lim_{k\to\infty}(P_2P_1)^k(z)$, which we assume that it exists, is also in S. We conclude from the previous lines that $\lim_{k\to\infty}(P_2P_1)^k(z)$ is a point of S and is also a fixed point of P_2P_1, and so, by the definition of S', this limit is in S'. Hence S' is nonempty.

To see that S' is also compact, we first show that it is closed. Suppose that $s'_k \in S'$ for each $k \in \mathbb{N}$ and $\lim_{k\to\infty} s'_k = s$ for some $s \in \mathcal{H}$. By the definition of S' we have $P_2P_1(s'_k) = s'_k$ for each $k \in \mathbb{N}$. Because $s'_k \in S' \subseteq S$ for each $k \in \mathbb{N}$ and because S is compact and hence closed, we conclude that the limit s is also in S. Since P_2 and P_1 are continuous, so is their composition, and so $s = \lim_{k\to\infty} s'_k = \lim_{k\to\infty} P_2P_1(s'_k) = P_2P_1(s)$, namely s is a fixed point of P_2P_1. Since we already know that $s \in S$, we have $s \in S'$ by the definition of S', and hence S' is closed. Thus S' is a closed subset of the compact set S, and hence, as is well-known, this implies that S' is compact.

Let $\hat{d} := \sup\{d(s, S') \mid s \in S\}$, where $d(s, S') := \inf\{\|s - s'\| \mid s' \in S'\}$. It is well-known and can easily be proved that the function $f(s) := d(s, E)$ is continuous (even nonexpansive) whenever $E \neq \emptyset$ (in particular, for $E := S'$), and so the compactness of S, and the Weierstrass Extreme Value Theorem, imply that f attains its maximum over S, namely $f(y) = \hat{d} = d(y, S')$ for some $y \in S$.

Since $y \in S \subseteq P_2P_1(S) = \{P_2P_1(x) \mid x \in S\}$, there is some $x \in S$ such that $y = P_2P_1(x)$. This fact, the continuity of the norm, the definition of $d(x, S')$, the compactness of S' and the Weierstrass Extreme Value Theorem, all imply that $d(x, S') = \|x - s'\|$ for some $s' \in S'$. Therefore,

$$\begin{aligned} \|x - s'\| &= d(x, S') \leq \sup\{d(s, S') \mid s \in S\} = \hat{d} \\ &= d(y, S') = \inf\{\|y - s''\| \mid s'' \in S'\} \leq \|y - s'\|. \end{aligned} \tag{5.44}$$

We claim that $x \in S'$. Indeed, suppose by way of nagation that this is not true. Since $s' \in S' \subseteq S \subseteq K_2$ and since any point in S' is a fixed point of P_2P_1, we know from [15, Theorem 2] that s' is a point in K_2 which is nearest to K_1. Thus $\|s' - P_1(s')\| = dist(K_2, K_1)$. On the other hand, since $x \in S \subseteq K_2$ and since we assume that $x \notin S'$, it follows that x is not a fixed point of P_2P_1 (otherwise it would be in S' by the definition of S', a contradiction). Therefore, [15, Theorem 2] implies that x cannot be a point in K_2 which is nearest to K_1, and so $dist(K_2, K_1) < \|x - P_1(x)\|$. We conclude from the previous lines that $\|s' - P_1(s')\| = dist(K_2, K_1) < \|x - P_1(x)\|$.

Consequently, from the necessary condition for equality in the nonexpansivness property of an orthogonal projection [15, Theorem 3] (see the second part of Proposition 3), we conclude that $\|P_1(x) - P_1(s')\| < \|x - s'\|$. This inequality, and the facts that P_2 is nonexpansive, that $s' = P_2P_1(s')$ and that $y = P_2P_1(x)$, all imply that

$$\|y - s'\| = \|P_2P_1(x) - P_2P_1(s')\| \leq \|P_1(x) - P_1(s')\| < \|x - s'\|, \tag{5.45}$$

which is a contradiction (5.44). Thus, the initial assumption that $x \notin S'$ cannot be true, namely $x \in S'$, and hence x is a fixed point of P_2P_1. Therefore, $x = P_2P_1(x) = y$ and hence, from the fact that $x \in S'$, we conclude that $y \in S'$. Thus, $d(y, S') = 0$ and so $\hat{d} = 0$. Since $0 \leq d(s, S') \leq \hat{d} = 0$ for all $s \in S$ by the definition of $\hat{d}$, it follows that each $s \in S$ satisfies $d(s, S') = 0$. But S' is closed, as we showed earlier, and hence every $s \in S$ is actually in S'. Therefore, $S \subseteq S'$, and since obviously $S' \subseteq S$, we have $S = S'$, that is, every point in S is a fixed point of P_2P_1, as claimed. From [15, Theorem 2] it follows that any point of S is a point of K_2 which is nearest to K_1.

Finally, if, in addition, either K_1 or K_2 is also compact, or one of these sets is finite-dimensional and $dist\,(K_2, K_1)$ is attained, then, as follows from [15, Theorem 4], $\lim_{k\to\infty} (P_2P_1)^k (z)$ exists for all $z \in \mathcal{H}$ and is a fixed point of P_2P_1, and hence, as shown above, $S \subseteq \mathrm{Fix}\,(P_2P_1)$ and any point of S is a point of K_2 which is nearest to K_1. □

Now we are able to present the A-S-HLWB algorithm.

Algorithm 1 The new alternating simultaneous HLWB (A-S-HLWB) algorithm

Input: The families $\{A_i\}_{i=1}^{I}$ and $\{B_j\}_{j=1}^{J}$ for given positive integers I and J and steering parameters sequences $\left\{\tau_t^{\mathcal{A}}\right\}_{t=1}^{\infty}$ and $\left\{\tau_s^{\mathcal{B}}\right\}_{s=1}^{\infty}$.

Initialization: Choose an arbitrary starting point $x^0 \in \mathcal{H}$.

Iterative Step: Given x^k, find the next iterate x^{k+1}, as follows:

(a) If k is even, define $r := \dfrac{k}{2}$ and use the iteration

$$x^{k+1} = x^{2r+1} := \widehat{Q}_{\mathcal{A},r}(x^{2r}). \tag{5.46}$$

(b) If k is odd, define $r := \dfrac{k-1}{2}$ and use the iteration

$$x^{k+1} = x^{2r+2} := \widehat{Q}_{\mathcal{B},r}(x^{2r+1}). \tag{5.47}$$

Starting from an initializtion point x^0, the first iterations out of the infinite sequence, generated by the A-S-HLWB algorithm, are as follows. Observe that the number of sweeps increase as iterations proceed.

$$x^1 = \widehat{Q}_{\mathcal{A},0}\left(x^0\right) = \prod_{t=0}^{0}\left(\tau_t^{\mathcal{A}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{A}}\right)\sum_{\ell=1}^{L} w_\ell P_{A_\ell}\right)\left(x^0\right). \tag{5.48}$$

$$x^2 = \widehat{Q}_{\mathcal{B},0}\left(x^1\right) = \prod_{t=0}^{0}\left(\tau_t^{\mathcal{B}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{B}}\right)\sum_{\ell=1}^{L} w_\ell P_{B_\ell}\right)\left(x^1\right). \tag{5.49}$$

$$x^3 = \widehat{Q}_{\mathcal{A},1}\left(x^2\right) = \prod_{t=0}^{1}\left(\tau_t^{\mathcal{A}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{A}}\right)\sum_{\ell=1}^{L} w_\ell P_{A_\ell}\right)\left(x^2\right). \tag{5.50}$$

$$x^4 = \widehat{Q}_{\mathcal{B},1}\left(x^3\right) = \prod_{t=0}^{1}\left(\tau_t^{\mathcal{B}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{B}}\right)\sum_{\ell=1}^{L} w_\ell P_{B_\ell}\right)\left(x^3\right). \tag{5.51}$$

$$x^5 = \widehat{Q}_{\mathcal{A},2}\left(x^4\right) = \prod_{t=0}^{2}\left(\tau_t^{\mathcal{A}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{A}}\right)\sum_{\ell=1}^{L} w_\ell P_{A_\ell}\right)\left(x^4\right). \tag{5.52}$$

$$x^6 = \widehat{Q}_{\mathcal{B},2}\left(x^5\right) = \prod_{t=0}^{2}\left(\tau_t^{\mathcal{B}}\mathrm{Id} + \left(1-\tau_t^{\mathcal{B}}\right)\sum_{\ell=1}^{L} w_\ell P_{B_\ell}\right)\left(x^5\right). \tag{5.53}$$

6. Convergence of the A-S-HLWB algorithm

In this section we present our main convergence result, namely Theorem 32 below. Its proof is inspired by the first part of the proof of [1, Theorem 1]. Before formulating the theorem, we need a lemma.

Lemma 31. *If there is some $\rho > 0$ such $A_i, B_j \subseteq B[0,\rho]$ for all $i \in \{1,2,\ldots,I\}$ and $j \in \{1,2,\ldots,J\}$, and if $x^0 \in B[0,\rho]$, then $x^k \in B[0,\rho]$ for all $k \in \mathbb{N}\cup\{0\}$.*

Proof. In order to prove this assertion, we first show by induction that if $x \in B[0,\rho]$, then $\widehat{Q}_{\mathcal{A},k}(x) \in B[0,\rho]$ for every $k \in \mathbb{N}\cup\{0\}$. Indeed, recall that $\widehat{Q}_{\mathcal{A},k}(x) = \prod_{t=0}^{k}\widehat{M}_{\mathcal{A},\tau_t^{\mathcal{A}}}(x)$. Since $x \in B[0,\rho]$ and A_i is contained in $B[0,\rho]$ for all $i \in I$, we have $P_{A_\ell}(x) \in A_\ell \subseteq B[0,\rho]$ for all $\ell \in \{1,2,\ldots,L\}$ and so, by the convexity of $B[0,\rho]$, also $\sum_{\ell=1}^{L} w_\ell P_{A_\ell}(x) \in B[0,\rho]$. Again by the convexity of $B[0,\rho]$ and the fact that $\tau_0^{\mathcal{A}} \in (0,1)$, also $\tau_0^{\mathcal{A}}x + \left(1-\tau_0^{\mathcal{A}}\right)\sum_{\ell=1}^{L} w_\ell P_{A_\ell}(x)$ is in $B[0,\rho]$, namely $\widehat{Q}_{\mathcal{A},0}(x) = \widehat{M}_{\mathcal{A},\tau_0^{\mathcal{A}}}(x) \in B[0,\rho]$. Assume now that $k \in \mathbb{N}$ and that the induction hypothesis holds for all $q \in \{0,1,\ldots,k-1\}$, that is, $\widehat{Q}_{\mathcal{A},q}(x) \in B[0,\rho]$ for all $q \in \{0,1,\ldots,k-1\}$. Since $\widehat{M}_{\mathcal{A},\tau_k^{\mathcal{A}}}(y) \in B[0,\rho]$ whenever $y \in B[0,\rho]$ by a similar argument to the one used in previous lines (we use the same argument which we used in order to show that $\widehat{M}_{\mathcal{A},\tau_0^{\mathcal{A}}}(x) \in B[0,\rho]$ whenever $x \in B[0,\rho]$, but now with $\widehat{M}_{\mathcal{A},\tau_k^{\mathcal{A}}}(y)$ and y instead of $\widehat{M}_{\mathcal{A},\tau_0^{\mathcal{A}}}(x)$ and x, respectively), we see that for $y := \widehat{Q}_{\mathcal{A},k-1}(x)$, we have $\widehat{Q}_{\mathcal{A},k}(x) = \widehat{M}_{\mathcal{A},\tau_k^{\mathcal{A}}}(\widehat{Q}_{\mathcal{A},k-1}(x)) = \widehat{M}_{\mathcal{A},\tau_k^{\mathcal{A}}}(y) \in B[0,\rho]$, since by the induction hypothesis for $k-1$ we have $y \in B[0,\rho]$. Hence the induction hypothesis holds for k as well, and so $\widehat{Q}_{\mathcal{A},k}(x) \in B[0,\rho]$ for every $k \in \mathbb{N}\cup\{0\}$, as required. Similarly, if $x \in B[0,\rho]$, then $\widehat{Q}_{\mathcal{B},k}(x) \in B[0,\rho]$ for every $k \in \mathbb{N}\cup\{0\}$.

Finally, in order to see that $x^k \in B[0,\rho]$ for all $k \in \mathbb{N}\cup\{0\}$, we apply induction on k. By our assumption $x^0 \in B[0,\rho]$. Assume that the induction hypothesis holds for all $q \in \{0,1,\ldots,k\}$, namely, that $x^q \in B[0,\rho]$ for all $q \in \{0,1,\ldots,k\}$. If k is even, then, according to (5.46), one has $x^{k+1} = \widehat{Q}_{\mathcal{A},k/2}(x^k)$, and so by previous lines and the induction hypothesis (that $x := x^k \in B[0,\rho]$) it follows that $x^{k+1} = \widehat{Q}_{\mathcal{A},k/2}(x) \in B[0,\rho]$, as well. If k is odd, then, according to (5.47), one has $x^{k+1} = \widehat{Q}_{\mathcal{B},(k-1)/2}(x^k)$, and so by previous lines and the induction hypothesis (that $x := x^k \in B[0,\rho]$) it follows that $x^{k+1} = \widehat{Q}_{\mathcal{B},(k-1)/2}(x) \in B[0,\rho]$, as well. Hence the induction hypothesis holds for $k+1$. Therefore indeed $x^k \in B[0,\rho]$ for every nonnegative integer k, as required. □

Theorem 32. *Given are two families of closed convex sets $\{A_i\}_{i=1}^{I}$ and $\{B_j\}_{j=1}^{J}$ in a Euclidean space $\mathcal{H}$, for some positive integers I and J and sequences of steering parameters $\left\{\tau_k^{\mathcal{A}}\right\}_{k=1}^{\infty}$ and $\left\{\tau_k^{\mathcal{B}}\right\}_{k=1}^{\infty}$. Assume that $A := \cap_{i=1}^{I}A_i \neq \emptyset$ and $B := \cap_{j=1}^{J}B_j \neq \emptyset$ but $A\cap B = \emptyset$. Assume also that there is some $\rho > 0$ such that $A_i, B_j \subseteq B[0,\rho]$*

for all $i \in \{1,2,\ldots,I\}$ and $j \in \{1,2,\ldots,J\}$. Also, assume that there is a unique best approximation pair relative to (A,B). Let $\{x^k\}_{k=0}^{\infty}$ be a sequence generated by Algorithm 1, where we assume that $x^0 \in B[0,\rho]$. Then

$$\lim_{k\to\infty} \left\|x^{k+1}-x^k\right\| = dist\,(A,B)\,, \tag{6.1}$$

and moreover, the odd subsequence $\{x^{2k+1}\}_{k=0}^{\infty}$ converges to a point $a \in A$, the even subsequence $\{x^{2k}\}_{k=0}^{\infty}$ converges to a point $b \in B$, and (a,b) is a best approximation pair relative to (A,B). In particular, the conclusion of the theorem holds when all the sets A_i and B_j, $i \in \{1,2,\ldots,I\}$ and $j \in \{1,2,\ldots,J\}$, are strictly convex.

Proof. Since A_i and B_j for $i \in \{1,2,\ldots,I\}$ and $j \in \{1,2,\ldots,J\}$ are closed subsets of the closed (hence compact) ball $B\,[0,\rho]$, they are compact and hence also their intersections $A = \cap_{i=1}^{I} A_i$ and $B = \cap_{j=1}^{J} B_j$ are. Thus, $dist\,(A,B)$ is attained by the continuity of the norm and the Weierstrass Extreme Value Theorem. Since $A_i, B_j \subseteq B\,[0,\rho]$ for all $i \in \{1,2,\ldots,I\}$ and $j \in \{1,2,\ldots,J\}$, by Theorem 28 we have $\lim_{k\to\infty}\left(\widehat{Q}_{\mathcal{B},k}\widehat{Q}_{\mathcal{A},k}\right) = P_B P_A$. Therefore, the conditions needed in Theorem 8 are satisfied, and hence $\operatorname{Fix}(P_B P_A)$ is nonempty. In addition, Lemma 31 ensures that $x^k \in B[0,\rho]$ for all nonnegative integer k.

Let S be the set of accumulation points of $\{x^{2r}\}_{r=0}^{\infty}$. By the Bolzano–Weierstrass theorem $S \neq \emptyset$. Moreover, since S is closed and is contained in the compact set $B\,[0,\rho]$, we have that S is compact too.

We claim that $P_B P_A(S) = S$. Indeed, let $s \in S$ be any point. Then there is a subsequence $\{x^{2r_k}\}_{k=0}^{\infty}$ such that $s = \lim_{k\to\infty} x^{2r_k}$, and so for a given $\varepsilon_1 > 0$ there is some natural number k_0 such that $\|s - x^{2r_k}\| \leq \varepsilon_1$ for all natural numbers $k > k_0$. Due to Theorem 28 and since the sequence $\{x^k\}_{k=0}^{\infty}$ is contained in $B\,[0,\rho]$, as proved above, for a given $\varepsilon_2 > 0$ there is a natural number r_0 such that for all natural numbers $r > r_0$ one has $\left\|P_B P_A(y) - \widehat{Q}_{\mathcal{B},r}\widehat{Q}_{\mathcal{A},r}(y)\right\| \leq \varepsilon_2$ for all $y \in B\,[0,\rho]$ and, in particular, for $y := x^{2r}$. Let $\varepsilon > 0$ and define $\varepsilon_1 := \dfrac{\varepsilon}{2}$ and $\varepsilon_2 := \dfrac{\varepsilon}{2}$. For ε_1 and ε_2 we can associate the natural numbers k_0 and r_0, respectively, mentioned above, and hence, if k is a natural number satisfying $k > k_0$ and $r_k > r_0$, then, by the nonexpansivity of $P_B P_A$, we have

$$\begin{aligned}
&\left\|P_B P_A(s) - x^{2r_k+2}\right\| = \left\|P_B P_A(s) - \widehat{Q}_{\mathcal{B},r_k}\widehat{Q}_{\mathcal{A},r_k}\left(x^{2r_k}\right)\right\| \\
&\leq \left\|P_B P_A(s) - P_B P_A\left(x^{2r_k}\right)\right\| + \left\|P_B P_A\left(x^{2r_k}\right) - \widehat{Q}_{\mathcal{B},r_k}\widehat{Q}_{\mathcal{A},r_k}\left(x^{2r_k}\right)\right\| \\
&\leq \left\|s - x^{2r_k}\right\| + \left\|P_B P_A\left(x^{2r_k}\right) - \widehat{Q}_{\mathcal{B},r_k}\widehat{Q}_{\mathcal{A},r_k}\left(x^{2r_k}\right)\right\| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{6.2}$$

This implies that $P_B P_A(s)$ is also an accumulation point of $\{x^{2r}\}_{r=0}^{\infty}$. Since s was an arbitrary point in S, we conclude that, $P_B P_A(S) \subset S$.

On the other hand, suppose that $s \in S$. Then s is the limit of a subsequence of $\{x^{2r}\}_{r=0}^{\infty}$, namely $s = \lim_{\ell\to\infty} x^{2r_\ell}$ for some subsequence $\{x^{2r_\ell}\}_{\ell=0}^{\infty}$. Since $\{x^{2r_\ell-2}\}_{\ell=1}^{\infty}$ is a subsequence of the sequence $\{x^t\}_{t=1}^{\infty}$ which is contained in the compact set

$B[0,\rho]$, it follows from the compactness of $B[0,\rho]$ that $\{x^{2r_\ell -2}\}_{\ell=1}^{\infty}$ has a limit point $s' \in B[0,\rho]$. Hence $s' = \lim_{k\to\infty} x^{2r_{\ell_k}-2}$ for some subsequence $\left\{x^{2r_{\ell_k}-2}\right\}_{k=1}^{\infty}$ of $\{x^{2r_\ell -2}\}_{\ell=1}^{\infty}$, and therefore, given $\varepsilon_1 > 0$, there is some $k_0 \in \mathbb{N}$ such that $\left\|s' - x^{2r_{\ell_k}-2}\right\| \leq \varepsilon_1$ for all $k > k_0$. In addition, since s' is the limit of a subsequence of the sequence $\{x^{2t}\}_{t=0}^{\infty}$, the definition of S implies that $s' \in S$. Due to Theorem 28 and since the sequence $\{x^t\}_{t=0}^{\infty}$ is contained in $B[0,\rho]$, as proved above, for a given $\varepsilon_2 > 0$ there is a natural number ℓ_0 such that for all natural numbers $\ell > \ell_0$ and for all $y \in B[0,\rho]$, one has $\left\|P_B P_A(y) - \widehat{Q}_{\mathcal{B},r_\ell -1}\widehat{Q}_{\mathcal{A},r_\ell -1}(y)\right\| \leq \varepsilon_2$. This inequality is true, in particular, for $y := x^{2r_\ell -2}$. Let $\varepsilon > 0$ and define $\varepsilon_1 := \dfrac{\varepsilon}{2}$ and $\varepsilon_2 := \dfrac{\varepsilon}{2}$. For ε_1 and ε_2 we can associate the natural numbers k_0 and ℓ_0 mentioned above, and hence, if k is a natural number satisfying $k > k_0$ and $\ell_k > \ell_0$, then, by the nonexpansivity of $P_B P_A$, we have

$$
\begin{aligned}
&\left\|P_B P_A(s') - x^{2r_{\ell_k}}\right\| = \left\|P_B P_A(s') - \widehat{Q}_{\mathcal{B},r_{\ell_k}-1}\widehat{Q}_{\mathcal{A},r_{\ell_k}-1}\left(x^{2r_{\ell_k}-2}\right)\right\| \\
&\leq \left\|P_B P_A(s') - P_B P_A\left(x^{2r_{\ell_k}-2}\right)\right\| + \left\|P_B P_A\left(x^{2r_{\ell_k}-2}\right) - \widehat{Q}_{\mathcal{B},r_{\ell_k}-1}\widehat{Q}_{\mathcal{A},r_{\ell_k}-1}\left(x^{2r_{\ell_k}-2}\right)\right\| \\
&\leq \left\|s' - x^{2r_{\ell_k}-2}\right\| + \left\|P_B P_A\left(x^{2r_{\ell_k}-2}\right) - \widehat{Q}_{\mathcal{B},r_{\ell_k}-1}\widehat{Q}_{\mathcal{A},r_{\ell_k}-1}\left(x^{2r_{\ell_k}-2}\right)\right\| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad (6.3)
\end{aligned}
$$

Therefore, $P_B P_A(s')$ is an accumulation point of the subsequence $\{x^{2r_\ell}\}_{\ell=1}^{\infty}$, and since we know that this subsequence also converges to s by the choice of $\{x^{2r_\ell}\}_{\ell=1}^{\infty}$, it follows that $s = P_B P_A(s') \in P_B P_A(S)$. Since s was an arbitrary point in S, we conclude that $S \subseteq P_B P_A(S)$, and since we already know that $P_B P_A(S) \subseteq S$, we conclude that $P_B P_A(S) = S$. By Lemma 30, S consists of points of B nearest to A.

Similarly, the set $\tilde{S}$ of all accumulation points of the odd subsequence $\left\{x^{2k+1}\right\}_{k=0}^{\infty}$ is contained in A and satisfies $\tilde{S} = P_A P_B\left(\tilde{S}\right)$, and so $\tilde{S}$ consists of points of A nearest to B by Lemma 30. Let $\tilde{a} \in \tilde{S}$. As explained before, Lemma 30 implies that $(\tilde{a}, P_B(\tilde{a}))$ is a best approximation pair relative to (A,B). Since we assume that there is a unique best approximation pair $(a,b) \in A \times B$ relative to (A,B), we conclude, in particular, that $\tilde{a} = a$. Hence, $\tilde{S} = \{a\}$, and similarly $S = \{b\}$. Therefore, $\left\{x^{2k}\right\}_{k=0}^{\infty}$ has a unique accumulation point, namely it converges to b, and $\left\{x^{2k+1}\right\}_{k=0}^{\infty}$ has a unique accumulation point, namely it converges to a. Now the continuity of the norm and the fact that (a,b) is a best approximation pair relative to (A,B) imply that $\lim_{k\to\infty}\left\|x^{2k} - x^{2k+1}\right\| = \|b-a\| = dist(A,B)$, as required. Finally, Proposition 16 implies that the conclusion of the theorem holds when all the sets A_i and B_j, $i \in \{1,2,\ldots,I\}$ and $j \in \{1,2,\ldots,J\}$, are strictly convex. □

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