Fighting Terrorism: How to position Rapid Response Teams?

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Abstract

In light of recent terrorist attacks, we introduce and study a Stackelberg game between a government and a terrorist. In this game, the government positions a number of heavily-armed rapid response teams on a line segment (e.g., a long boulevard or shopping avenue) and then the terrorist attacks a location with the highest damage. This damage is the product of the time it takes the closest rapid response team to react and the damage caused per time unit, which is modelled via a damage rate function. We prove that there exists a subgame perfect Nash equilibrium that balances the possible damage on all intervals of the line segment that result from positioning the rapid response teams. We discuss the implications for various types of damage rate functions.

Keywords: Counter-terrorism, Stackelberg Game, Resource Allocation

1. Introduction

Terrorism has claimed hundreds of innocent lives in Europe over the past years. Examples are the Bataclan attack in Paris, the van attack on Barcelona's la Rambla, and the truck incident in Nice. These attacks have also affected the economy by destruction of property, increased market uncertainty and loss of tourism resulting in monetary losses in the order of millions.

Improving the protectability against terrorism is an important societal concern. To address this concern, governmental agencies have formulated several strategic counter terrorism agendas throughout the years (Bossong, 2021). A recent initiative in these strategic counter terrorism agendas is the deployment of rapid response teams, which we refer to as "response teams" in this paper. These heavily-armed and highly-trained response teams are located at high potential attack locations and are capable to respond to attacks within minutes. It is important to select the location of these response teams carefully: on the one hand they should be positioned close enough to high risk spots (i.e., places where many people cluster), on the other hand they should also be sufficiently dispersed to cover many risk spots within reasonable time. Models and methods that are closely related to this positioning problem can be found in location theory.

The theory of locations focuses on the positioning of resources, such as supermarkets and warehouses (Eiselt et al., 2004), as well as the allocation of resources to incidents, such as ambulances and snow removal vehicles (Wang and Liu, 2019). In this theory, it is typically assumed

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that demand (e.g., consumers that need to go to a supermarket or patients who need an ambulance) is independent of the position of the resources. However, in our setting, demand is a result of the decision of a terrorist. For instance, a terrorist might observe potential attack locations and security measures upfront, and based on that information decides on which location(s) to attack. As a consequence, results from the field of location theory cannot be directly applied to our setting.

To account for dependent/strategic demand one can make use of game theory, which analyses and models the strategic behaviour of decision makers (Osborne, 2004). One specific type of game that we can find in this field is a sequential game. Sequential games represent situations where players make a move one after the other. These games have been applied to terrorism protection before, including protection of critical infrastructure (see, e.g., Bier et al. (2007), Zhuang and Bier (2007), Scaparra and Church (2008), Powell (2009), Brown et al. (2011), Hausken and Zhuang (2011), Holzmann and Smith (2019), Jiao and Luo (2019) and Musegaas et al. (2022)) and security of airports (see, e.g., Pita et al. (2009) Korzhyk et al. (2011) and Shieh et al. (2012)). The majority of these papers do not explicitly model the geographical positioning of protection resources. Berman and Gavious (2007) is an exception in that regard. This paper studies a Stackelberg game where the leader first allocates a number of supportive resources (e.g., ambulances) over a network and the follower subsequently attacks one node in this network: the one that leads to the highest expected damage for the leader. This damage depends on the initial value of the nodes as well as the time to transport the closest resource to the attack node. The paper solves this problem efficiently for the case where one resource needs to be positioned somewhere in the network (i.e., on a node or arc). The paper also considers the setting where multiple resources need to be positioned somewhere on nodes only. This problem is formulated as an integer linear programming problem and is illustrated via a case study of the 20 largest cities in the US. A similar type of problem has been addressed in Meng et al. (2013), except that multiple resources, each with capacity constraints, can be used in response to an attack. The paper formulates the problem as an integer linear program and applies the program to a case study with 19 districts in Shanghai.

Clearly, Meng et al. (2013) and Berman and Gavious (2007) could be useful to our setting, especially when realizing that the supportive resources have a similar role as our response teams. However, both papers assume the number of potential attack and resource locations to be limited when multiple supportive resources need to positioned (around 20 nodes). This is in sharp contrast to our setting, in which attack locations and response team locations are not limited to a finite set of points. As such, we decided to model our setting by a continuous space environment rather than using the discrete network setting as suggested in Berman and Gavious (2007) and Meng et al. (2013). Continuous location problems are, however, typically notoriously hard problems (see, e.g., García and Marín (2015)). Therefore, in this paper, we focus on a continuous line segment only, rather than on, for example, a 2-dimensional space environment.

In particular, we propose a leader follower game in which the leader (e.g., government) positions response teams on a line segment. This line segment could, for instance, represent a shopping avenue or boulevard such as the Ocean drive in Miami, La Rambla in Barcelona, or the strip in Las Vegas. We assume that after the response teams are positioned, one terrorist attack takes place. This attack will occur at a location that maximizes the damage to the government. Here, a proxy for the damage is the product of the time it takes the closest response team to react and the damage that an attack causes per time unit per location. We represent this damage per time unit per location by a damage rate function over the interval of the line segment. The related damage of an attack can be seen as a composition of the amount of people that are affected and the damage to the surroundings such as buildings or statues. The leader wants to position the response teams such that the damage of the attack is minimized. Thus, the leader minimizes the maximum damage of an anticipated attack, which boils down to solving a min-max problem. We prove that there exists a subgame perfect Nash equilibrium such that the strategy of the leader in this equilibrium balances the damage between each pair of response team locations. We call the strategy of the leader in this subgame perfect Nash equilibrium an optimal strategy of the leader. We prove that an optimal strategy of the leader balances the possible damage on all intervals of the line segment that result from positioning the response teams. We do so by decomposing the inner maximization problem into several local subproblems, where every subproblem resembles the local maximal damage on an interval. By using Berge's maximum theorem, we are able to prove that we can always reposition a single response team such that the damages of the two neighbouring intervals coincide. We exploit this idea to the first two intervals and subsequently balance the local damage of the third interval with the first two intervals, while keeping them balanced as well. By iteratively using this procedure, we can ultimately prove the existence of an optimal strategy that balances the local damage of all intervals. We also discuss the implications for various types of damage rate functions. In particular, we study a setting where (i) damage is more or less equal over the line segment, (ii) damage increases from one side to the other, and (iii) damage is centered around a specific point. For each of these cases, we derive expressions for the optimal positioning of response teams. Moreover by analyzing these three cases, we learn that the response teams are positioned closer to each other in more centered areas. It also turns out that if the people or buildings are more centered, one can better protect against the consequences of a terrorist attack. Moreover, we illustrate that in most cases damage reduces when the number of response teams increases. This is, however, not always the case in the strict sense. It turns out that in some specific scenarios adding an extra response team will not reduce the damage of an attack!

From a mathematical perspective, our paper has some overlap with the continuous p-center problem. A p-center consists of a set of p resources that minimizes the maximum distance between a demand point and a closest resource belonging to that set (Calik, Labbé, et al., 2015). In the classical p-center problem, the resources and the demand points are positioned only on the nodes of the graph (see, e.g., Mladenovic et al. (2003), Tansel (2011), Calik and Tansel (2013) and Davidovic et al. (2011)). Additionally, the p-center problem has multiple variants, one of them being the continuous p-center problem, where the resources can be positioned anywhere on the graph and the demand can be anywhere on the graph (see, e.g., Chandrasekaran and Tamir (1980), Tamir (1987), Tamir (1988) and Hansen et al. (1991)). Another variant of the p-center problem is the weighted p-center problem in which the resources can also be positioned anywhere, but the demand points are positioned on the nodes of the graph and have different weights (see, e.g., Averbakh and Berman (1997), Burkard and Dollani (2003), Calik and Tansel (2013) and Bhattacharya and Shi (2007)). Similar to the p-center problem, we also consider a min-max problem to position resources. However, the continuous p-center problem considers distance as the only objective, while we focus on the product of distance and a damage rate function. Additionally, the weighted p-center problem considers the demand to be only on the nodes of the graph, while we consider demand to be anywhere on the line segment. Thus, our problem can be seen as a combination of the continuous p-center problem and the weighted p-center problem for a graph with only two nodes and one edge, which is outlined in Table 1. Our problem with a uniform damage rate function is the same as the continuous p-center problem with two nodes and one edge in between the nodes. More specifically, the optimal value of the objective function in Chandrasekaran and Tamir (1980) and Tamir (1987) for a graph with two nodes and one edge of length one, is the same as the optimal value of the objective function in our problem with a uniform damage rate function.

	Discrete	Continuous
Not weighted	Mladenovic et al. (2003) Tansel (2011) Calik and Tansel (2013) Davidovic et al. (2011)	Chandrasekaran and Tamir (1980) Tamir (1987) Tamir (1988)
Weighted	Averbakh and Berman (1997) Burkard and Dollani (2003) Calik and Tansel (2013) Bhattacharya and Shi (2007)	Our model

Table 1: The p-center problem

The rest of the paper is organized as follows. Section 2 provides definitions and results which will be used frequently in several proofs of this paper. Section 3 introduces the Stackelberg protection location game and Section 4 studies a best response of the follower and the leader, respectively. In Section 5 different instances of the game are evaluated. Finally, Section 6 concludes.

2. Preliminaries

In this section we provide definitions and results essential for several proofs in the paper. Let S be a subset of $\mathbb{R}^m$ and T be a subset of $\mathbb{R}^n$. A *correspondence* is a relation $\varphi : S \to T$ such that $\varphi(x) \neq \emptyset$ for every $x \in S$. A correspondence φ is *compact-valued* if $\varphi(x)$ is *compact* at all $x \in S$. Moreover, a correspondence φ is *continuous* if it is *upper hemicontinuous* and *lower hemicontinuous* at all $x \in S$. Below, we provide a necessary and sufficient condition for a correspondence to be upper (lower) hemicontinuous at $x \in S$. In this paper it suffices to use these conditions, see Herings (1996) for an explicit definition of upper (lower) hemicontinuity.

Theorem 2.1 (Herings (1996))**.** *Let S be a subset of $\mathbb{R}^m$, let T be a subset of $\mathbb{R}^n$, let $\bar{x}$ be an element of S, and let $\varphi : S \to T$ be a compact-valued correspondence. Then the correspondence φ is upper hemicontinuous at $\bar{x}$ if and only if for every sequence $(x^n)_{n\in\mathbb{N}}$ in S converging to $\bar{x}$ and every sequence $(y^n)_{n\in\mathbb{N}}$ in T with $y^n \in \varphi(x^n)$, for all $n \in \mathbb{N}$, converging to $\bar{y} \in \mathbb{R}^n$ it holds that $\bar{y} \in \varphi(\bar{x})$.*

Theorem 2.2 (Herings (1996))**.** *Let S be a subset of $\mathbb{R}^m$, let T be a subset of $\mathbb{R}^n$, and let $\bar{x}$ be an element of S. Then the correspondence φ is lower hemicontinuous at $\bar{x}$ if and only if for every sequence $(x^n)_{n\in\mathbb{N}}$ in S converging to $\bar{x}$ and for every element $\bar{y}$ of $\varphi(\bar{x})$ there exists a sequence $(y^n)_{n\in\mathbb{N}}$ in T such that $y^n \in \varphi(x^n)$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} y^n = \bar{y}$.*

We now provide a specific case of a correspondence that is compact-valued.

Lemma 2.3. *Let S be a subset of $\mathbb{R}^m$ and let T be a subset of $\mathbb{R}^n$. Additionally, let $p : S \to \mathbb{R}$ and $q : S \to \mathbb{R}$ be two continuous and bounded functions such that $p(x) \leq q(x)$ for all $x \in S$. Then the correspondence $\varphi : S \to T$ with $\varphi(x) = [p(x), q(x)]$ for all $x \in S$ is continuous and compact-valued, and $\varphi(x)$ is non-empty for all $x \in S$.*

Proof. See Appendix. □

Finally, we provide Berge's theorem which gives sufficient conditions for the continuity of a specific class of optimized functions.

Theorem 2.4 (Berge's maximum theorem (Herings (1996))). *Let S be a subset of $\mathbb{R}^m$, let T be a subset of $\mathbb{R}^n$, and let $\varphi : S \to T$ be a continuous, compact-valued correspondence. Let $h : S \times T \to \mathbb{R}$ be a continuous function and let the relation $g : S \to \mathbb{R}$ be defined by $g(x) = \max_{y \in \varphi(x)} h(x, y)$, for all $x \in S$. Then g is a continuous function.*

3. Model

We introduce Stackelberg protection location games (SPL games) where a government is the leader and the opponent (e.g., a terrorist) the follower. In such a game, first the leader has to position $n \in \mathbb{N}$ response teams on line segment $[0, 1]$. We denote the location of these response teams by $d = (d_1, \dots, d_n) \in D^n$, where $D^n = \{d \in [0, 1]^n \mid d_1 \leq d_2 \leq \dots \leq d_n\}$. The speed of these response teams is assumed to be constant and is denoted by $v \in \mathbb{R}_{>0}$. Once the leader has positioned the response teams at $d \in D^n$, the follower selects an attack location on the interval denoted by $a \in A$, where $A = [0, 1]$. This decision is based on the amount of damage that an attack causes. The damage is determined by two components. The first component is the time it takes the closest response team to be at a given attack location, which equals

$$\min_{i \in \{1,\dots,n\}} \frac{|d_i - a|}{v} \text{ for all } d \in D^n.$$

The second component is the continuous damage rate function $f : [0, 1] \to \mathbb{R}_{\geq 0}$ which represents the damage of an attack per time unit. The product of these two components equals the damage of an attack and is denoted by

$$D(d, a) = \min_{i \in \{1,..,n\}} \frac{|d_i - a|}{v} \cdot f(a) \text{ for all } a \in A \text{ and all } d \in D^n.$$

The follower tries to maximize this damage, while the leader tries to minimize it. Formally, the leader is interested in solving the following optimization problem: [3]

$$\mathscr{P} := \quad \min_{d \in D} \max_{a \in A} D(d, a). \tag{1}$$

Please, note that the speed of the response teams is a constant. Consequently, it does not have an impact on the strategies of both the leader and follower and thus can be disregarded in the model. For that reason, in the remainder of this paper, we assume that the speed of the response teams equals $v = 1$. We denote a specific SPL situation by $\theta = (n, f)$ with n the number of response teams and f the damage rate function. We denote the set of all SPL situations by Θ.

We conclude this section with an example of an SPL situation. Note that the parameters are not set to represent reality, but to keep the calculations simple and easy to follow.

[3] Formally, the maximum and minimum should be reformulated as a supremum and infimum, respectively. We restrict attention to the maximum and the minimum as they are both well-defined.

Example 1. *Let $\theta \in \Theta$ with $n = 2$ and $f(x) = 1$ for all $x \in [0, 1]$. Suppose the leader positions its response teams at $d = (0.2, 0.8)$ and the follower attacks at $a = 0.4$. Then, $D((0.2, 0.8), 0.4) = \min\{|0.2 - 0.4|, |0.8 - 0.4|\} \cdot 1 = 0.2$. This situation is visualized in Figure 1.*

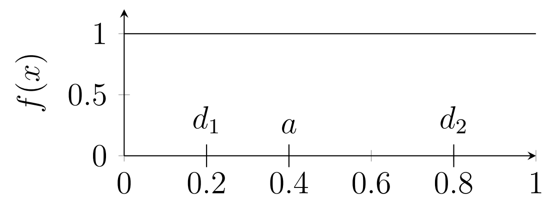


Figure 1: Visual representation of θ with $n = 2$, $f(x) = 1$ for all $x \in [0, 1]$, $d = (0.2, 0.8)$, and $a = 0.4$.

◇

4. Analysis of SPL games

Given the sequential order in which the leader and follower make a decision in this Stackelberg game, we start in Section 4.1 with studying a specific best response for the follower. Thereafter, in Section 4.2, we focus on a strategy of the leader within a specific subgame perfect Nash equilibrium. We refer to this as an optimal strategy of the leader or an optimal solution.

4.1. Response follower

In this section, we study the response of the follower. In doing so, let us first reconsider Example 1.

Example 2. *Reconsider θ of Example 1. In order to investigate an optimal response of the follower, we first visualize $D(d, a)$ with $d = (0.2, 0.8)$ and for $a \in A$. This is represented in Figure 2.*

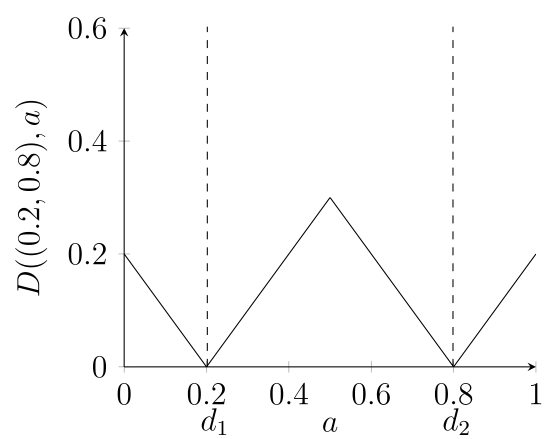


Figure 2: $D(0.2, 0.8), a)$ for $a \in A$.

To identify an optimal response of the follower, we divide Figure 2 in three areas; to the left of d_1, in between d_1 and d_2 and to the right of d_2. In the left area it is optimal to attack at $a = 0$, for which $D(d, 0) = 0.2$. In the middle area it is optimal to attack at $a = 0.5$, for which $D(d, 0.5) = 0.3$, and *in the right area it is optimal to attack at* $a = 1$*, for which* $D(d, 1) = 0.2$*. Comparing the associated damages per area, we conclude that the damage is maximized when the follower attacks at* $a = 0.5$ *with corresponding damage* $D(d, 0.5) = 0.3$*.*

◇

In Example 2, we separated the maximization problem of the follower into several local maximization problems. This turns out to be useful in general. For every SPL situation $\theta \in \Theta$ and $d \in D^n$, we separate the maximization problem of the follower, i.e., the inner maximization problem in equation (1), into $n + 1$ local maximization problems. We refer to them as *local damage problems* and call their optimal values the *local damages*. The first local damage problem, which identifies the maximal damage to the left of the first response team, is denoted by $\mathscr{L} : [0, 1] \to \mathbb{R}$ with

$$\mathscr{L}(d_1) = \max_{a \in [0,d_1]} f(a) \cdot (d_1 - a) \text{ for all } d_1 \in [0, 1].$$

Then, we introduce $n - 1$ local damage problems that each identify the maximal damage between two adjacent response teams. Let $i \in \{1, 2, ..., n - 1\}$ and denote $D^{i,i+1} = \{(d_i, d_{i+1}) \in [0, 1]^2 \mid d_i \leq d_{i+1}\}$, which is the set of feasible locations of response teams i and $i + 1$. Then, the local damage problem between response team i and $i + 1$ is given by $\mathscr{I}^i : D^{i,i+1} \to \mathbb{R}$ with

$$\mathscr{I}^i(d_i, d_{i+1}) = \max_{a \in [d_i,d_{i+1}]} f(a) \cdot \min\{a - d_i, d_{i+1} - a\} \text{ for all } (d_i, d_{i+1}) \in D^{i,i+1}.$$

The last local damage problem, which identifies the maximal damage to the right of the last response team, is denoted by $\mathscr{R} : [0, 1] \to \mathbb{R}$ with

$$\mathscr{R}(d_n) = \max_{a \in [d_n,1]} f(a) \cdot (a - d_n) \text{ for all } d_n \in [0, 1].$$

The damage realized by the follower can be determined by taking the maximum over all local damages. That is, for every $d \in D^n$, the damage realized under a best response reads

$$\max_{a \in A} D(d, a) = \max \left\{ \mathscr{L}(d_1), \mathscr{I}^1(d_1, d_2), \mathscr{I}^2(d_2, d_3), \ldots, \mathscr{I}^{n-1}(d_{n-1}, d_n), \mathscr{R}(d_n) \right\}. \tag{2}$$

We now illustrate this by means of our leading example.

Example 3. *Reconsider* θ *of Example 2. We can separate the inner maximization problem of the follower in the following three local damage problems:*

$$\begin{aligned} \mathscr{L}(d_1) &= \max_{a \in [0,d_1]} 1 \cdot (d_1 - a) = d_1 \textit{ for all } d_1 \in [0, 1], \\ \mathscr{I}^1(d_1, d_2) &= \max_{a \in [d_1,d_2]} 1 \cdot \min\{a - d_1, d_2 - a\} = \frac{d_2 - d_1}{2} \textit{ for all } (d_1, d_2) \in D^{1,2}, \\ \mathscr{R}(d_2) &= \max_{a \in [d_2,1]} 1 \cdot (a - d_2) = 1 - d_2 \textit{ for all } d_2 \in [0, 1]. \end{aligned}$$

For $d = (0.2, 0.8)$ *we obtain* $\mathscr{L}(0.2) = 0.2$*,* $\mathscr{I}^1(0.2, 0.8) = 0.3$*, and* $\mathscr{R}(0.8) = 0.2$*. So, the damage under a best response equals* $\max_{a \in A} D((0.2, 0.8), a) = \max\{0.2, 0.3, 0.2\} = 0.3$*. Note that this formalizes what has been established in Example 2.* ◇

4.2. Strategy leader

In this section, we focus on a leader's strategy within a subgame perfect Nash equilibrium. As already mentioned in the introduction of this paper, we refer to this as an optimal strategy of the leader. We start by characterising an optimal strategy of the leader for our leading example.

Example 4. *Reconsider θ of Example 3. To find an optimal strategy of the leader, we solve $\mathscr{P}$ in two steps. First, we fix a location $d_2' \in [0,1]$ and subsequently optimize the location of d_1. Since the local damage of $\mathscr{R}(d_2')$ is not affected by this decision, our subproblem reads*

$$\begin{aligned} \min_{d_1 \in [0,d_2]} \max\{\mathscr{L}(d_1), \mathscr{I}^1(d_1, d_2'), \mathscr{R}(d_2')\} &= \min_{d_1 \in [0,d_2']} \max\left\{d_1, \frac{d_2' - d_1}{2}, 1 - d_2'\right\} \\ &= \max\{\frac{1}{3}d_2', 1 - d_2'\}. \end{aligned} \tag{3}$$

The minimum is attained at $d_1^* = \frac{1}{3}d_2'$ where $\mathscr{L}(d_1^*) = \mathscr{I}^1(d_1^*, d_2') = \frac{1}{3}d_2'$ (i.e., where the local damages of the first two local damage problems are equal to each other). Note that this is not always a unique minimum, if $\frac{1}{3}d_2' \leq 1 - d_2'$ there exist multiple points where the minimum is attained. In Figure 3b, we illustrate the outcome of the optimization of d_1 given that $d_2' = 0.8$. By comparing this figure with the one of Figure 3a, we see that the local damage of $\mathscr{I}^1$ has decreased from 0.3 to $\frac{4}{15}$ and the local damage of $\mathscr{L}$ has increased from 0.2 to $\frac{4}{15}$. Since the local damage of $\mathscr{R}$ remains constant, we can conclude that the damage of $\mathscr{P}$ decreased from 0.3 to $\frac{4}{15}$.

The second step is to optimize d_2', while keeping an optimized solution $d_1^* = \frac{1}{3}d_2'$ [4]. This leads to

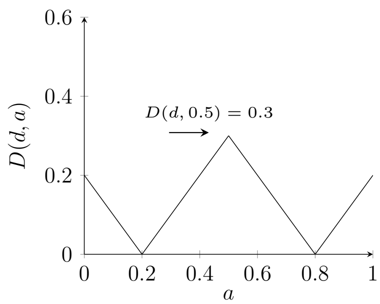
$$\min_{d_2' \in [0,1]} \max\left\{\mathscr{L}(d_1^*), \mathscr{I}^1(d_1^*, d_2'), \mathscr{R}(d_2')\right\} = \min_{d_2' \in [0,1]} \max\left\{\frac{1}{3}d_2', \frac{1}{3}d_2', 1 - d_2'\right\} = \frac{1}{4}.$$

The minimum is attained at $d_2^* = \frac{3}{4}$, and so $d_1^* = \frac{1}{4}$. For this solution, we also have an equivalence between the local damages of $\mathscr{I}^1$ and $\mathscr{R}$. Consequently, we obtain

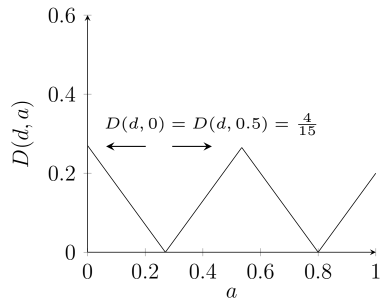
$$\mathscr{L}(d_1^*) = \mathscr{I}^1(d_1^*, d_2^*) = \mathscr{R}(d_2^*) = \frac{1}{4}. \tag{4}$$

Hence, under optimal $d^* = (\frac{1}{4}, \frac{3}{4})$, the local damage of each local damage problem is the same, implying that the follower is indifferent between attacking in the interval $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$. In Figure 3c, we illustrate the setting with $(d_1^*, d_2^*) = (\frac{1}{4}, \frac{3}{4})$. By comparing this figure with the one of Figure 3b, we see that local damages of $\mathscr{L}$ and $\mathscr{I}^1$ have been reduced simultaneously until they meet with the local damage of the $\mathscr{R}$, leading to a (local) damage of $\frac{1}{4}$ for $\mathscr{L}$, $\mathscr{I}^1$, and $\mathscr{R}$.

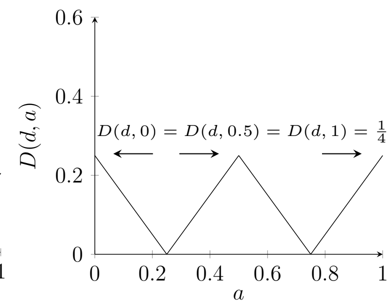
[4] Formally $d_1^*(d_2') = \frac{1}{3}d_2'$ as d_1^* is dependent on d_2'.



(a) $d = (0.2, 0.8)$.



(b) $d = (\frac{4}{15}, 0.8)$.



(c) $d = (0.25, 0.75)$.

Figure 3: The damage of an attack when d_1 and d_2 are repositioned

◇

In Example 4, we illustrated the existence of an optimal strategy $d^* \in D^n$ for which $\mathscr{L}(d_1^*) = \mathscr{I}^1(d_1^*, d_2^*) = \mathscr{R}(d_2^*)$. In the remainder of this section, we show that there always exists such an optimal strategy. That is, there always exists an optimal strategy for the leader for which the local damages of all local damage problems are exactly the same. We refer to such a strategy as a *balanced strategy*. We now explain how we prove existence and optimality of a balanced strategy.

For every $\theta \in \Theta$ and $i \in \{2, 3, \dots, n\}$ we introduce a function $\mathscr{L}^i : [0, 1] \to \mathscr{R}$. This function $\mathscr{L}^i$ determines the minimal damage to the left of response team i and as such can be recognized as a subproblem of $\mathscr{P}$, restricted to the left of d_i. Formally, function $\mathscr{L}^i$ is recursively defined as follows:

$$\mathscr{L}^i(d_i) = \min_{d_{i-1} \in [0, d_i]} \max\{\mathscr{L}^{i-1}(d_{i-1}), \mathscr{I}^{i-1}(d_{i-1}, d_i)\} \text{ for all } d_i \in [0, 1], \tag{5}$$

with $\mathscr{L}^1 = \mathscr{L}$.

By exploiting the recursive structure of $\mathscr{L}^i$, our problem $\mathscr{P}$ can be reformulated as

$$\min_{d_n \in [0,1]} \max\{\mathscr{L}^n(d_n), \mathscr{R}(d_n)\}. \tag{6}$$

Now we prove the existence of a balanced strategy and the optimality of all balanced strategies by showing the following four steps:

1. There exists a $d' \in D^n$ that solves the following system of equations:

$$\begin{aligned} &\mathscr{L}^n(d'_n) = \mathscr{R}(d'_n) \\ &\mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i) \text{ for all } i \in \{2, 3, \dots, n\} \end{aligned} \tag{7}$$

2. If d' solves (7), then the minimum in (6) is attained at d'_n and the minimum in (5) with $d_i = d'_i$ is attained at d'_{i-1} for all $i \in \{2, 3, \dots, n\}$. Thus, d' is optimal.

3. If d' solves (7), then d' is balanced.

4. All balanced strategies have the same damage.

For the first step, we need to prove that there exists a $d' \in D^n$ that solves (7). Since $\mathscr{R}(1) = \mathscr{L}^n(0) = 0$, it suffices to show that $\mathscr{R}$ is continuous and non-increasing and $\mathscr{L}^n$ continuous and non-decreasing. Because $\mathscr{L}^n$ has a recursive structure, we also need to show continuity and non-decreasing/increasing properties of $\mathscr{L}^i$ for all $i \in \{1, 2, \ldots, n\}$ and $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$.

We start by showing that $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$ are continuous. we do so by using Berge's maximum theorem (see Theorem 2.4). This theorem provides sufficient conditions for continuity of an optimized function and we exactly check for these properties to prove our case.

Lemma 4.1. *Let $\theta = (n, f) \in \Theta$. $\mathscr{L}$, $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$ and $\mathscr{R}$ are continuous.*

Proof. See Appendix. □

We also show the non-increasing/non-decreasing behavior of $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$. To prove these results, we each time evaluate the functions on two different response team locations. We then show that an optimal solution of the optimization problem associated to the function of one of these response team locations is a feasible solution of the optimization problem associated to the function of the other response team location.

Lemma 4.2. *Let $\theta = (n, f) \in \Theta$. $\mathscr{L}$ is non-decreasing, $\mathscr{R}$ is non-increasing, and for all $i \in \{1, ..., n-1\}$ the following holds*

(i) $\mathscr{I}^i(\cdot, d_{i+1})$ *is non-increasing for all* $d_{i+1} \in [0, 1]$

(ii) $\mathscr{I}^i(d_i, \cdot)$ *is non-decreasing for all* $d_i \in [0, 1]$.

Proof. See appendix. □

By combining Lemma 4.1 and Lemma 4.2, we can show that $\mathscr{L}^i$ is continuous and non-decreasing for all $i \in \{2, 3, \ldots, n\}$. For continuity, we apply Berge's maximization problem taking into account that (i) a minimization problem can be reformulated as a maximization problem and (ii) the maximum of two continuous functions (using an inductive argument on $\mathscr{L}^{i-1}$ and $\mathscr{I}^{i-1}$) is still continuous. Non-decreasing can be shown using similar arguments as in Lemma 4.2.

Lemma 4.3. *Let $\theta = (n, f) \in \Theta$. $\mathscr{L}^i$ is continuous and non-decreasing for all $i \in \{1, 2, ..., n\}$.*

Proof. See Appendix. □

Since $\mathscr{R}$ is continuous and non-increasing (Lemma 4.1 and Lemma 4.2), $\mathscr{L}^n$ is continuous and non-decreasing (Lemma 4.3), and $\mathscr{R}(1) = \mathscr{L}^n(0) = 0$, there exists a $d'_n \in [0, 1]$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$. Additionally, via an inductive way, it holds that there exists a $d'_{i-1} \in [0, d'_i]$ such that $\mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i)$ for all $i \in \{2, 3, \ldots, n\}$. Thus, the first step of our four step plan holds and we formalize this in the upcoming lemma.

Lemma 4.4. *Let $\theta = (n, f) \in \Theta$. There exists a $d \in D^n$ that solves (7).*

Proof. See Appendix. □

By Lemma 4.4, we know that there exists a $d \in D^n$ that solves (7). As $\mathscr{L}^i$ is non-decreasing for all $i \in \{1, \ldots, n\}$ and $\mathscr{I}^{i-1}$ is non-increasing in its first argument for all $i \in \{2, \ldots, n\}$, we know that (7) leads to a minimum for (5) and (6) with $d_i = d'_i$ is attained at d'_{i-1} for all $i \in \{2, 3, \ldots, n\}$. This is shown formally in the following lemma.

Lemma 4.5. *Let $\theta = (n, f) \in \Theta$. If $d' \in D^n$ solves (7), the minimum of (6) is attained at d'_n. Thus, d'_n is optimal.*

Proof. See Appendix. □

For the third step, we prove that if d' solves (7), then d' is balanced. We, again, make use of the fact that the minimum of (6) is attained at d'_n and the minimum of (5) with $d_i = d'_i$ is attained at d'_{i-1} for all $i \in \{2, 3, \ldots, n\}$. From this, we can conclude that $\mathscr{L}^i(d'_i) = \mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i)$ for all $i \in \{2, 3, \ldots, n\}$. By exploiting this relationship several times we show that d' is balanced.

Lemma 4.6. *Let $\theta = (n, f) \in \Theta$. If $d' \in D^n$ solves (7), then d' is balanced.*

Proof. See Appendix. □

For the fourth step, we prove that all balanced strategies lead to the same damage. We do so by showing that it cannot be true that two balanced strategies exist that lead to different damages.

Lemma 4.7. *Let $\theta = (n, f) \in \Theta$. All balanced strategies lead to the same damage.*

Proof. See Appendix. □

By combining Lemma 4.4, Lemma 4.5, and Lemma 4.6, we can conclude that there always exists a balanced strategy that is optimal. Combining this with the fact that all balanced strategies lead to the same damage (Lemma 4.7), we know that all balanced strategies are optimal. Together, this results in the following main result.

Theorem 4.8. *Let $\theta = (n, f) \in \Theta$. There exists a balanced strategy and any balanced strategy is optimal.*

Proof. See Appendix. □

We would like to stress that balancedness is a sufficient, but not a necessary condition for an optimal strategy. This is illustrated in the upcoming example.

Example 5. *Let $\theta \in \Theta$ with $n = 4$, and damage rate function f:*

$$f(x) = \begin{cases} 20x & x \in [0, 0.05] \\ 2 - 20x & x \in (0.05, 0.1] \\ 0 & x \in (0.1, 0.4) \cup (0.6, 0.9) \\ x - 0.4 & x \in [0.4, 0.45] \\ 0.5 - x & x \in (0.45, 0.5] \\ x - 0.5 & x \in (0.5, 0.55] \\ 0.6 - x & x \in (0.55, 0.6] \\ 20x - 18 & x \in [0.9, 0.95] \\ 20 - 20x & x \in (0.95, 1]. \end{cases}$$

Moreover, let $d = (0.05, 0.2, 0.8, 0.95)$. A visual representation of f and d is presented in Figure 4.

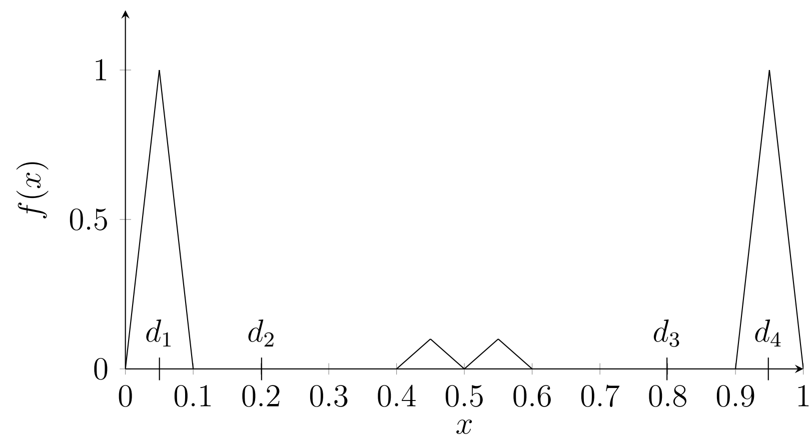


Figure 4: Visual representation of θ with $n = 4$ and $d = (0.05, 0.2, 0.8, 0.95)$.

For this situation, the first local damage problem equals

$$\mathscr{L}(0.05) = \max_{a\in[0,0.05]} 20a \cdot (0.05 - a) = \frac{1}{80},$$

with $a^* = 0.025$. The second local damage problem reads

$$\mathscr{I}^1(0.05, 0.2) = \max_{a\in[0.05,0.1]} (a - 0.05) \cdot (2 - 20a) = \frac{1}{80},$$

with $a^* = 0.075$. Please, note that we did not consider interval $(0.1, 0.2]$ in the optimization problem, because $f(a) = 0$ for all $a \in (0.1, 0.2]$. The third local damage problem equals:

$$\mathscr{I}^2(0.2, 0.8) = \max\Big\{ \max_{a_1\in[0.4,0.45]} (a_1 - 0.4) \cdot (a_1 - 0.2), \max_{a_2\in[0.45,0.5]} (0.5 - a_2) \cdot (a_2 - 0.2), \max_{a_3\in[0.5,0.55]} (a_3 - 0.5) \cdot (0.8 - a_3), \max_{a_4\in[0.55,0.6]} (0.6 - a_4) \cdot (0.8 - a_4) \Big\} = \frac{1}{80},$$

with $a_1^* = a_2^* = 0.45$ and $a_3^* = a_4^* = 0.55$. Note that for this local damage problem we divide the objective function into four different functions, such that we take into account the different cases in the damage rate function and we still assume that the closest response team reacts to the attack while getting rid of the minimization sign in the objective function. We did not consider intervals $[0.2, 0.4)$ and $(0.6, 0.8]$, because $f(a) = 0$ for $a \in [0.2, 0.4) \cup (0.6, 0.8]$. The fourth local damage problem is similar to the second local damage problem and equals

$$\mathscr{I}^3(0.8, 0.95) = \max_{a\in[0.9,0.95]} (0.95 - a) \cdot (20a - 18) = \frac{1}{80},$$

with $a^* = 0.925$. Note that we did not consider interval $[0.8, 0.9)$ in the optimization problem, because $f(a) = 0$ for all $a \in [0.8, 0.9)$. The final local damage problem reads

$$\mathscr{R}(0.95) = \max_{a\in[0.95,1]} (20 - 20a) \cdot (a - 0.95) = \frac{1}{80}.$$

By Theorem 4.8, strategy d is optimal for the leader. Now, let $d' = (0.05, 0.4, 0.6, 0.95)$. Then, again $\mathscr{L}(0.05) = \mathscr{R}(0.95) = \frac{1}{80}$. Moreover, because $f(a) = 0$ for $a \in (0.2, 0.4) \cup (0.6, 0.8)$, we have $\mathscr{I}^1(0.05, 0.4) = \mathscr{I}^3(0.6, 0.95) = \frac{1}{80}$. However, for the remaining local damage problem, we obtain

$$\begin{aligned}\mathscr{I}^2(0.4, 0.6) &= \max\Big\{\max_{a_1\in[0.4,0.45]}(a_1 - 0.4)\cdot(a_1 - 0.4), \max_{a_2\in[0.45,0.5]}(0.5 - a_2)\cdot(a_2 - 0.4),\\ &\qquad \max_{a_3\in[0.5,0.55]}(a_3 - 0.5)\cdot(0.6 - a_3), \max_{a_4\in[0.55,0.6]}(0.6 - a_4)\cdot(0.6 - a_4)\Big\}\\ &= \frac{1}{400},\end{aligned}$$

with (also) $a_1^* = a_2^* = 0.45$ and $a_3^* = a_4^* = 0.55$. Because $\mathscr{I}^2(0.4, 0.6) = \frac{1}{400} < \frac{1}{80} = \mathscr{L}(0.05)$, d' is not a balanced strategy. However, the damage under strategy d', which is $\max\{\frac{1}{80}, \frac{1}{400}\} = \frac{1}{80}$, coincides with the damage under optimal strategy d. So, d' is optimal as well. ◇

5. Insights for different damage rate functions

In this section we apply the result of Theorem 4.8 to various damage rate functions and study the position of response teams. We start with a uniform damage rate function, then discuss a linear damage rate function and conclude with a triangular damage rate function. For each of them, we also study the impact on damage of an increasing number of response teams.

5.1. Uniform damage rate function

Let the damage rate function be *uniform*, i.e., we have $f(x) = c$ for a given $c \in \mathbb{R}_{\geq 0}$ and for all $x \in [0, 1]$. This could, for instance, represent a district where the number of people is more or less the same at every location, such as a crowded marketplace or a beach boulevard.

In Example 4, we also discussed a setting with a uniform damage rate function. There, we showed that response teams should be positioned such that the distance between an optimal attack location and the closest team is equal everywhere. This structure holds in general. That is, the distance between two subsequent response teams is constant ($\frac{1}{n}$) and the distance from the first (last) team to its closest boundary is half of this distance ($\frac{1}{2n}$). This result is now formalized.[5]

Theorem 5.1. *Let $\theta = (n, f) \in \Theta$ with f being uniform. An optimal location of the response teams is given by $d_i^* = \frac{2i-1}{2n}$ for all $i \in \{1, .., n\}$, with associated damage $\frac{c}{2n}$.*

Proof. See appendix. □

In Figure 5, we visualise this damage for $c = 1$ and various numbers of response teams. Note that the marginal reduction in damage is decreasing in the number of response teams.

[5]The optimal value of the objective function for this specific instance of our problem is the same as the optimal value of the objective function in Chandrasekaran and Tamir (1980) and Tamir (1987) for a graph with two nodes and one edge of length one.

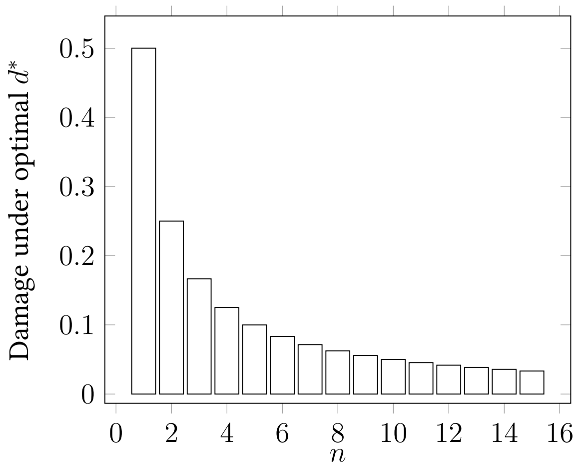


Figure 5: Damage under optimal d^* per number of response teams

5.2. Linear f

Let the damage rate function be *linearly increasing*, i.e., we have $f(x) = c \cdot x$ for a given $c \in \mathbb{R}_{\geq 0}$ and for all $x \in [0, 1]$. This could, for instance, represent a district where the number of people increases more or less linearly when going from the beginning to the end of the district, such as a shopping avenue with a market square at the end of the street. We would like to stress that this function is also a building block for the triangle function discussed later on.

We start with a useful property for linear increasing damage rate functions. It tells us how to position the response teams to the left of an arbitrary response team. In the proof, we use that the best response of the attacker is to attack exactly in the middle of any two response teams. Using this property allows us to find a relationship between the response teams, once allocated optimally.

Lemma 5.2. *Let $\theta = (n, f) \in \Theta$ with f being linearly increasing, $d \in D^n$ and $k \in \{1, 2, \ldots, n\}$. If $d_i = \sqrt{\frac{i}{k}} d_k$ for all $i \in \{1, ..., k\}$, then $\mathscr{L}(d_1) = \mathscr{I}^1(d_1, d_2) = \ldots = \mathscr{I}^k(d_{k-1}, d_k)$.*

Proof. See Appendix. □

By applying Lemma 5.2 to the n-th response team, we can characterize an optimal position of all response teams. We do so by identifying the position of the n-th response team for which the minimized damage to the left of the n-th team equals the local damage to the right of the n-th response team.

Theorem 5.3. *Let $\theta = (n, f) \in \Theta$ with f being linearly increasing. Then, an optimal location of the response teams is given by $d_i^* = 2\sqrt{i}(\sqrt{n+1} - \sqrt{n})$ for all $i \in \{1, ..., n\}$, with associated damage $\frac{c}{(\sqrt{n+1}+\sqrt{n})^2}$.*

Proof. See Appendix. □

Theorem 5.3 tells us that an optimal position of the response teams can be described via a square root relationship. This means that relatively more response teams are positioned towards the end of the line segment, which is natural since also more damage can be realized there.

Figure 6 visualises the damage under d^* for $c = 2$ and various numbers of response teams. Note that the marginal reduction in damage is decreasing in the number of teams, again.

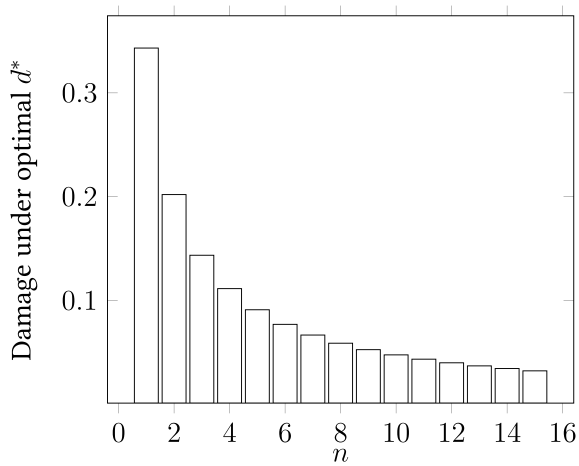


Figure 6: Damage under optimal d^* per number of response teams

Please, also note that the damages are smaller than the ones in Figure 5 (i.e., the uniform case) for each number of teams.[6] This is interesting, because the *average damage rate* is the same for both cases, namely 1.[7] This might indicate that the government is able to better protect a line segment with a linear than uniform damage rate function. The intuition is that damage is more centered for the linear case, namely to the right, which can be better protected.

5.3. Triangular f

Let the damage rate function be *triangular*, i.e., we have:

$$f(x) = \begin{cases} c \cdot x & \text{for all } x \in [0, \beta] \\ c \cdot \frac{\beta}{1-\beta} \cdot (1 - x) & \text{for all } x \in (\beta, 1] \end{cases}$$

for a given $c \in \mathbb{R}_{\geq 0}$ and $\beta \in [0, 1]$, which is the top of the triangle. This function could, for instance, represent a boulevard with one specific touristic attraction at location β. An example of this damage rate function (with $c = 1$ and $\beta = \frac{1}{3}$) is given in Figure 7.

[6] This can also be concluded from the damage expressions: $\frac{1}{2n} \geq \frac{2}{(\sqrt{n+1}+\sqrt{n})^2}$ for all $n \in \mathbb{N}_+$.

[7] We define the average damage rate as $\int_0^1 f(x)\mathrm{d}x$. For both uniform and linear increasing, we obtain 1.

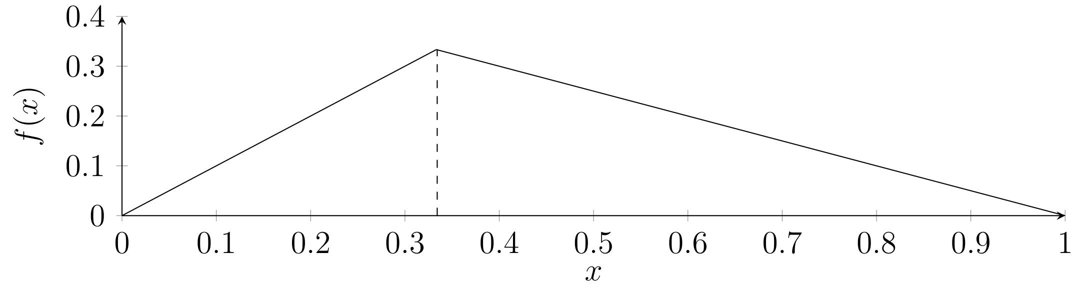


Figure 7: The damage function $f(x) = \begin{cases} x & \text{if } x \leq \frac{1}{3} \\ \frac{1}{2} \cdot (1-x) & \text{if } x > \frac{1}{3} \end{cases}$

To analyse the triangular case, we use that the triangular damage rate function is described by two linear functions. From the previous section, we already learned how to position teams optimally if the damage rate function is linearly increasing. Similarly, we can also identify an optimal position of the teams for a linear decreasing function. We do so by using our results of the linear increasing case and some relabeling of d and transformation on x.

We consider a damage rate function f to be *linearly decreasing* if $f(x) = c \cdot (1-x)$ for a given $c \in \mathbb{R}_{\geq 0}$ and for all $x \in [0, 1]$. We now present two results for this damage rate function.

Corollary 5.4. *Let $\theta = (n, f) \in \Theta$ with f being linearly decreasing, $d \in D^n$ and $k \in \{1, \ldots, n-1\}$. If $d_i = 1 - \sqrt{\frac{n+1-i}{n+1-k}} \cdot (1-d_k)$ for all $i \in \{k, \ldots, n\}$, then $\mathscr{I}^k(d_k, d_{k+1}) = \ldots = \mathscr{I}^{n-1}(d_{n-1}, d_n) = \mathscr{R}(d_n)$.*

Corollary 5.5. *Let $\theta = (n, f) \in \Theta$ with f being linearly decreasing. Then, an optimal location of the response teams is given by $d_i^* = 1 - 2\sqrt{n+1-i}(\sqrt{n+1} - \sqrt{n})$ for all $i \in \{1, ..., n\}$.*

By using Corollary 5.4 and Lemma 5.2, we can identify an optimal location of the response teams for a very specific instance of the triangular damage rate function. Namely the one where, if the teams are positioned such that there is balancedness, exactly one of the response teams is positioned at the top (i.e., at β). In total, there exist n of such instances, the one where the first response team is on the top, the second one is on the top, and so on.

We identify the n instances (i.e., we describe β) as follows. We first assume that response team $k \in \{1, 2, \ldots, n\}$ is positioned at the top. Then, we know how to optimally position the teams to the left of k (using Lemma 5.2), as well as how to optimally position the response teams to the right of k (using Corollary 5.4). Consequently, we can also identify the local damage to the left of response team k as well to the right of k. Subsequently, if the location of β is such that the local damages are equal to each other, we have found an optimal solution. It turns out that $\beta = \frac{k}{n+1}$.

Theorem 5.6. *Let $\theta = (n, f) \in \Theta$ with f being triangular. If $\beta = \frac{k}{n+1}$ for some $k \in \{1, \ldots, n\}$, then an optimal location of the response teams is given by (8), with associated damage $\frac{c \cdot k}{4(n+1)^2}$.*

$$d_i^* = \begin{cases} \sqrt{\frac{i}{k}} \cdot \beta & \text{for all } i \in \{1, \ldots, k-1\} \\ \beta & \text{for } i = k \\ 1 - \sqrt{\frac{n+1-i}{n+1-k}} \cdot (1-\beta) & \text{for all } i \in \{k+1, \ldots, n\} \end{cases} \tag{8}$$

Proof. See Appendix. □

Figure 8 shows the damage per number of response teams when $\beta = \frac{1}{2}$ and $c = 4$, so when the average damage rate equals 1. In this case, the damage can be described by $\frac{1}{2(n+1)}$ as we take $k = \beta \cdot (n+1)$ which is according to the if condition in Theorem 5.6.

Consequently, also in this case the marginal reduction in damage is decreasing in the number of teams, which can be seen in Figure 8. Note that only the damage for an odd number of teams is included in Figure 8 as Theorem 5.6 can only be applied to these cases. When we compare the damages with the ones of Figure 5 and Figure 6, we see that the triangular damage rate function is the best performing one[8]. This might indicate that the government is able to more efficiently protect a line segment with a triangular damage rate function than a uniform or linear damage rate function. The intuition is that damage is even more centered, namely to the top (at β), which can be protected better.

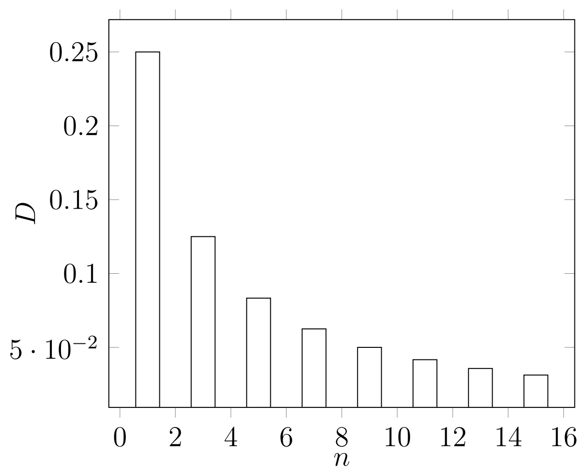


Figure 8: Damage under optimal d^* per number of response teams

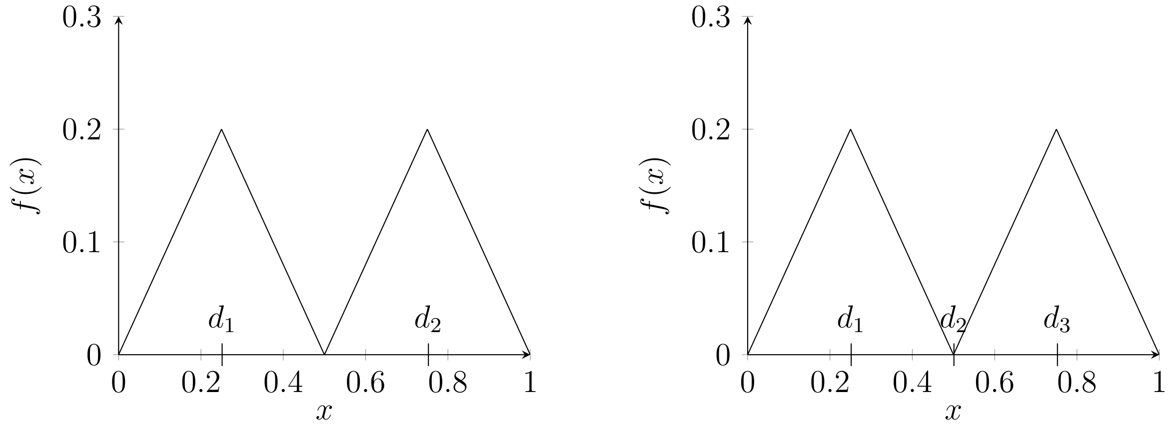
We would like to conclude this section with an interesting insight for the setting where multiple symmetric triangles are positioned next to each other. In particular, via an example, we illustrate that damage is not always strictly decreasing in the number of response teams.

Example 6. *Let $\theta \in \Theta$ with $n = 3$ and damage rate function f:*

$$f(x) = \begin{cases} 0.8 \cdot x & x \in [0, 0.25] \\ 0.4 - 0.8 \cdot x & x \in (0.25, 0.5] \\ 0.8 \cdot x - 0.4 & x \in (0.5, 0.75] \\ 0.8 - 0.8 \cdot x & x \in (0.75, 1]. \end{cases}$$

Moreover, let $d = (0.25, 0.75)$. A visual representation of f and d is presented in Figure 9(a).

[8]This can also be concluded from the damage expressions: $\frac{1}{2(n+1)} \leq \frac{1}{(\sqrt{n+1}+\sqrt{n})^2} \leq \frac{1}{2n}$ for all $n \in \mathbb{N}_+$



(a) $d = (0.25, 0.75)$.

(b) $d = (0.25, 0.5, 0.75)$.

Figure 9: Damage rate function consisting of two symmetric triangles with optimal locations for $n = 2$ and $n = 3$.

One can easily verify that d *is optimal (i.e.,* $\mathscr{L}(0.25) = \mathscr{I}^1(0.25, 0.75) = \mathscr{R}(0.75) = \frac{1}{80}$*). However, now suppose that the leader can position one extra response team (i.e.,* $n = 3$*). If this extra response team is positioned in the middle (see Figure 9(b)), we can conclude that* $d' = (0.25, 0.5, 0.75)$ *is optimal with the same damage (i.e.,* $\mathscr{L}(0.25) = \mathscr{I}^1(0.25, 0.5) = \mathscr{I}^2(0.5, 0.75) = \mathscr{R}(0.75) = \frac{1}{80}$*). Hence, adding an extra response team does not always lead to a reduction of the damage.* ◇

6. Conclusions and further research

In light of recent terrorist attacks, we study a Stackelberg game between a government and a terrorist. The government positions response teams on a line segment and thereafter the terrorist decides where to attack. We model the situation by a min-max problem in which the terrorist maximizes the damage and the government minimizes this maximum damage. Our model thus reflects a situation where the terrorist observes attack locations and security measures upfront, and based on that information decides where to attack. At the same time, it also reflects a situation where the government is not sure about the information the terrorist has obtained, or the motives of the terrorist, and therefore wants to hedge against the worst possible outcome.

One of the main contributions is that a balanced strategy is optimal for the leader. Such a strategy balances the possible damage on all intervals that result from positioning the rapid response teams. A balanced strategy makes the attacker indifferent: it is equally interesting to attack either to the left of the response teams, to the right of the response teams or between any two consecutive response teams. It also turns out that the leader can better protect against a terrorist attack if the people or buildings are more centered. Furthermore, it can be seen that often damage reduces when the number of response teams increases. However, this is not always the case in the strict sense. It turns out that in some specific scenarios adding an extra response team cannot strictly reduce the damage of an attack.

This study has been carried out under some conditions which can be relaxed in further studies. For instance, we assume that the response teams can be positioned on a line segment only. A logical generalization would be to change the one-dimensional line segment to a two-dimensional space. Here the response teams can not only be located on a busy shopping avenue or boulevard, but also on a more complex street network. Such a situation could be modelled by a damage rate function ranging over a two-dimensional space where the response teams can travel through the space according to Manhattan distance. For such a setting it is not immediately clear what a balanced strategy would be and it is likely that not all balanced strategies lead to the same damage, which severely complicates the analysis. In our study, we also assume that the terrorist will only attack at one location. In practice, we have seen that terrorists might plan multiple attacks, closely after each other. As such it is interesting to generalize our model by analyzing the possibility of multiple attacks by the terrorist and the possibility of relocating and re-assigning the available response teams by the government.

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7. Appendix

Proof of Lemma 2.3

Proof. First we prove that $\varphi(x)$ is compact and non-empty for all $x \in S$. Observe that $p(x) \in \varphi(x)$ for all $x \in S$. Hence, $\varphi(x)$ is non-empty for all $x \in S$. Additionally, as p and q are bounded and $\varphi(x)$ is defined by a closed interval for all $x \in S$, $\varphi(x)$ is closed and bounded and thus compact for all $x \in S$. Consequently, φ is compact-valued.

Second, we prove that φ is upper hemicontinuous. Let $\bar{x} \in S$ and let $(x^n)_{n\in\mathbb{N}}$ be a sequence in S converging to $\bar{x}$. Let $(y^n)_{n\in\mathbb{N}}$ be a sequence in T with $y^n \in \varphi(x^n)$ for all $n \in \mathbb{N}$, converging to $\bar{y} \in \mathbb{R}$. We know by continuity of p and q that $\lim_{n\to\infty} p(x^n) = p(\bar{x})$ and $\lim_{n\to\infty} q(x^n) = q(\bar{x})$. Since $\lim_{n\to\infty} y^n = \bar{y}$ and $y^n \in [p(x^n), q(x^n)]$ for all $n \in \mathbb{N}$, we know that $p(\bar{x}) \leq \bar{y} \leq q(\bar{x})$ and so $\bar{y} \in \varphi(\bar{x})$. Note that φ is compact-valued by the first part of this proof. By Theorem 2.1, it proves that φ is upper hemicontinuous at $\bar{x}$ and so it proves that φ is upper hemicontinuous in general.

Third, we prove that φ is lower hemicontinuous. Let $\bar{x} \in S$ and let $(x^n)_{n\in\mathbb{N}}$ be a sequence in S converging to $\bar{x}$. Let $\bar{y} \in \varphi(\bar{x})$. We know that $p(\bar{x}) \leq \bar{y} \leq q(\bar{x})$. Thus, there exists an $\alpha \in [0,1]$ such that $\bar{y} = \alpha \cdot p(\bar{x}) + (1-\alpha) \cdot q(\bar{x})$. We set $y^n = \alpha \cdot p(x^n) + (1-\alpha) \cdot q(x^n)$ for all $n \in \mathbb{N}$. As $p(x^n) \leq q(x^n)$ we know that $y^n \leq q(x^n)$ and $y^n \geq p(x^n)$ for all $n \in \mathbb{N}$. Consequently, $y^n \in \varphi(x^n)$ for all $n \in \mathbb{N}$. Moreover, we know that $\lim_{n\to\infty} p(x^n) = p(\bar{x})$ and $\lim_{n\to\infty} q(x^n) = q(\bar{x})$. Thus, $y^n = \alpha \cdot p(x^n) + (1-\alpha) \cdot q(x^n)$ and $\lim_{n\to\infty} \alpha \cdot p(x^n) + (1-\alpha) \cdot q(x^n) = \alpha \cdot p(\bar{x}) + (1-\alpha) \cdot q(\bar{x}) = \bar{y}$. Consequently there exists a sequence, namely $(y^n)_{n\in\mathbb{N}}$ in T such that $y^n \in \varphi(x^n)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} y^n = \bar{y}$. By Theorem 2.2, it proves that φ is lower hemicontinuous at $\bar{x}$ and so it proves that φ is lower hemicontinuous in general. As φ is upper hemicontinuous and lower hemicontinuous, φ is continuous. □

Proof of Lemma 4.1

Proof. First, we focus on $\mathscr{L}$, then on $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$ and finally on $\mathscr{R}$.

Let $\varphi : [0,1] \to [0,1]$ be a correspondence with $\varphi(d_1) = [0, d_1]$ for all $d_1 \in [0,1]$. Let $p : [0,1] \to \mathbb{R}$ be a function with $p(d_1) = 0$ for all $d_1 \in [0,1]$ and let $q : [0,1] \to \mathbb{R}$ be a function with $q(d_1) = d_1$ for all $d_1 \in [0,1]$. Let $g : [0,1]^2 \to \mathbb{R}$ be a function with $g(d_1, a) = f(a) \cdot (d_1 - a)$ for all $(d_1, a) \in [0,1]^2$. Note that g is continuous as f is continuous and the product of two continuous functions remains continuous. Since p and q are continuous and bounded functions and $p(d_1) \leq q(d_1)$ for all $d_1 \in [0,1]$, by Lemma 2.3 it is true that φ is continuous and compact-valued. Moreover, g is continuous and thus by Theorem 2.4 it is true that $\mathscr{L}(d_1) = \max_{a\in[0,d_1]} g(d_1, a)$ is continuous for all $d_1 \in [0,1]$.

If $n \geq 2$, let $\varphi : D^{1,2} \to [0,1]$ be a correspondence with $\varphi(d_1, d_2) = [d_1, d_2]$ for all $(d_1, d_2) \in D^{1,2}$. Let $p : D^{1,2} \to \mathbb{R}$ be a function with $p(d_1, d_2) = d_1$ for all $(d_1, d_2) \in D^{1,2}$ and let $q : D^{1,2} \to \mathbb{R}$ be a function with $q(d_1, d_2) = d_2$ for all $(d_1, d_2) \in D^{1,2}$. Let $g : D^{1,2} \to \mathbb{R}$ be a function defined by $g(d_1, d_2) = \max_{a\in[d_1,d_2]} f(a) \cdot \min\{a - d_1, d_2 - a\}$ for all $(d_1, d_2) \in D^{1,2}$. Note that g is a continuous function as the product as well as the minimum of two continuous functions remains continuous. Since p and q are continuous and bounded functions and $p(d_1, d_2) \leq q(d_1, d_2)$ for all $(d_1, d_2) \in D^{1,2}$, by Lemma 2.3 is is true that φ is continuous and compact-valued. Moreover, g is continuous and thus by Theorem 2.4 it is true that $\mathscr{I}^1(d_1, d_2) = \max_{a\in[d_1,d_2]} g(d_1, d_2)$ is continuous for all $(d_1, d_2) \in D^{1,2}$. As $\mathscr{I}^{i+1}(d_i, d_{i+1}) = \mathscr{I}^i(d_i, d_{i+1})$ for all $(d_i, d_{i+1}) \in D^{i,i+1}$ for all $i \in \{1, 2, \ldots, n-2\}$, we conclude that $\mathscr{I}^i$ is continuous for all $i \in \{1, .., n-1\}$.

Let $\varphi : [0,1] \to [0,1]$ be a correspondence with $\varphi(d_n) = [d_n, 1]$ for all $d_n \in [0,1]$. Let $p : [0,1] \to \mathbb{R}$ be a function with $p(d_n) = d_n$ for all $d_n \in [0,1]$ and let $q : [0,1] \to \mathbb{R}$ be a function with $q(d_n) = 1$ for all $d_n \in [0,1]$. Let $g : [0,1]^2 \to \mathbb{R}$ be a function with $g(d_n, a) = f(a) \cdot (a - d_n)$ for all $(d_n, a) \in [0,1]^2$. Note that g is continuous as f is continuous and the product of two continuous functions remains continuous. Since p and q are continuous and bounded functions and $p(d_n) \leq q(d_n)$ for all $d_n \in [0,1]$, by Lemma 2.3 it is true that φ is continuous and compact-valued. Moreover, g is continuous and thus by Theorem 2.4 it is true that $\mathscr{R}(d_n) = \max\limits_{a \in [d_n, 1]} g(d_n, a)$ is continuous for all $d_n \in [0,1]$. □

Proof of Lemma 4.2

Proof. First we focus on $\mathscr{L}$, then on $\mathscr{R}$ and finally on $\mathscr{I}^i$ for all $i \in \{1, 2, \ldots, n-1\}$.

Let $d_1 \in [0,1]$ and $d_1' \in [0,1]$ such that $d_1' > d_1$. Let $a^* \in \arg\max\limits_{a \in [0, d_1]} f(a) \cdot (d_1 - a)$. We derive the following:

$$\mathscr{L}(d_1) = f(a^*) \cdot (d_1 - a^*) \leq f(a^*) \cdot (d_1' - a^*)) \leq \max_{a \in [0, d_1']} f(a) \cdot (d_1' - a) = \mathscr{L}(d_1').$$

The first inequality holds as $d_1' > d_1$. The second inequality holds as we know that $a^* \in [0, d_1]$. Consequently, a^* is a feasible solution to the optimization problem $\max\limits_{a \in [0, d_1']} f(a) \cdot (d_1' - a)$. Hence, $\mathscr{L}$ is non-decreasing.

Let $d_n \in [0,1]$ and $d_n' \in [0,1]$ such that $d_n' < d_n$. Let $a^* \in \arg\max\limits_{a \in [d_n, 1]} f(a) \cdot (a - d_n)$. Then,

$$\mathscr{R}(d_n) = f(a^*) \cdot (a^* - d_n) \leq f(a^*) \cdot (a^* - d_n') \leq \max_{a \in [d_n', 1]} f(a) \cdot (a - d_n') = \mathscr{R}(d_n').$$

The first inequality holds as $d_n' < d_n$. The second inequality holds as we know that $a^* \in [d_n, 1]$. Consequently, a^* is a feasible solution to the optimization problem $\max\limits_{a \in [d_n', 1]} f(a) \cdot (a - d_n')$. Hence, $\mathscr{R}$ is non-increasing.

Let $i \in \{1, ..., n-1\}$ and $d_{i+1} \in [0,1]$. First, we prove that $\mathscr{I}^1(\cdot, d_{i+1})$ is non-increasing. Let $d_i \in [0, d_{i+1}]$ and $d_i' \in [0, d_{i+1}]$ such that $d_i < d_i'$. Let $a^* \in \arg\max\limits_{a \in [d_i', d_{i+1}]} f(a) \cdot \min\{a - d_i', d_{i+1} - a\}$. We derive the following:

$$\begin{aligned}
\mathscr{I}^1(d_i', d_{i+1}) &= \max_{a \in [d_i', d_{i+1}]} f(a) \cdot \min\{a - d_i', d_{i+1} - a\} \\
&= f(a^*) \cdot \min\{a^* - d_i', d_{i+1} - a^*\} \\
&\leq f(a^*) \cdot \min\{a^* - d_i, d_{i+1} - a^*\} \\
&\leq \max_{a \in [d_i, d_{i+1}]} f(a) \cdot \min\{a - d_i, d_{i+1} - a\} \\
&= \mathscr{I}^1(d_i, d_{i+1}).
\end{aligned}$$

The first inequality holds as $d_i < d_i'$. The second inequality holds as we know that $a^* \in [d_i', d_{i+1}]$. Consequently, a^* is a feasible solution to the optimization problem $\max\limits_{a \in [d_i, d_{i+1}]} f(a) \cdot \min\{a - d_i, d_{i+1} - a\}$. Hence, $\mathscr{I}^1(\cdot, d_{i+1})$ is non-increasing. As by definition it holds that $\mathscr{I}^i(d_i, d_{i+1}) = \mathscr{I}^1(d_i, d_{i+1})$ for all $i \in \{1, ..., n-1\}$, it follows that $\mathscr{I}^i(\cdot, d_{i+1})$ is non-increasing for all $i \in \{1, ..., n-1\}$.

Let $i \in \{1, ..., n-1\}$ and $d_i \in [0, 1]$. First, we prove that $\mathscr{I}^1(d_i, \cdot)$ is non-decreasing. Let $d_{i+1} \in [d_i, 1]$ and $d'_{i+1} \in [d_i, 1]$ such that $d_{i+1} < d'_{i+1}$. Let $a^* \in \arg\max_{a \in [d_i, d_{i+1}]} f(a) \cdot \min\{a - d_i, d_{i+1} - a\}$. We derive the following:

$$\begin{aligned}
\mathscr{I}^1(d_i, d_{i+1}) &= \max_{a \in [d_i, d_{i+1}]} f(a) \cdot \min\{a - d_i, d_{i+1} - a\} \\
&= f(a^*) \cdot \min\{a^* - d_i, d_{i+1} - a^*\} \\
&\leq f(a^*) \cdot \min\{a^* - d_i, d'_{i+1} - a^*\} \\
&\leq \max_{a \in [d_i, d'_{i+1}]} f(a) \cdot \min\{a - d_i, d'_{i+1} - a\} \\
&= \mathscr{I}^1(d_i, d'_{i+1}).
\end{aligned}$$

The first inequality holds as $d_{i+1} < d'_{i+1}$. The second inequality holds as we know that $a^* \in [d_i, d_{i+1}]$. Consequently, a^* is a feasible solution to the optimization problem $\max_{a \in [d_i, d'_{i+1}]} f(a) \cdot \min\{a - d_i, d'_{i+1} - a\}$. Hence, $\mathscr{I}^1(d_i, \cdot)$ is non-decreasing. As by definition it holds that $\mathscr{I}^i(d_i, d_{i+1}) = \mathscr{I}^1(d_i, d_{i+1})$ for all $i \in \{1, ..., n-1\}$, it follows that $\mathscr{I}^i(d_i, \cdot)$ is non-decreasing for all $i \in \{1, ..., i-1\}$. □

Proof of Lemma 4.3

Proof. We prove this lemma by induction.

First, we show continuity. By lemma 4.1 we know that $\mathscr{L}^1$ is continuous. Take a $i \in \{2, ..., n-1\}$ and suppose that $\mathscr{L}^i$ is continuous. Note that for all $d_{i+1} \in [0, 1]$ the following holds:

$$\begin{aligned}
\mathscr{L}^{i+1}(d_{i+1}) &= \min_{d_i \in [0, d_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\} \\
&= -\max_{d_i \in [0, d_{i+1}]} -\max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\}
\end{aligned}$$

Let $\varphi : [0, 1] \to [0, 1]$ be a correspondence with $\varphi(d_{i+1}) = [0, d_{i+1}]$ for all $d_{i+1} \in [0, 1]$. Let $p : [0, 1] \to \mathbb{R}$ be a function with $p(d_{i+1}) = 0$ for all $d_{i+1} \in [0, 1]$ and let $q : [0, 1] \to \mathbb{R}$ be a function with $q(d_{i+1}) = d_{i+1}$ for all $d_{i+1} \in [0, 1]$. Let $g : D^{i,i+1} \to \mathbb{R}$ be a function with $g(d_i, d_{i+1}) = -\max\{\mathscr{L}^i(d_i), \mathscr{I}^{i+1}(d_i, d_{i+1})\}$ for all $(d_i, d_{i+1}) \in D^{i,i+1}$. Note that $\mathscr{L}^i$ is continuous, $\mathscr{I}^i$ is continuous by Lemma 4.1 and the maximum of two continuous functions is continuous. As a continuous function multiplied by -1 is still continuous, it can be concluded that g is continuous. Since p and q are continuous and bounded functions and $p(d_{i+1}) \leq q(d_{i+1})$ for all $d_{i+1} \in [0, 1]$, by Lemma 2.3 it is true that φ is upper and lower hemicontinuous and thus continuous and that φ is compact-valued. Since φ is a continuous, compact-valued correspondence and g is continuous, by Theorem 1.1 it is true that $\mathscr{L}^{i+1}$ given by $-\max_{d_i \in [0, d_{i+1}]} g(d_i, d_{i+1})$ is continuous. We conclude by the principle of induction that $\mathscr{L}^i$ is continuous for all $i \in \{1, ..., n\}$.

Secondly, we prove by induction that $\mathscr{L}^i$ is non-decreasing for all $i \in \{1, .., n\}$. As $\mathscr{L}^1 = \mathscr{L}$ by definition and $\mathscr{L}$ is non-decreasing by Lemma 4.2, we conclude that $\mathscr{L}^1$ is non-decreasing.

Now let $i \in \{1, ..., n-1\}$ and assume that $\mathscr{L}^i$ is non-decreasing. We prove that $\mathscr{L}^{i+1}$ is also non-decreasing. By definition, we know that $\mathscr{L}^{i+1}(d_{i+1}) = \min_{d_i \in [0, d_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\}$ for all $d_{i+1} \in [0,1]$. Take a $d_{i+1} \in [0,1]$ and a $d'_{i+1} \in [0,1]$ such that $d_{i+1} < d'_{i+1}$. Let $d_i^* \in \arg\min_{d_i \in [0, d'_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d'_{i+1})\}$. We distinguish between two cases.

Case 1: $d_i^* \leq d_{i+1}$
We derive the following:

$$\begin{aligned}
\mathscr{L}^{i+1}(d'_{i+1}) &= \max\{\mathscr{L}^i(d_i^*), \mathscr{I}^i(d_i^*, d'_{i+1})\} \\
&\geq \max\{\mathscr{L}^i(d_i^*), \mathscr{I}^i(d_i^*, d_{i+1})\} \\
&\geq \min_{d_i \in [0, d_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\} \\
&= \mathscr{L}^{i+1}(d_{i+1}).
\end{aligned}$$

Note that $\max\{\mathscr{L}^i(d_i^*), \mathscr{I}^i(d_i^*, d_{i+1})\}$ is well-defined as $d_i^* \in [0, d_{i+1}]$. The first inequality holds as Lemma 4.2 states that $\mathscr{I}^i(d_i, \cdot)$ is non-decreasing. The second inequality holds as we know that d_i^* is a feasible solution to the optimization problem $\min_{d_i \in [0, d_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\}$.

Case 2: $d_i^* > d_{i+1}$

We derive the following:

$$\begin{aligned}
\mathscr{L}^{i+1}(d'_{i+1}) &= \max\{\mathscr{L}^i(d_i^*), \mathscr{I}^i(d_i^*, d'_{i+1})\} \\
&\geq \mathscr{L}^i(d_i^*) \\
&\geq \mathscr{L}^i(d_{i+1}) \\
&= \max\{\mathscr{L}^i(d_{i+1}), \mathscr{I}^i(d_{i+1}, d_{i+1})\} \\
&\geq \min_{d_i \in [0, d_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\} \\
&= \mathscr{L}^{i+1}(d_{i+1}).
\end{aligned}$$

The second inequality holds as $d_i^* > d_{i+1}$ and we have assumed that $\mathscr{L}^i$ is non-decreasing. The second equality holds as we know that $\mathscr{L}^i(d_{i+1}) \geq 0$ and $\mathscr{I}^i(d_{i+1}, d_{i+1}) = 0$. The third inequality holds as d_{i+1} is a feasible solution to the optimization problem $\min_{d_i \in [0, d_{i+1}]} \max\{\mathscr{L}^i(d_i), \mathscr{I}^i(d_i, d_{i+1})\}$.

Thus, we have proven that $\mathscr{L}^{i+1}$ is non-decreasing. Consequently, by induction we have shown that $\mathscr{L}^i$ is non-decreasing for all $i \in \{1, ..., n\}$. □

Proof of Lemma 4.4

Proof. We prove by backwards induction that there exists a $d \in D^n$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$ and $\mathscr{L}^{i-1}(d_{i-1}) = \mathscr{I}^{i-1}(d_{i-1}, d_i)$ for all $i \in \{2, 3, \ldots, n\}$.

First we show that there exists a $d_n \in [0,1]$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$. Observe that $\mathscr{R}$ is continuous (Lemma 4.1) and non-increasing (Lemma 4.2), $\mathscr{L}^n$ is continuous and non-decreasing

(Lemma 4.3), and $\mathscr{R}(1) = \mathscr{L}^n(0) = 0$. Hence, there exists a $d_n \in [0,1]$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$. Fix a $d_n \in [0,1]$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$.

Now let $k \in \{1, 2, \ldots, n-1\}$ and assume that there exists a $(d_{n-k}, \ldots, d_n) \in [0,1]^{k+1}$ with $d_{n-k} \leq \ldots, \leq d_n$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$ and $\mathscr{L}^{i-1}(d_{i-1}) = \mathscr{I}^{i-1}(d_{i-1}, d_i)$ for all $i \in \{n-k+1, \ldots, n\}$. Observe that $\mathscr{L}^{n-k-1}$ is continuous and non-decreasing (Lemma 4.3), $\mathscr{I}^{n-k-1}$ is continuous and non-increasing in its first argument (Lemma 4.1 and Lemma 4.2), and $\mathscr{L}^{n-k-1}(0) = \mathscr{I}^{n-k-1}(d_{n-k}, d_{n-k}) = 0$. Thus, there exists a $(d_{n-k-1}, \ldots, d_n) \in [0,1]^{k+2}$ with $d_{n-k-1} \leq \ldots, \leq d_n$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$ and $\mathscr{L}^{i-1}(d_{i-1}) = \mathscr{I}^{i-1}(d_{i-1}, d_i)$ for all $i \in \{n-k, \ldots, n\}$. Fix a $(d_{n-k-1}, \ldots, d_n) \in [0,1]^{k+2}$ with $d_{n-k-1} \leq \ldots, \leq d_n$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$ and $\mathscr{L}^{i-1}(d_{i-1}) = \mathscr{I}^{i-1}(d_{i-1}, d_i)$ for all $i \in \{n-k, \ldots, n\}$.

Hence, by backwards induction we obtain that there exists a $d \in D$ such that $\mathscr{L}^n(d_n) = \mathscr{R}(d_n)$ and $\mathscr{L}^{i-1}(d_{i-1}) = \mathscr{I}^{i-1}(d_{i-1}, d_i)$ for all $i \in \{2, 3, \ldots, n\}$. □

Proof of Lemma 4.5

Proof. Take a $d' \in D^n$ with $\mathscr{L}^n(d'_n) = \mathscr{R}(d'_n)$ and $\mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i)$ for all $i \in \{2, 3, \ldots, n\}$. By Lemma 4.4 such a d' exists. Note that the minimum of (6) is attained at d'_n as $\mathscr{R}$ is continuous (Lemma 4.1) and non-increasing (Lemma 4.2), $\mathscr{L}^n$ is continuous and non-decreasing (Lemma 4.3) and $\mathscr{R}(1) = \mathscr{L}^n(0) = 0$. Additionally, note that for all $i \in \{2, 3, \ldots, n\}$ the minimum of (5) with $d_i = d'_i$ is attained at d'_{i-1} as $\mathscr{L}^{i-1}$ is continuous and non-decreasing (Lemma 4.3) and $\mathscr{I}^{i-1}$ is continuous and non-increasing in its first argument (Lemma 4.1 and Lemma 4.2) and $\mathscr{L}^{i-1}(0) = \mathscr{I}^{i-1}(d_i, d_i) = 0$. As the minimum of (6) is attained at d'_n and the minimum of (5) with $d_i = d'_i$ is attained at d'_{i-1} for all $i \in \{2, 3, \ldots, n\}$, d' is optimal. □

Proof of Lemma 4.6

Proof. Take a $d' \in D^n$ with

$$\mathscr{L}^n(d'_n) = \mathscr{R}(d'_n) \tag{9}$$

and $\mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i)$ for all $i \in \{2, 3, \ldots, n\}$. By Lemma 4.5, we know that the minimum of (6) is attained at d'_n and the minimum of (5) with $d_i = d'_i$ is attained at d'_{i-1} for all $i \in \{2, 3, \ldots, n\}$. Thus, for all $i \in \{2, 3, \ldots, n\}$ we have

$$\mathscr{L}^i(d'_i) = \mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i). \tag{10}$$

By combining (9) with (10), we obtain

$$\mathscr{R}(d'_n) = \mathscr{I}^{n-1}(d'_{n-1}, d'_n) = \ldots = \mathscr{I}^1(d'_1, d'_2) = \ldots = \mathscr{L}(d'_1).$$

So, d' is a balanced strategy. □

Proof of Lemma 4.7

Proof. Take a $d \in D^n$ and a $d' \in D^n$ such that d and d' are balanced. Suppose $\max_{a \in A} D(d', a) \neq \max_{a \in A} D(d, a)$ and assume without loss of generality that $\max_{a \in A} D(d', a) > \max_{a \in A} D(d, a)$. Note

$$\begin{aligned} \mathscr{L}(d'_1) &= \mathscr{I}^1(d'_1, d'_2) = \ldots = \mathscr{I}^{n-1}(d'_{n-1}, d'_n) = \mathscr{R}(d'_n) = \max_{a \in A} D(d', a) \\ &> \max_{a \in A} D(d, a) = \mathscr{L}(d_1) = \mathscr{I}^1(d_1, d_2) = \ldots = \mathscr{I}^{n-1}(d_{n-1}, d_n) = \mathscr{R}(d_n). \end{aligned} \tag{11}$$

As $\mathscr{L}(d'_1) > \mathscr{L}(d_1)$ and $\mathscr{L}$ is non-decreasing (Lemma 4.2), we know that $d'_1 > d_1$.

Let $k \in \{1, 2, \ldots, n-1\}$ and assume that $d'_k > d_k$. As $\mathscr{I}^k$ is non-increasing in its first argument (Lemma 4.2) and $d'_k > d_k$, we know that $\mathscr{I}^k(d'_k, d_{k+1}) \leq \mathscr{I}^k(d_k, d_{k+1})$. Moreover, by (11) we have $\mathscr{I}^k(d'_k, d'_{k+1}) > \mathscr{I}^k(d_k, d_{k+1})$. Consequently, $\mathscr{I}^k(d'_k, d_{k+1}) < \mathscr{I}^k(d'_k, d'_{k+1})$. Since $\mathscr{I}^k$ is non-decreasing in its second argument (Lemma 4.2), $d'_{k+1} > d_{k+1}$

Hence, by induction we obtain that $d'_i > d_i$ for all $i \in \{1, 2, \ldots, n\}$. From this we conclude that $d'_n > d_n$. Since $\mathscr{R}$ is non-increasing (Lemma 4.2), we have $\mathscr{R}(d'_n) \leq \mathscr{R}(d_n)$, which contradicts (11). Thus, $\max\limits_{a \in A} D(d', a) = \max\limits_{a \in A} D(d, a)$. $\square$

Proof of Theorem 4.8

Proof. As by Lemma 4.4 there exists a $d' \in D^n$ such that $\mathscr{L}^n(d'_n) = \mathscr{R}(d'_n)$ and $\mathscr{L}^{i-1}(d'_{i-1}) = \mathscr{I}^{i-1}(d'_{i-1}, d'_i)$ for all $i \in \{2, 3, \ldots, n\}$, by Lemma 4.5 this d' is optimal and by Lemma 4.6 this d' is also balanced, we can conclude that there exists a balanced strategy that is optimal.

By Lemma 4.7, we know that all balanced strategies have the same damage, and so, all balanced strategies are optimal. $\square$

Proof of Theorem 5.1

Proof. Let $d \in D^n$. The local damage functions are given by the functions below.

$$\mathscr{L}(d_1) = \max_{a \in [0, d_1]} c \cdot (d_1 - a) = c \cdot d_1$$

$$\mathscr{I}^i(d_i, d_{i+1}) = \max_{a \in [d_i, d_{i+1}]} c \cdot \min\{a - d_i, d_{i+1} - a\} = c \cdot \frac{d_{i+1} - d_i}{2} \text{ for all } i \in \{1, \ldots, n-1\}$$

$$\mathscr{R}(d_n) = \max_{a \in [d_n, 1]} c \cdot (a - d_n) = c \cdot (1 - d_n).$$

Note that the first result follows since the objective function is decreasing in a, hence its maximum is attained at $a = 0$. The second result follows since the objective function is increasing in a on the interval $a \in [d_i, \frac{d_i + d_{i+1}}{2}]$ and the objective function is decreasing on the interval $a \in [\frac{d_i + d_{i+1}}{2}, d_{i+1}]$, hence the maximum is attained at $a = \frac{d_i + d_{i+1}}{2}$. The third result follows since the objective function is increasing in a, thus the maximum is attained at $a = 1$.
Substituting $d^*_i = \frac{2i-1}{2n}$ for all $i \in \{1, .., n\}$ results in the following:

$$\mathscr{L}(d^*_1) = \mathscr{I}(d^*_i, d^*_{i+1}) = \mathscr{R}(d^*_n) = c \cdot \frac{1}{2n}.$$

As the optimal values of all problems are the same, by Theorem 4.8 it is true that the location of the response teams given by $d^*_i = \frac{2i-1}{2n}$ for all $i \in \{1, .., n\}$ is optimal. $\square$

Proof of Lemma 5.2

Proof. Suppose that $d_i = \sqrt{\frac{i}{k}} d_k$ for all $i \in \{1, \ldots, k\}$. Then, for all $i \in \{2, \ldots, k-1\}$, we have

$$\begin{aligned}\mathscr{L}(d_1) &= \max_{a \in [0, d_1]} c \cdot a \cdot (d_1 - a) = \frac{c \cdot d_1^2}{4} = \frac{c \cdot d_k^2}{4 \cdot k} = c \cdot \frac{d_{i+1}^2 - d_i^2}{4} \\ &= \max_{a \in [d_i, d_{i+1}]} c \cdot a \cdot \min\{a - d_i, d_{i+1} - a\} = \mathscr{I}(d_i, d_{i+1})\end{aligned}$$

Note that the second equality follows because the objective function is convex in a with its maximum attained at $a = \frac{d_1}{2}$. Note that the objective function in the fifth line is equal to $c \cdot a \cdot (a - d_i)$ if $a \in [d_i, \frac{d_i + d_{i+1}}{2}]$ and is equal to $c \cdot a \cdot (d_{i+1} - a)$ if $a \in [\frac{d_i + d_{i+1}}{2}, d_{i+1}]$. Consequently, the sixth equality follows since the objective function is increasing in a on the interval $a \in [d_i, \frac{d_i + d_{i+1}}{2}]$ and the objective function is decreasing on the interval $a \in [\frac{d_i + d_{i+1}}{2}, d_{i+1}]$, hence the maximum is attained at $a = \frac{d_i + d_{i+1}}{2}$. From the above equations, we can conclude that $\mathscr{L}(d_1) = \mathscr{I}(d_1, d_2) = \ldots = \mathscr{I}(d_{k-1}, d_k)$. □

Proof of Theorem 5.3

Proof. Let $i \in \{1, \ldots, n\}$, we show that $d_i^* = \sqrt{\frac{i}{n}} d_n^*$.

$$d_i^* = 2\sqrt{i}(\sqrt{n+1} - \sqrt{n}) = \sqrt{\frac{i}{n}} \cdot 2\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \sqrt{\frac{i}{n}} d_n^*$$

By Lemma 5.2 it is true that $\mathscr{L}(d_1^*) = \mathscr{I}(d_1^*, d_2^*) = \ldots = \mathscr{I}(d_{n-1}^*, d_n^*)$. Additionally, we show that $\mathscr{L}(d_1^*) = \mathscr{R}(d_n^*)$, given by the following expressions:

$$\begin{aligned}
\mathscr{L}(d_1^*) &= \max_{a \in [0, d_1^*]} c \cdot a \cdot (d_1^* - a) = c \cdot \frac{{d_1^*}^2}{4} = c \cdot (1 - 2\sqrt{n}\sqrt{n+1} + 2n) \\
&= c \cdot (1 - d_n^*) = \max_{a \in [d_n^*, 1]} c \cdot a \cdot (a - d_n^*) = \mathscr{R}(d_n^*)
\end{aligned}$$

Note that the second equality follows as the objective function is convex in a with its maximum attained at $a = \frac{d_1}{2}$. The third equality follows from substituting $d_1^* = 2\sqrt{1}(\sqrt{n+1} - \sqrt{n})$ and the fourth equality follows from substituting $d_n^* = 2\sqrt{n}(\sqrt{n+1} - \sqrt{n})$. Finally, the sixth result follows because the objective function is increasing in a with its maximum attained at $a = 1$. We can conclude the following:

$$\mathscr{L}(d_1^*) = \mathscr{I}(d_1^*, d_2^*) = \ldots = \mathscr{I}(d_{n-1}^*, d_n^*) = \mathscr{R}(d_n^*).$$

By Theorem 4.8 we conclude that the location of the resources given by $d_i^* = 2\sqrt{i}(\sqrt{n+1} - \sqrt{n})$ for all $i \in \{1, \ldots, n\}$ is optimal. For the associated damage of this strategy, we obtain

$$\mathscr{L}(d_1^*) = c \cdot \frac{{d_1^*}^2}{4} = c \cdot \frac{(2\sqrt{n+1} - 2\sqrt{n})^2}{4} = \frac{c}{(\sqrt{n+1} + \sqrt{n})^2}.$$

□

Proof of Theorem 5.6

Proof. Let $k \in \{1, \ldots, n\}$ and let $\beta = \frac{k}{n+1}$. Additionally, let $d_i = \sqrt{\frac{i}{k}} \cdot \beta$ for all $i \in \{1, \ldots, k-1\}$, $d_k = \beta$ and $d_i = 1 - \sqrt{\frac{n+1-i}{n+1-k}} \cdot (1-\beta)$ for all $i \in \{k+1, \ldots, n\}$. As the first k resources are positioned on a linearly increasing damage rate function, and $d_i = \sqrt{\frac{i}{k}} d_k$, by a natural variant of Lemma 5.2 for which f is linearly increasing on $[0, d_k]$, it holds that $\mathscr{L}(d_1) = \mathscr{I}(d_1, d_2) = \ldots = \mathscr{I}(d_{k-1}, d_k)$. As the last $n-k$ resources are positioned on a linearly decreasing damage rate function, and $d_i = 1 - \sqrt{\frac{n+1-i}{n+1-k}} \cdot (1-\beta)$ for all $i \in \{k+1, \ldots, n\}$, by a natural variant of Corollary 5.3.2 for which f is linearly decreasing on $[d_k, 1]$ it holds that $\mathscr{I}(d_k, d_{k+1}) = \ldots = \mathscr{I}(d_{n-1}, d_n) = \mathscr{R}(d_n)$.

Additionally, we show that $\mathscr{L}(d_1) = \mathscr{R}(d_n)$, given by the following expressions.

$$\begin{aligned}
\mathscr{L}(d_1) &= \max_{a \in [0, d_1]} c \cdot a \cdot (d_1 - a) = c \cdot \frac{d_1^2}{4} = c \cdot \frac{k}{4(n+1)^2} \\
&= \frac{c}{4} \cdot \frac{\frac{k}{n+1}}{1 - \frac{k}{n+1}} \cdot \left(\sqrt{\frac{1}{n+1-k}} \cdot \frac{n+1-k}{n+1}\right)^2 = c \cdot \frac{\beta \cdot (1-d_n)^2}{4(1-\beta)} \\
&= c \cdot \frac{\beta}{1-\beta} \left(1 - \frac{1+d_n}{2}\right)\left(\frac{1+d_n}{2} - d_n\right) = \max_{a \in [d_n, 1]} c \cdot \frac{\beta}{1-\beta} \cdot (1-a) \cdot (a - d_n) \\
&= \mathscr{R}(d_n)
\end{aligned}$$

Note that the second equality follows since the objective function is convex in a with its maximum attained at $a = \frac{d_1}{2}$. The fifth equality follows since $\beta = \frac{k}{n+1}$ and $d_n = 1 - \sqrt{\frac{1}{n+1-k}} \cdot (1-\beta)$. The seventh equality follows since the objective function is convex in a with its maximum attained at $a = \frac{d_n+1}{2}$. As the local damage functions are all equal to each other, by Theorem 4.6 we conclude that this location of the resources is optimal. □