Worst-Case Conditional Value at Risk for Asset Liability Management: A Novel Framework for General Loss Functions

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ABSTRACT

Asset-liability management (ALM) is a challenging task faced by pension funds due to the uncertain nature of future asset returns and interest rates. To address this challenge, this paper presents a new mathematical model that uses a Worst-case Conditional Value-at-Risk (WCVaR) constraint to ensure that the funding ratio remains above a regulator-mandated threshold with a high probability under the worst-case probability distribution that plausibly explains historical sample data. A tractable reformulation of this WCVaR constraint is developed based on the definition and a new reformulation/approximation of the Worst-case Lower Partial Moment (WLPM) for a general loss function. Additionally, a new data-driven moment-based ambiguity set is developed to capture uncertainty in the moments of random variables in the ALM problem. The proposed approach is evaluated using real-world data from the Canada Pension Plan (CPP) and is shown to outperform classical ALM models, based on either CVaR or WCVaR with fixed moments, on out-of-sample data. The proposed framework for handling correlated uncertainty using WCVaR with nonlinear loss functions can be used in other application areas.

1. Introduction

Financial institutions like pension funds and insurance companies are mandated to prudently manage large amounts of assets and liabilities. Decision-makers in these institutions have to maintain a delicate balance between maximizing return and controlling risk to ensure their long-term financial sustainability. The Asset-liability Management (ALM) problem aims to achieve this goal by optimally allocating available funds to different assets such that profit is maximized while current and future liabilities are covered and any regulatory requirements are satisfied (Zenios, 1995). This problem is of particular concern for pension funds that must guarantee pre-defined payback to retirees (*i.e.,* defined benefit pension plans) (Bodie, Marcus and Merton, 1988).

Pension funds control a sizable portion of global financial assets, in excess of \$60.6 trillion by the end of 2021, which represents 33% of the global assets [1]. At that time, the pension funds in 9 out of the 38 Organisation for Economic Co-operation and Development (OECD) countries had assets exceeding their respective GDPs. Furthermore, pension assets have grown by 5.7% in the last decade (2010-2020) [2] which exceeds the GDP growth rate of 2.6% over the same period [3], signifying the increasing importance of retirement savings globally. However, as large segments of the population have been reaching their retirement ages, outflows from pension funds to pay their benefits are also accelerating. The ratio of total benefits paid from retirement savings plans to GDP varies across OECD countries, ranging from 0.5% to 8% [4].

To meet their future obligations, pension funds need to invest collected contributions in diversified portfolios of assets (*i.e.,* fixed-income, public/private equities, real estate, and infrastructures) to generate sufficient returns. However, these investments come with inherent risks that can affect the portfolio's value and the fund's ability to meet its commitments. Like other investment portfolios, pension funds are exposed to asset price variations over time due to market, sector-specific, and company-specific risks. In contrast to classical investment portfolios, pension funds have defined future obligations and are subject to additional regulatory requirements that stipulate a minimum acceptable ratio between current assets and the present value of future liabilities (*i.e.,* the funding ratio). Hence, pension funds are

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also exposed to interest rate risks that severely affect the liabilities' present value, rendering classical portfolio selection problem (PSP) techniques unsuitable to manage them. Instead, ALM models that jointly consider asset returns and interest rate risks are used.

Several risk measures have been proposed in the literature to quantify this risk (Chen, Yang and Yin, 2008; Chiu and Li, 2006; Chiu and Wong, 2012; Leippold, Trojani and Vanini, 2004; Shen, Wei and Zhao, 2020; Ferstl and Weissensteiner, 2011), among which is the Conditional Value at Risk (CVaR), which was first used in the context of ALM by Bogentoft, Romeijn and Uryasev (2001). CVaR combines the risk level and the probability of an asset or portfolio's return falling below a specified threshold. In order to develop CVaR as a risk measure in the ALM problem, a loss function that considers the losses resulting from mismatches between asset returns and liabilities is required. The use of CVaR enables pension funds to control their risk exposure by managing the tail risk of their investments. Despite its usefulness, the ALM formulation proposed by Bogentoft et al. (2001) uses a sample-average-approximation (SAA) approach to model the uncertainty about asset returns, thus not capturing the full extent of variability of returns and interest rates and resulting in intractable formulations.

The solvency of funded pension plans is highly sensitive to the assumptions embedded in the expected returns and interest/discount rates (parameters of CVaR), as emphasized by Konstantin (2018). The discount rate is crucial in determining the funding status of pension plans. As bond yields have fallen over the past few decades, the discount rate should be adjusted downwards, while it remains highly aggressive in US public pension plans, the major holder of pension plans globally. Additionally, the expected fund performance varies significantly among different entities, even without necessarily different allocations. This variability in returns makes it challenging for pension managers to determine the optimal asset allocation to cover future liabilities. Furthermore, D'Addio, Seisdedos and Whitehouse (2009) highlighted the significant impact of uncertainty in asset returns on pension funds, indicating the need for a conservative approach to investment based on asset returns uncertainty.

Managing the uncertainty associated with the ALM problem is critical for institutions to make better investment decisions and manage risk effectively. Therefore, research on ALM problems under uncertainty has focused on developing models and methods that can quantify and manage the various sources of uncertainty. As highlighted by Birge and Louveaux (2011), the most common approaches to this problem are stochastic programming (SP) and robust optimization (RO). SP is a risk-neutral approach that aims to find a solution that optimizes the expected value of the loss function. Various studies, such as (Klaassen, 1997; Kouwenberg, 2001; Consigli, 2008; Duarte, Valladão and Veiga, 2017; Kopa, Moriggia and Vitali, 2018; Barro, Consigli and Varun, 2022), have applied SP to ALM problems. However, SP requires that the distribution function of the random variables be known. On top of that, the method is risk-neutral, meaning there is no immunity against scenarios that are worse than expected. Additionally, SP solutions may be infeasible for some scenarios. Despite its limitations, SP remains an intuitive approach with favorable convergence properties.

Another appealing method proposed for addressing uncertainty in ALM problems is RO. Researchers such as (Iyengar and Ma, 2016; Platanakis and Sutcliffe, 2017; Gülpinar and Pachamanova, 2013; Gülpınar, Pachamanova and Çanakoğlu, 2016) have used RO to develop ALM models under uncertainty. Despite the advantages of RO over SP models, such as being a risk-averse method and not requiring knowledge of the distribution function of uncertain parameters, the solutions produced by RO are usually overly conservative. This can increase the opportunity cost of ALM problems by basing the decisions on the worst-case scenario. Interested readers can refer to (Ben-Tal, El Ghaoui and Nemirovski, 2009; Bertsimas, Brown and Caramanis, 2011; Gabrel, Murat and Thiele, 2014; Ghahtarani, Saif and Ghasemi, 2022) for more information about RO methods.

While RO and SP have been proposed for the ALM problem, there is currently no research in the literature that considers the combination of risk measures with the uncertainty of probability distribution in ALM optimization. This combination has several advantages. First, it allows for a more comprehensive risk modeling in pension fund management by considering a risk measure. Second, it enables pension fund managers to make more informed decisions on asset allocation, taking into account the uncertainty of returns and the associated risk. Third, it provides a more accurate representation of the underlying probability distribution by using a set of possible distribution functions for random variables called the *ambiguity set*, which can lead to better risk management and improved long-term financial stability. Finally, the combination of risk measures with uncertainty in ALM optimization can lead to more robust and reliable solutions, which are essential for ensuring the long-term financial health of pension funds. This gap in the literature and the benefits of the combination of a risk measure and the ambiguity of distribution function motivates us to adopt distributionally robust optimization (DRO) approaches for the ALM problem. DRO considers the worst-case distribution within a set of candidate distributions that are compatible with available data. By using a risk measure (*e.g.,* CVaR) and accounting for ambiguity in the probability distribution through a DRO approach, more realistic solutions leading to better long-term financial outcomes for pension funds can be achieved. Combining CVaR with DRO leads to the worst-case CVaR (WCVaR) risk measure.

Although the literature suggests WCVaR as a valuable tool for PSPs, there is a gap in the theoretical framework that limits its applicability to more complex loss functions like that of the ALM problem. The loss function in the ALM problem is more intricate than that of the regular PSP due to the uncertainty of both asset returns and the present value of liabilities. On the other hand, the majority of research on CVaR in portfolio selection problems (PSP) assumes the availability of full knowledge of the distribution function of portfolio losses. However, the distribution functions of uncertain asset returns and the present value of liabilities in the ALM problem are not fully known due to the changing parameters based on market conditions. To address this issue, we have developed a novel theoretical framework that proposes the use of WCVaR for linear and nonlinear loss functions of random variables. Our theoretical development not only addresses the gap in the literature but also offers promising possibilities for extending WCVaR to other problem domains such as supply chain management and engineering design. With its enhanced versatility and applicability, WCVaR has the potential to become a go-to tool for a wider range of decision-making scenarios.

The remaining sections of this paper are structured as follows. Section 2 provides a review related to the optimization formulation of the ALM problem using CVaR. In Section 3, we present an extension for the worst-case lower partial moment (WLPM) for functions of random variables. This extension is crucial in developing the WCVaR for more complex loss functions. Furthermore, in Section 3, we propose a formulation for WCVaR that is applicable to general loss functions. Section 4 delves into how to develop WCVaR for the ALM problem, along with an explanation of how to extend the data-driven moment-based ambiguity set. To test the proposed formulation on real data of the Canada Pension Plan (CPP), numerical experiments are conducted, and the results are presented in Section 5. Finally, Section 6 offers some conclusions and suggests potential areas for future research.

2. The ALM problem with CVaR constraints

In pension funds, premiums are collected from sponsors or currently active employees, and pensions are paid to retired employees. Moreover, available funds are invested in assets, which should be managed so that at each time period, the total value of all assets exceeds the fund's liabilities. The goal is to minimize the contribution rate by the sponsor and active employees of the fund (see Bogentoft et al. (2001)). Hence, the ALM problem for a pension fund tries to find the optimal contribution rate and allocation of funds in assets during an investment horizon of length T, which is divided into a set of decision moments $t = 0, \dots, T$. At each time t, decisions are made on the value of contributions to the fund and portfolio allocation. Let y_t be the contribution rate at period t, which is a fraction of the sponsor and/or active employee's wage w_t at time t. Besides, $x_{n,t}$ are decision variables of money invested in asset n in the t^{th} period. The value of assets held by the fund at time t is denoted by A_t. Payments made by the fund to retirees at time t are liabilities and denoted by l_t. The present value of liabilities at period t is calculated by $L_t = \sum_t^T \frac{l_t}{(1+\gamma)^t}$, $\forall t = 0, \dots, T$, where γ is the discount rate. We consider a case in which benefit payments, *i.e.,* liabilities, are fixed and predefined. These kinds of pension funds are called *defined-benefit plans*. The present value of liabilities, L_t, is a random variable since the discount rate used to calculate it is, itself, a random variable. The funding ratio is defined as the ratio of the value of assets in period t to the present value of liabilities in period t. Finally, ψ is the minimum threshold of the funding ratio and is normally imposed by regulations. Model (1) shows the mathematical formulation of the ALM problem introduced by Bogentoft et al. (2001):

$$\min_{y_t, x_{n,t}} \quad h(y_1, \dots, y_T), \tag{1a}$$

$$\text{s.t.} \quad \sum_{n=0}^{N} x_{n,t} = A_t + w_t y_t - l_t, \quad t = 0, \dots, T-1, \tag{1b}$$

$$A_t \geq \psi L_t, \qquad t = 0, \dots, T, \tag{1c}$$

$$A_t = \sum_{n=0}^{N} x_{n,t-1}(1 + \xi_{n,t}), \quad t = 0, \dots, T, \tag{1d}$$

$$x_{n,t} \in \mathcal{X}, y_t \in \mathcal{Y}, \qquad t = 0, \dots, T, n = 0, \dots, N. \tag{1e}$$

In their paper, Bogentoft et al. (2001) introduced a function denoted by $h(y_0, \dots, y_T)$, which serves as the objective function for the ALM problem expressed in (1). The function is defined in terms of the contribution rate and plays a crucial role in determining the optimal ALM strategy. The objective function (1a) can be the average contribution rate or the present value of all contributions. In this formulation, we consider the present value of contributions as the objective function, expressed as $h(y_0, \dots, y_T) = \sum_{t=0}^{T} \frac{w_t y_t}{(1+\gamma)^t}$. Constraint (1b), called the balance constraint, ensures that the sum of all investments at period t is equal to the assets held by the fund plus the contributions gathered at period t minus liabilities in this period. Constraint (1c), called the funding ratio, guarantees that the ratio of assets owned by the fund to the present value of liabilities at period t is greater than a minimum threshold ψ. Constraint (1d) calculates the value of assets owned by the fund at time t. In this formulation, the asset returns $\xi_{n,t}$ and the discount rate γ are uncertain parameters. Uncertainty of the discount rate γ leads to uncertainty in the present values of liabilities and future contributions. Finally, $\mathcal{X}$ and $\mathcal{Y}$ in (1e) are sets of regulatory constraints for the investment allocation and the contribution rate.

To make the formulation easier, we define $W_t = \frac{w_t}{(1+\gamma)^t}$, representing the present value of the sponsor and/or active employee's wages, which is also uncertain because it depends on the uncertain discount rate γ. The objective function of model (1) can be transformed into $\mathrm{W}^\intercal \mathrm{y}$, where $\mathrm{W} = \{W_0, \dots, W_T\} \in \mathbb{R}^{T+1}$ and $\mathrm{y} = \{y_0, \dots, y_T\} \in \mathbb{R}^{T+1}$ are the vectors of the present value of the active employee's wages and decision variables related to the contribution rates, respectively. We also define the vector $\mathrm{r}_t = \mathrm{e} + \xi_t$, $t = 0, \dots, T$, where e is an all-ones vector of size $N+1$. Additionally, the investment decision variable is defined as a vector in each period, $\mathrm{x}_t = \{x_{0,t}, \dots, x_{n,t}\}$. Using these notations, the ALM problem (1) can be transformed into a vector representation as follows:

$$
\begin{aligned}
\min_{\mathrm{y},\mathrm{x}_t} \quad & \mathrm{W}^\intercal \mathrm{y}, && && \text{(2a)}\\
\text{s.t.} \quad & \mathrm{e}^\intercal \mathrm{x}_t = \mathrm{r}_t^\intercal \mathrm{x}_{t-1} + w_t y_t - l_t, && t = 0, \dots, T-1, && \text{(2b)}\\
& \mathrm{r}_t^\intercal \mathrm{x}_{t-1} \geq \psi L_t, && t = 0, \dots, T, && \text{(2c)}\\
& \mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} && t = 0, \dots, T. && \text{(2d)}
\end{aligned}
$$

In order to quantify the risk associated with an investment portfolio using the CVaR measure, it is essential to establish a loss function that captures the potential losses. Based on (Bogentoft et al., 2001), the loss function for problem (2) for each period, t, is defined as $f_\psi(\mathrm{x}_t; \mathrm{r}_t, L_t) = \psi L_t - \mathrm{r}_t^\intercal \mathrm{x}_{t-1}$ as per constraint (2c). Note that the loss function and the CVaR are defined for each period t. However, to simplify the formulations, we suppress the t subscript. The probability that $f_\psi(\mathrm{x}; \mathrm{r}, L)$ is not exceeding a threshold α is calculated as $\Psi(\mathrm{x}, \alpha) = \int_{f_\psi(\mathrm{x};\mathrm{r},L) \leq \alpha} p(\mathrm{r}, L) d(\mathrm{r}, L)$, where $p(\mathrm{r}, L)$ is the joint distribution function of the present value of liabilities and asset returns as random variables. It is worth noting that $p(\mathrm{r})$ is the marginal distribution function of asset returns and $p(L)$ is the marginal distribution of the present value of liabilities.

Value-at-Risk (VaR) is a measure of financial losses over a given time horizon under normal market conditions and a specified level of confidence. It provides an estimate of the maximum loss that an investor could expect to suffer over a given time horizon assuming that the portfolio is held to maturity and that market conditions remain stable. For a confidence level β and a fixed x, the VaR is formally represented as $VaR_\beta(\mathrm{x}) = \min\{\alpha \in \mathbb{R} : \Psi(\mathrm{x}, \alpha) \geq \beta\}$. CVaR is then defined as the expected loss that exceeds VaR, and is calculated as $CVaR_\beta(\mathrm{x}) = \frac{1}{1-\beta} \int_{f_\psi(\mathrm{x};\mathrm{r},L) \geq VaR_\beta(\mathrm{x})} f_\psi(\mathrm{x}; \mathrm{r}, L) p(\mathrm{r}, L) d(\mathrm{r}, L)$. Borrowing the approach proposed by Rockafellar, Uryasev et al. (2000), we introduce an auxiliary function $G_\beta(\mathrm{x}, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\mathrm{r} \in \mathbb{R}^{n+1}, L \in \mathbb{R}} [f_\Psi(\mathrm{x}; \mathrm{r}, L) - \alpha]^+ p(\mathrm{r}, L) d(\mathrm{r}, L)$, where $[.]^+ = max\{., 0\}$, and then $CVaR_\beta(\mathrm{x}) = \min_{\alpha \in \mathbb{R}} G_\beta(\mathrm{x}, \alpha)$.

To calculate $G_\beta(\mathrm{x}, \alpha)$, it is necessary to have full knowledge about the joint distribution function of asset returns and the present value of liabilities, $p(\mathrm{r}, L)$. However, in reality, full knowledge about this joint distribution function may not be available. Therefore, we apply a DRO approach that considers the ambiguity about the distribution function of these random variables. DRO offers a powerful framework for dealing with uncertainty in ALM by avoiding the assumption of a single distribution for randomly distributed variables. In this context, we have two key random variables: the present value of wages of active employees, W, and random variables in the loss function $f_\Psi(\mathrm{x}; \mathrm{r}, L)$ of the ALM problem. These variables have distinct distribution functions, namely q, the distribution function of the present value of the active employee's wages, and $p(\mathrm{r}, L)$, the joint distribution function of asset returns and the present value of liabilities, respectively.

To account for the investor's ambiguity regarding the true distribution of the loss function and the present value of pension active employee wages, we introduce ambiguity sets of distributions. More specifically, we define Q as the ambiguity set of the distribution function of the present value of active employees' wages and $P(\mathrm{r}, L)$ as the ambiguity set of the joint distribution function of asset returns and the present value of liabilities. Finally, $P(\mathrm{r})$ and $P(L)$ are the ambiguity sets of marginal distribution functions of asset returns and the present value of liabilities, respectively. Using these ambiguity sets, we can formulate the DRO version of the ALM problem (2) as follows:

$$\min_{\mathrm{y},\mathrm{x}_t} \sup_{q\in Q} \quad \mathbb{E}_q[\mathrm{W}]^\intercal \mathrm{y}, \tag{3a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = \inf_{p(\mathrm{r})\in P(\mathrm{r})} \mathbb{E}_{p(\mathrm{r})}[\mathrm{r}_t]^\intercal \mathrm{x}_{t-1} + w_t y_t - l_t, \quad t = 0,\dots,T-1, \tag{3b}$$

$$\sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} \min_{\alpha\in\mathbb{R}} G_\beta(\mathrm{x}_t,\alpha) \le 0, \qquad t = 0,\dots,T, \tag{3c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \qquad t = 0,\dots,T. \tag{3d}$$

The goal is to minimize the worst-case expected present value of future contributions to the fund, represented by the objective function (3a), subject to the balance constraint (3b), the WCVaR constraint (3c), and the regulatory constraint (3d). In the objective function (3a), the minimization is over the contribution rate y and the investment allocation in each period x_t, while the maximization is over all probability distributions in the ambiguity set Q. The expected value is taken with respect to the probability distribution $q \in Q$. In the balance constraint (3b), the worst-case expected value is over the marginal distribution function of asset returns. The maximization of WCVaR is over the joint distribution function of asset returns and the present value of liabilities, and the minimization is over α, which is VaR here. By doing so, we obtain more robust results that are less sensitive to specific assumptions about the underlying probability distributions, making it particularly well-suited for managing financial risks in uncertain environments.

The subsequent task is to introduce WCVaR for ALM. However, the loss function for the ALM problem is more intricate than that of PSPs since the loss function of ALM has two random variables, asset returns and the present value of liabilities, while the loss function of PSP has just one random variable, asset returns. Therefore, an extension of the theoretical framework for the Worst-case Lower Partial Moment (WLPM) and WCVaR is necessary to apply them to more extensive loss functions.

3. WLPM and WCVaR for linear loss functions

Chen, He and Zhang (2011) proposed WLPM as a risk measure that has a close connection with WCVaR. Let ξ be a univariate random variable, with μ and σ being the first and second moments of ξ, and α is a fixed target. Chen et al. (2011) proved that $\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[(\alpha-\xi)^+\right] = \frac{\alpha-\mu+\sqrt{\sigma^2+(\alpha-\mu)^2}}{2}$ and showed that WCVaR can be defined based on WLPM. In particular, for a regular PSP with the loss function $-\mathrm{r}^\intercal\mathrm{x}$, where $\mathrm{r} \in \mathbb{R}^n$ is the asset returns vector, $\mathrm{x} \in \mathbb{R}^n$ is the vector of decision variables which is the proportion of investment in each asset, and $P(\mathrm{r})$ is an ambiguity set of the distribution function of asset returns, the WCVaR is defined as:

$$WCVaR_\beta(\mathrm{x}) = \sup_{p(\mathrm{r})\in P(\mathrm{r})} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[\left(-\mathrm{r}^\intercal\mathrm{x}-\alpha\right)^+\right], \tag{4}$$

where $\sup_{p(\mathrm{r})\in P(\mathrm{r})} \mathbb{E}\left[(-\mathrm{r}^\intercal\mathrm{x}-\alpha)^+\right]$ is the WLPM. Here, the vector of asset returns, $\mathrm{r} \in \mathbb{R}^n$, is a random variable with mean $\hat{\mu}$ and covariance $\hat{\Sigma} \succ 0$ that belongs to a family of distributions

$$P(\mathrm{r}) = \left\{p \in M_+ | P(\mathrm{r} \in \Omega) = 1, \mathbb{E}_p(\mathrm{r}) = \hat{\mu},\ Cov_p(\mathrm{r}) = \hat{\Sigma}\right\},$$

where M_+ is the set of all probability measures on the measurable space $(\mathbb{R}^n, B)$ with the σ-algebra B on $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ is a closed convex set known to contain the support of the random vector r. By using this ambiguity set, as proven by Chen et al. (2011), the WCVaR evaluates to:

$$\max_{p(\mathrm{r})\in P(\mathrm{r})} CVaR_\beta(\mathrm{x},p) = -\hat{\mu}^\intercal\mathrm{x} + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\mathrm{x}^\intercal\hat{\Sigma}\mathrm{x}}. \tag{5}$$

The WCVaR formulation (5) is based on the assumption that the first two moments of the uncertain distribution function are known. However, there might be uncertainty about the moments when they are estimated using limited data samples. Kang, Li, Li and Zhu (2019) proposed the WCVaR with uncertain moments based on a data-driven moment-based ambiguity set defined as follows:

$$D_{P(\mathrm{r})}\left(\gamma_1,\gamma_2\right) = \left\{p \in M_+ | P\left(\mathrm{r} \in \Omega\right) = 1, \left(\mathbb{E}_p\left(\mathrm{r}\right) - \hat{\mu}\right)^\intercal \hat{\Sigma}^{-1}\left(\mathbb{E}_p\left(\mathrm{r}\right) - \hat{\mu}\right) \leq \gamma_1, ||Cov_p\left(\mathrm{r}\right) - \hat{\Sigma}||_F \leq \gamma_2, Cov_p\left(\mathrm{r}\right) \succ 0\right\},$$

which is originally introduced by Delage and Ye (2010). In this ambiguity set, $\hat{\mu}$ and $\hat{\Sigma}$ are estimates of the mean vector and the covariance matrix of the random variable r, respectively. Kang et al. (2019) proved that WCVaR under moment uncertainty (as defined in $D_{P(\mathrm{r})}\left(\gamma_1,\gamma_2\right)$) is as follows:

$$\max_{p(\mathrm{r})\in D_{P(\mathrm{r})}\left(\gamma_1,\gamma_2\right)} CVaR_\beta\left(\mathrm{x},p\right) = -\hat{\mu}^\intercal \mathrm{x} + \sqrt{\gamma_1}\sqrt{\mathrm{x}^\intercal\hat{\Sigma}\mathrm{x}} + k\sqrt{\mathrm{x}^\intercal\left(\hat{\Sigma} + \gamma_2 I_n\right)\mathrm{x}}, \tag{6}$$

where I_n is the identity matrix of size n, and $k = \sqrt{\frac{\beta}{1-\beta}}$.

The WCVaR reformulations (5) and (6) use the facts that the PSP loss function is a linear function of x and that r is the only random variable. However, the loss function can be more complex. As shown in Section 2, the loss function of the ALM problem includes a linear function of asset returns and the present value of future liabilities as random variables. To propose a tractable reformulation of the WCVaR constraint in the ALM problem, we are extending the WLPM and WCVaR formulations for the linear loss function of multiple random variables. For more clarity, we start with a linear loss function of a univariate random variable, then extend it to a linear function of multivariate random variables.

Lemma 1. *Let ξ be a univariate random variable, where $\mathbb{E}\left[\xi\right] = \mu$, $Var\left(\xi\right) = \sigma^2$, and $f\left(.\right)$ is a linear function of the random variable ξ that $f : \mathbb{R} \to \mathbb{R}$. Then, WLPM is as follows:*

$$\sup_{\xi\sim\left(\mu,\sigma^2\right)} \mathbb{E}\left[\left(\alpha - f\left(\xi\right)\right)^+\right] = \frac{\alpha - f\left(\mu\right) + \sqrt{f'\left(\mu\right)^2\sigma^2 + \left(\alpha - f\left(\mu\right)\right)^2}}{2}.$$

Proof. The exact second-order Taylor expansion of $f\left(\xi\right)$ around $\mu = \mathbb{E}\left[\xi\right]$ for a linear function is as follows:

$$\mathbb{E}[f(\xi)] = \mathbb{E}\left[f\left(\mu\right) + f'\left(\mu\right)\left(\xi - \mu\right) + \frac{1}{2}f''\left(\mu\right)\left(\xi-\mu\right)^2\right].$$

It is known that $\mathbb{E}(a+b) = \mathbb{E}(a) + \mathbb{E}(b)$. Then:

$$\mathbb{E}\left[f\left(\xi\right)\right] = \mathbb{E}\left[f\left(\mu\right)\right] + f'\left(\mu\right)\mathbb{E}\left[\xi-\mu\right)] + \frac{1}{2}f''\left(\mu\right)\mathbb{E}\left[\xi-\mu\right]^2,$$

where $\mathbb{E}\left[f\left(\mu\right)\right] = f\left(\mu\right)$, and $\mathbb{E}\left[\xi-\mu\right] = \mathbb{E}\left[\xi\right] - \mu = \mu - \mu = 0$. Then:

$$\mathbb{E}\left[f\left(\xi\right)\right] = f\left(\mu\right) + \frac{1}{2}f''\left(\mu\right)\mathbb{E}\left[\xi-\mu\right]^2.$$

Since $\mathbb{E}\left[\xi-\mu\right]^2 = Var\left(\xi\right) = \sigma^2$, then:

$$\mathbb{E}\left[f\left(\xi\right)\right] = f\left(\mu\right) + \frac{1}{2}f''\left(\mu\right)\sigma^2$$

Because $f(.)$ is a linear function then $f''\left(.\right) = 0$, consequently $\mathbb{E}\left[f(\xi)\right] = f\left(\mu\right)$.

Next, we need to find $Var\left(f\left(\xi\right)\right)$. The first order Taylor expansion of $f\left(\xi\right)$ around $\mu = \mathbb{E}\left[\xi\right]$ is $f\left(\mu\right) + f'\left(\mu\right)\left(\xi-\mu\right)$. Then, $Var\left(f\left(\xi\right)\right)$ is as follows:

$$Var\left[f\left(\xi\right)\right] = Var\left[f\left(\mu\right) + f'\left(\mu\right)\left(\xi-\mu\right)\right] = Var\left[f\left(\mu\right) + f'\left(\mu\right)\xi - f'\left(\mu\right)\mu\right].$$

The first term, $f(\mu)$, is constant; then $Var(f(\mu)) = 0$. The third term, $Var\left(f'(\mu)\mu\right)$, is also a constant with a variance equal to zero. Consequently, the variance of $f(.)$ is as follows:

$$Var[f(\xi)] = Var\left[f'(\mu)\xi\right] = \left(f'(\mu)\right)^2 Var[\xi] = f'(\mu)^2 \sigma^2.$$

By substituting $\mathbb{E}[f(\xi)]$ and $Var(f(\xi))$ into $\frac{\alpha-\mathbb{E}[f(\xi)]+\sqrt{Var(f(\xi))+(\alpha-\mathbb{E}[f(\xi)])^2}}{2}$, the WLPM is as follows:

$$\mathbb{E}_{\xi\sim(\mu,\sigma^2)}\left[(\alpha - f(\xi))^+\right] = \frac{\alpha - f(\mu) + \sqrt{f'(\mu)^2\sigma^2 + (\alpha - f(\mu))^2}}{2}.$$

□

Theorem 2. *Let ξ be a univariate random variable with mean μ and variance σ^2, and define the ambiguty set $P = \{p \in M_+ | P(\xi \in \Omega) = 1, \xi \sim \left(\mu, \sigma^2\right)\}$. Moreover, $f(\xi)$ is a linear loss function, where $f : \mathbb{R} \to \mathbb{R}$. Then WCVaR can be calculated as follows :*

$$WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right] = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{f'(\mu)^2\sigma^2}.$$

PROOF. Based on its definition, $WCVaR_\beta = \sup_{p(.)\in P}\min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right]$. To reformulate the WCVaR, we need to calculate the WLPM term in the WCVaR definition. In Lemma 1, the LPM is in the form $\sup_{p(.)\in P}\mathbb{E}\left[(\alpha - f(\xi))^+\right]$. Hence, rearrange the WLPM term in CVaR as follows:

$$\sup_{p(.)\in P}\mathbb{E}\left[(f(\xi) - \alpha)^+\right] = \sup_{p(.)\in P}\mathbb{E}\left[(-\alpha - (-f(\xi)))^+\right].$$

By substituting the WLPM from Lemma 1 into WCVaR, we have:

$$WCVaR_\beta = \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\frac{-\alpha + f(\mu) + \sqrt{f'(\mu)^2\sigma^2 + (-\alpha + f(\mu))^2}}{2}.$$

The optimal value of α (α^*) can be calculated using the first-order optimality condition $\frac{\partial WCVaR_\beta}{\partial\alpha} = 0$. With that, we have:

$$\alpha^* = f(\mu) + \frac{2\beta - 1}{2\sqrt{\beta(\beta-1)}}\sqrt{f'(\mu)^2\sigma^2}.$$

By substituting α^* back, the WCVaR reduces to:

$$WCVaR_\beta = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{f'(\mu)^2\sigma^2}.$$

□

Now let us consider the case when the loss function is a linear function of multivariate random variables, which is applicable in the context of the ALM problem.

Lemma 3. *Let $\xi = \{\xi_1, \ldots, \xi_n\}$ be a multivariate random variable, where $\mathbb{E}[\xi_i] = \mu_i$, $Var(\xi_i) = \sigma_i^2$, $Cov(\xi_i, \xi_j) = \sigma_{ij}$, and $f(.)$ is a linear function of the random variable ξ, that $f : \mathbb{R}^n \to \mathbb{R}$. Then, WLPM is as follows:*

$$\sup_{\xi\sim(\mu,\Sigma_\xi)}\mathbb{E}\left[(\alpha - f(\xi))^+\right] = \frac{\alpha - f(\mu) + \sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j \sigma_{ij} + (\alpha - f(\mu))^2}}{2},$$

where $\mu = \{\mu_1, \ldots, \mu_n\}$ is the mean vector, and $d_i = \frac{\partial f(\xi)}{\partial \xi_i} |_{\xi=\mu}$, in which $|_{\xi=\mu}$ means to evaluate the expression with μ_i replacing ξ_i.

Proof. Based on the second-order Taylor series expansion of $f(.)$ around $\mu = \{\mu_1, \dots, \mu_n\}$, the expected value of $f(\xi)$ is as follows:

$$\mathbb{E}[f(\xi)] = \mathbb{E}[f(\mu)] + \mathbb{E}[\nabla f(\mu)(\xi - \mu)] + \mathbb{E}\left[\frac{1}{2}(\xi - \mu)^{\intercal} H_f(\mu)(\xi - \mu)\right],$$

where $H_f = \frac{\partial^2 f(\xi)}{\partial \xi_i \partial \xi_j}$ is the Hessian matrix of f, and ∇f is the gradient of f. Since $f(.)$ is a linear function, then its second derivation is zero. Moreover, the second term of Taylor approximation is zero since $\mathbb{E}[\xi - \mu] = \mathbb{E}[\xi] - \mu = \mu - \mu = 0$. Hence, $\mathbb{E}[f(\xi)] = f(\mu)$. Moreover, the variance of $f(.)$ has to be calculated. Based on the first-order Taylor expression, the variance of $(f(\xi))$ is as follows:

$$Var(f(\xi)) = Var\left(f(\mu) + \nabla f(\mu)^{\intercal}(\xi - \mu)\right) = Var(f(\mu) + \nabla f(\mu)'\xi - \nabla f(\mu)'\mu).$$

Since $f(\mu)$, and $\nabla f(\mu)\mu$ are constants, their variances are zero. Hence, $Var(f(\xi)) = Var\left(\nabla f(\mu)'\xi\right)$ which is equivalent to $\nabla f'(\mu)^2 \Sigma_\xi$, where Σ_ξ is the covariance matrix. This formulation can be expanded as follows:

$$Var(f(\xi)) = \sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij},$$

where $d_i = \frac{\partial f(\xi)}{\partial \xi_i} |_{\xi=\mu}$.

By substituting $\mathbb{E}[f(\xi)]$ and $Var(f(\xi))$ into $\frac{\alpha - \mathbb{E}[f(\xi)] + \sqrt{Var(f(\xi)) + (\alpha - \mathbb{E}[f(\xi)])^2}}{2}$, then $\sup_{\xi \sim (\mu, \Sigma_\xi)} \mathbb{E}\left[(\alpha - f(\xi))^+\right]$ is calculated as follows:

$$\frac{\alpha - f(\mu) + \sqrt{\sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij} + (\alpha - f(\mu))^2}}{2}.$$

□

Theorem 4. *Let $\xi \in \mathbb{R}^n$ be a multivariate random variable with mean vector μ and covariance matrix Σ_ξ, where the ambiguity set is $P = \{p \in M_+ | P(\xi \in \Omega) = 1, \xi \sim (\mu, \Sigma_\xi)\}$. Moreover, $f(\xi)$ is a linear loss function, where $f : \mathbb{R}^n \to \mathbb{R}$. Then WCVaR is defined as $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right]$ which is calculated by:*

$$WCVaR_\beta = \sup_{p(.)\in P} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right] = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij}}.$$

Proof. WCVaR is defined as $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right]$. In Lemma 3, we showed how to calculate the WLPM of a linear function of multivariate random variables as: $\sup_{p(.)\in P} \mathbb{E}\left[(f(\xi) - \alpha)^+\right] = \sup_{p(.)\in P} \mathbb{E}\left[(-\alpha - (-f(\xi)))^+\right]$, which is calculated by:

$$\frac{-\alpha + f(\mu) + \sqrt{\sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij} + (-\alpha + f(\mu))^2}}{2}, \tag{7}$$

to be substituted in WCVaR formulation. Then, the optimal value of α (α^*) is calculated using the first-order optimality condition $\frac{\partial WCVaR_\beta}{\partial \alpha} = 0$. With that, we have:

$$f(\mu) + \frac{2\beta - 1}{2\sqrt{\beta(\beta - 1)}}\sqrt{\sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij}}.$$

By substituting α^* back in (7), we get:

$$WCVaR_\beta = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij}}.$$

□

The theorems presented in this paper, namely Theorems 2 and 4, offer a means of computing WCVaR for linear loss functions. However, WCVaR is not only applicable to financial problems but also to a variety of other fields where it is used as a risk measure for more general nonlinear loss functions. This paper also extends the theorems to accommodate nonlinear loss functions and provides lemmas and proofs in the Appendix.

The extended WCVaR presented in this paper has a wide range of potential applications, such as in supply chain and engineering problems, safety analysis, and healthcare. For readers interested in further exploring these applications, we recommend the following references: (Tao, Liu, Xie and Javed, 2021; Chaudhuri, Kramer, Norton, Royset and Willcox, 2022; Zhu, Wen, Ji and Qiu, 2020; Chaudhuri, Norton and Kramer, 2020; Chapman, Bonalli, Smith, Yang, Pavone and Tomlin, 2021; von Schantz, Ehtamo and Hostikka, 2020; Dehlendorff, Kulahci, Merser and Andersen, 2010).

In the next section, we develop the WCVaR formulation for the ALM problem, in which a linear function of random variables is used as a loss function. We are using the theoretical results derived in this section to tractably reformulate the ALM problem with a WCVaR constraint.

4. WCVaR for ALM problem

In this section, we use Theorem 4 to derive a tractable reformulation of the ALM problem with the WCVaR constraint (3c). This constraint ensures that the asset-liability mismatch is controlled in each period, in the sense that the funding ratio remains above ψ with high probability, while accounting for the ambiguity surrounding the joint probability distribution of the asset returns and the present values of liabilities. Since the loss function $f_{\psi}(x; \mathrm{r}, L) = \psi L_t - \mathrm{r}_t^{\intercal} \mathrm{x}_{t-1}$ in this set of constraints is linear in the random variables r and L, Theorem 4 applies and the reformulation is exact. Recall that the random variables are defined as $L \sim \left(\bar{L}, \bar{\Sigma}_L\right)$, $\mathrm{r} \sim \left(\hat{\mathrm{r}}, \bar{\Sigma}_{\mathrm{r}}\right)$, and $Cov(L, \mathrm{r}) = \bar{\sigma}_{L,\mathrm{r}}$, where $\bar{L} \in \mathbb{R}$, $\bar{\Sigma}_L \in \mathbb{R}$, $\bar{\mathrm{r}} \in \mathbb{R}^{n+1}$, $\bar{\Sigma}_{\mathrm{r}} \in \mathbb{R}^{n+1\times n+1}$, and $\bar{\sigma}_{L,\mathrm{r}} \in \mathbb{R}^{n+1}$. With that, we prove the following proposition.

Proposition 1. *For a given $t \in \{0, \dots, T\}$, and using the ambiguity set $P(\mathrm{r}, L) = \{p(\mathrm{r}, L) \in M_+ | P(\mathrm{r}, L \in \Omega) = 1, \mathrm{r} \sim \left(\bar{\mathrm{r}}, \bar{\Sigma}_{\mathrm{r}}\right), L \sim \left(\bar{L}, \bar{\Sigma}_L\right), Cov(L, \mathrm{r}) = \bar{\sigma}_{L,\mathrm{r}}\}$, the left hand side (LHS) of the WCVaR constraint (3c) can be tractably reformulated as follows:*

$$\sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[\left(-\alpha - \left(\mathrm{x}^{\intercal}\mathrm{r} - \psi L\right)\right)^{+}\right] = -\bar{\mathrm{r}}^{\intercal}\mathrm{x} + \psi\bar{L} + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\psi^2\bar{\Sigma}_L + \mathrm{x}^{\intercal}\bar{\Sigma}_{\mathrm{r}}x - 2\mathrm{x}^{\intercal}\bar{\sigma}_{(L,\mathrm{r})}}$$

PROOF. Using the basic properties of mean and variance, it is easy to show that $\mathrm{r}^{\intercal}\mathrm{x} - \psi L \sim (\bar{r}^{\intercal}\mathrm{x} - \psi\bar{L}, \psi^2\bar{\Sigma}_L + \mathrm{x}^{\intercal}\bar{\Sigma}_{\mathrm{r}}\mathrm{x} - 2\mathrm{x}^{\intercal}\bar{\sigma}_{L,\mathrm{r}})$. Then WCVaR is defined as:

$$WCVaR_{\beta} = \sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[\left(-\alpha - \left(\mathrm{x}^{\intercal}\mathrm{r} - \psi L\right)\right)^{+}\right] \tag{8}$$

Based on Lemma 3, the WLPM is calculated as follows:

$$\sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} \mathbb{E}\left[\left(-\alpha - \left(\mathrm{r}^{\intercal}\mathrm{x} - \psi L\right)\right)^{+}\right] = \left[\frac{1}{2}\sqrt{\psi^2\bar{\Sigma}_L + \mathrm{x}^{\intercal}\bar{\Sigma}_{\mathrm{r}}\mathrm{x} - 2\mathrm{x}^{\intercal}\bar{\sigma}_{L,\mathrm{r}} + \left(\bar{\mathrm{r}}^{\intercal}\mathrm{x} - \psi\bar{L} - \alpha\right)^2} + \frac{-\alpha - \left(\bar{\mathrm{r}}^{\intercal}\mathrm{x} - \psi\bar{L}\right)}{2}\right]. \tag{9}$$

By substituting (9) into the WCVaR formula (8), we obtain:

$$WCVaR_{\beta}(\mathrm{x}) = \alpha + \frac{1}{1-\beta}\left[\frac{1}{2}\sqrt{\psi^2\bar{\Sigma}_L + \mathrm{x}^{\intercal}\bar{\Sigma}_{\mathrm{r}}\mathrm{x} - 2\mathrm{x}^{\intercal}\bar{\sigma}_{L,\mathrm{r}} + \left(\bar{\mathrm{r}}^{\intercal}\mathrm{x} - \psi\bar{L} - \alpha\right)^2} + \frac{-\alpha - \left(\bar{\mathrm{r}}^{\intercal}\mathrm{x} - \psi\bar{L}\right)}{2}\right].$$

In Theorem 4, we showed that $\alpha^*_{\mathrm{x}} = \frac{2\beta-1}{2\sqrt{\beta(1-\beta)}}\sqrt{\psi^2\bar{\Sigma}_L + \mathrm{x}^{\intercal}\bar{\Sigma}_{\mathrm{r}}\mathrm{x} - 2\mathrm{x}^{\intercal}\bar{\sigma}_{L,\mathrm{r}}} - \bar{\mathrm{r}}^{\intercal}\mathrm{x} + \psi\bar{L}$. By substituting it back, the LHS of the WCVaR constraint (3c) can be written as follows:

$$WCVaR_{\beta}(\mathrm{x}) = -\bar{r}^{\intercal}\mathrm{x} + \psi\bar{L} + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\psi^2\bar{\Sigma}_L + \mathrm{x}^{\intercal}\bar{\Sigma}_{\mathrm{r}}\mathrm{x} - 2\mathrm{x}^{\intercal}\bar{\sigma}_{(L,\mathrm{r})}}. \tag{10}$$

□

It should be noted that the WCVaR reformulation (10) is based on the assumption that the moments of random variables, asset returns, and the present value of future liabilities, are fixed and known. With $\bar{\mathrm{W}} = \mathbb{E}_p[\mathrm{W}]$ and $\bar{\mathrm{r}}_t = \mathbb{E}_{p(r)}[\mathrm{r}_t]$, model (3) is transformed into model (11) by substituting the WCVaR formula (10) in constraint (3c) to obtain:

$$
\begin{aligned}
\min_{\mathrm{y},\mathrm{x}_t} \quad & \bar{\mathrm{W}}^\intercal \mathrm{y}, && && \text{(11a)}\\
\text{s.t.} \quad & \mathrm{e}^\intercal \mathrm{x}_t = \bar{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} + \omega_t y_t - l_t, && t = 0,\dots,T-1, && \text{(11b)}\\
& \bar{r}_t^\intercal \mathrm{x}_{t-1} + \psi \bar{L}_t + \sqrt{\frac{\beta}{1-\beta}} \sqrt{\psi^2 \bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal \bar{\Sigma}_{\mathrm{r}_t} \mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal \bar{\sigma}_{(L_t,\mathrm{r}_t)}} \le 0, && t = 0,\dots,T, && \text{(11c)}\\
& \mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} && t = 0,\dots,T, && \text{(11d)}
\end{aligned}
$$

which is a nonlinear program.

Even though we assumed that the moments of the uncertain distribution functions are known, the moments themselves might be uncertain. Moments of asset returns are uncertain because they depend on a variety of factors, such as market conditions, economic trends, and company performance. Moreover, moments of liabilities are also uncertain because they are affected by a variety of factors, such as interest rates, inflation, and changes in demographics. To address this case, we extend the moment-based ambiguity set of Delage and Ye (2010) as follows:

$$
D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\gamma_5^t\right) = \left\{ p \in M_+ | P\left(\mathrm{r}_t, \mathrm{L}_t \in \Omega\right) = 1, \quad
\begin{array}{ll}
U_{\mathrm{r}_t} = \left(\mathrm{r}_t - \hat{\mathrm{r}}_t\right)^\intercal \hat{\Sigma}_{\mathrm{r}_t}^{-1} \left(\mathrm{r}_t - \hat{\mathrm{r}}_t\right) \le \gamma_1^t, & \\
U_{\Sigma_{\mathrm{r}_t}} = ||Cov_P\left(\mathrm{r}_t\right) - \hat{\Sigma}_{\mathrm{r}_t}||_F \le \gamma_2^t, & Cov_P\left(\mathrm{r}_t\right) \ge 0\\
U_{L_t} = (L_t - \hat{L}_t)^\intercal \hat{\Sigma}_{L_t}^{-1} (L_t - \hat{L}_t) \le \gamma_3^t, & \\
U_{\Sigma_{L_t}} = ||Cov_P\left(L_t\right) - \hat{\Sigma}_{L_t}||_F \le \gamma_4, & Cov_P(L_t) \ge 0\\
U_{\sigma_{(L_t,\mathrm{r}_t)}} = ||\bar{\sigma}_{(L_t,\mathrm{x}_t)}||_\infty \le \gamma_5^t, &
\end{array}
\right\},
$$

where $\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t \in \mathbb{R}$, and $\gamma_5^t \in \mathbb{R}^{n+1}$. Moreover, $\hat{\mathrm{r}}_t$ and $\hat{L}_t$ are estimates of the mean of asset returns and the present value of future liabilities in period t, respectively. Similarly, $\hat{\Sigma}_{\mathrm{r}_t}$ and $\hat{\Sigma}_{L_t}$ are estimates of the variance-covariance matrix of asset returns, and the present value of liabilities, respectively.

The proposed ambiguity set is designed to capture the uncertainty of moments in a data-driven manner. It consists of two ellipsoidal uncertainty sets for each time period t: U_{r_t} and U_{L_t}. The former represents the uncertainty set of the mean of asset returns, while the latter characterizes the uncertainty set of the mean of present values of future liabilities. To quantify the size of these sets, we use the parameters γ_1^t and γ_3^t. To capture the uncertainty of the second moments, the Frobenius norm is used to define two uncertainty sets: $U_{\Sigma_{\mathrm{r}_t}}$ and $U_{\Sigma_{L_t}}$. These sets represent possible variations in the real variance-covariance matrices of asset returns and the present value of future liabilities, respectively. Intuitively, the Frobenius norm measures the "size" of the matrices, and the uncertainty sets ensure that the real matrices are close to their estimates, up to a certain radius. The sizes of the uncertainty sets are determined by γ_2^t and γ_4^t, which represent the second and fourth moments of the estimation errors, respectively. Additionally, $U_{\sigma(L,\mathrm{r}_t)}$ denotes the box uncertainty set for the covariance of asset returns and the present value of future liabilities in each period t. γ_5^t is the size of this uncertainty set. Finally, $Cov_P(\mathrm{r}_t)$ and $Cov_P(L_t)$ represent the actual variance-covariance matrices of asset returns and the present value of future liabilities that should be positive semi-definite.

The present value of active employee wages is also a random variable. Consequently, the data-driven moment-based ambiguity set for the present value of active employee wages is defined as follows:

$$
Q(\gamma_6,\gamma_7) = \{p \in M_+ | P\left(\mathrm{W} \in \Omega\right) = 1, U_{\mathrm{W}} = \left(\mathrm{W} - \hat{\mathrm{W}}\right)^\intercal \hat{\Sigma}_{\mathrm{W}}^{-1} (\mathrm{W} - \hat{\mathrm{W}}) \le \gamma_6, U_{\Sigma_{\mathrm{W}}} = ||Cov_P\left(\mathrm{W}\right) - \hat{\Sigma}_{\mathrm{W}}||_F \le \gamma_7\},
$$

where an ellipsoidal uncertainty set U_{W} is used to represent the possible variations in the mean of the present value of active employee wages. Similarly, the uncertainty set $U_{\Sigma_{\mathrm{W}}}$ captures the variations in the variance-covariance matrix of the present value of active employee wages. To specify the sizes of these uncertainty sets, we use the parameters γ_6 and γ_7, where these parameters determine the radius of the uncertainty sets. Finally, $\bar{\mathrm{W}}$ and $\hat{\Sigma_{\mathrm{W}}}$ denote estimates of the mean and the variance-covariance matrices of the present value of active employee wages, respectively.

A tractable reformulation of the LHS of the WCVaR constraint (3c) with the proposed data-driven ambiguity set $D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\gamma_5^t\right)$ is developed in proposition 2.

Proposition 2. *Considering that $p(\mathrm{r}, L) \in D^t_{P(\mathrm{r},L)}\left(\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t, \gamma_5^t\right)$, the LHS of the WCVaR constraint (3c) with uncertain moments can be reformulated as follows:*

$$\sup_{p(\mathrm{r},L)\in D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\gamma_5^t\right)} CVaR_\beta\left(\mathrm{x}_{t-1}\right) = -\left[\hat{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} - \sqrt{\gamma_1^t}\sqrt{\mathrm{x}_{t-1}^\intercal \hat{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1}}\right] + \psi\left[\hat{L}_t + \sqrt{\gamma_3^t}\sqrt{\psi^2\hat{\Sigma}_{L_t}}\right] +$$

$$k\sqrt{\mathrm{x}_{t-1}^\intercal\left(\hat{\Sigma}_{\mathrm{r}_t} + \gamma_2^t I_{n+1}\right)\mathrm{x}_{t-1} + \psi^2\left(\hat{\Sigma}_{L_t} + \gamma_4^t\right) + 2\mathrm{x}_{t-1}^\intercal\gamma_5^t},$$

where $k = \sqrt{\frac{\beta}{1-\beta}}$.

Proof. In proposition (1), by fixing x_t, it was shown that:

$$\sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} CVaR_\beta(\mathrm{x}) = -\bar{\mathrm{r}}_t{}^\intercal\mathrm{x}_{t-1} + \psi\bar{L}_t + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(L_t,\mathrm{r}_t)}}.$$

Now, let us consider the case $p(\mathrm{r}, L) \in D^t_{P(\mathrm{r},L)}$, then the WCVaR $\sup_{p(\mathrm{r},L)\in D^t_{P(\mathrm{r},L)}} CVaR_\beta\left(\mathrm{x}_t\right)$ evaluates to:

$$\max_{\bar{\mathrm{r}}_t\in U_{\mathrm{r}_t}} -\bar{\mathrm{r}}_t^\intercal\mathrm{x}_{t-1} + \max_{\bar{L}_t\in U_{L_t}}\psi\bar{L}_t + k\max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}}\sqrt{\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}}.$$

The first term can be written as follows:

$$\max_{\bar{\mathrm{r}}_t\in U_{\mathrm{r}_t}} -\bar{\mathrm{r}}_t^\intercal\mathrm{x}_{t-1} = -\min_{\bar{\mathrm{r}}_t\in U_{\mathrm{r}_t}}\bar{\mathrm{r}}_t^\intercal\mathrm{x}_{t-1},$$

which is a classical robust optimization problem when an ellipsoidal uncertainty set is used for the uncertain parameter $\bar{\mathrm{r}}_t$. Consequently, its tractable reformulation is:

$$\min_{\bar{\mathrm{r}}_t\in U_{\mathrm{r}_t}}\bar{\mathrm{r}}_t^\intercal\mathrm{x}_{t-1} = \hat{\mathrm{r}}_t{}^\intercal\mathrm{x}_{t-1} - \sqrt{\gamma_1^t}\sqrt{\mathrm{x}_t^\intercal\hat{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1}}. \tag{12}$$

Likewise, the second term, related to the present value of future liabilities, can be tractably reformulated as follows:

$$\max_{\bar{L_{t]}}\in U_{L_t}}\psi\bar{L}_t = \psi\left[\hat{L}_t + \sqrt{\gamma_3^t}\sqrt{\psi^2\hat{\Sigma}_{L_t}}\right]. \tag{13}$$

Since the square root is a monotonically increasing function, then $\max_{z\in\mathcal{Z}}\sqrt{f(z)} = \sqrt{\max_{z\in\mathcal{Z}} f(z)}$. Hence:

$$\max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}}\sqrt{\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}} =$$

$$\sqrt{\max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}}\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}}. \tag{14}$$

Also, because the terms under the square root depend on different uncertainty sets, they are separable. Then, the expression (14) is equivalent to:

$$\sqrt{\max_{\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}}}\psi^2\bar{\Sigma}_{L_t} + \max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}}}\mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - \min_{\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}}2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}}.$$

Kang et al. (2019) showed that $\max_{\bar{\Sigma}_{r_t} \in U_{\Sigma_{r_t}}} x_{t-1}^{\intercal} \bar{\Sigma}_{r_t} x_{t-1} = x_{t-1}^{\intercal} \left(\hat{\Sigma}_{r_t} + \gamma_2^t I_{n+1} \right) x_{t-1}$, *i.e.,* the worst-case is obtained by perturbing the nominal variance-covariance matrix by the radius of the ambiguity set. By using the same proof developed in (Kang et al., 2019, Proposition 2.2), $\max_{\bar{\Sigma}_{L_t} \in U_{\Sigma_{L_t}}} \psi^2 \bar{\Sigma}_{L_t} = \psi^2 \left(\hat{\Sigma}_{L_t} + \gamma_4^t \right)$. Finally, $\min_{\bar{\sigma}_{(L_t, r_t)} \in U_{\sigma_{(L_t, r_t)}}} 2x_{t-1}^{\intercal} \bar{\sigma}_{(r_t, L_t)}$ is a robust optimization problem with a box uncertainty set, which evaluates to $-2x_{t-1}' \gamma_5^t$. With that, the third term can be tractably reformulated as follows:

$$\sqrt{\max_{\bar{\Sigma}_{L_t} \in U_{\Sigma_{L_t}}} \psi^2 \bar{\Sigma}_{L_t} + \max_{\bar{\Sigma}_{r_t} \in U_{\Sigma_{r_t}}} x_{t-1}^{\intercal} \bar{\Sigma}_{r_t} x_{t-1} - \min_{\bar{\sigma}_{(L_t, r_t)} \in U_{\sigma_{(L_t, r_t)}}} 2x_{t-1}^{\intercal} \bar{\sigma}_{(r_t, L_t)}} =$$

$$\sqrt{x_{t-1}^{\intercal} \left(\hat{\Sigma}_{r_t} + \gamma_2^t I_{n+1} \right) x_{t-1} + \psi^2 \left(\hat{\Sigma}_{L_t} + \gamma_4^t \right) + 2x_{t-1}^{\intercal} \gamma_5^t} \tag{15}$$

Now, by combining (12), (13), and (15), the LHS of constraint (3c) is equivalent to:

$$\sup_{p(r,L) \in D_{P(r,L)}^t \left(\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t, \gamma_5^t \right)} CVaR_\beta \left(x_{t-1} \right) = - \left[\hat{r}_t^{\intercal} x_{t-1} - \sqrt{\gamma_1^t} \sqrt{x_{t-1}^{\intercal} \hat{\Sigma}_{r_t} x_{t-1}} \right] + \psi \left[\hat{L}_t + \sqrt{\gamma_3^t} \sqrt{\psi^2 \hat{\Sigma}_{L_t}} \right] +$$

$$k \sqrt{x_{t-1}^{\intercal} \left(\hat{\Sigma}_{r_t} + \gamma_2^t I_{n+1} \right) x_{t-1} + \psi^2 \left(\hat{\Sigma}_{L_t} + \gamma_4^t \right) + 2x_{t-1}^{\intercal} \gamma_5^t},$$

where $k = \sqrt{\frac{\beta}{1-\beta}}$. □

Since $p(r, L) \in D_{P(r,L)}^t \left(\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t, \gamma_5^t \right)$, $q \in Q \left(\gamma_6, \gamma_7 \right)$, and based on proposition 2, the robust counterpart of model (3) is as follows:

$$min_{y \in \mathcal{Y}, x_t \in \mathcal{X}} \quad \hat{W}^{\intercal} y + \sqrt{\gamma_6} \sqrt{y^{\intercal} \hat{\Sigma}_W y}, \tag{16a}$$

$$\text{s.t.} \quad e^{\intercal} x_t = \hat{r}_t^{\intercal} x_{t-1} - \sqrt{\gamma_1^t} \sqrt{x_{t-1}^{\intercal} \hat{\Sigma}_{r_t} x_{t-1}} + \omega_t y_t - l_t, \quad t = 0, \ldots, T-1, \tag{16b}$$

$$\begin{aligned} & - \hat{r}_t^{\intercal} x_{t-1} + \sqrt{\gamma_1^t} \sqrt{x_{t-1}^{\intercal} \hat{\Sigma}_{r_t} x_{t-1}} + \psi \hat{L}_t + \psi^2 \sqrt{\gamma_3^t} \sqrt{\hat{\Sigma}_{L_t}} + \\ & k \sqrt{x_{t-1}^{\intercal} (\hat{\Sigma}_{r_t} + \gamma_2^t I_{n+1}) x_{t-1} + \psi^2 (\hat{\Sigma}_{L_t} + \gamma_4^t) + 2x_{t-1}^{\intercal} \gamma_5^t} \leq 0 \leq 0, \qquad t = 0, \ldots, T. \end{aligned} \tag{16c}$$

Model (16) represents a DRO version of the ALM model that accounts for moment uncertainty. This model is more complex than the original ALM problem, which was a linear programming model. The nonlinear nature of the model and the incorporation of moment-based ambiguity sets allow for a more accurate representation of the uncertainty inherent in the ALM problem. In the next section, we will evaluate the proposed model using real-world data, through which we can assess its effectiveness in providing robust solutions that improve the long-term financial outcomes of pension funds.

5. Numerical results

In this research, we use data from the Canada pension plan (CPP) to conduct numerical experiments/tests. Contributions to CPP are compulsory for all working Canadians aged 18-70. Based on CPP information [5]. Also, around 5.8 million individuals are receiving retirement benefits from CPP each month. On average $737.88 are paid in July

[5] https://open.canada.ca/data/en/dataset/1fab2afd-4f3c-4922-a07e-58d7bed9dcfc

2022 to retired Canadians [6]. Moreover, $14,371,853$ individuals are contributing to CPP based on CPP investments report [7].

CPP is investing in 5 asset classes [8]: fixed income, private equity, public equity, infrastructure, and real estate. Moreover, CPP investments are geographically diversified in North America, Europe, and Asia. In our analysis, we use data from 10 major indexes from 2012 to 2022: $S\&P500$ index is used for public equities, Private Equity Index (PRIVEXD) is used for private equities, SP/TSX Capped Real Estate Index (GSPRTRE) is used for the real estate sector, Treasury Yield 10 Years (TNX) is used for fixed-income assets, and finally, $S\&P$ Global Infrastructure TR (SPGTINTR) is used for infrastructure investment. $S\&P\ TSX$ Composite is the index of the Canadian market. For public equities in Europe, FTSEurofirst 300 is used. STOXX Europe 20 is used for the private equity index in Europe. Shanghai Stock Exchange (SSE) and Nikkei-225 indexes are used as representatives of investment in Asia. The value of the total asset in CPP is \$539 B in 2022. Based on the most recent report of CPP, the projected earnings of contributors for 2022 have been $\$585,498$ M, where about %9.9 of that, $\$57,964$ M, is the contribution to CPP [9].

In order to apply model (16), individual WCVaR constraints are required for each period. As a result, it is necessary to determine the moments of the uncertain distribution function of random variables for each period. However, it is possible for asset returns to follow the same distribution in each period and for the mean/variance differences among periods to lack statistical significance. Consequently, the uncertain parameters in each period may exhibit the same moments. Statistical analysis is conducted to find the distribution function and the first two moments of asset returns in each period. The Individual Distribution Identification (IDI) feature in Minitab was used to conduct goodness-of-fit tests to identify the distribution function of returns with the maximum likelihood among a standard set of distribution functions. Table 1 shows the results of the goodness of fit for testing the distribution function of asset returns in each period. Based on the results illustrated in Table 1, we can conclude that the return of assets in most periods follows a normal distribution since the p-values of the goodness-of-fit tests are greater than the significant level, $\alpha = 0.05$, in most periods.

We next test whether there are significant differences between the mean/variance values among different periods (months). Consequently, we test the equality of the mean/variance of asset returns in each period for all assets by using a one-way ANOVA test. Table 2 shows the results of this test. The null hypothesis for equality of variance is "All variances of an asset class in each period are equal", while its alternative is "At least one variance is different". Similarly, the null hypothesis for equality of the mean is "All means of an asset class in each period are equal" and the alternative one is "At least one mean is different". The Significance level for this test is 0.05. Based on the p-values illustrated in Table 2, we fail to reject the null hypotheses. Hence, we do not have any evidence to support the assumption of different means/variances across periods for the return of assets.

For solving the ALM problem, we consider a set of regulatory constraints. The contribution rate in each period is required to be between 5% to 10%. The investment in the US market cannot be greater than 60% of the whole fund. Investment in Canada must be at least 20% of the fund. At least 10% of the fund must be invested in fixed-income assets. Investment in Asia cannot be greater than 15% of the fund. Finally, the funding ratio should be at least 1.05. We provide in-sample and out-of-sample performance analyses to compare the results of the proposed DRO formulation in two cases, uncertain moments WCVaR (UMWCVaR) (16) and fixed moments WCVaR (FMWCVaR) (11), in addition to the stochastic programming (SP) reformulation of the ALM problem with CvaR constraints (SPCVaR). In-sample performance analysis refers to evaluating the performance of a model on the same data that it was trained on. We are using historical data of CPP for in-sample analysis. On the other hand, out-of-sample performance analysis refers to evaluating the performance of a model on data that it has not seen during the training phase. We are using the simulation to generate data for out-of-sample analysis. Both in-sample and out-of-sample comparisons are based on the funding ratio and the fund return in each period.

Table 3 displays the in-sample performance of the funding ratio and fund return of the ALM problem under two different proposed approaches: UMWCVaR and the FMWCVaR, as well as the risk-neutral approach of SPCVaR. It consists of 11 periods (columns 3-14), each representing a specific time point. For the UMWCVaR model, the highest funding ratio is 1.13 in the final period, while the lowest funding ratio is 1.09 in the first period. The corresponding fund return ranges from 0.002 to 0.02. The overall return in this investment horizon is 5.1%. For the FMWCVaR model, the funding ratio ranges from 1.09 to 1.14, and the fund return ranges from 0.007 to 0.019 with an overall return of 9.9%.

[6] https://www.canada.ca/en/services/benefits/publicpensions/cpp/cpp-benefit/amount.html
[7] https://www.cppinvestments.com/the-fund/our-performance/financial-results/f2022-annual-results
[8] https://ca.investing.com/
[9] https://www.osfi-bsif.gc.ca/Eng/oca-bac/ar-ra/cpp-rpc/Pages/cpp30.aspx

Table 1
p-value of the goodness of fit for testing the distribution function of asset returns in each period

Index	Distribution	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
PRIVEXD	Normal	0.688	0.343	<0.005	0.117	0.868	0.367	0.835	0.206	0.504	0.771	<0.005	0.127
	2-Parameter Exponential	0.053	0.035	<0.010	<0.010	0.035	<0.010	0.024	<0.010	0.011	0.05	0.146	<0.010
	3-Parameter Weibull	>0.500	0.298	0.072	>0.500	>0.500	>0.500	>0.500	0.182	>0.500	>0.500	0.093	0.466
	Smallest Extreme Value	>0.250	>0.250	0.124	>0.250	>0.250	>0.250	>0.250	0.089	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	0.208	<0.010	<0.010	>0.250	0.048	>0.250	0.094	0.231	>0.250	0.086	0.024
	Logistic	>0.250	>0.250	0.032	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.012	0.202
$S\&P500$	Normal	0.69	0.341	<0.005	0.118	0.869	0.367	0.835	0.206	0.504	0.77	<0.005	0.185
	2-Parameter Exponential	0.053	0.035	<0.010	<0.010	0.035	<0.010	0.024	<0.010	0.011	0.05	0.124	<0.010
	3-Parameter Weibull	>0.500	0.295	0.072	>0.500	>0.500	>0.500	>0.500	0.181	>0.500	>0.500	0.071	>0.500
	Smallest Extreme Value	>0.250	>0.250	0.124	>0.250	>0.250	>0.250	>0.250	0.088	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	0.208	<0.010	<0.010	>0.250	0.048	>0.250	0.094	0.231	>0.250	0.062	0.033
	Logistic	>0.250	>0.250	0.032	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.008	>0.250
GSPRTRE	Normal	0.11	0.137	<0.005	0.552	0.513	0.105	0.378	0.119	0.464	0.727	<0.005	0.124
	2-Parameter Exponential	0.041	<0.010	<0.010	0.03	0.023	<0.010	0.048	0.025	0.12	>0.250	0.159	>0.250
	3-Parameter Weibull	0.206	0.124	0.01	>0.500	>0.500	>0.500	>0.500	0.308	0.332	>0.500	0.115	0.389
	Smallest Extreme Value	0.05	0.2	0.024	>0.250	>0.250	>0.250	0.053	>0.250	>0.250	>0.250	<0.010	0.092
	Largest Extreme Value	0.208	0.078	<0.010	0.229	0.244	<0.010	>0.250	0.042	>0.250	>0.250	0.143	0.107
	Logistic	0.085	0.139	<0.005	>0.250	>0.250	>0.250	>0.250	0.12	>0.250	>0.250	0.022	0.098
$S\&P$/TSX Composite	Normal	0.008	0.197	<0.005	0.088	0.698	0.168	0.207	0.269	0.337	0.791	0.078	0.253
	2-Parameter Exponential	>0.250	<0.010	<0.010	<0.010	0.033	<0.010	>0.250	<0.010	>0.250	0.021	>0.250	<0.010
	3-Parameter Weibull	>0.500	0.244	0.072	0.093	>0.500	0.4	0.38	>0.500	0.367	>0.500	>0.500	>0.500
	Smallest Extreme Value	<0.010	>0.250	0.123	0.017	>0.250	>0.250	0.106	>0.250	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	0.054	<0.010	0.08	>0.250	0.04	>0.250	0.049	>0.250	>0.250	>0.250	0.081
	Logistic	0.083	0.238	0.021	0.149	>0.250	0.235	0.169	>0.250	>0.250	>0.250	0.211	>0.250
TNX	Normal	0.29	0.366	0.073	0.169	0.011	0.725	0.131	0.835	0.858	0.077	0.022	0.237
	2-Parameter Exponential	0.148	0.021	<0.010	>0.250	0.017	<0.010	0.023	0.017	0.13	0.097	>0.250	<0.010
	3-Parameter Weibull	0.226	0.479	0.054	>0.500	0.062	>0.500	0.181	>0.500	>0.500	0.408	>0.500	0.295
	Smallest Extreme Value	0.212	0.087	0.059	0.018	<0.010	>0.250	>0.250	>0.250	>0.250	<0.010	<0.010	>0.250
	Largest Extreme Value	0.244	>0.250	0.016	>0.250	0.108	0.173	0.063	>0.250	>0.250	>0.250	>0.250	0.035
	Logistic	0.23	>0.250	0.098	>0.250	0.056	>0.250	0.118	>0.250	>0.250	0.21	0.13	>0.250
SPGTINTR	Normal	0.005	0.026	<0.005	0.163	0.638	0.907	0.179	0.262	0.253	0.495	0.163	0.794
	2-Parameter Exponential	0.202	<0.010	<0.010	<0.010	0.024	0.017	0.018	0.012	<0.010	0.034	>0.250	>0.250
	3-Parameter Weibull	0.201	0.18	0.04	0.136	>0.500	>0.500	0.44	0.111	0.306	>0.500	>0.500	>0.500
	Smallest Extreme Value	<0.010	>0.250	0.075	0.04	>0.250	>0.250	>0.250	0.187	>0.250	>0.250	0.034	>0.250
	Largest Extreme Value	>0.250	<0.010	<0.010	0.077	>0.250	>0.250	0.043	0.201	0.037	0.228	>0.250	>0.250
	Logistic	0.052	0.042	0.009	>0.250	>0.250	>0.250	0.223	0.213	>0.250	>0.250	0.247	>0.250
FTSEurofirst 300	Normal	0.407	0.662	0.006	0.422	0.038	0.794	0.752	0.337	0.734	0.922	0.113	0.623
	2-Parameter Exponential	0.033	0.015	<0.010	0.101	<0.010	0.099	>0.250	<0.010	0.045	0.144	>0.250	0.063
	3-Parameter Weibull	>0.500	>0.500	0.099	0.485	0.299	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500
	Smallest Extreme Value	0.095	>0.250	0.174	0.159	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.013	>0.250
	Largest Extreme Value	>0.250	>0.250	<0.010	>0.250	<0.010	>0.250	>0.250	0.064	>0.250	>0.250	>0.250	>0.250
	Logistic	>0.250	>0.250	0.046	>0.250	0.101	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250
STOXX Europe 20	Normal	0.262	0.511	<0.005	0.731	0.892	0.393	0.79	0.467	0.05	0.901	<0.005	0.613
	2-Parameter Exponential	0.114	0.012	<0.010	0.021	0.057	<0.010	0.058	0.015	<0.010	0.031	0.246	0.046
	3-Parameter Weibull	>0.500	>0.500	0.038	>0.500	>0.500	>0.500	>0.500	0.439	0.203	>0.500	0.188	>0.500
	Smallest Extreme Value	0.046	0.138	0.071	>0.250	>0.250	>0.250	>0.250	0.227	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	>0.250	<0.010	>0.250	>0.250	0.061	>0.250	0.23	0.014	>0.250	0.178	>0.250
	Logistic	>0.250	>0.250	0.017	>0.250	>0.250	>0.250	>0.250	>0.250	0.07	>0.250	0.023	>0.250
SSE	Normal	0.045	0.146	0.048	<0.005	0.33	0.213	0.607	0.115	0.554	0.373	0.706	0.029
	2-Parameter Exponential	<0.010	0.043	>0.250	0.07	<0.010	<0.010	<0.010	<0.010	>0.250	0.016	>0.250	>0.250
	3-Parameter Weibull	>0.500	0.403	>0.500	0.049	0.174	>0.500	>0.500	0.476	0.482	0.413	>0.500	0.417
	Smallest Extreme Value	>0.250	0.011	0.016	<0.010	0.246	>0.250	>0.250	>0.250	>0.250	0.062	>0.250	<0.010
	Largest Extreme Value	<0.010	>0.250	0.205	0.082	0.167	0.069	0.169	0.019	>0.250	>0.250	>0.250	>0.250
	Logistic	0.226	>0.250	0.049	0.015	>0.250	0.221	>0.250	0.232	>0.250	>0.250	>0.250	0.136
Nikkei 225	Normal	0.487	0.57	0.435	0.676	0.544	0.152	0.892	0.78	0.055	0.451	0.35	0.102
	2-Parameter Exponential	0.09	0.057	<0.010	>0.250	<0.010	<0.010	0.018	<0.010	0.018	<0.010	0.023	<0.010
	3-Parameter Weibull	0.37	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500	0.083	>0.500	0.495	0.278
	Smallest Extreme Value	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.145	>0.250	0.067	>0.250
	Largest Extreme Value	0.194	0.224	0.054	>0.250	0.097	0.014	>0.250	0.249	0.022	0.17	>0.250	0.015
	Logistic	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.046	>0.250	>0.250	0.226

The funding ratios are slightly different in WCVaR models, which suggests that the uncertainty of moments affects the funding ratio and fund return.

For the SPCVaR model, the funding ratio ranges from 1.09 in the first period to 1.43 in the 11th period, and the fund return ranges from 0.017 to 0.032 overall return of 33%. The funding ratio and fund return of the SPCVaR model are higher than the UMWCVaR and FMWCVaR models, which indicates that the risk-neutral approach of SP is more optimistic than the WCVaR of ALM with fixed and uncertain moments.

Figure 1 shows the in-sample performance of the funding ratio of the SPCVaR, UMWCVaR, and FMWCVaR models. It illustrates that the SPCVaR has better performance than the FMWCVaR and UMWCVaR models based on funding ratio, which is predictable since the FMWCVaR and UMWCVaR models are more conservative than SPCVaR.

Table 2
Hypothesis test for equality of the mean/variance of asset returns in each period

Test	Equality of variances	Equality of means
PRIVEXD	0.971	0.407
$S\&P500$	0.974	0.422
GSPRTRE	0.831	0.965
$S\&P$/TSX Composite	0.813	0.496
TNX	0.275	0.868
SPGTINTR	0.797	0.733
FTSEurofirst 300	0.644	0.401
STOXX Europe 20	0.755	0.632
SSE	0.978	0.861
Nikkei 225	0.925	0.407

Table 3
In-sample performance of the ALM models

	UMWCVaR		FMWCVaR		SPCVaR	
Periods	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return
1	1.09	0.020	1.09	0.019	1.09	0.02
2	1.10	0.004	1.11	0.006	1.13	0.032
3	1.10	0.004	1.12	0.006	1.16	0.032
4	1.10	0.003	1.11	0.007	1.19	0.019
5	1.11	0.003	1.12	0.008	1.22	0.031
6	1.11	0.003	1.12	0.008	1.26	0.031
7	1.12	0.003	1.13	0.008	1.29	0.018
8	1.12	0.003	1.13	0.008	1.32	0.029
9	1.12	0.003	1.14	0.008	1.36	0.030
10	1.12	0.003	1.12	0.008	1.40	0.030
11	1.13	0.002	1.13	0.008	1.43	0.017

Figure 2 demonstrates the fund return in each period. Although the SPCVaR has a higher return in each period than the two other models, it also has higher volatility. The UMWCVaR and FMWCVaR models show slightly different trends in the funding ratio and fund returns, which indicates that the uncertainty of moments has an impact on the performance of the ALM problem. Meanwhile, the SPCVaR model provides an optimistic scenario for the system's future performance with higher volatility of fund return in each period.

Asset allocation is a crucial decision in the ALM problem. It involves deciding how to distribute investments across different asset classes to achieve the desired level of return while minimizing risk. Figure 3 compares the optimal asset allocation of three models over the investment horizon, which is represented on the horizontal axis. The vertical axis in Figure 3 shows the proportion of investment in each asset class. In each period, there are three bars representing the asset allocation of the different models. The first bar corresponds to the UMWCVaR model, the second bar represents the optimal asset allocation of the FMWCVaR model, and the last bar shows the optimal asset allocation of the SPCVaR model. As shown in Figure 3, the WCVaR models provide more diversified portfolios than the SPCVaR model, which leads to a less risky portfolio. The WCVaR models consider the probability distribution of returns and estimate the risk of the portfolio based on the worst-case scenario. As a result, the WCVaR models provide more robust and stable asset allocation over time. In contrast, the SPCVaR model does not account for the uncertainty of the distribution function and can lead to more volatile asset allocation over the investment horizon. The comparison of the optimal

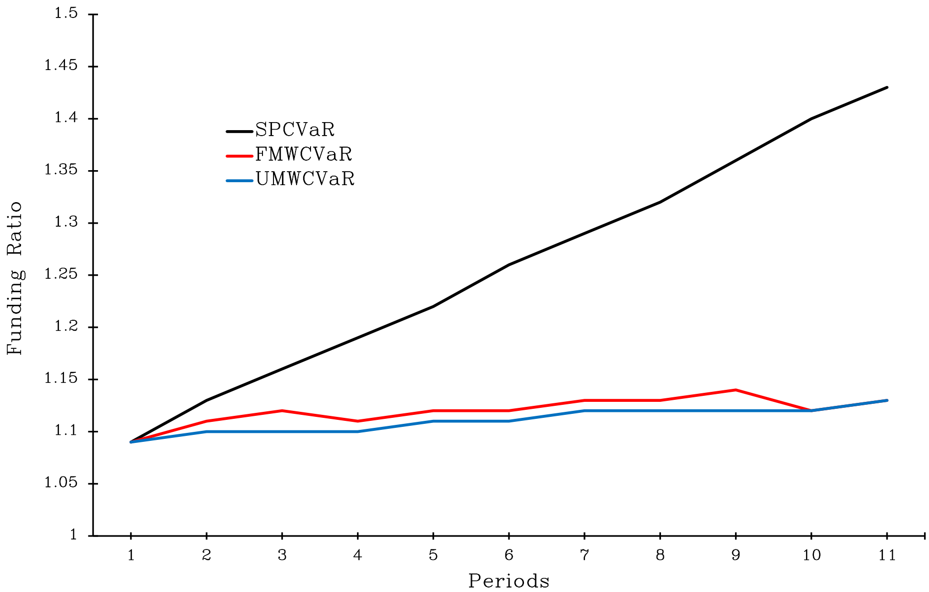

Figure 1: In-sample performance of funding ratio

Table 4
Optimal contribution rates of three models based on funding ratio

Models	FR=1.02	FR=1.05	FR=1.07	FR=1.1	FR=1.15
UMWCVaR	3.7%	5.7%	6.6%	7.7%	10.2%
FMWCVaR	0.9%	2.4%	3.3%	4.8%	7.1%
SPCVaR	0.1%	2.3%	3.2%	4.6%	7.1%

asset allocation of the different models in Figure 3 highlights the advantages of using the WCVaR models, which provide more diversified and less risky portfolios compared to the SPCVaR model.

Another point of comparison is the contribution rate, which changes based on the funding ratio (FR) threshold. Table 4, shows the comparison of the optimal contribution rates of three models (UMWCVaR, FMWCVaR, and SPCVaR). As shown in Table 4, the optimal contribution rates of the three models differ depending on the FR parameter. For instance, when FR=1.02, the optimal contribution rates for the UMWCVaR, FMWCVaR, and SPCVaR models are 3.7%, 0.9%, and 0.1%, respectively. However, as FR is increased, the optimal contribution rates of all three models also increase. Furthermore, the UMWCVaR model has the highest optimal contribution rates among the three models for all FR values. This suggests that this model may be the most conservative in managing risk under different FR scenarios. In contrast, the SPCVaR model has the lowest optimal contribution rates for FR values up to 1.1. However, when FR=1.15, the optimal contribution rates of the SPCVaR model become equal to that of the FMWCVaR model.

Besides in-sample analysis, we are comparing the out-of-sample performance of the above-mentioned models using simulation. 1000 scenarios of asset returns are generated based on distribution functions of asset returns in Table 1. Then, the optimal investment strategies of the UMWCVaR, FMWCVaR, and SPCVaR models are used to compare the

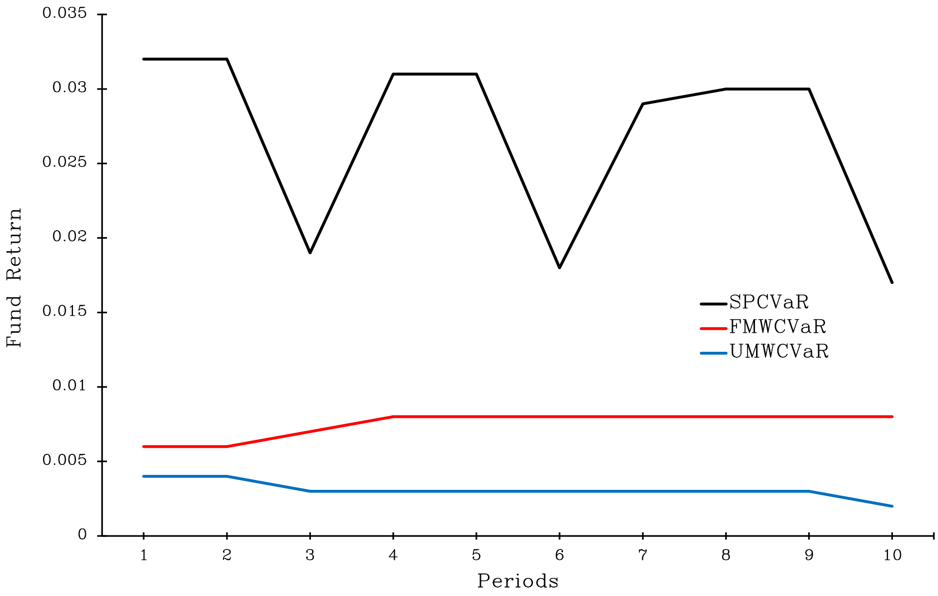

Figure 2: In-sample performance of fund return

funding ratio and value of assets in each period. Table 5 presents the out-of-sample performance of three ALM models. In the first two columns, we have the results of the UMWCVaR model, showing that the funding ratio ranges from 0.96 to 1.13 and the fund return ranges from 0.004 to 0.038 with an overall return of 9% in the investment horizon. The next two columns present the results of the FMWCVaR model, where the funding ratio ranges from 0.93 to 0.97 and the fund return ranges from -0.013 to 0.026 with an overall return of -5%. Finally, the last two columns present the results of the SPCVaR model, where the funding ratio ranges from 0.8 to 1.02 and the fund return ranges from -0.157 to 0.134 with an overall return of -2% with very high volatility.

Figure 4 compares the out-of-sample performance of the UMWCVaR, FMWCVaR, and SPCVaR models based on the fund return in each period. When comparing the fund return, we can observe that the FMWCVaR and SPCVaR models have 5 periods with a negative return rate. Moreover, the funding return of the SPCVaR model shows high volatility in comparison to other two models. The overall return of these two models, SPCVaR and FMWCVaR, are negative: -2% and -5%, respectively. On the other hand, the UMWCVaR model has a positive return in all periods with an overall average return of 9% which is very similar to the actual fund return of CPP last year which was 10% [10]. This indicates that the UMWCVaR model is more effective in generating return compared to the FMWCVaR and the SPCVaR models.

Figure 5 demonstrates the out-of-sample performance of models based on the funding ratio. Comparing the three models based on the funding ratio, we can see that the UMWCVaR model has higher funding ratios compared to the FMWCVaR and SPCVaR models. Moreover, the FMWCVaR model has better performance than the SPCVaR model except in the 6th period. This suggests that the UMWCVaR model is more stable and has a better ability to meet its

[10]https://www.cppinvestments.com/the-fund/our-performance/financial-results/f2022-annual-results

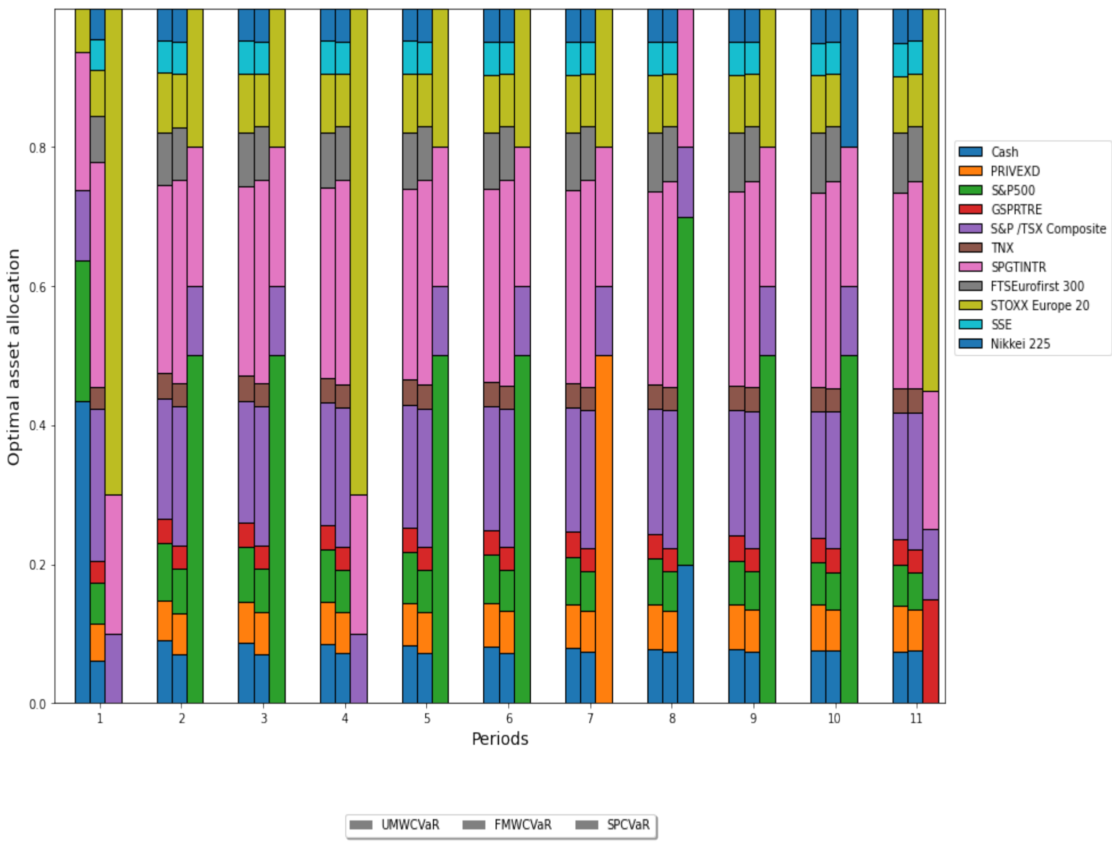

Figure 3: Comparision of optimal asset allocation

obligations than the two other models. On the other hand, the SPCVaR model has a lower funding ratio, indicating a higher risk of not being able to meet its obligations.

In conclusion, based on the results presented in Table 5, it appears that the UMWCVaR model outperforms the FMWCVaR and SPCVaR models in terms of funding ratio and fund return, implying better stability and asset management performance.

6. Conclusions

In this paper, we proposed a theoretical foundation for developing the WCVaR formulation for the ALM problem. The proposed theoretical development can be used in any problem with general loss functions. Based on the proposed theoretical foundation of WCVaR, we introduced the DRO reformulation of the ALM problem where the loss function is a linear function of asset returns and the present value of liabilities. The DRO reformulation of the ALM problem is proposed in two cases. First, the moments of the uncertain distribution function are fully known and fixed. Second, the moments of the distribution function of random variables are uncertain and belong to the uncertainty set.

Real data of CPP are used to test and analyze the performance of optimal investment strategies obtained by solving the DRO reformulations. The analysis was based on the in-sample and out-of-sample performance of the models. The results showed that the SP reformulation of the ALM has better in-sample performance than the DRO reformulation of the ALM models with respect to the fund return and funding ratio in each period. However, out-of-sample performance analysis revealed that the investment strategy of the DRO formulation of the ALM problem with uncertain moments has a better funding ratio and higher overall average fund return than the DRO with fixed moments and SP models. Consequently, we can conclude that the investment strategy achieved from the DRO reformulation of the ALM problem

Table 5
Out-of-sample performance of the ALM models

Periods	UMWCVaR		FMWCVaR		SPCVaR	
	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return
1	0.96	0.017	0.95	0.007	0.80	-0.157
2	0.97	0.006	0.94	-0.01	0.90	0.134
3	0.98	0.013	0.94	-0.004	0.93	0.028
4	1.02	0.038	0.94	0.004	0.85	-0.079
5	1.02	0.006	0.95	0.004	0.94	0.098
6	1.03	0.011	0.93	-0.013	0.94	0.00
7	1.04	0.011	0.96	0.026	0.92	-0.02
8	1.05	0.008	0.95	-0.01	0.91	-0.015
9	1.09	0.038	0.94	-0.003	0.85	-0.063
10	1.10	0.004	0.95	0.009	0.94	0.103
11	1.13	0.031	0.97	0.012	1.02	0.094

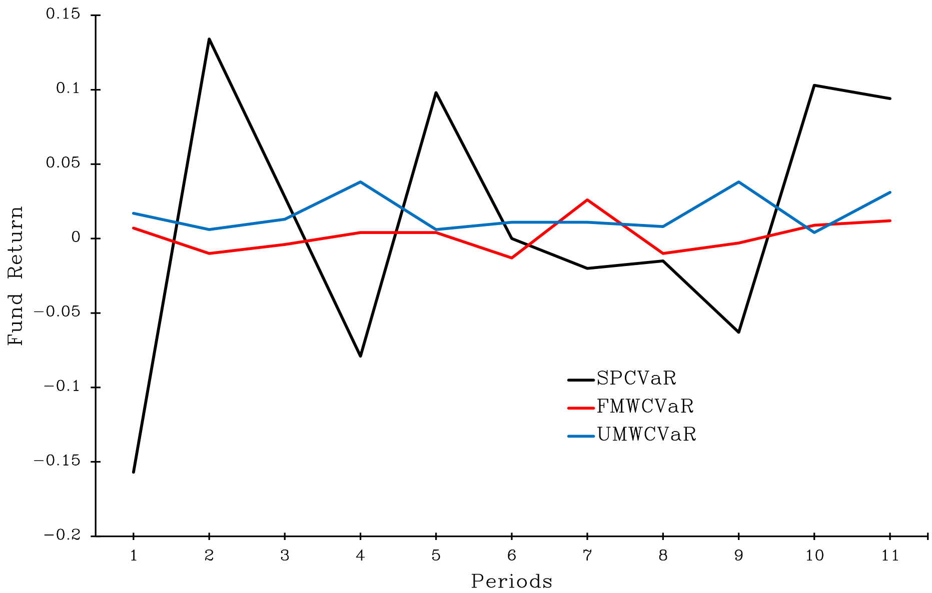

Figure 4: Out-of-sample performance of fund return

with uncertain moments can handle the asset and liability balance of pension funds better than the investment strategies of the DRO with fixed moments and SP models.

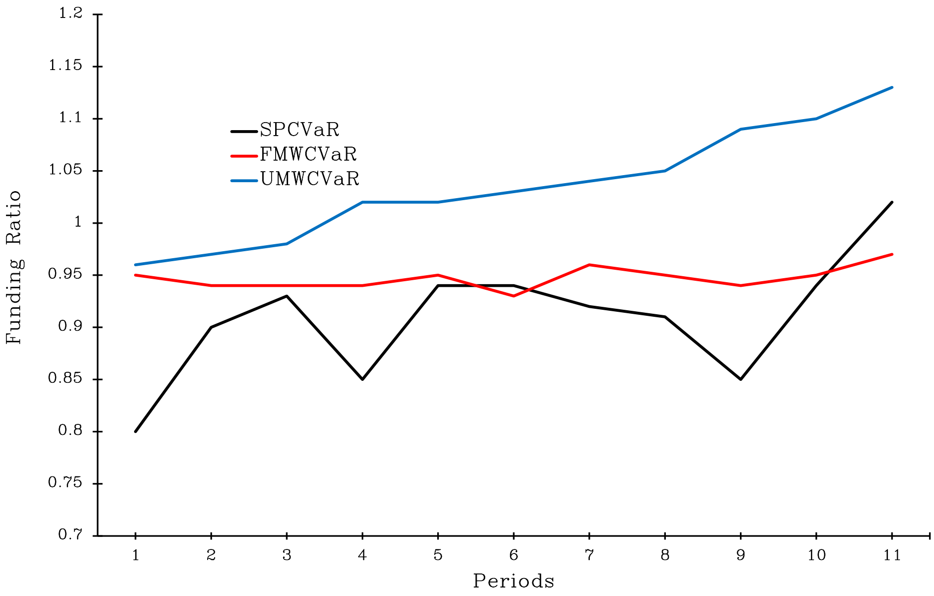

Figure 5: Out-of-sample performance of funding ratio

There are several avenues for future research that can build upon the contributions of this work. Firstly, while we have demonstrated the efficacy of the DRO reformulation of the ALM problem with moment-based ambiguity sets, it is important to investigate the performance of investment strategies obtained from the DRO formulation of ALM with statistical-distance-based ambiguity sets. This can provide insights into the impact of different types of ambiguity sets on the performance of the model. Secondly, the current study has focused on the application of DRO reformulation of ALM under uncertainty with respect to the moments of the distribution function of random variables. However, there is a need to investigate the impact of more general types of uncertainty, such as scenario-based uncertainty. Such investigations can shed light on how to develop investment strategies that can perform well under a range of uncertain scenarios. Thirdly, the current study has analyzed the performance of the proposed models using real-world data of CPP. However, further testing on a broader range of pension funds can provide a better understanding of the generalizability of the proposed models. Finally, the proposed models can be extended to incorporate other important considerations in pension fund management, such as taxes, transaction costs, and regulatory constraints. Such extensions can provide a more comprehensive framework for pension fund management that can handle a wider range of real-world constraints.

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Appendix A

Lemma 5. *Let ξ be a univariate random variable, where $\mathbb{E}[\xi]=\mu$, $Var(\xi)=\sigma^2$, and $f(.)$ is a nonlinear function of random variabnle ξ that $f:\mathbb{R}\rightarrow\mathbb{R}$. Then:*

$$\sup_{\xi\sim(\mu,\sigma^2)}\mathbb{E}\left[(\alpha-f(\xi))^+\right]\approxeq\frac{\alpha-\left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)+\sqrt{f'(\mu)^2\sigma^2+\left(\alpha-\left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)\right)^2}}{2},$$

where $f'(.)$ and $f''(.)$ are first and second derivation of $f(.)$, respectively.

PROOF. First, we need to find the expected value of $f(\xi)$. The second-order Taylor approximation of $f(\xi)$ around μ is:

$$\mathbb{E}[f(\xi)]\approxeq\mathbb{E}\left[f(\mu)+f'(\mu)(\xi-\mu)+\frac{1}{2}f''(\mu)(\xi-\mu)^2\right].$$

It is known that $\mathbb{E}(a+b)=\mathbb{E}(a)+\mathbb{E}(b)$. Then we can expand the proposed second-order Taylor approximation as:

$$\mathbb{E}[f(\xi)]\approxeq\mathbb{E}[f(\mu)]+f'(\mu)\mathbb{E}[\xi-\mu]+\frac{1}{2}f''(\mu)\mathbb{E}[\xi-\mu]^2,$$

where $\mathbb{E}[f(\mu)]=f(\mu)$, and $\mathbb{E}[\xi-\mu]=\mu-\mu=0$. Then:

$$\mathbb{E}[f(\xi)]\approxeq f(\mu)+\frac{1}{2}f''(\mu)\mathbb{E}[\xi-\mu]^2.$$

Since $\mathbb{E}[\xi-\mu]^2=Var(\xi)=\sigma^2$, then:

$$\mathbb{E}[f(\xi)]\approxeq f(\mu)+\frac{1}{2}f''(\mu)\sigma^2.$$

Now, we need to approximate $Var(f(\xi))$. The first order Taylor approximation of $f(\xi)$ around μ is:

$$f(\mu)+f'(\mu)(\xi-\mu).$$

Then $Var(f(\xi))$ can be approximated as:

$$Var[f(\xi)]\approxeq Var\left[f(\mu)+f'(\mu)(\xi-\mu)\right]=Var\left[f(\mu)+f'(\mu)\xi-f'(\mu)\mu\right].$$

The first term, $f(\mu)$, is constant then $Var(f(\mu))=0$. The third term $Var\left(f'(\mu)\mu\right)$ is also constant with a variance of zero. Consequently:

$$Var[f(\xi)]\approxeq Var\left[f'(\mu)\xi\right]=\left(f'(\mu)\right)^2Var[\xi]=f'(\mu)^2\sigma^2.$$

By substituting $\mathbb{E}[f(\xi)]$ and $Var(f(\xi))$ into the WLPM reformulation by Chen et al. (2011),

$$\sup_{\xi\sim(\mu,\sigma^2)}\mathbb{E}\left[(\alpha-f(\xi))^+\right]=\frac{\alpha-\mathbb{E}[f(\xi)]+\sqrt{Var(f(\xi))+(\alpha-\mathbb{E}[f(\xi)])^2}}{2},$$

we obtained the desired result. □

Theorem 6. *Let ξ be a univariate random variable with mean μ and variance σ^2, and define the ambiguty set $P=\{p\in M_+|P(\xi\in\Omega)=1,\xi\sim\left(\mu,\sigma^2\right)\}$. Moreover, $f(\xi)$ is a loss function, where $f:\mathbb{R}\rightarrow\mathbb{R}$. Then WCVaR can be approximated as follows:*

$$WCVaR_\beta\approxeq f(\mu)+\frac{1}{2}f''(\mu)\sigma^2+\sqrt{\frac{\beta}{1-\beta}}\sqrt{f'(\mu)^2\sigma^2}$$

PROOF. Based on the definition, $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi)-\alpha)^+\right]$. To reformulate the WCVaR, we need to calculate the WLPM term in the WCVaR definition. In Lemma 5, the LPM is in the form $\sup_{p(.)\in P} \mathbb{E}\left[(\alpha - f(\xi))^+\right]$. Hence we need to rearrange the LPM in CVaR as:

$$\sup_{p(.)\in P} \mathbb{E}\left[(f(\xi)-\alpha)^+\right] = \sup_{p(.)\in P} \mathbb{E}\left[(-\alpha-(-f(\xi)))^+\right].$$

based on Lemma 5:

$$\sup_{p(.)\in P} \mathbb{E}\left[(-\alpha-(-f(\xi)))^+\right] \approxeq \frac{-\alpha+\left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)+\sqrt{f'(\mu)^2\sigma^2+\left(-\alpha+\left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)\right)^2}}{2}.$$

Now, we can substitute this WLPM into the WCVaR formulation. Consequently:

$$WCVaR_\beta \approxeq \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\frac{-\alpha+\left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)+\sqrt{f'(\mu)^2\sigma^2+\left(-\alpha+\left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)\right)^2}}{2}.$$

To evaluate the minimization over α in the WCVaR definition we use the first-order optimality condition $\frac{\partial WCVaR_\beta}{\partial\alpha} = 0$, resulting in:

$$\alpha^* = f(\mu) + \frac{1}{2}f''(\mu)\sigma^2 + \frac{2\beta-1}{2\sqrt{\beta(\beta-1)}}\sqrt{f'(\mu)^2\sigma^2}.$$

By substituting α^* back in the definition of WCAR, we obtain the desired result. □

Lemma 5 and Theorem 6 are based on a function of a univariate random variable, while in many cases, loss functions are functions of multivariate random variables such as engineering design problems. Consequently, we extend this lemma/theorem to multivariate random variables.

Lemma 7. *Let $\boldsymbol{\xi} = \{\xi_1,\ldots,\xi_n\}$ be a multivariate random variable, where $\mathbb{E}\left[\xi_i\right] = \mu_i$, $Var\left(\xi_i\right) = \sigma_i^2$, $Cov\left(\xi_i,\xi_j\right) = \sigma_{ij}$, and $f(.)$ is a nonlinear function of random variable $\boldsymbol{\xi}$ that $f:\mathbb{R}^n\to\mathbb{R}$. Then $\sup_{\boldsymbol{\xi}\sim(\boldsymbol{\mu},\Sigma_{\boldsymbol{\xi}})}\mathbb{E}\left[(\alpha-f(\boldsymbol{\xi}))^+\right]$ can be approximated by:*

$$\frac{1}{2}\left(\alpha-\left(f(\boldsymbol{\mu})+\sum_i e_i\frac{\sigma_i^2}{2}+\sum_i\sum_{j>i}e_{ij}\sigma_{ij}\right)+\sqrt{\sum_i d_i^2\sigma_i^2+2\sum_i\sum_{j>i}d_id_j\sigma_{ij}+\left(\alpha-\left(f(\boldsymbol{\mu})+\sum_i e_i\frac{\sigma_i^2}{2}+\sum_i\sum_{j>i}e_{ij}\sigma_{ij}\right)\right)^2}\right),$$

where $\boldsymbol{\mu} = \{\mu_1,\ldots,\mu_n\}$ is the mean vector, $e_i = \frac{\partial^2 f(\boldsymbol{\xi})}{\partial^2\xi_i}\mid_{\boldsymbol{\xi}=\boldsymbol{\mu}}$, $e_{ij} = \frac{\partial^2 f(\boldsymbol{\xi})}{\partial\xi_i\partial\xi_j}\mid_{\boldsymbol{\xi}=\boldsymbol{\mu}}$, $d_i = \frac{\partial f(\boldsymbol{\xi})}{\partial\xi_i}\mid_{\boldsymbol{\xi}=\boldsymbol{\mu}}$ and $\mid_{\boldsymbol{\xi}=\boldsymbol{\mu}}$ means to evaluate the expression with μ_i replacing ξ_i.

PROOF. Based on the second-order Taylor series expansion of $f(.)$ around $\boldsymbol{\mu} = \{\mu_1,\ldots,\mu_n\}$, the expected value of $f(.)$ is approximated by:

$$\mathbb{E}\left[f(\boldsymbol{\xi})\right] \approxeq \mathbb{E}\left[f(\boldsymbol{\mu})\right] + \mathbb{E}\left[\nabla f(\boldsymbol{\mu})(\boldsymbol{\xi}-\boldsymbol{\mu})\right] + \mathbb{E}\left[\frac{1}{2}(\boldsymbol{\xi}-\boldsymbol{\mu})^\intercal H_f(\boldsymbol{\mu})(\boldsymbol{\xi}-\boldsymbol{\mu})\right],$$

where H_f is the Hessian matrix of f. The second term is zero since $\mathbb{E}\left[\boldsymbol{\xi}-\boldsymbol{\mu}\right] = \mathbb{E}\left[\boldsymbol{\xi}\right]-\boldsymbol{\mu} = \boldsymbol{\mu}-\boldsymbol{\mu} = 0$. In the last term, $\mathbb{E}\left[(\boldsymbol{\xi}-\boldsymbol{\mu})^2\right] = \Sigma_{\boldsymbol{\xi}}$ is the variance-covariance matrix of $\boldsymbol{\xi}$, then $\mathbb{E}\left[f(\boldsymbol{\xi})\right] \approxeq f(\boldsymbol{\mu}) + \frac{1}{2}H_f(\boldsymbol{\mu})\Sigma_{\boldsymbol{\xi}}$. Expansion of this formulation is:

$$\mathbb{E}\left[f(\boldsymbol{\xi})\right] \approxeq f(\boldsymbol{\mu}) + \sum_i e_i\frac{\sigma_i^2}{2} + \sum_i\sum_{j>i}e_{ij}\sigma_{ij}.$$

Moreover, based on the first-order Taylor approximation, the variance of $f(.)$ is:

$$Var(f(\xi)) \approxeq Var\left(f(\mu) + \nabla f(\mu)^{\intercal}(\xi - \mu)\right) = Var\left(f(\mu) + \nabla f(\mu)^{\intercal}\xi - \nabla f(\mu)^{\intercal}\mu\right).$$

Since $f(\mu)$, and $\nabla f(\mu)\mu$ are constants, their variances are zero. Hence, $Var(f(\xi)) \approxeq Var\left(\nabla f(\mu)^{\intercal}\xi\right)$ which is equivalent to $\nabla f(\mu)^2 \Sigma_{\xi}$. This formulation can be expanded as $Var(f(\xi)) \approxeq \sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j \sigma_{ij}$. By substituting $\mathbb{E}[f(\xi)]$ and $Var(f(\xi))$ into $\frac{\alpha - \mathbb{E}[f(\xi)] + \sqrt{Var(f(\xi)) + (\alpha - \mathbb{E}[f(\xi)])^2}}{2}$, we obtain the desired result. □

Theorem 8. *Let $\xi \in \mathbb{R}^n$ be a multivariate random variable with mean vector μ and covariance matrix Σ_{ξ}, where the ambiguity set is $P = \{p \in M_+ | P(\xi \in \Omega) = 1, \xi \sim \left(\mu, \Sigma_{\xi}\right)\}$. Moreover, $f(\xi)$ is a loss function, where $f : \mathbb{R}^n \to \mathbb{R}$. Then WCVaR is defined as $WCVaR_{\beta} = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right]$ which is approximated by:*

$$f(\mu) + \sum_i e_i \frac{\sigma_i^2}{2} + \sum_i\sum_{j>i} e_{ij}\sigma_{ij} + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j \sigma_{ij}}.$$

PROOF. WCVaR is defined as $WCVaR_{\beta} = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi) - \alpha)^+\right]$. In Lemma 7, we showed how to approximate the WLPM of a function of multivariate random variables as $\sup_{p(.)\in P}\mathbb{E}\left[(f(\xi) - \alpha)^+\right] = \sup_{p(.)\in P}\mathbb{E}\left[(-\alpha - (-f(\xi)))^+\right]$ which is approximated by:

$$\frac{1}{2}\left(-\alpha + \left(f(\mu) + \sum_i e_i\frac{\sigma_i^2}{2} + \sum_i\sum_{j>i} e_{ij}\sigma_{ij}\right) + \sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j\sigma_{ij} + \left(-\alpha + \left(f(\mu) + \sum_i e_i\frac{\sigma_i^2}{2} + \sum_i\sum_{j>i} e_{ij}\sigma_{ij}\right)\right)^2}\right),$$

which can be substituted in WCVaR formulation instead of WLPM.

Minimization of WCVaR is over α. Then its optimal value α^* is needed. Optimal α^* can be calculated by using the first-order optimality condition $\frac{\partial WCVaR_{\beta}}{\partial \alpha} = 0$. The α^* is as follows:

$$\alpha^* = f(\mu) + \sum_i e_i\frac{\sigma_i^2}{2} + \sum_i\sum_{j>i} e_{ij}\sigma_{ij} + \frac{2\beta - 1}{2\sqrt{\beta(\beta-1)}}\sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j\sigma_{ij}}.$$

Finally, α^* can be used in the formulation of WCVaR instead of α which leads to the desired result. □

A quadratic function is a special case of a nonlinear function. A loss function can be defined based on a quadratic function of a random variable. For example, tracking errors in index-tracking PSPs is an example of a quadratic function that can be used as a loss function. In Lemma 5 and theorem 6, both the mean and variance of the loss function are approximated by the Taylor approximation method. However, by using a quadratic function of a random variable as a loss function, the variance of the loss function should be approximated while the expected value of the loss function can be calculated based on exact formulation. Remark 1 shows how to calculate the WLPM for a quadratic function of a random variable.

Remark 1. Let ξ be an univariate random variable, where $\mathbb{E}[\xi] = \mu$, and $Var(\xi) = \sigma^2$. Then, WLPM is approximated as follows:

$$\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[\left(\alpha - \xi^2\right)^+\right] \approxeq \frac{\alpha - \left(\sigma^2 - \mu^2\right) + \sqrt{4\mu^2\sigma^2 + \left(\alpha - \left[\sigma^2 - \mu^2\right]\right)^2}}{2}.$$

PROOF. Based on definition of the first two moments of ξ, $Var(\xi) = \mathbb{E}\left[\xi^2\right] - \mathbb{E}[\xi]^2 = \sigma^2$. Then, $\mathbb{E}\left[\xi^2\right] = \sigma^2 - \mu^2$. The first-order Taylor approximation of ξ^2, around $\mathbb{E}[\xi]$, is $\mathbb{E}[\xi]^2 + 2\mathbb{E}[\xi](\xi - \mathbb{E}[\xi])$. Consequently, variance of ξ^2 is approximated as follows:

$$Var\left(\xi^2\right) \approxeq Var\left(\mathbb{E}[\xi]^2 + 2\mathbb{E}[\xi](\xi - \mathbb{E}[\xi])\right)$$

$$= Var\left(\mathbb{E}\left[\xi\right]^2 + 2\mathbb{E}\left[\xi\right]\xi - 2\mathbb{E}\left[\xi\right]\mathbb{E}\left[\xi\right]\right) = Var\left(\mu^2 + 2\mu\xi - 2\mu^2\right).$$

The first and third terms are constants, then their variances are zero. Hence,

$$Var\left(\xi^2\right) \approxeq Var\left(2\mu\xi\right) = 4\mu^2 Var\left(\xi\right).$$

By using $\mathbb{E}\left[\xi^2\right]$ and $Var\left(\xi^2\right)$, WLPM is as follows:

$$\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[\left(\alpha - \xi^2\right)^+\right] \approxeq \frac{\alpha - \left(\sigma^2 - \mu^2\right) + \sqrt{4\mu^2\sigma^2 + \left(\alpha - \left[\sigma^2 - \mu^2\right]\right)^2}}{2}.$$

□

The WCVaR for quadratic loss function is defined based on remark 2.

Remark 2. Let ξ be a univariate random variable with mean μ and variance σ^2, where the ambiguity set is $P = \{p \in M_+ | P\left(\xi \in \Omega\right) = 1, \xi \sim \left(\mu, \sigma^2\right)\}$. Moreover, ξ^2 is a loss function. Then WCVaR is defined as:

$$WCVaR_\beta \approxeq \sigma^2 - \mu^2 + 2\mu\sigma\sqrt{\frac{\beta}{1-\beta}}$$

PROOF. Based on definition, $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[\left(\xi^2 - \alpha\right)^+\right]$. The WLPM of ξ^2 is defined based on remark 1. Hence:

$$\sup_{p(.)\in P} \mathbb{E}\left[\left(\xi^2 - \alpha\right)^+\right] = \sup_{p(.)\in P} \mathbb{E}\left[(-\alpha - (-\xi^2))^+\right] \approxeq \frac{-\alpha + \left(\sigma^2 - \mu^2\right) + \sqrt{4\mu^2\sigma^2 + \left(-\alpha + \left[\sigma^2 - \mu^2\right]\right)^2}}{2}.$$

By using the approximation of WLPM in WCVaR formulation, WCVaR of the quadratic loss function is as follows:

$$WCVaR_\beta \approxeq \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\frac{-\alpha + \left(\sigma^2 - \mu^2\right) + \sqrt{4\mu^2\sigma^2 + \left(-\alpha + \left[\sigma^2 - \mu^2\right]\right)^2}}{2}.$$

Minimization of WCVaR is over α, hence its optimal value is needed which can be calculated by solving the first-order optimality condition, $\frac{\partial WCVaR_\beta}{\partial\alpha} = 0$. The optimal α^* is as follows:

$$\alpha^* = \sigma^2 - \mu^2 + \frac{2\beta - 1}{2\sqrt{\beta(\beta-1)}}\sqrt{4\mu^2 Var(\xi)}.$$

By using α^* in $WCVaR$ instead of α, WCVaR is approximated as desired result.

□