Democratization of Complex-Problem Solving to Enhance Participation, Transparency, Accountability, and Fairness: An Optimization Perspective

Mikhail A. Bragin[1,*], Karmel S. Shehadeh[b]

[a]Department of Electrical and Computer Engineering, University of Connecticut, Storrs, CT, USA
[b]Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, USA

Abstract

Operations in critical areas of importance to society, such as healthcare, transportation and logistics, power systems, and emergency response, profoundly affect multiple stakeholders with diverse perspectives. These operations are often modeled using discrete programming methods to capture the various decision-making factors through centrally-selected objectives and constraints. Unfortunately, centralized modeling and solution methodologies may overlook the perspectives and needs of certain stakeholders, potentially leading to the exclusion of certain stakeholders. Additionally, discrete programming problems suffer from the curse of combinatorial complexity, which can result in suboptimal solutions and difficulties in achieving a transparent, intuitive, fair, and equitable outcome. To address these challenges and foster inclusive synergy, we propose an approach to democratize complex problem-solving through distributed modeling and computation methods to enable participation as well as to increase fairness, accountability, and transparency. Our approach employs the latest versions of Lagrangian Relaxation to decompose complex problems into subproblems, empowering stakeholders to actively and autonomously participate in independent decision-making by incorporating constraints and preferences in accordance with their values. The fast coordination of subproblems based on the economic "supply and demand" principle ensures that the optimization outcomes are economically efficient. In addition, this approach harnesses "cyber-human" collective intelligence to enable efficient decision-making.

Keywords: Discrete Programming; Complex Problem Solving; Democratization; Participatory Decision-Making; Collective Intelligence; Optimization; Fairness; Transparency; Accountability

[*]Corresponding author.

Email addresses: mikhail.bragin[AT]uconn.edu, mikhail_a_bragin[AT]yahoo.com (Mikhail A. Bragin), kshehadeh[AT]lehigh.edu, kas720[AT]lehigh.edu (Karmel S. Shehadeh)

1. Introduction

Operations in critical sectors of societal importance, such as healthcare, transportation and logistics, power systems, and emergency response, encompass multiple stakeholders with diverse constraints and objectives, which complicates and challenges the decision-making process. The decisions made in these areas have substantial repercussions for individuals and society as a whole. Stakeholders may vary across sectors, including customers, providers, medical professionals, patients, emergency responders, transportation operators, power grid operators, generator owners, and local authorities, among others. Hence, it is essential to ensure fair, transparent, and equitable decision-making that addresses the needs and concerns of all stakeholders while promoting beneficial outcomes for society. This can be accomplished by engaging all stakeholders in the decision-making process and implementing policies that foster fairness, transparency, and equity. Inclusive and collaborative approaches, combined with computationally efficient algorithms, can lead to more robust and sustainable solutions that benefit all parties involved and contribute to the overall well-being and resilience of society. Utilizing efficient computational methods enables stakeholders to quickly analyze various scenarios and make informed decisions, ultimately enhancing the effectiveness and responsiveness of decision-making processes in these critical sectors.

Decision-making in the above application domains often involves inherently discrete decisions, such as assignment, location, allocation, commitment decisions, etc. Consequently, the associated optimization problems are frequently modeled as discrete programming problems that capture the various decision-making aspects. Traditional centralized approaches formulate corresponding systems of interest from the perspectives of centralized decision-makers that generally gather or define data, objectives, and constraints. Such formulations, however, are based on assumptions that could potentially overlook stakeholders' needs and often do not accurately or fairly capture all stakeholders' perspectives, thereby leading to exclusion, inequality, as well as a lack of fairness, transparency, and participation. Modern systems, such as smart cities, transportation networks, and power grids, grow ever more complex and interconnected, traditional approaches to problem-solving, which often rely on centralized methods, are therefore proving increasingly inadequate. Specifically, as the number of stakeholders increases (e.g., distributed energy owners) and the needs of stakeholders are recognized to be increasingly valuable (e.g., patients in healthcare systems or evacuees requiring immediate attention), centralized methods may overlook the perspectives and the needs of multiple stakeholders, leading to unfairness, exclusion, and inequality. Furthermore, including stakeholders' needs may require sharing private information with the centralized decision-maker to incorporate into the decision-making process. This may compromise those stakeholders' privacy and violate privacy rights, potentially leading to biased solutions that can negatively impact vulnerable populations. On the other hand, excluding such information and stakeholders' needs from the decision-making process (through centralized formulation) leads to solutions that could be unfair, meaningless, and irrelevant to many stakeholders. Scalability is also a significant concern for discrete programming problems. Unfortunately, most real-life problems have complex combinatorial nature; thus, it is often difficult to obtain high-quality solutions within a reasonable time.

Giving stakeholders more decision-making power to formulate their (sub)problems according to their needs and solve them can potentially improve fairness, transparency, and equity. In this paper, we explore the application of advanced Lagrangian Relaxation (LR) techniques, which efficiently "reverse" combinatorial complexity through problem decomposition and accelerate coordination, guided by market principles of "supply and demand." Our paper discusses how these methods can enhance participation, transparency, accountability, and fairness in optimization contexts. This is achieved by empowering stakeholders to make their own decisions, broadly defined as formulating their own subproblems based on their needs, preferences, and moral principles, as well as solving the resulting subproblems.

These key features of Lagrangian Relaxation enable decentralized decision-making and collaboration through harnessing collective intelligence and efficient coordination. The key difference from the traditional idea of LR, which primarily involves decomposing centrally-formulated problems and the associated subproblem solving and coordination, is the ability to incorporate stakeholders' objectives and constraints into global decision-making. After decomposing the overall problem into stakeholder subproblems, each stakeholder has complete, unbiased control to optimize the subproblem they designed, eliminating the need to share private and sensitive information with the central decision maker. By doing so, our proposed idea empowers local communities to participate in fair decision-making by allowing them to formulate their own subproblems and solve them to arrive at more equitable solutions. This approach also leads to computationally effective decision-making that harnesses collective intelligence and utilizes fast convergence principles, thereby enabling the democratization of complex-problem solving.

By involving multiple stakeholders in the problem-solving process, our proposed approach ensures that all perspectives and needs are fairly considered, leading to more inclusive and equitable outcomes fostering trust and collaboration, satisfaction, and desire for the participation of stakeholders. While the decision-making is done at the stakeholder level, the coordination can be done either centrally or distributedly. Important to note that central coordination is not tantamount to centralized decision-making with the difficulties mentioned above. In the case of centralized coordination, a deeper understanding of the needs and concerns of different stakeholders can be gained through the sensitivity nature of Lagrangian multipliers. This aspect of the method benefits stakeholders and society as a whole, as it leads to deeper insights and better decision-making at policy-making levels.

The paper is structured as follows. In Section 2, we review relevant literature and discuss the democratization of problem-solving from a multidisciplinary perspective not necessarily limited to discrete programming problems. In Section 3, we present our proposed democratization approach in detail and describe our system-optimization framework. In Section 4, we discuss limitation with future directions delineated. In Section 5, conclusions are provided.

2. Literature Review

In this section, we discuss relevant work. Specifically, in subsection 2.1, we review recent research on promoting the democratization of decision-making through the promotion of inclusivity, diversity, transparency, collective intelligence, and equity in various domains. Then, in subsection 2.2, we briefly introduce a specific type of discrete programming problem, separable mixed-integer linear programming (MILP), from a centralized perspective. In addition, we discuss some of the related applications of MILP in various domains, such as power systems, healthcare, transportation, and humanitarian work. Finally, in subsection 2.3, we review Lagrangian Relaxation (LR) methods for discrete optimization problems, as well as non-smooth optimization arising in LR. Overall, these subsections emphasize various problem-solving aspects at different levels, including philosophical, societal, civic, legal, managerial, ethical, fair and equitable, centralized, distributed, technological, and computational approaches.

2.1. Advancing Inclusive and Ethical Decision-Making through Democratization and Collective Intelligence

There has been growing interest in democratizing knowledge creation and collaboration in recent years. This trend reflects a broader societal shift towards promoting democratization, inclusivity, diversity, transparency, and equity in various domains, including science, politics, and culture. Scholars from various fields, including philosophy, law, management, social media, computer science, mathematics, anthropology, ethnography, and sociology, have looked at the democratization aspects from different angles and have explored different strategies for achieving these goals. In this section, we will review some of the key approaches and insights that have emerged from this body of literature, focusing on their implications for knowledge production and collaboration.

Epistemic Injustice and Inclusive Knowledge Production: The Philosophers' Perspective. Nuccetelli's works (Nuccetelli and Seay, 2007; Nuccetelli, 2021) focus primarily on epistemic (i.e., related to knowledge acquisition and understanding) virtues and values in the context of ethics and philosophy. One can adopt some of the principles in Nuccetelli's work to democratize complex-problem solving. For example, intellectual humility and open-mindedness, as advocated by Nuccetelli, can encourage stakeholders to recognize the limits of their knowledge and be more receptive to alternative perspectives and solutions, fostering collaboration and participation. Furthermore, the philosophical idea that problem-solving should not be confined to a particular group or class of experts can be inspired by G.E. Moore's work (Moore, 1903). Moore's emphasis on the open-ended nature of ethical questions, as suggested by his assertion that "*all things which are good are also something else*," implies that independent evaluation and multiple perspectives might be essential for effective (democratic) decision-making. The exploration of ethical considerations at the intersection of technology has been a crucial subject of debate, attracting the attention of numerous researchers as discussed next.

Ethical Considerations and Democratization in Emerging Technologies and Algorithmic Decision-Making. The works of Brey (Brey, 2012), Coeckelbergh (Coeckelbergh, 2012), Mittelstadt et al. (Mittelstadt et al., 2016), and Pak-Hang Wong (Wong, 2020) collectively address ethical, social, and political challenges, such as unpredictability (Brey, 2012) of emerging technologies and algorithmic decision-making. As "technology displays arrogance and lack of humility" (Coeckelbergh, 2012) frequently taking on the role of the sole decision maker (Mittelstadt et al., 2016), the emphasis on the importance of public engagement, algorithmic mediation, transparency, accountability, democratization/democracy, and fairness in the development and deployment of emerging technologies is crucial. Wong (2020), in particular, argues for a framework that democratizes algorithmic fairness, underlining the significance of considering various stakeholders' concerns and interests when designing and implementing algorithms. This approach ensures greater fairness and inclusivity, while the other works contribute to the broader conversation on ethical implications and the need for anticipatory ethics, data ethics, and moral responsibility in technology.

Democracy as Problem-Solving: Civic Capacity in Communities Across the Globe. Xavier de Souza Briggs (Briggs, 2008) explores the notion of democracy as a way to solve societal problems, emphasizing the importance of citizen participation and collaboration in developing a collective capacity in decision-making processes. The author argues that democracy is not only about voting and representation but also about involving stakeholders in finding solutions to complex problems such as urban planning, environmental sustainability, and public health. Through case studies from around the world, the book showcases various examples of how communities have come together to address these challenges, highlighting the importance of building civic capacity and creating spaces for meaningful dialogue and deliberation. Overall, Briggs' work provides a compelling argument for the value of participatory democracy, which is reflected in the democratization of complex-problem solving proposed in this paper.

Potential of Decentralized Systems for Democratizing Problem-Solving: Legal Scholar's Approach. Yochai Benkler's work (Benkler, 2000, 2002, 2004, 2006, 2011) emphasizes the potential of decentralized systems and distributed forms of collaboration in creating more participatory and equitable problem-solving. He highlights the role of open-source software, peer production, and commons-based approaches (i.e., approaches promoting cooperation, horizontal and decentralized decision-making as well as networks) in transforming economic and social relations. Benkler's work demonstrates that decentralized networks challenge traditional hierarchies and centralized power structures, promoting greater participation and diverse viewpoints in decision-making processes.

Collective Intelligence and Online Collaboration: The Management Perspective. Malone et al. (2009) identified diversity, independence, decentralization, and aggregation as key factors for collective intelligence and crowdsourcing (obtaining information by using a large number of people). Thomas Malone's book (Malone, 2018) explores the idea of combining the cognitive abilities of humans with the computational power of machines to create what he calls the "Supermind." This concept refers to a "powerful combination of many individual minds" working together, where humans and machines collaborate in a "cyber-human" fashion to solve complex problems and make better decisions. Malone emphasizes the potential for technology to enhance collective intelligence and improve democratic decision-making through this collaboration. Malone and Bernstein (2022) provide a comprehensive overview of the field with contributions from multiple experts.

Technology and Society: Issues of Privacy, Surveillance, Transparency, and Accountability. Danah Boyd's work emphasizes the need for greater attention to technology's social and ethical implications, specifically related to privacy, surveillance, transparency, and accountability (Boyd, 2014; Boyd and Crawford, 2012; Boyd, 2019). Specifically, Boyd provides insights into the importance of considering various user groups' diverse needs and perspectives when designing technology solutions (Boyd, 2014). In addition, her focus on transparency and ethical oversight can mitigate potential negative consequences of technology and promote the equitable distribution of its benefits (Boyd and Crawford, 2012). Moreover, Boyd's work addresses issues of power and control in the technology industry, promoting a more equitable distribution of power and ensuring technology benefits society as a whole (Boyd and Crawford, 2012).

Protecting Privacy and Promoting Transparency in Distributed Systems: Computer Science and Mathematics Approach. Cynthia Dwork's research on privacy and security in distributed systems emphasizes individual rights and data processing transparency (Dwork, 2008; Dwork et al., 2012, 2014). Her work on differential privacy provides a framework for protecting individuals' privacy in statistical databases and quantifies privacy risks in the data release. Dwork also proposes algorithms for achieving fairness in machine learning, including the "fairness through awareness" framework that combines differential privacy with statistical discrimination measures (Dwork et al., 2012). Her work is important for promoting ethical and fair problem-solving, protecting privacy rights, and addressing issues of bias and discrimination. Overall, Cynthia Dwork's work highlights the importance of protecting individual rights and promoting transparency in developing and deploying technological systems, particularly in the context of privacy and fairness.

Understanding Technology in Social and Cultural Contexts: Ethical Considerations. Annette Markham's ethnographically-grounded research explores the social and cultural contexts shaping technology's development and use, emphasizing its role in forming social relationships and cultural practices (Markham, 1998). Her work, Markham (1998) reveals the intersection of technology with social and cultural contexts, addressing authenticity and self-presentation in online environments. In collaboration with Nancy K. Baym (Markham and Baym, 2008), Markham provides an exploration of methodologies for conducting research in online environments, discussing challenges, opportunities, and ethical considerations in studying internet-based phenomena. Her work is crucial for understanding democratization, decision-making access, and the online/offline interplay in modern society, which informs effective collaboration in problem-solving. Overall, Markham's research underscores the need to consider these factors when developing inclusive, equitable problem-solving methods.

AI and Data-Driven Decision-Making: Ethical and Social Implications. Kate Crawford emphasized the need for transparency, accountability, diversity, and inclusivity in the development and deployment of AI systems (Crawford, 2013; Whittaker et al., 2018; Crawford and Paglen, 2021). Her works, including (Crawford, 2013; Whittaker et al., 2018; Crawford and Paglen, 2021), discuss potential biases and errors in AI and the importance of understanding the social and cultural context of training data. Crawford's work provides valuable insights into the ethical and social implications of AI and data-driven decision-making. It can help ensure that AI is used in ways that are fair, transparent, and accountable.

FAT and Privacy-Aware AI Modeling Approaches: Advancing Business Decision Making. Dmitry Zhdanov et al. (2022) explore the potential of incorporating fairness, accountability, and transparency (FAT) and privacy-aware artificial intelligence (AI) modeling approaches into business decision-making frameworks. By considering AI's ethical and social implications, this approach can mitigate the risks of bias and discrimination, enhance transparency and accountability, and foster trust and acceptance among stakeholders. The paper provides a review of relevant literature and proposes a framework for integrating FAT and privacy-aware AI modeling approaches into existing business decision-making processes, highlighting the potential benefits of improving decision-making quality, reducing risk, and promoting social responsibility.

Addressing Energy Justice and Electricity Tariff Design: Equitable Power Grid Modernization. Energy justice is a priority in power systems. Grid modernization, for example, aims to achieve social and economic equity while addressing the disproportionate harms (social, economic, and health burdens) caused by the energy system on historically affected communities (U.S. Department of Energy, 2022). Incorporating energy justice objectives in grid planning can benefit disadvantaged communities by reducing pollution, mitigating infrastructure impacts, and enhancing resiliency and energy security while supporting decarbonization. By considering energy justice in electricity tariff design, system planners can minimize energy burdens on vulnerable households while maintaining system cost recovery and fairness (Khan et al., 2023).

Addressing Equity and Justice in Zero-Emission Freight and Drayage Truck Deployment. Polluting drayage trucks powered by older diesel engines emit high levels of harmful pollutants, particularly in disadvantaged communities, posing significant threats to public health (Ramirez-Ibarra and Saphores, 2023). Policymakers aim to deploy zero-emission freight and drayage trucks to mitigate negative impacts and ensure equitable access to zero-emission transportation by 2045 (Brown et al., 2021). However, addressing equity and justice concerns in the transition is critical for an equitable distribution of benefits and burdens (Enriquez, 2019). Targeted investments in disadvantaged communities and stakeholder engagement are essential for identifying and addressing unique challenges and opportunities (Enriquez, 2019). Promoting equity and justice in deploying of zero-emission freight and drayage trucks is essential to realizing a sustainable transition that benefits all communities (Enriquez, 2019).

Fairness and Decision-Making. Various studies across multiple disciplines, including economics, decision theory, and operations research explored the issue of fairness and equity in decision-making models and algorithms in diverse contexts. We refer to (Karsu and Morton, 2015; Shehadeh and Snyder, 2022; McGregor et al., 2019; Young, 1995) for comprehensive surveys, discussions, and applications.

2.2. Centralized Perspective for Separable Mixed-Integer Programming Problems

In this section, we will first briefly introduce separable mixed-integer linear programming (MILP) problems, taking a centralized perspective. We will then discuss the limitations of centralized decision-making.

Centralized Mixed-Integer Linear Programming. Mixed-Integer Linear Programming (MILP) has been a powerful paradigm across various scientific disciplines, including mathematics, operations research, engineering, and computer science. For example, MILP has found application in various areas of societal importance, such as healthcare (Kopanos et al., 2010; Stefansson et al., 2011; Kim and Mehrotra, 2015; VonAchen et al., 2016; Zhu and Ursavas, 2018; Khlif Hachicha and Zeghal Mansour, 2018; Shehadeh et al., 2020; Dastgoshade et al., 2020; Ge and Yuan, 2021; Kayvanfar et al., 2021; Prabhu et al., 2021; Tsang and Shehadeh, 2023; Wickett et al., 2023), transportation and logistics (Freund et al., 2017; Jiang et al., 2020; Karolemeas et al., 2021; Dong and Leng, 2021; Archetti et al., 2021; Kamyabniya et al., 2021; Balakrishnan et al., 2021; Reddy et al., 2022; Gupta et al., 2022; Shao et al., 2022; Shehadeh and Tucker, 2022; Yan et al., 2022), humanitarian applications (Smalley et al., 2015; Aghaei et al., 2019; Hamdan and Diabat, 2020; Ahani et al., 2021; Kamyabniya et al., 2021), and power and energy systems (Afshar et al., 2008; Pozo et al., 2012; Morales-España et al., 2013; Bischi et al., 2014; Wierzbowski et al., 2016; Theo et al., 2016; Schill et al., 2017; Nikoobakht et al., 2018; Damchi et al., 2018; Rafinia et al., 2020; Sun et al., 2018; Chen et al., 2020; Li et al., 2020; Wu et al., 2021; Shen et al., 2021; Wu et al., 2023). A centralized decision maker frequently formulates these MILP problems by using cost components related to each stakeholder (or a "subsystem"), with the corresponding objective functions being additive. Mathematically, the objective takes the following form:

$$\min_{(x^c,y^c):=\left\{x_i^c,y_i^c\right\}_{i=1}^{I}} \left\{ \sum_{i=1}^{I^c} \left((c_i^{c,x})^T \cdot x_i^c + (c_i^{c,y})^T \cdot y_i^c \right) \right\}. \tag{1}$$

Furthermore, coupling constraints are additive in terms of I^c subsystems:

$$s.t.\ \ \sum_{i=1}^{I^c} A_i^{c,x} \cdot x_i^c + \sum_{i=1}^{I^c} A_i^{c,y} \cdot y_i^c - b^c = 0, \{x_i^c, y_i^c\} \in \mathcal{F}_i^c, i = 1, \ldots, I^c. \tag{2}$$

The *primal* problem (1)-(2) is assumed to be feasible and the feasible region $\mathcal{F}^c = \prod_{i=1}^{I^c} \mathcal{F}_i^c$ with $\mathcal{F}_i^c \subset \mathbb{Z}^{n_i^{c,x}} \times \mathbb{R}^{n_i^{c,y}}$ is assumed to be bounded and finite. The information about b^c is assumed to be public or known to the decision maker with certainty; in the case of stochastic modeling, the decision maker is assumed to know the underlying probabilistic descriptions affecting b^c.

Limitations of Centralized Decision Making. While formulating the problem, the central decision maker (hence the superscript "c") decides on the number of stakeholders to be included I^c, and the constraints (2) that couple stakeholders' subsystems. However, the centralized approaches may include the following issues:

1. **Expert Limitations.** Relying solely on experts (i.e., the traditional centralized decision-makers) can be problematic and inconsistent with the ideas of Nuccetelli and Moore reviewed in subsection 2.1, because even experts may still have a narrow perspective and may not consider all stakeholders' diverse needs and interests.

2. **Stakeholder Exclusion.** The assumption is that I^c represents an upper limit on the number of stakeholders the central decision-maker is willing to include, which may be lower than the actual number of stakeholders.

3. **Compromised Stakeholder Autonomy.** Even the [centrally] optimal decision to (1)-(2) may have a binding effect on individual stakeholders forcing them to make moves that are inconsistent with their moral values and free will. Stakeholder willingness to participate may be compromised. Each stakeholder's careful evaluation of their principles is needed as related to fairness, justice, and the common good.

4. **Privacy Violation.** Even in the above scenario, a stakeholder would need to share private information with the centralized decision-maker regarding the values of $c_i^{c,x}$, $c_i^{c,y}$, $A_i^{c,x}$, and $A_i^{c,y}$, thereby violating privacy.

5. **Inaccurate Approximations.** Alternatively, the central decision maker may resort to approximations or assumptions to address the lack of information. However, such approximations may lead to an inaccurate characterization of the stakeholders' needs and preferences.

6. **Violation of Fairness. Bias.** In the former case, the privacy of some stakeholders may be compromised, whereas, in the latter, the problem may suffer from a lack of accuracy leading to the issues of the violation of fairness and individual moral values as well as the potential for biases in centralized decision-making processes, as the central decision-maker may have their own personal or professional biases, which could inadvertently affect the decision outcome. These biases may lead to less equitable or less efficient solutions that do not fully account for the needs and values of all stakeholders.

7. **Perpetuation of Power Structures and Inequalities.** Furthermore, centralized decision-making may inadvertently perpetuate existing power structures and inequalities. Centralized decision-makers may, consciously or unconsciously, prioritize the interests of more powerful or influential stakeholders, while marginalized or underrepresented groups may have their needs overlooked.

The presence of the decision variables x^c generally makes the problem NP-hard, perhaps, with a few exceptional cases, suffering from the curse of combinatorial complexity, thereby bringing other layers of difficulty. As more stakeholders' information is included, the problem size increases, and the number of feasible solutions grows super-linearly, typically in an exponential fashion, which leads to difficulties in achieving optimality or even near-optimal solutions in practical applications in a computationally efficient manner. Once again, suboptimal solutions may be unfair, and stakeholders may view such solutions as non-transparent and limiting their rights and freedoms. Furthermore, for operational optimization problems that need to be solved sufficiently fast to react to rapidly changing circumstances and new information, centralized approaches may not be flexible enough due to the abovementioned computational and information-exchange-related issues. In rapidly evolving situations, centralized decision-makers may struggle to keep up with the changes in stakeholders' preferences and needs. This can be detrimental to such applications as ambulance relocation (Lee et al., 2022), efficient failure detection in large-scale distributed systems (Er-Rahmadi and Ma, 2022), home healthcare routing (Dastgoshade et al., 2020), location and inventory prepositioning of disaster relief supplies (Shehadeh and Tucker, 2022), oral cholera vaccine distribution (Smalley et al., 2015), pharmaceutical distribution (Zhu and Ursavas, 2018), plant factory crop scheduling (Huang et al., 2020), post-disaster blood supply (Hamdan and Diabat, 2020; Kamyabniya et al., 2021), reducing vulnerability to human trafficking (Kaya et al., 2022), urgent surgery scheduling (Kayvanfar et al., 2021), and many others.

In light of the above challenges, there is a growing need for a new paradigm of problem-solving that emphasizes participation, transparency, and accountability to harness the full potential of collective intelligence and enable more effective and equitable decision-making.

2.3. Toward Fast Coordination (Linear Convergence) and Superlinear Reduction of Complexity through Lagrangian Relaxation.

Combinatorial difficulties have been resolved by exploiting the separability through the *dual* "price-based" decomposition and coordination Lagrangian Relaxation technique with super-linear reduction of complexity upon decomposition. After relaxing coupling constraints (2), the optimal coordination of subproblems amounts to the maximization of a concave continuous non-smooth dual function:

$$\max_{\lambda}\{q(\lambda) : \lambda \in \Omega \subset \mathbb{R}^m\}, \tag{3}$$

where

$$q(\lambda) = \min_{(x,y):=\{x_i,y_i\}_{i=1}^{I}} L(x, y, \lambda), \{x_i, y_i\} \in \mathcal{F}_i, i = 1, \ldots, I. \tag{4}$$

Here $L(x, y, \lambda) \equiv \sum_{i=1}^{I}(c_i^x)^T \cdot x_i + \sum_{i=1}^{I}(c_i^y)^T \cdot y_i + (\lambda)^T \cdot \left(\sum_{i=1}^{I} A_i^x \cdot x_i + \sum_{i=1}^{I} A_i^y \cdot y_i - b\right)$ is the Lagrangian function. The Lagrangian multipliers λ ("dual" variables) are the decision variables with respect to the dual problem (3). The minimization within (4) with respect to $\{x, y\}$ is referred to as the "relaxed problem." Through the rest of this subsection, the superscript "c" is dropped since most of the formulas presented are generic for centralized as well as distributed methods in terms of dual function definitions, multipliers updates, etc.

Lagrangian Relaxation significantly reduces the complexity of solving a discrete optimization problem by decomposing it into smaller subproblems to be solved at a time:

$$\min_{x_i,y_i} \left\{(c_i^x)^T \cdot x_i + (c_i^y)^T \cdot y_i + (\lambda)^T \cdot (A_i^x \cdot x_i + A_i^y \cdot y_i), \{x_i, y_i\} \in \mathcal{F}_i\right\}. \tag{5}$$

The decomposition feature of Lagrangian Relaxation is powerful for solving a broad range of optimization problems that would otherwise be impractical to solve due to combinatorial complexity. Moreover, the scalability of decomposition enables the extension of problem-solving efforts to support larger populations, thereby supporting the promotion of democratization, as discussed in more detail in Section 3.

To contextualize the development of Lagrangian Relaxation, we will first provide historical perspectives, which will pave the way for more sophisticated approaches that will be unveiled during the discussion. The coordination of subproblems has been traditionally accomplished by iteratively updating the Lagrangian multipliers λ using a series of steps s^k along subgradients $g(x^k, y^k)$. Specifically, the update is performed according to the following equation:

$$\lambda^{k+1} = \lambda^k + s^k \cdot g(x^k, y^k). \tag{6}$$

Here, $\{x^k, y^k\}$ is a compact notation for an optimal solution $\{x^*(\lambda^k), y^*(\lambda^k)\}$ to the relaxed problem (4), which can be obtained by optimally solving all subproblems (5) with multipliers equal to λ^k. Within Lagrangian Relaxation, subgradients are defined as levels of constraint violations $g(x^k, y^k) \equiv \left(\sum_{i=1}^I A_i^x \cdot x_i^k + \sum_{i=1}^I A_i^y \cdot y_i^k - b\right)$. If inequality constraints $\sum_{i=1}^I A_i^x \cdot x_i + \sum_{i=1}^I A_i^y \cdot y_i \leq b$ are present, they are generally converted into equality constraints by introducing non-negative real-valued slack variables z such that $\sum_{i=1}^I A_i^x \cdot x_i + \sum_{i=1}^I A_i^y \cdot y_i + z = b$. Multipliers are then updated per (5) with subsequent projection onto the positive orthant - a set delineated by constraints $\lambda \geq 0$.

Throughout the rest of the paper, the general formula (6) for multiplier update will be kept while detailing specific methodologies to improve coordination either through better choices of stepsizes, multiplier-updating directions, or both.

Minimization of Non-smooth Functions. Efficient optimization of general non-smooth convex functions stems from the seminal work of Polyak (Polyak, 1969, p. 15). Intending to achieve *geometric* (also referred to as *linear*) rate of convergence so that $\|\lambda^k - \lambda^*\|$ is monotonically decreasing, Polyak developed the stepsizing formula, which, for the problem under consideration, is presented in the following way:

$$0 < s^k < \gamma \cdot \frac{q(\lambda^*) - q(\lambda^k)}{\left\|g(x^k, y^k)\right\|^2}, \gamma < 2. \tag{7}$$

Convergence by using Polyak's stepsizing is of significant, yet theoretical importance for the convergence in the context of Lagrangian Relaxation, since neither the dual function $q(\lambda)$ nor the optimal dual value $q(\lambda^*)$ is known beforehand. When solving complex optimization problems using Lagrangian Relaxation, obtaining a closed-form expression for the dual function is unrealistic due to the function's complex facial structure, where each facet corresponds to a particular solution to the relaxed problem. Optimizing the relaxed problem can be computationally demanding, especially when dealing with a large number of subsystems. The computational effort involved in obtaining subgradients $g(x^k, y^k)$ may accordingly be significant. Moreover, the optimal dual value $q(\lambda^*)$ must be obtained through optimization, making it unknown before optimization.

The Subgradient-Level Method. The Subgradient-Level Method (Goffin and Kiwiel, 1999) addressed the issue of the lack of knowledge about $q(\lambda^*)$ required to compute Polyak's step-size

(7). This method adaptively adjusts "level" estimates of $q(\lambda^*)$ by detecting sufficient ascent of dual values and oscillations of the dual solutions (multipliers). However, the threshold for such detection is set heuristically, and the method assumes that the dual function is available.

The Surrogate Subgradient Method. The Surrogate Sub-gradient Method (Zhao et al., 1999), on the other hand, addressed the computational challenge by updating the multipliers after solving one subproblem (5) at a time rather than solving all the subproblems as in subgradient methods. This significantly reduces computational effort, especially for problems with a large number of subsystems. After solving one subproblem, the multipliers are updated as

$$\lambda^{k+1} = \lambda^k + s^k \cdot g(\tilde{x}^k, \tilde{y}^k), \tag{8}$$

by using the following version of the Polyak formula

$$0 < s^k < \gamma \cdot \frac{q(\lambda^*) - L(\tilde{x}^k, \tilde{y}^k, \lambda^k)}{\left\| g(\tilde{x}^k, \tilde{y}^k) \right\|^2}, \quad \gamma < 1. \tag{9}$$

Here and thereafter "tilde" indicates that the relaxed problem is optimized with respect to one subproblem. Within this method, as well as within methods subsequently discussed in this subsection, to further reduce computational complexity, the "surrogate optimality condition" is used:

$$L(\tilde{x}^k, \tilde{y}^k, \lambda^k) < L(\tilde{x}^{k-1}, \tilde{y}^{k-1}, \lambda^k). \tag{10}$$

Instead of finding the optimal solution for a subproblem (5)—a process that necessitates exploring multiple potential subproblem solutions—the condition (10) is leveraged. This condition speeds up the overall solution process, requiring the identification of only one solution $\{\tilde{x}^k, \tilde{y}^k\}$ that meets (10). Verifying this single solution's satisfaction of (10) is a much less complex operation than the need for finding the exact optimal solution to a subproblem. According to Zhao et al. (1999), the Surrogate Subgradient Method guarantees convergence to λ^*. In addition to the reduction of computational effort, a concomitant reduction of multiplier zigzagging has been observed. Unlike the Subgradient-Level Method, the Surrogate Subgradient Method needs the optimal dual value to compute stepsizes, but it does not require the dual values (only "surrogate" dual values $L(\tilde{x}^k, \tilde{y}^k, \lambda^k)$ are required) to achieve convergence.

Exploiting Distributed Computing with Distributed and Asynchronous Surrogate Lagrangian Relaxation. The rise of technologies supporting distributed computational capabilities and the communication enabled by the Internet of Things and Industry 4.0 has opened new avenues for solving complex problems more efficiently.

Assuming a single coordinator, the Distributed and Asynchronous Surrogate Lagrangian Relaxation (DA-SLR) methodology (Bragin et al., 2020) has been developed to efficiently coordinate distributed subsystems in an asynchronous manner, eliminating the overhead associated with synchronization. Within the method, the same multiplier updating formula (8) is used every time a coordinator receives a subproblem solution without waiting for other subproblems. The step sizing formula obtained through the "contraction mapping" concept is computed as:

$$s^k = \alpha^k \cdot \frac{s^{k-1} \cdot \|g(\tilde{x}^{k-1}, \tilde{y}^{k-1})\|}{\|g(\tilde{x}^k, \tilde{y}^k)\|}, \alpha^k = 1 - \frac{1}{M \cdot k^{1-1/k^r}}, M > 1, 0 < r < 1. \tag{11}$$

This approach is well-suited for the democratization of complex problem solving since after decomposition into smaller subproblems, each subproblem can be formulated and solved very efficiently by distributed computational resources, with a coordinator responsible for updating the Lagrange multipliers and managing communication between the subsystems without sharing private information - only subproblem solutions $\{\tilde{x}^k, \tilde{y}^k\}$ are required by the coordinator.

Compared to its sequential version (Bragin et al., 2015), DA-SLR exhibits faster convergence based on empirical evidence. For example, in one instance of the generalized assignment problem, DA-SLR achieved a 12 times speed-up to reach a gap of 0.03%.

Leveraging distributed computing resources with DA-SLR has the following potential benefits: (1) Enhanced computational efficiency and scalability with a potential for harnessing collective intelligence to solve complex problems whereby each subproblem solver can independently and asynchronously perform decision-making; and (2) Leveraging distributed computing resources, each fully controlled by a corresponding stakeholder, instead of relying on a single (central) computer, the method has the potential to democratize decision-making. A more detailed discussion on democratization is delegated to Section 3.

We refer to (Bragin, 2023) for a more comprehensive review of other coordination methods. The latest development aims to reduce the computational effort required to obtain multiplier-updating directions, alleviate multiplier zigzagging, and achieve linear convergence without requiring dual values as explained next.

Surrogate Level-Based Lagrangian Relaxation. To exploit the linear convergence potential inherent to Polyak's steps sizing formula, the Surrogate "Level-Based" Lagrangian Relaxation (SLBLR) method has been recently developed (Bragin and Tucker, 2022). The high-level idea is to reset "level-values" (estimates of the optimal dual value) by detecting the divergence of multipliers rather than by detecting "significant oscillations" of multipliers or "significant ascent" of dual value as in Subgradient-Level methods (Goffin and Kiwiel, 1999). In comparison, the multiplier-divergence detection procedure (Bragin and Tucker, 2022) is beneficial in several aspects:

1. Multiplier oscillation is a natural, yet undesirable, phenomenon when optimizing non-smooth functions. Within Subgradient-Level methods, the oscillation detection is operationalized after multipliers have traveled a heuristically predetermined distance. Hence, multipliers divergence may either go undetected for a significant number of iterations, or level values may be reset prematurely. In contrast, the multiplier divergence (Bragin and Tucker, 2022) is detected whenever it occurs;

2. Because of the decision-making procedure involved in the detection of multiplier divergence, the number of hyperparameters to be adjusted is drastically reduced;

3. Sufficient ascent of dual functions (Goffin and Kiwiel, 1999) cannot be generally operationalized in an efficient way since the computations of dual functions are computationally costly in the first place as argued above.

Specific details of operationalization of multipliers detection and resetting of level values can be found in (Bragin and Tucker, 2022).

In the method, the following version of the Polyak stepsize formula is used

$$s^k = \zeta \cdot \gamma \cdot \frac{\overline{q}_j - L(\tilde{x}^k, \tilde{y}^k, \lambda^k)}{\|g(\tilde{x}^k, \tilde{y}^k)\|^2}, \zeta < 1. \tag{12}$$

The stepsize (12) is reduced through ζ every time a divergence of multipliers is detected, which also leads to the reduction of level values $\overline{q}_j$ approaching $q(\lambda^*)$ from above. The significant advantage of SLBLR is its efficient and operationalizable decision-based procedure for determining level values without the need for estimation or heuristic adjustments of optimal dual value estimates. In a sense, SLBLR is user-friendly and significantly reduces the need for the domain knowledge to determine hyperparameters $\overline{q}_j$; robustness with respect to other hyperparameters has also been demonstrated (Bragin and Tucker, 2022).

Table 1 briefly summarizes key requirements for each of the methods reviewed above. Numerical results from (Bragin and Tucker, 2022) indicate that the SLBLR method has solved generalized assignment problems to optimality, achieved over two orders of magnitude computational improvements compared to branch-and-cut for job-shop and pharmaceutical scheduling, and demonstrated scalability. This method has successfully overcome major issues of previous methods, making it

Table 1: Comparison of key methods for non-smooth optimization

Method	Dual Values	Optimal Dual Value	Comput. Effort	Potential Privacy Violation
Polyak (1969)	Needed	Needed	High†	Limited††
Goffin and Kiwiel (1999)	Needed	Not Needed	High†	Limited††
Zhao et al. (1999)	Not Needed	Needed	Low	Limited††
Bragin et al. (2020)	Not Needed	Not Needed	Low	No††
Bragin and Tucker (2022)	Not Needed	Not Needed	Low	Limited††

† Polyak (1969) as well as Goffin and Kiwiel (1999) deal with non-smooth optimization irrespective of Lagrangian Relaxation, and the high computational effort is hypothesized since the effort to obtain subgradient directions requires solving all the subproblems unlike that within (Zhao et al., 1999), (Bragin et al., 2020) and (Bragin and Tucker, 2022).

†† Privacy revelation is not discussed in any of the above papers and the privacy requirements are inferred assuming decentralized subproblem solving. For example, within (Polyak, 1969; Goffin and Kiwiel, 1999; Zhao et al., 1999; Bragin and Tucker, 2022) stepsize computations require either dual functions or "surrogate dual functions," which, in turn, are affine combinations of subproblem costs. In contrast, (Bragin et al., 2020) only requires subproblem solutions to update multipliers and stepsizes.

suitable for coordinating multiple subsystems and providing a solid platform to support the democratization of complex problem decision-making, as explained in the next section.

We conclude this section by noting that specific requirements and restrictions within a system influence the choice of the method for a particular system. For example, within (Bragin and Tucker, 2022), while the coordinator does not require exact stakeholder information, it does need subproblem solutions and costs to update the Lagrange multipliers; the advantage of this method is its faster convergence compared to other techniques. On the other hand, the method proposed by Bragin et al. (2020) offers lower privacy revelation requirements. Still, its convergence is governed by the "non-summable" stepsize given in (11), resulting in a slower convergence rate. The decision to use one method over the other depends on the specific needs and priorities of the system being addressed. Faster convergence might be desirable in situations where time is of the essence while preserving privacy might be more important in scenarios where sensitive information is involved.

3. Democratization of Combinatorial Problem-Solving

This section discusses using the Lagrangian Relaxation-based methods as a potential step toward democratizing problem-solving. Specifically, it allows optimization to occur at a local level by involving stakeholders with diverse perspectives to formulate their subproblems (considering unique demographics or socioeconomic status). This can help ensure that decisions (e.g., allocation of resources) are more equitable.

Consider a high-level example of allocating resources, whereby b^c can represent the "demand" or the "supply" depending on the application. As argued, the satisfaction of the following constraints

$$s.t.\ \ \sum_{i=1}^{I^c} A_i^{c,x} \cdot x_i^c + \sum_{i=1}^{I^c} A_i^{c,y} \cdot y_i^c = b^c, \{x_i^c, y_i^c\} \in \mathcal{F}_i^c, i = 1, \ldots, I^c, \tag{13}$$

in a centralized fashion may lead to exclusion, lack of privacy, and to unfairness. To avoid these difficulties, the goal is to allow $I^d > I^c$ ("d" stands for "decentralized") communities/stakeholders to participate in the decision-making in a broad sense–to formulate their own objectives

$$\min_{x_i^d, y_i^d} \left\{ (c_i^{d,x})^T \cdot x_i^d + (c_i^{d,y})^T \cdot y_i^d \right\} \tag{14}$$

and local constraints $\left\{x_i^d, y_i^d\right\} \in \mathcal{F}_i^d$ as well as to solve their subproblems–to collectively satisfy the following coupling constraint:

$$s.t.\ \ \sum_{i=1}^{I^d} A_i^{d,x} \cdot x_i^d + \sum_{i=1}^{I^d} A_i^{d,y} \cdot y_i^d = b^c, \left\{x_i^d, y_i^d\right\} \in \mathcal{F}_i^d, i = 1, \ldots, I^d. \tag{15}$$

The intent is to include the needs of stakeholders from various socioeconomic backgrounds or marginalized communities to have more access to resources and participate in decision-making. Note that the objective of the overall system is:

$$\min_{(x^d, y^d) := \left\{x_i^d, y_i^d\right\}_{i=1}^{I^d}} \left\{ \sum_{i=1}^{I^d} \left((c_i^{d,x})^T \cdot x_i^d + (c_i^{d,y})^T \cdot y_i^d \right) \right\}, \tag{16}$$

which is structured in the same way as (1), yet having a different significance as being more inclusive and privacy-preserving, since the perspective of the centralized decision maker is superseded by those of I^d stakeholders, who do not have to reveal specific values for $c_i^{d,x}$ and $c_i^{d,y}$.

The current form is not entirely set up yet for participatory decision-making. Because of the "hard" constraint (15), stakeholder i's objective (14) cannot be optimized independently from other stakeholders' objectives. Participatory decision-making is operationalized by using Lagrangian

Relaxation methods. Specifically, after relaxing constraints (15), each stakeholder has full control over the decision-making of its own subproblem, which is formulated as follows:

$$\min_{x_i^d, y_i^d} \Big\{ (c_i^{d,x})^T \cdot x_i^d + (c_i^{y,d})^T \cdot y_i^d + \lambda^T \cdot \Big(A_i^{d,x} \cdot x_i^d + A_i^{y,d} \cdot y_i^d \Big), \Big\{ x_i^d, y_i^d \Big\} \in \mathcal{F}_i^d \Big\}. \tag{17}$$

Decentralized decision-making, as proposed in (17), aligns with the concepts of decentralized systems and stakeholder empowerment proposed by Briggs and Benkler (Briggs, 2008; Benkler, 2006). Our method further emphasizes and highlights privacy preservation, transparency, accountability, and the capacity of information and communication technologies to create a more inclusive and equitable approach to problem-solving. Specifically, in this framework, private information ($c_i^{d,x}$, $c_i^{y,d}$, $A_i^{d,x}$, and $A_i^{y,d}$) needs to be known only to the stakeholder/community i, and decision variables (x_i^d and y_i^d) are fully controlled by the stakeholder. Feasible regions ($\mathcal{F}_i^d$) are delineated by local constraints, which are also under the stakeholder's purview.

By empowering communities and promoting democratic participation, decentralized decision-making can potentially enhance the effectiveness and impartiality of resource allocation, which is governed not only by "local" decision-making but also by the market "supply and demand" principle behind the update of Lagrangian multipliers. Moreover, from an ethical and moral standpoint, Moore's and Nuccetelli's ideas (Nuccetelli and Seay, 2007; Nuccetelli, 2021; Moore, 1903) can be interpreted to suggest that one might be open to the idea of involving multiple perspectives and recognizing the intrinsic value of diverse viewpoints, which can be in line with the decentralized decision-making essential for ensuring that diverse perspectives and interests are taken into account and that decisions are made in an inclusive and fair manner. Building on the works of Brey, Coeckelbergh, Mittelstadt, and Pak-Hang Wong, which collectively address the ethical, social, and political challenges of emerging technologies and algorithmic decision-making (Brey, 2012; Coeckelbergh, 2012; Mittelstadt et al., 2016; Wong, 2020), our approach strives to create inclusive, ethical, and equitable technology development and decision-making processes. Considering Crawford's research (Crawford, 2013; Crawford and Paglen, 2021; Whittaker et al., 2018), we address fairness and equity concerns by fostering inclusivity and reducing bias in the decision-making process. The decentralized decision-making is operationalized in this paper through the use of Lagrangian Relaxation, which leverages the strengths of both humans and computers, in accordance with Malone's ideas (Malone, 2018). Inspired by Dwork's dedication to protecting individual rights and promoting fairness in technological systems (Dwork, 2008; Dwork et al., 2012, 2014), we consider social and cultural contexts to develop a more inclusive optimization process that respects diverse perspectives. Humans collectively define private information and local constraints suitable for their locality while computers solve subproblems and provide updated multipliers ("price-signals") updated by using formulas (8) and (11) or (12). Ultimately, this integrated approach has the potential to contribute to effective and democratic decision-making by leveraging the strengths of both humans and computers in a decentralized decision-making framework.

Key Features and Takeaways. The proposed method empowers local stakeholders by actively involving them in the decision-making process (**participation and accountability**), fostering more community-centered and tailored solutions that align with each locality's unique needs and circumstances. This approach also enables stakeholders to contribute to the overall solution by formulating and solving their own subproblems in an open and accessible manner (**transparency**). Moreover, transparency gives stakeholders access to the information they need to make informed decisions. By allowing stakeholders to formulate and solve their subproblems, Lagrangian Relaxation methods promote fairness, accountability, and transparency, addressing potential power imbalances in centralized decision-making (**fairness**).

Transparency, explainability, and interpretability are crucial yet distinct concepts for a successful method. While transparency focuses on the accessibility and understanding of the decision-making process, **explainability** involves stakeholders' ability to articulate how pricing signals (multipliers) influence decisions and the final solution. This allows stakeholders to explain their decisions to local communities, fostering trust and credibility in the decision-making process. **Interpretability** entails understanding the relationships between pricing signals (multipliers) and resource supply and demand, helping stakeholders make informed decisions.

Coordinating and integrating local solutions into a global solution ensures feasible and **near-optimal** global solution. The Lagrangian Relaxation approach, grounded in economic theory, employs Lagrangian multipliers as "shadow prices" to discourage less economically viable decisions, ensuring fairness, equity, and **economic viability and efficiency**.

The **sensitivity** nature of Lagrangian multipliers, defined as "rates of change of the optimal cost as the levels of constraint changes," (Bertsekas, 1999) can help reveal how, for example, resource allocation adapts to changes in demand or supply, providing valuable insights for both local stakeholders and governments. This information assists stakeholders in understanding the effects of global constraints on local resource allocation as well as the associated costs and supports coordinated decision-making in dynamic circumstances. Below, we discuss examples of how sensitivity analysis can be leveraged to assist stakeholders in understanding the effects of global constraints on resource allocation and associated costs.

1. When Lagrangian multipliers λ converge to a high value, it signals a higher demand for resources (with a fixed supply, say, b^c) and a need for adjustments in resource allocation. Local stakeholders can use this information to anticipate potential shortages and make informed decisions. Simultaneously, the increased multipliers indicate the necessity for government intervention to provide additional resources/supply $(b^c + \Delta b^c)$ to achieve the optimally global solution of $(x^* + \Delta x, y^* + \Delta y)$ thereby mitigating resource scarcity as well as bringing the costs down by $-\left(\lambda^*\right)^T \cdot \Delta b^c$ (derivation is similar to that provided in (Bertsekas, 1999, p. 308)). Such intervention can help ensure that a sufficient amount of goods is supplied to stakeholders, thereby maintaining a balanced and equitable distribution of resources in response to the increased demand.

2. When multipliers decrease, it signifies a lower demand for resources or an increased supply, easing the pressure on resource allocation. For example, in the case where multipliers become zero, which can happen when constraints are not binding (e.g., fewer suppliers are needed than available), this can signal that the supplies can be taken out from the problem without increasing the overall cost since the corresponding part $-\left(\lambda^*\right)^T \cdot \Delta b^c$ will be zero. This way, the supplies can be repurposed, enabling more efficient resource allocation and better utilization of available resources in other areas of need.

The sensitivity of Lagrangian multipliers supports democratization by empowering local stakeholders with information on global constraint impacts on local solutions. This enables them to participate knowledgeably in decision-making processes, fostering a more collaborative approach to resource allocation. Understanding the sensitivity of Lagrangian multipliers can lead to significant improvements in resource allocation, including:

1. Efficiency: Optimal utilization of available resources, leading to better resource allocation strategies and reduced waste.

2. Equitability: Ensuring a balanced and equitable distribution of resources in response to changes in demand.

3. Adaptability: Allowing stakeholders to quickly respond to changes in demand or supply, anticipating potential shortages, and repurposing supplies as needed.

For example, to address efficiency, equitability, and adaptability in the above resource allocation scenario with a fixed supply, stakeholders can take the following possible actions:

1. Analyze the Lagrangian multipliers to identify the most critical constraints or disparities in resource allocation, as well as the constraints most sensitive to changes in demand.
2. Collaborate with other stakeholders, such as local government agencies, businesses, non-profit organizations, or community representatives, to address the identified constraints or disparities collectively and share information about anticipated or ongoing changes in demand.
3. Develop strategies that address the identified constraints or disparities and adapt to changes in demand. These strategies could include adjusting policies or regulations, investing in additional resources or infrastructure, implementing innovative technologies, promoting co-operation and information-sharing, reallocating resources, or securing additional supplies.

By understanding the sensitivity of Lagrangian multipliers and following these steps, stakeholders can address current resource allocation challenges and build a robust foundation for future decision-making. This collaborative and data-driven approach fosters a resilient resource allocation system that can better respond to changing circumstances and emerging needs, ultimately promoting sustainable and equitable growth for all stakeholders involved.

The Lagrangian Relaxation methods are marked by **flexibility** by allowing communities to participate or withdraw as needed while still delivering results. The method's **adaptability** lies in the continuous adjustment of Lagrangian multipliers in response to changing demands or resource supply. In addition, decentralized approaches can distribute communication and coordination burdens more evenly among stakeholders, leading to more efficient information sharing and decision-making.

4. Future Directions

This paper introduces a new application of the Lagrangian Relaxation (LR) method for participatory decision-making in complex systems. While it presents a general framework and potential benefits, further research is required to evaluate its effectiveness and limitations. Future studies could employ real-world examples and case studies to investigate practical applications and identify areas for improvement. Several directions for future research include:

1. **Game-Theoretical Stakeholder Analysis.** Investigating stakeholder behavior under the centralized and the decentralized frameworks using game-theoretical analyses, highlighting the potential for fair and inclusive decision-making.

2. **Stakeholder Engagement in Different Sectors.** Studying the impact of engaging stakeholders by applying the proposed and other methods in various sectors, such as transportation, power systems, and healthcare, to promote stakeholder participation, transparency, accountability, and fairness.

3. **Revenue vs. Customer Satisfaction Trade-offs.** Assessing trade-offs between revenue maximization and customer satisfaction, applying the Myerson-Satterthwaite theorem, and evaluating the impact of customer-centric objectives on pricing decisions.

4. **Developing Fully Distributed and Decentralized Decision-Making Methods.** Investigating technical, computational, and algorithmic challenges and potential benefits of developing fully distributed and decentralized decision-making methods without the need for a coordinator to even coordinate stakeholders. In this paradigm, stakeholders coordinate with one another by sharing updated multipliers and solutions without broadcasting private technical information.

5. **Moral Responsibilities in Decentralized Decision-Making.** Investigating the operationalization of moral responsibilities in fully decentralized decision-making and exploring the potential of blockchain and distributed computing technologies for maintaining trust and accountability.

6. **Regulatory Implications.** Examining the regulatory implications of decentralized decision-making, including challenges posed by collusion, where two or more stakeholders cooperate to gain an unfair advantage over others, often through secret agreements or other forms of coordination that violate ethical or legal norms. Developing regulations that discourage such behavior and ensure equitable outcomes is a key goal of this research direction.

7. **Democratization Examples in Various Domains.** Providing examples of the democratization of problem-solving in various domains, such as logistics and transportation (Freund et al., 2017; Jiang et al., 2020; Karolemeas et al., 2021; Dong and Leng, 2021; Archetti et al., 2021; Kamyabniya et al., 2021; Balakrishnan et al., 2021; Reddy et al., 2022; Gupta et al., 2022; Shao et al., 2022; Yan et al., 2022), including decarbonization efforts through logistics

network design (Jiang et al., 2020), power systems (Afshar et al., 2008; Pozo et al., 2012; Morales-España et al., 2013; Bischi et al., 2014; Wierzbowski et al., 2016; Theo et al., 2016; Schill et al., 2017; Nikoobakht et al., 2018; Damchi et al., 2018; Rafinia et al., 2020; Sun et al., 2018; Chen et al., 2020; Li et al., 2020; Wu et al., 2021; Shen et al., 2021; Wu et al., 2023) including generation and transmission expansion planning (Pozo et al., 2012) as well as combines cooling, heat and power (Bischi et al., 2014) and hybrid power systems planning (Theo et al., 2016), healthcare (Kopanos et al., 2010; Stefansson et al., 2011; Kim and Mehrotra, 2015; VonAchen et al., 2016; Zhu and Ursavas, 2018; Khlif Hachicha and Zeghal Mansour, 2018; Shehadeh et al., 2020; Dastgoshade et al., 2020; Ge and Yuan, 2021; Kayvanfar et al., 2021; Prabhu et al., 2021; Tsang and Shehadeh, 2023; Wickett et al., 2023) including operating room scheduling, nurse management (Kim and Mehrotra, 2015), and patient appointment scheduling (Khlif Hachicha and Zeghal Mansour, 2018; Shehadeh et al., 2020), and humanitarian applications (Smalley et al., 2015; Aghaei et al., 2019; Hamdan and Diabat, 2020; Ahani et al., 2021; Kamyabniya et al., 2021) such as disaster relief operations (Kamyabniya et al., 2021) and refugee settlement (Ahani et al., 2021). These applications encompass both operational and planning optimization problems, influencing short-term and long-term societal well-being.

8. **Extensions to Stochastic Settings.** Optimization problems involving uncertain parameters occur in almost all areas of science and engineering. Thus, future efforts should consider extending the LR framework to stochastic formulations, where both the master and subproblem involve uncertain parameters. In addition, one should ensure that the resulting stochastic formulation maintains feasibility, efficiency, and fairness.

9. **Societal Acceptance of Decentralized Decision-Making.** Investigating the factors affecting societal acceptance of decentralized decision-making processes in various contexts is crucial. This research could explore the role of communication, trust, and perceived fairness in promoting acceptance and assess barriers and challenges to implementing such approaches. Identifying strategies to increase public awareness and preparation for future challenges could foster greater engagement and support for democratic problem-solving in complex systems.

Overall, the democratization of solution methodologies offers a promising avenue for addressing complex problems across various domains. As research in this area evolves, advancements in participatory decision-making frameworks are expected to emerge, empowering stakeholders and promoting transparency and fairness in decision-making processes. Such advancements can foster collaboration, equitable decision-making, and more effective, sustainable solutions for complex societal challenges. The decentralized approach using Lagrangian Relaxation encourages stakeholder participation, transparency, accountability, and fairness, ultimately improving decision-making in diverse systems and sectors. More inclusive and comprehensive solutions can be created by considering multiple perspectives and allowing stakeholders to define objectives and metrics that reflect diverse needs and values. The incorporation of equity concerns into decision-making processes contributes to a more just and equitable world, where the needs and perspectives of all individuals and communities are considered and valued.

5. Conclusion

In this paper, we propose an approach to democratizing decision-making in complex-problem solving using Lagrangian Relaxation. Our approach emphasizes the importance of inclusivity, diversity, transparency, collective intelligence, and equity in the decision-making process. By decentralizing the problem-solving process, we enable stakeholders to actively contribute their constraints, preferences, and knowledge, leading to more effective and equitable outcomes. The proposed method facilitates stakeholder collaboration by decomposing complex problems into subproblems that can be solved separately while still coordinating efforts to achieve a near-optimal global solution. This decentralization aligns with the vision of Benkler, Briggs, and others, promoting stakeholder participation, transparency, accountability, and fairness in the decision-making process. Moreover, our approach addresses fairness and equity concerns by incorporating FAT principles and privacy-aware techniques in the optimization process. Through this, we create an equitable and socially responsible problem-solving environment that is sensitive to the ethical, legal, and social implications of the decisions being made. We reviewed relevant literature in various domains that inspired this paper.

As future directions, we envision exploring game-theoretical stakeholder analysis, examining the impact of strategic behavior on the proposed framework, engaging stakeholders in various sectors, assessing trade-offs between revenue maximization and customer satisfaction, investigating moral responsibilities in decentralized decision-making, considering regulatory implications, providing examples of democratization in various domains, and transitioning to stochastic formulations in subproblem formulations. These studies can help evaluate the effectiveness and limitations of the Lagrangian Relaxation method for participatory decision-making in complex systems and facilitate the integration of our approach in real-world applications. In conclusion, the proposed method offers a promising path toward a more inclusive, transparent, and equitable approach to complex problem-solving. By harnessing the collective intelligence of diverse stakeholders and leveraging the power of decentralized collaboration, pressing societal challenges can be addressed, and advancements toward a more just and sustainable future can be achieved.

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