ON THE EQUILIBRIUM PRICES OF A REGULAR LOCALLY LIPSCHITZ EXCHANGE ECONOMY

TRUONG XUAN DUC HA

Abstract. We extend classical results by Debreu and Dierker about equilibrium prices of a regular economy with continuously differentiable demand functions/excess demand function to a regular exchange economy with these functions being locally Lipschitz. Our concept of a regular economy is based on Clarke's concept of regular value and we show that such a regular economy has a finite, odd number of equilibrium prices, the set of economies with infinite number of equilibrium prices has Lebesgue measure zero and there exist locally Lipschitz selections of equilibrium prices around a regular economy.

1. Introduction

Many economic problems can be reduced to solving and analyzing solutions to an equation

$$f(x) = y,$$

around a regular value y, where $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is a function and Ω is an open set. Recall that a regular point of a differentiable map is a point where the derivative is onto and a value is regular if all pre-images are regular points. When f is continuously differentiable (briefly, C^1) and y is regular, the inverse function theorem implies the finiteness of the solution set and Sard's theorem implies that the set of values y such that the solution set is infinite has the Lebesgue measure zero.

An economy is defined by demand functions (which in its turn is defined by a utility function) or an excess demand function. It is of interest to know what the set of equilibrium prices of a given economy looks like and how it depends on the parameters describing this economy. This can be done in the case of regular economies, see [1], [5], [7], [17] and the references in [1] for a survey on contributions to the regular exchange economies. Roughly speaking, one assumes an economy to be regular in the sense that some value is a regular value of some map associated to this economy (for instance, the endowment is a regular value of a map involving demand functions, see [1], [5], [17], or zero is a regular value of an

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excess demand function, see [7]) and deduce properties of the set of equilibrium prices from the ones of the solution set to an equation given by this map.

In [5], Debreu studied a regular economy with C^1 demand functions. He proved that the number of equilibrium prices is finite and locally constant and the set of endowments for which the associated economies have a infinite number of equilibrium prices is of Lebesgue measure zero. Using index of fixed point, Dierker precised Debreu's result by showing that the number of equilibrium prices of a regular economy is odd [7].

When demand functions are merely continuous, Shannon [17] uses Rader's concept of regularity [15], namely a point is a Rader's regular point if the derivative exists at this point and is onto. Biheng and Bonnisseau [1] consider a special case when the preferences of consumers are represented by utility functions satisfying some natural conditions so that demand functions are locally Lipschitz and C^1 on an open set of full Lebesgue measure and the concept of regular economy takes into consideration only points at which some projection map is C^1. It has been established that set of equilibrium prices of regular economies in these nonsmooth cases retains most properties of the C^1 case. Debreu's result on generic finiteness of equilibrium prices has been extended to economies with concave definable utility functions by Blume and Zame [2] and to economies with semi-algebraic utility functions by Ioffe [11, Theorem 9.65].

Continuing this direction of research, we extend results by Debreu and Dierker to a regular economy demand functions or excess demand function of which are locally Lipschitz. Here, we use Clarke's concept of regularity to define and study a "regular economy"and our results can be applied to some economies which may not be considered in [17] and [1].

We would like to mention that obtained a generalization of one of their results in which the utility function is assumed strictly concave (we drop the concavity requirement).

The note is organized as follows. In Section 2, we recall concepts of Clarke's regular values, Sard's theorem and Brower's degree (motivated by Shannon's approach [17], we use the latter instead of index of fixed points). Section 3 is devoted to the behaviour of a locally Lipschitz map around a Clarke's regular value. In the last section, we formulate results about equilibrium prices of a regular economy with locally Lipschitz data.

2. Clarke's regular values, Sard's theorem and Brower's degree

Notations Let be given a normed space X. We denote by $\mathbb{B}$ and $\mathbb{B}(x, r)$ the open unit ball of X and the open ball centered at $x \in X$ with radius r, respectively. For a nonempty set $U \subset X$, $\bar{U}$ and ∂U stand for the closure and the boundary of U. Let $\mathbb{R}^n$ be the n-dimensional euclidean space and $\mathbb{R}^{n\times p}$ be the space of $n \times p$-matrices equipped with the norm $\|(a_{ij})\|_{\mathbb{R}^{n\times p}} := (\sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij}^2)^{1/2}$.

Let $\Omega \subseteq \mathbb{R}^n$ be an open nonempty set and $f = (f_1, ..., f_p) : \bar{\Omega} \to \mathbb{R}^p$ $(n \geq p)$ be a map. If each f_i (and hence f) is locally Lipschitz on Ω, then it follows from the Rademacher theorem that f is Fréchet differentiable (each f_i is Fréchet differentiable) a.e. on any neighborhood of x in Ω. The Fréchet derivative $f'(x)$ coincides with the $n \times p$-matrix of partial derivatives of f at x. *Clarke's subdifferential* of f at $x \in \Omega$, denoted by $\partial f(x)$, is the convex hull of all $n \times p$-matrices obtained as the limit of a sequence of the form $f'(x_i)$ where $x_i \to x$ and $f'(x_i)$ is defined

$$\partial f(x) := \text{clconv } \{v \in \mathbb{R}^{n \times p} : v = \lim_{x_i \to x, f'(x_i) \text{ exists}} f'(x_i)\}$$

[4, Definition 2.6.1]. The set $\partial f(x)$ is nonempty compact convex for each $x \in \Omega$. We refer an interested reader to [4, Propositions 2.2.2, 2.2.4. 2.6.2] for other properties of Clarke's subdifferential.

Recall that a Clarke's regular point of a locally Lipschitz map is a point where all elements in Clarke's subdifferential are onto [4] and a value is Clarke's regular if all pre-images are Clarke's regular points. We provide some examples illustrating that the set of Clarke's regular values and the sets of Rader's regular values are not contained one in other. In what follows, a point is critical in some sense if it is not regular in this sense and a value is critical in some sense if there exists one preimage which is critical in this sense.

Example 2.1. (i) See [12, Proposition (1.9)]. Let M be a measurable subset of $\mathbb{R}$, which intersects every nonempty open interval $I \subset \mathbb{R}$ in a set of positive measure $0 < \text{mes}(M \cap I)$, and let g be the indicator function of M. Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined on $\mathbb{R}$ by $f(x) := \int_0^x g(t)dt$. The function f is well-defined, strictly increasing, and locally Lipschitz on $\mathbb{R}$. The derivative of f is almost everywhere 0 or 1 and each value is achieved on a dense subset of $\mathbb{R}$. Thus, $\partial f(x) = [0, 1]$ for all $x \in \mathbb{R}$ and f is nowhere C^1.

Let $h : \mathbb{R} \to \mathbb{R}$ be the continuous function defined on $\mathbb{R}$ by $h(x) := f(x) + x$. Since $\partial h(x) = [1, 2]$ for all $x \in \mathbb{R}$, all values of h are Clarke's regular. Meanwhile, since $f'(x)$ exists only a.e., not every value of h is Rader's regular.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

see [13, p.324]. This function is locally Lipschitz on $\mathbb{R}$ and

$$\partial f(x) = \begin{cases} \{2x \sin \frac{1}{x} - \cos \frac{1}{x}\} & \text{if } x \neq 0 \\ [-1, 1] & \text{if } x = 0 \end{cases}$$

Thus, f is C^1 everywhere except at $x = 0$ (it has Fréchet derivative $f'(0) = 0$).

Let $h : \mathbb{R} \to \mathbb{R}$ be the continuous function defined on $\mathbb{R}$ by $h(x) := f(x) + x$. It is easy to see that h is locally Lipschitz on $\mathbb{R}$,

$$\partial h(x) = \begin{cases} \{2x \sin \frac{1}{x} - \cos \frac{1}{x} + 1\} & \text{if } x \neq 0 \\ [0, 2] & \text{if } x = 0 \end{cases}$$

and the Fréchet derivative $h'(0) = 1$. Note that $h^{-1}(0) = \{0\}$. Thus, 0 is a Rader's regular value of h, but it is not a Clarke's regular value. Moreover, this map is not one-to-one in any neighborhood of zero. Note that the function h is due to Andrew McLennan, see [17, p. 152] and [18, p. 2756].

(iii) For the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (|x| + y, 2x + |y|)$, it holds

$$\partial f(0,0) = \left\{ \begin{bmatrix} s & 1 \\ 2 & t \end{bmatrix} \; : \; |s| \leq 1, |t| \leq 1 \right\},$$

see [4, Remarks 7.1.2.(iii)]). Since $f^{-1}(0,0) = \{(0,0)\}$ and for any $A \in \partial f(0,0)$, we have $\det A \leq -1$, it follows that $(0,0)$ is a Clarke's regular value of f. Note that $(0,0)$ is not a Rader's regular value because f is not differentiable at $(0,0)$.

Classical Sard's theorem [16] in the special case $n = p$ states that the set of critical value of a C^1 map from $\Omega \subseteq R^n$ to R^n has the Lebesgues measure zero. There have been obtained refinements of Sard's theorem for the set of Rader's critical value of a continuous map [15, Lemma 2], for the set of Clarke's critical values of piecewise essentially smooth Lipschitz map [9, Theorem 3.2] and [10, Theorem 4.1]. Recall that a continuous map $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is said to be *essentially smooth* if it is locally Lipschitz on Ω and is C^1 a.e. on Ω [3, p.68] and f is said to be *piecewise essentially smooth Lipschitz* if it is a continuous selection of a finite number of essentially smooth Lipschitz maps $g_i : \Omega \to \mathbb{R}^p$, $i \in I$, I is a finite index set , i.e. $f(x) \in \{g_i(x), i \in I\}, \;\; \forall x \in \Omega$ [10]. Essentially smooth Lipschitz maps on an open subset of $\mathbb{R}^n$ form a broad linear space. In particular, the demand functions considered in [1] are an example of essentially smooth Lipschitz maps, see Proposition 3.1 in this paper.

We conclude this section with *Brower's degree*. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}^n$ be a continuous map, $U \subset \Omega$ is an open bounded set such that $\bar{U} \subset \Omega$ and $y \in \mathbb{R}^n \setminus f(\partial U)$. Brower's degree of f on U at y, denoted by $d(f, U, y)$, is a function with values in $\mathbb{Z}$ satisfying some basic properties (see, e.g., [6, p.5]). We recall here two properties that will be used later.

(1) (homotopy invariance) If $h : [0,1] \times \bar{U} \to \mathbb{R}^n$ is continuous, $y : [0,1] \to \mathbb{R}^n$ is continuous, and $y(t) \notin h(t, \partial U)$ for every $t \in [0,1]$, then $d(h(t,.), U, y(t))$ is independent of $t \in [0,1]$.

(2) If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a nonsingular matrix and $A^{-1}y \in U$, then $d(A, U, y) = \text{sign} \det A$.

A classical result states that if f is a C^1 map and y is a regular value of f, then

$$d(f,U,y) = \sum_{x\in U, f(x)=y} \text{sign } \det f'(x).$$

This formula remains true in the case f is merely continuous and y is a Rader's regular value [17, Theorem 9]. When f is locally Lipschitz and y is a Clarke's regular value of f, Pourciau proved that the following equality holds [14, Theorem 3.5]

$$d(f,U,y) = \sum_{x\in U, f(x)=y} \text{sign } \det\partial f(x), \tag{1}$$

where sign $\det\partial f(x) := \operatorname{sign}\det A$ for any $A \in \partial f(x)$ (it is known that in this case, for each $x \in f^{-1}(y)$, one has $\det A \neq 0$ and $\operatorname{sign}\det A \equiv$ constant for all $A \in \partial f(x)$ [14]). This formula will be used to prove that the set of equilibrium prices

3. Behaviour of a locally Lipschitz map around a Clarke's regular value

Let us recall Shannon's result on the behaviour of f around a Rader's regular value.

Theorem 3.1. [17, Theorem 12] *Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set, $f : \Omega \to \mathbb{R}^n$ is a locally Lipschitz map, y is a Rader's regular value of f and $d(f,\Omega,y) \neq 0$. Then there exist $x_1, ..., x_k$ in Ω and neighborhoods W_i of x_i for each i such that $f^{-1}(y) = \{x_1, ..., x_k\}$ and $W_i \cap f^{-1}(y) = \{x_i\}$. Moreover, there exist $\lambda > 0$ and a neighborhood V of y such that for all $v \in V$ one has $W_i \cap f^{-1}(v) \neq \emptyset$ and if $u_i \in W_i \cap f^{-1}(v)$, then $\|u_i - x_i\| \leq \lambda\|v - y\|$.*

We will prove a result about the behaviour of f around a Clarke's regular value.

Theorem 3.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set and a map $f : \bar{\Omega} \to \mathbb{R}^n$ is locally Lipschitz on Ω and continuous on $\partial\Omega$ (the latter means that if $x \in \partial\Omega$ and $x_i \in \Omega$, $x_i \to x$, then $f(x_i) \to f(x)$). Let $y \in \mathbb{R}^n \setminus f(\partial\Omega)$ be a Clarke's regular value of f. Then*

- (i) *There exist $x_1, ..., x_k$ in Ω, open neighborhoods W_i of x_i and V_i of y, locally Lipschitz maps $g_i : V_i \to W_i$ $(i = 1, ..., k)$ such that $f^{-1}(y) = \{x_1, ..., x_k\}$, $\bar{W}_i \cap f^{-1}(y) = \{x_i\}$, $g_i(y) = x_i$, and W_i and V_i are lipeomorphic by the map f and g_i (here, $f^{-1}(y) := \{x \in \bar{\Omega} \ : \ f(x) = y\}$).*
- (ii) *There exists a neighborhood V of y such that $V \subset \cap_{i=1}^k V_i$ and for all $v \in V$, v is a Clarke's regular value of f and the equation $f(x) = v$ has exact k solutions, namely, $f^{-1}(y) = \{g_1(v), ..., g_k(v)\}$.*

To prove Theorem 3.2, we need the inverse function theorem and some auxiliary results.

Proposition 3.1. [4, Theorem 7.1.1] *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set, $f : \Omega \to \mathbb{R}^p$ is Lipschitz near $x \in \Omega$. If $\partial f(x)$ is of maximal rank, i.e., any $A \in \partial f(x)$ has the maximal* *rank, then there exist open neighborhoods U and V of x and $f(x)$, resp., and a Lipschitz map $g : V \to \mathbb{R}^n$ such that $g(f(u)) = u$ for all $u \in U$ and $f(g(v)) = v$ for all $v \in V$.*

Proposition 3.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set, $f : \Omega \to \mathbb{R}^n$ is Lipschitz near $x \in \Omega$. If $\partial f(x)$ is of the maximal rank, then there exists an open neighborhood U of x such that for all $u \in U$, $\partial f(u)$ is of the maximal rank, and*

$$\text{sign det } \partial f(u) \equiv \text{sign det } \partial f(x), \forall u \in U.$$

Proof. Since $\partial f(x)$ is of the maximal rank, it follows from [14] that either sign det $\partial f(x) = -1$ or sign det $\partial f(x) = 1$. This means that either sign det$A = -1$ for all $A \in \partial f(x)$ or sign det$A = 1$ for all $A \in \partial f(x)$. Consider the first case. The continuity of the function det(.) implies that for any $A \in \partial f(x)$, there exists $\delta_A > 0$ such that det$B = -1, \forall B \in \mathbb{B}(A, \delta_A)$. It is clear that $\partial f(x) \subset_{A\in\partial f(x)} \mathbb{B}(A, \delta_A/2)$. Since the set $\partial f(x)$ is compact, there exists a finite number of matrices, say $A_1, ..., A_q$, such that

$$\partial f(x) \subset \cup_{i=1}^{q} \mathbb{B}(A_i, \frac{1}{2}\delta_{A_i}).$$

Let $\epsilon := \min\{\frac{1}{3}\delta_{A_i}, i = 1, .., q\}$. The upper-semicontinuity of the set-valued map $\partial F(.)$ implies the existence of a scalar $\delta > 0$ such that

$$\partial f(u) \subseteq \partial f(x) + \epsilon\mathbb{B},\ \forall u \in \mathbb{B}(x, \delta).$$

We show that $U := \mathbb{B}(x, \delta)$ is the desired neighborhood. Indeed, take $u \in U$ and let $T \in \partial f(u)$. Then we have

$$T \in \partial f(x) + \epsilon\mathbb{B} \subset \cup_{i=1}^{q} \mathbb{B}(A_i, \frac{1}{2}\delta_{A_i}) + \epsilon\mathbb{B}.$$

Hence, $T \in \mathbb{B}(A_i, \frac{1}{2}\delta_{A_i}) + \epsilon\mathbb{B}$ for some $i \in \{1, ..., q\}$. It follows that

$$T \in \mathbb{B}(A_i, \frac{5}{6}\delta_{A_i}) \subset \mathbb{B}(A_i, \delta_{A_i})$$

and therefore, det$T = -1$. Thus, sign det $\partial f(u) = -1 =$ sign det $\partial f(x)$, as it was to be shown. The second case can be considered similarly. □

Proposition 3.3. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set, $f : \Omega \to \mathbb{R}^n$ is Lipschitz near $x \in \Omega$ and $y = f(x)$. If $\partial f(x)$ is of the maximal rank, then there exists an open neighborhood W of x such that $f^{-1}(y) \cap \bar{W} = \{x\}$.*

Proof. Let U be an open neighborhood of x as in Proposition 3.2. Without loss of generality, we may assume that the open ball $\mathbb{B}(x, r)$ with r sufficiently small is included in U. We show that $f^{-1}(y) \cap \mathbb{B}(x, r) = \{x\}$ and therefore, $W := \mathbb{B}(x, \frac{1}{2}r)$ is the set with the desired property. Indeed, suppose to the contrary that for some $u \in \mathbb{B}(x, r)$, $u \neq x$ one has $f(u) = y$. By Leburg's mean value theorem [4, Theorem 2.3.7], there exist $t \in [x, u] =: \{\lambda x + (1 - \lambda)u :$

$0 \le \lambda \le 1\}$ and $A \in \partial f(t)$ such that $f(u) - f(x) = A(u - x)$. Since $t \in \mathbb{B}(x, r) \subset U$, $\partial f(t)$ is of maximal rank and A is therefore nonsingular. As $f(x) = f(u) = y$, it follows that $A(u - x) = 0$. This is a contradiction because $u - x \neq 0$ and A is nonsingular. $\square$

We are ready to prove Theorem 3.2.

Proof. (i) First we show that $f^{-1}(y)$ is a compact subset of Ω. Let $\{x_i\}$ be a sequence in $\Omega \cap f^{-1}(y)$ converging to some $x \in \bar{\Omega}$. As $f(x_i) = y$, we get $f(x) = y$. The assumption $y \notin f(\partial\Omega)$ implies $x \in \Omega$. Thus, $f^{-1}(y)$ is a closed subset of Ω. Since Ω is bounded, it follows that $f^{-1}(y)$ is a compact subset of Ω.

Next, we show that $f^{-1}(y)$ is a finite set. Let $x \in f^{-1}(y)$ be an arbitrary point. Since y is a Clarke's regular value of f, $\partial f(x)$ is of maximal rank. The inverse function theorem stated in Proposition 3.1 implies the existence of open neighborhoods $\tilde{W}$ and $\tilde{V}$ of x and y, resp., and a Lipschitz map $g : \tilde{V} \to \mathbb{R}^n$ such that $\tilde{W} \cap f^{-1}(y) = \{x\}$ and f and g are one-to-one and onto on $\tilde{W}$ and $\tilde{V}$, resp. Thus, all points of the compact set $f^{-1}(y)$ are isolated. Hence, this set consists of a finite number of points, say $f^{-1}(y) = \{x_1, ..., x_k\}$.

By Propositions 3.2 and 3.3, we can find open neighborhoods U_i of x_i for each $i = 1, ..., k$ such that for all $u \in U_i$, $\partial f(u)$ is of the maximal rank and $\bar{U}_i \cap f^{-1}(y) = \{x_i\}$. Applying Proposition 3.1 we can find for each $i = 1, ..., k$ open neighborhoods $W_i \subseteq U_i$ of x_i, open neighborhoods V_i of y and locally Lipschitz maps $g_i : V_i \to \mathbb{R}^n$ such that W_i and V_i are lipeomorphic by the map f and g_i, namely, $g_i(f(u)) = u$ for all $u \in W_i$ and $f(g_i(v)) = v$ for all $v \in V_i$.

(ii) Let $V := \mathbb{B}(y, \rho)$ be an open ball such that $\mathbb{B}(y, \rho) \subset \cap_{i=1}^k V_i$ and $v \in V$. Then $v \in V_i$ and for $w_i = g_i(v) \in W_i$ one has $f^{-1}(v) \cap W_i = \{w_i\}$ for all $i = 1, ..., k$. Thus, the equation $f(x) = v$ has exactly k solutions w_i on the set $\cup_{i=1}^k W_i$ and we claim that for ρ sufficiently small, this equation has no other solution outside this set. Suppose to the contrary that one can find sequences $\{v_j\}$ converging to y and $\{u_j\}$ such that $f(u_j) = v_j$ and $u_j \notin \cup_{i=1}^k W_i$ for all $j = 1, 2, ...$ Since the sequence $\{u_j\}$ is bounded, we may assume that it converges to some x and therefore, $f(x) = y$. On the other hand, since $u_j \notin \cup_{i=1}^k W_i$, we get $x \notin \cup_{i=1}^k W_i$ and hence $x \notin f^{-1}(y)$, a contradiction. Finally, since $W_i \subset U_i$, it follows that $\partial f(w_i)$ is of the maximal rank for all $i = 1, ..., k$ and therefore, v is a Clarke's regular value of f. $\square$

Remark 3.1. Let us provide some comments about applications of Theorem 3.2 in the cases of Clarke's regular values and Rader's regular values.

(i) The assertion (ii) in Theorem 3.2 implies that the set of Clarke's regular values is open. In contrast, the set of Rader's regular values may not be open. To see this, let us consider the function h in Example 2.1 (ii). Note that $h^{-1}(0) = \{0\}$. Recall that $\bar{y} = 0$ is a Rader's regular value but it is not a Clarke's regular value. Let $x_k = 1/(k\pi)$

and $y_k := h(x_k) = 1/(k\pi)$. Since $h'(x_k) = 0$, y_k is a Rader's critical value. On the other hand, we have $y_k \to 0$.

(ii) If y only is a Rader's regular value, then it may happen that $f^{-1}(v) = \emptyset$ for v near y and f may not be one-to-one in any neighborhood of y and $x \in f^{-1}(y)$ (see the function h in Example 2.1(ii)).

(iii) Let $f : \Omega \to \mathbb{R}$ with $\Omega =]-2, 2[$ be the function given by $f(x) = |x|$ if $|x| \leq 1$, $f(x) = 2|x| - 1$ if $|x| > 1$. Then each value $y \neq 0$ is a Clarke's regular value of f, and one can apply Theorem 3.2 to study the equation $f(x) = y$. Meanwhile, although each $y \notin \{0, 1\}$ is a Rader's regular value of f, Theorem 3.1 cannot be applied because the assumption $d(f, \Omega, y) \neq 0$ is not satisfied; in fact, we have $d(f, \Omega, y) \equiv 0$.

4. Regular exchange economies

Our aim in this section is to extend Debreu's and Dierker's results about equilibrium prices of an exchange economy to the case the demand functions or the excess demand function are locally Lipschitz.

Let us first recall some concepts about an exchange economy, see for instance [5] and [8]. Consider a pure exchange economy with l goods/commodities and m consumers whose needs and preferences are fixed and whose resources vary as in [5]. Let

$$P := \{p \in \mathbb{R}^l_{++} \ : \ \sum_{i=1}^{l} p_i = 1\} \text{ and } Q := \{q = (p_1, ..., p_{l-1}) \in \mathbb{R}^{l-1}_{++} \ : \ \sum_{i=1}^{l-1} p_i < 1\}$$

be the price simplex and the open price simplex. Points of P and Q are in one-to-one correspondence, and in what follows, we associate to a point $q = (p_1, ..., p_{l-1}) \in Q$ the point $p = (p_1, ..., p_l) \in P$ with $p_l = 1 - \sum_{i=1}^{l-1} p_i$.

It is convenient to specify the preferences of the ith consumer by his *demand function* f_i, $f_i : P \times \mathbb{R}_{++} \to \mathbb{R}^l_+$. Given the price vector p in P and his wealth v_i in $\mathbb{R}_+$, the ith consumer demands the commodity vector $f_i(p, v_i)$ in $\mathbb{R}^l_+$. The demand functions are supposed to fulfil $pf_i(p, v) - v = 0$ for any price-wealth pair (p, v). The preferences of the ith consumer can also be represented by a *utility function* $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ and the demand function f_i is the solution of a (maximizing) optimization problem with the objective map being the utility function u_i.

An *economy* $\mathcal{E}$ is defined by $(f_1, ..., f_m, \omega_1, ..., \omega_m)$, an m-tuple $(f_1, ..., f_m)$ of demand functions, and an m-tuple $\omega = (\omega_1, ..., \omega_m)$ of *initial endowment vectors* in $\mathbb{R}^{lm}_{++}$ (each $\omega_i \in \mathbb{R}^l_{++}$, $i \in \{1, ..., m\}$). Since the demand functions f_i are fixed, this economy is actually defined by $\omega \in \mathbb{R}^{lm}_{++}$ and we then denote it by $\mathcal{E}_\omega$. The space of economies is $\mathbb{R}^{lm}_{++}$.

The economy $\mathcal{E}$ can also be characterized by an *excess demand function* $\eta = (\eta_1, .., \eta_l) : P \to \mathbb{R}^l$, which satisfies Walras' Law $p\eta(p) = 0$. For our purpose, we will use a function $\phi : Q \to \mathbb{R}^{l-1}$ defined by $\phi(q) = (\eta_1(p), .., \eta_{l-1}(p))$ for any $q \in Q$.

An element p in P is an *equilibrium price vector* (shortly, an equilibrium price) of the economy $\mathcal{E}$ if it is a zero of the excess demand function η, i.e.,

$$\eta(p) = 0$$

(and hence, a zero of the map ϕ). Given an initial endowment vector $\omega \in \mathbb{R}^{lm}_{++}$, we say that p in P is an *equilibrium price* of the economy $\mathcal{E}_\omega$ if

$$\sum_{i=1}^{m} f_i(p, p\omega_i) = \sum_{i=1}^{m} \omega_i.$$

It is clear that these definitions coincide when η is of the form $\eta(p) = \sum_{i=1}^{m}(f_i(p, p\omega_i)) - \omega_i)$.

Denote by $W(\omega)$ the set of equilibrium prices of the economy $\mathcal{E}_\omega$

$$W(\omega) := \{p \in P \mid \sum_{i=1}^{m} f_i(p, p\omega_i) = \sum_{i=1}^{m} \omega_i\}.$$

Debreu proved [5, p.390] that $W(\omega)$ *is nonempty* for every $\omega \in \mathbb{R}^{lm}_{++}$ if demand functions are continuous and satisfy the following desirability assumption.

Assumption (A) [5] *If the sequence $\{(p^k, v^k)\}_{k=1}^{\infty}$ in $P \times \mathbb{R}_{++}$ converges to an element (p, v) in $\partial P \times \mathbb{R}_{++}$, then $\lim_{k\to\infty} \|f_i(p^k, v^k)\| = +\infty$.*

Assumption (A) expresses the idea that every commodity is desired by the ith consumer and is satisfied provided each consumer has a strictly monotone utility function, see [17].

Let us formulate the first result of this section, which is an application of Theorem 3.2.

Theorem 4.1. *Assume that the demand functions f_i $(i = 1, ..., m)$ are locally Lipschitz and one function, say f_1, satisfies Assumption (A).*

(i) *Assume that the economy $\mathcal{E}_{\bar{\omega}}$ is regular in the sense that $\bar{\omega}$ is a Clarke's regular value of the map $F : U \to \mathbb{R}^{lm}$ defined as follows: for $u = (q, v, z_2, ..., z_m) \in U$*

$$F(u) = (f_1(p, v) + \sum_{i=2}^{m} f_i(p, p \cdot z_i) - \sum_{i=2}^{m} \omega_i, z_2, ..., z_m),$$

where $U := Q \times \mathbb{R}_{++} \times \mathbb{R}^{l(m-1)}_{++}$. Then there are an open neighborhood V of $\bar{\omega}$ and k locally Lipschitz functions $g_1, ..., g_k$ from V to Q such that for every $\omega \in V$, the set $W(\omega)$ consists of the k distinct elements $g_1(\omega), ..., g_k(\omega)$. The set of regular economies (i.e., the set of the endowments ω which is a regular value of F) is open in $\mathbb{R}^{lm}_{++}$.

(ii) *If the functions* f_i $(i = 1, ..., m)$ *are piecewise essentially smooth, then the set of economies* $\mathcal{E}_\omega$ *with infinite equilibrium (i.e., the set of* $\omega \in \mathbb{R}^{lm}_{++}$ *for which* $W(\omega)$ *is infinite) has a Lebesgue measure zero.*

Proof. We follow the scheme of the proof of [5, Theorem 1 and Remark, p.390]. Observe that for a given endowment $\omega = (\omega_1, ..., \omega_m) \in \mathbb{R}^{lm}_{++}$, the equality $F(u) = \omega$ is satisfied if and only if $v = p \cdot \omega_1$, $z_i = \omega_i$ for $i = 2, ..., m$ and $p \in W(\omega)$. In other words,

$$p \in W(\omega) \Longleftrightarrow F(q, p\omega_1, \omega_2, ..., \omega_m) = \omega.$$

Thus, points of $W(\omega)$ are in one-to-one correspondence with points of $F^{-1}(\omega)$ and one can deduce properties of the set $W(\omega)$ from the ones of the solution sets of the equation $F(u) = \omega$.

(i) First, we show that for any $\omega \in \mathbb{R}^{lm}_{++}$, the set $F^{-1}(\omega)$ is a compact subset of U. Observe that U is an open subset of $\mathbb{R}^{lm}$. Recall that

$$F^{-1}(\omega) = \{(q, p\omega_1, \omega_2, ..., \omega_m) \in U \ : \sum_{i=1}^{m} f_i(p, p\omega_i) - \sum_{i=1}^{m} \omega_i = 0\}.$$

Note that $F^{-1}(\omega)$ is bounded. We claim that $F^{-1}(\omega)$ stays away from the boundary of U. Suppose to the contrary that there exists a sequence $\{u^j\}$ in $F^{-1}(\omega)$, $u^j = (q^j, p^j\omega_1, \omega_2, ..., \omega_m)$, such that $u^j \to \tilde{u} \in \partial U$. Then we have $\tilde{u} = (\tilde{q}, \tilde{p}\omega_1, \omega_2, ..., \omega_m)$. It is clear that $(p^j, p^j\omega_1)$ tends to $(\tilde{p}, \tilde{p}\omega_1) \in \partial P \times \mathbb{R}_{++}$. Since the function f_1 satisfies Assumption (A), we get $\lim_{j\to\infty} \|f_1(p^j, p^j\omega_1)\| = \infty$. On the other hand, since $f_1(p^k, p^k\omega_1) = \sum_{i=1}^m \omega_i - \sum_{i=2}^m f_i(p^k, p^k\omega_i)$ and f_i $(i = 1, ..., m)$ take values in $\mathbb{R}^l_+$, the sequence $\{\|f_1(p^k, p^k\omega_1)\|\}$ is bounded. This contradiction implies that $\tilde{u} \notin \partial U$. Now, it is easy to see that $F^{-1}(\omega)$ is closed.

Next, since $F^{-1}(\bar{\omega})$ is a compact subset of U, we can choose a scalar $\rho > 0$ such that $F^{-1}(\bar{\omega}) \subset \rho\mathbb{B}$ and $\bar{\omega} \notin F(\partial(U \cap \rho\mathbb{B})$. Theorem 3.2 implies the existence of an open neighborhood V of $\bar{\omega}$ and k locally Lipschitz functions $\hat{g}_1, ..., \hat{g}_k$ from V to U such that for every $\omega \in V$, the set $F^{-1}(\omega)$ consists of the k distinct elements $\hat{g}_1(\omega), ..., \hat{g}_k(\omega)$. Denote by $g_i(\omega) \in Q$ the coordinate of $\hat{g}_i(\omega)$ which corresponds to the variable belonging to Q. One can easily see that $g_i(\omega)$, $i = 1, .., k$ are the desired locally Lipschitz functions.

Theorem 3.2 also implies that the set of regular economies is open in $\mathbb{R}^{lm}_{++}$.

(ii) Since the functions f_i $(i = 1, ..., m)$ are piecewise essentially smooth, so is the map F. The version of Sard's theorem for a piecewise essentially smooth map [10, Theorem 4.1] implies that the set of critical values of F has a Lebesgue measure zero. As the set $W(\omega)$ is infinite if and only if ω is a Clarke's critical value of the map F (see the assertion (i)), it follows that the set of $\omega \in \mathbb{R}^{lm}_{++}$ for which $W(\omega)$ is infinite has a Lebesgue measure zero. $\square$

Next, we show that a regular economy with a locally Lipschitz excess demand function has an odd number of equilibrium prices. The definition of a regular economy is motivated by the one introduced by Dierker in [7].

Theorem 4.2. *Assume that the excess demand function η is locally Lipschitz and satisfies the following desirability assumption*

Assumption (D) [7] *If the sequence $\{q^k\}_{k=1}^{\infty}$ in Q converges to an element q in ∂Q, then there exists an $h \in \{1,..,l\}$ such that $\{p_h^k\}$ (p_h^k is the h-th coordinate of p^k) converges to zero and $\overline{\lim}_{k\to\infty}\eta_h(p^k) = +\infty$.*

If the economy $\mathcal{E}$ is regular in the sense that zero is a Clarke's regular point of ϕ, then it has an odd number of equilibrium prices.

Note that this definition of a regular economy is independent of the order in which the commodities are indexed.

Proof. Our proof is motivated by the ones for [7, Theorem 1] and [17, Theorem 15]. Recall that $\phi : Q \to \mathbb{R}^{l-1}$ is the map defined by $\phi(q) = (\eta_1(p), ..., \eta_{l-1}(p))$. Let $g : Q \to \mathbb{R}^{l-1}$ be the map defined by $g(q) = \frac{1}{l}\mathbf{e_{l-1}} - q$, where $\mathbf{e_{l-1}}$ is the vector in $\mathbb{R}^{l-1}$ with all components being 1. For any $t \in [0,1]$, we define a map $H_t : Q \to \mathbb{R}^{l-1}$ by

$$H_t(q) = (1-t)\phi(q) + tg(q).$$

Denote

$$S := \{q \in Q \;:\; \exists t \in [0,1] \text{ such that } H_t(q) = 0\}.$$

We claim that S is a compact subset of Q. Since S is bounded, it suffices to show that S stays away from the boundary of Q. Suppose to the contrary that there exists a sequence $\{q^k\}$ in Q converging to $\bar{q} \in \partial Q$. By Assumption (D), there exists an $h \in \{1, ..., l\}$ such that $\{p_h^k\}$ tends to zero and $\overline{\lim}_{k\to\infty}\eta_h(p^k) = \infty$. Then for any integer N sufficiently large, we have

$$\frac{1}{l} - p_h^N > 0 \text{ and } \eta_h(p^N) > 0 \tag{2}$$

(here and in what follows, N stands for an index and not for a power). Recall that since $q^N \in S$, there exists $t^N \in [0,1]$ such that $(1-t^N)\phi(q^N) + t^N g(q^N) = 0$ or equivalently,

$$(1-t^N)\eta_j(p^N) + t^N(\frac{1}{l} - p_j^N) = 0, \quad \forall j \in \{1, ..., l-1\}. \tag{3}$$

If $h \in \{1, ..., l-1\}$, the inequalities in (2) imply

$$(1-t^N)\eta_h(p^N) + t^N(\frac{1}{l} - p_h^N) > 0,$$

which is a contradiction to (3).

Next, assume that $h = l$. In this case we have $\sum_{j=1}^{l-1}\bar{p}_j = 1$ and $\bar{p}_l = 0$. Multiplying j-th equality in (3) with p_j^N for each $j = 1, ..., l-1$ and summarizing from $j = 1$ up to $j = l-1$, we get

$$(1-t^N)\sum_{j=1}^{l-1} p_j^N \eta_j(p^N) + t^N \sum_{j=1}^{l-1} p_j^N (\frac{1}{l} - p_j^N) = 0.$$

By Walras' Law, we have $\sum_{j=1}^{l-1} p_j^N \eta_j(p^N) = -p_l^N \eta_l(p^N)$. Hence,

$$-(1-t^N)p_l^N \eta_l(p^N) + t^N \sum_{j=1}^{l-1} p_j^N (\frac{1}{l} - p_j^N) = 0. \tag{4}$$

It follows from the equalities

$$\min_{p_j>0, \sum_{j=1}^{l-1} p_j = 1} \sum_{j=1}^{l-1} (p_j)^2 = \frac{1}{l-1}$$

(see [17, p.162]) and $\sum_{j=1}^{l-1} \bar{p}_j = 1$ that

$$\sum_{j=1}^{l-1} \bar{p}_j (\frac{1}{l} - \bar{p}_j) = \frac{1}{l} - \sum_{j=1}^{l-1} (\bar{p}_j)^2 \leq \frac{1}{l} - \frac{1}{l-1} = -\frac{1}{l(l-1)} < 0.$$

Since $\sum_{j=1}^{l-1} p_j^k (\frac{1}{l} - p_j^k)$ converges to $\sum_{j=1}^{l-1} \bar{p}_j (\frac{1}{l} - \bar{p}_j)$, we may assume without lost of generality that for this sufficiently large N it holds

$$\sum_{j=1}^{l-1} p_j^N (\frac{1}{l} - p_j^N) < 0. \tag{5}$$

Since $q^N \in Q$, we have $p_l^N > 0$. Note that the second inequality in (2) in the case $h = l$ has the form $\eta_l(p^N) > 0$. Therefore,

$$p_l^N \eta_l(p^N) > 0. \tag{6}$$

Now, we get a contradiction because if $t^N = 0$, then (4) and (6) imply $0 = -p_l^N \eta_l(p^N) < 0$ and if $t^N \neq 0$, then (4) - (6) imply $0 = -(1-t^N)p_l^N \eta_l(p^N) + t^N \sum_{j=1}^{l-1} p_j^N (\frac{1}{l} - p_j^N) < 0$.

Since S is a compact subset of Q, we can find an open bounded set $\mathcal{U} \subset Q$ such that $\bar{\mathcal{U}} \subset Q$, $S \subset \mathcal{U}$ and $0 \notin H_t^{-1}(\partial \mathcal{U})$ for all $t \in [0,1]$. By the homotopy invariance, we get $d(\phi, \mathcal{U}, 0) = d(g, \mathcal{U}, 0)$. Since det $g'(q) = (-1)^{l-1}$ for all $q \in Q$, we get $d(g, \mathcal{U}, 0) = (-1)^{l-1}$. Therefore, $d(\phi, \mathcal{U}, 0) = (-1)^{l-1}$. On the other hand, since the set of zeros of ϕ in Q is contained in S and 0 is a Clarke's regular value of ϕ, Theorem 3.2 implies that this set consists of k vectors u_j with $u_j \in \mathcal{U}$ for $j = 1, ..., k$. Applying (1), we get

$$d(\phi, \mathcal{U}, 0) = \sum_{j=1}^{k} \text{sign det} \partial \phi(u_j).$$

As $d(\phi, \mathcal{U}, 0) = (-1)^{l-1}$, it follows that k is an odd number. Since p is an equilibrium price of $\mathcal{E}$ iff q is a zero of ϕ, the economy $\mathcal{E}$ has an odd number of equilibrium prices. $\square$

Remark 4.1. (a) Recall that there are values which are regular in the sense of Clarke but not in the sense of Rader (see Example 2.1). Hence, Theorems 4.1 and 4.2 can be applied to some economies that may not be not considered in [17] and [1].

(b) In this note, we consider a locally Lipschitz economy, which is a case much simpler than the ones with definable or semi-algebraic economy studied in [2] and [11]. This allows us to have simple proofs with direct use of generalization of classical results such as inverse function theorem, Sard's theorem and Brower's degree.

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Thang long Institute for Mathematics and Applied Sciences, Thang Long University