ON THE NUMBER OF PIVOTS OF DANTZIG'S SIMPLEX METHODS FOR LINEAR AND CONVEX QUADRATIC PROGRAMS

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Abstract. Refining and extending works by Ye and Kitahara-Mizuno, this paper presents new results on the number of pivots of simplex-type methods for solving linear programs of the Leontief kind, certain linear complementarity problems of the P kind, and nonnegative constrained convex quadratic programs. Our results contribute to the further understanding of the complexity and efficiency of simplex-type methods for solving these problems.

Keywords. Simplex methods, linear programs, linear complementarity problems, convex quadratic programs, number of pivots, Leontief matrices, Z-matrices, complexity

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1. Introduction

This paper is a continuation of our sustained interest in identifying classes of linear and quadratic programs and linear complementarity problems (LPs, QPs, and LCPs, respectively) for which the number of pivots in simplex-type methods can be upper-bounded by certain quantities derived from the problem input vectors and matrices. In favorable cases, such upper bounds would be strongly polynomially bounded in the number of variables and constraints of the problems, hence rendering their strongly polynomial solvability. Thus, this interest is different from the worst-case exponential behavior or from the probabilistic average-case analysis of this kind of methods for these problems, or from the polynomial analysis of the ellipsoid method or interior-point approaches where the worst-case complexity is bounded by the input size of the problem data. In what follows, we summarize some old

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and new results in the literature which provide the motivation for the kind of results we aim to obtain.

Pre-2000: Since the seminal paper [30], there has been an extensive literature on the study of Leontief systems and associated linear programs (LPs):

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\textbf{minimize}} & c^\top x \\ \textbf{subject to} & Ax = b \ \ \textbf{and} \ \ x \geq 0, \end{array} \tag{1}$$

where $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given vectors and the matrix $A \in \mathbb{R}^{m \times n}$ satisfies the following conditions: (a) each column of A has at most one positive entry, (b) $b \in \mathbb{R}^m_{++}$, and (c) a feasible solution of (1) exists. A matrix A satisfying condition (a) is called *pre-Leontief*; we call A *pre-Leontief-plus* if each column of A has exactly one positive entry. A significant amount of this literature is motivated by network-type problems [22, 14, 21] and their solution by Dantzig's simplex method. In particular, the reference [21] shows that when A is a *Leontief flow matrix*, i.e., its entries are all integral and the positive entries are all equal to one, then the LP (1) can be solved in $\mathrm{O}(n^2 U \log(npU))$ pivots by the simplex method using Dantzig's rule for choosing the entering variable, where p is the largest entry of A in absolute value, and U is a valid upper bound on any extreme-point solution. An interesting observation remarked in the paragraph before Theorem 3 in [21] is that this complexity for (1) can be obtained by considering the special case where A is a *Leontief flow matrix* and the vector b is the vector of ones.

Post-2000: With no mention of the references [22, 14, 21], Ye [31] showed that when the LP (1) is derived from a Markov decision problem with a fixed discount rate, then the number of simplex pivots with the least reduced cost rule to select the entering nonbasic variable is a low-order polynomial in (m, n). Extending this special result to a general LP, the paper [16] shows that for the general LP (1) the number of distinct basic solutions generated by the simplex method with the same least reduced cost rule is $\mathrm{O}\Big(n \left\lceil m \frac{\gamma}{\delta} \log \Big(m \frac{\gamma}{\delta} \Big) \right\rceil \Big)$, where γ and δ are the maximum and minimum, respectively, of the positive components of all the basic feasible solutions. A companion reference [15] performs a similar analysis for the simplex method for LPs with upper bounds. In a subsequent work [18], the authors show that the constant γ can be computed by solving a linear program while the computation of δ is NP-hard in general.

Strong polynomiality: The authors of the above references recognized that their results can be used to infer the strong polynomiality of the simplex method if polynomiality of the key constants in the respective bounds can be established. In general, however, a nondegeneracy assumption is needed to ensure that the number of different basic solutions generated by the method equals the number of pivots.

The role of the Z-property: A square matrix with nonpositive off-diagonal entries is called a Z-matrix. If a Z-matrix further has a nonnegative inverse, it is called a Minkowski matrix. It is known that a Z-matrix M is Minkowski if and only if there exists a positive vector d such that $Mx = d$ has a nonnegative solution (Theorem 3.11.10 [6]). Thus if the system (1) is feasible and A is a pre-Leontief-plus matrix, then subject to a proper permutation of the columns, any feasible basis B of such a system with a positive right-hand side must be a Minkowski matrix. Therefore, the nondegeneracy assumption is satisfied by any feasible Leontief system because with b being positive, it follows that $B^{-1}b > 0$ is nondegenerate.

The matrix-theoretic Z-property has been responsible for the strongly polynomial complexity of pivoting-type methods for solving various classes of linear complementarity problems and bound-variable convex quadratic programs. The first result in this area is due to Chandrasekaran [4] for LCPs with a Z-matrix. The most recent addition to this literature is the article [25] that establishes the strongly polynomial solvability by parametric principal pivoting for a certain class of convex quadratic programs with bounded variables where some such bounds may be infinite. A brief summary of the preceding results for LCPs with "H-matrices" with positive diagonals and extensions can be found in the last reference.

Goals of this short note: Inspired by the analysis in [16] for linear programs, our work makes several important contributions that add to our understanding of the performance of pivoting methods, and importantly, the bottlenecks that may cause a large number of pivots in such methods.

• We give practical bounds on the two key constants γ and δ in the cited reference for the case where A is a pre-Leontief-plus matrix; these bounds are in terms of some constants derived from the Leontief properties of A and the right-hand side vector b; see part B(ii) in Proposition 1.

• We apply the above results to a class of matrices for which the associated LCPs are solvable by the Simplex Method, a subject first advanced by Mangasarian's pioneering idea of solving LCPs as LPs [20].

• We analyze the iteration count of a pivoting method for quadratic programming due independently to Dantzig [8, 7] and to van de Panne-Whinston[28, 29]; this method is formally described as Algorithm 4.2.11 in [6, pages 248–251] where a proof of the finite-step termination of the algorithm can be found. Subsequently, we will abbreviate this as the DvPW algorithm. Moreover, we provide two applications and show how to estimate the complexity-related bounds in those contexts.

2. The Leontief LP

Let $A \in \mathbb{R}^{m\times n}$ be pre-Leontief-plus. As such, we may assume with no loss of generality that the matrix A is structured as follows: For each row $i = 1, \cdots, m$, there is a group of columns $\mathcal{G}_i$ such that $a_{ij} > 0$ for all $j \in \mathcal{G}_i$; note that $\mathcal{G}_i \cap \mathcal{G}_{i'} = \emptyset$ because each column of A has exactly one positive entry. Let k_g be the number of elements in group $\mathcal{G}_g$ for $g = 1, \cdots, m$ so that $\displaystyle\sum_{g=1}^{m} k_g = n$. We remark that if $b > 0$ and the problem (1) is feasible, then $\mathcal{G}_g \neq \emptyset$ for all g. By permuting the columns of A if needed, we may assume that the elements in these m groups of columns are consecutively labelled in the same order as $\{1, \cdots, n\}$ so that with $\mathcal{G}_1 = \{1, \cdots, k_1\}$, we have, inductively, $\mathcal{G}_g = \left\{ \left(\displaystyle\sum_{j=1}^{g-1} k_j \right) + 1, \; \cdots, \; \displaystyle\sum_{j=1}^{g} k_j \right\}$, for $g = 2, \cdots, m$. Below is an example of a 3×11 matrix A with 3 groups each with 4, 3, and 4 columns respectively arranged with the column labels being the same as the labels of the variables:

$$
\begin{array}{c}
\begin{array}{cccc|ccc|cccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11}
\end{array} \\
\left[\begin{array}{cccc|ccc|cccc}
+ & + & + & + & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus & + & + & + & \ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & + & + & + & +
\end{array}\right]
\end{array}
$$

With A structured as displayed, and $\mathcal{G}_g \neq \emptyset$ for all $g = 1, \cdots, m$, we define an $m \times m$ matrix $\overline{A}$ with entries

$$
\bar{a}_{ig} \triangleq \begin{cases} \displaystyle\min_{j \in \mathcal{G}_g} a_{ij} & \text{if } i = g \\ -\displaystyle\max_{j \in \mathcal{G}_g} |\, a_{ij} \,| & \text{if } i \neq g. \end{cases}
$$

The off-diagonal entries of $\overline{A}$ is clearly nonpositive; so $\overline{A}$ is a Z-matrix. In what follows we assume that the matrix $\overline{A}$ is Minkowski.

Proposition 1. Let $A \in \mathbb{R}^{m\times n}$ be pre-Leontief. The following two statements (A) and (B) hold:

(A) If the system $Ax = b,\ x \geq 0$ has a solution for some $b \in \mathbb{R}^m_{++}$, then

(i) every basic feasible solution is nondegenerate;

(ii) for every feasible basis $B \in \mathbb{R}^{m\times m}$, there exists a vector $d_B \in \mathbb{R}^m_{++}$ such that $Bd_B > 0$;

(iii) the system $Ax = b',\ x \geq 0$ has a solution for all $b' \in \mathbb{R}^m_+$.

(B) Conversely, if A is pre-Leontief-plus and the matrix $\overline{A}$ is Minkowski, then

(i) the system $Ax = b,\ x \geq 0$ has a solution for all vectors $b \in \mathbb{R}^n_+$; moreover,

(ii) if $\bar{x}$ is any basic feasible solution of the the system $Ax = b,\ x \geq 0$, where $b \in \mathbb{R}^m_{++}$, then

$$\underline{\delta} \triangleq \min_{1\leq i\leq m} \left[\frac{b_i}{\max\limits_{j\in\mathcal{G}_i} a_{ij}} \right] \leq \delta_{\bar{x}} \leq \gamma_{\bar{x}} \leq \max_{1\leq i\leq m} \left[(\overline{A})^{-1} b \right]_i \triangleq \overline{\gamma},$$

where $\delta_{\bar{x}}$ and $\gamma_{\bar{x}}$ are the smallest positive element of $\bar{x}$ and the largest element of $\bar{x}$, respectively.

(iii) Let $C \triangleq \dfrac{\overline{\gamma}}{\underline{\delta}}$. Then for any vector $c \in \mathbb{R}^n$ for which the LP (1) with $b \in \mathbb{R}^n_{++}$ has an optimal solution, the simplex method with the least reduced cost rule will solve the LP (1) in no more than $\mathrm{O}(\, n \lceil m\, C \log (m\, C) \rceil\,)$ pivots.

Proof. We prove only the bounds on $\bar{x}$. Let $B \in \mathbb{R}^{m\times m}$ be a feasible basis corresponding to $\bar{x}$; thus $B\,\bar{x}_\beta = b$, where $\bar{x}_\beta$ consists of the basic components of $\bar{x}$ corresponding to the basis B. By the above arrangement of the columns of A, each column of B has exactly one positive elements, which are the diagonals of B; moreover, we have $\beta_i \in \mathcal{G}_i$ for all $i = 1, \cdots, m$. On one hand, for all $i = 1, \cdots, m$, we have $b_i = (\, B\,\bar{x}_\beta\,)_i \leq B_{ii}\,\bar{x}_{\beta_i} = a_{i\beta_i}\,\bar{x}_{\beta_i}$. Thus $\bar{x}_{\beta_i} \geq \dfrac{b_i}{a_{i\beta_i}} \geq \dfrac{b_i}{\max\limits_{j\in\mathcal{G}_i} a_{ij}}$. This establishes the lower bound for $\delta_{\bar{x}}$. On the other hand, we have, $b_i = (B\,\bar{x}_\beta)_i = a_{i\beta_i}\,\bar{x}_{\beta_i} + \displaystyle\sum_{g\neq i}^{m} a_{i\beta_g}\,\bar{x}_{\beta_g} \geq \sum_{g=1}^{m} \bar{a}_{ig}\,\bar{x}_{\beta_g}$ for all $i = 1, \cdots, m$. Thus $b \geq \overline{A}\bar{x}_\beta$. The upper bound for $\gamma_{\bar{x}}$ follows readily by the nonnegativity of the inverse of $\overline{A}$. The last statement of the proposition is due to [16] and requires no proof. □

In the next section, we give a class of LPs for which the constant C can be identified more explicitly in terms of the problem data.

2.1. **Mangasarian's class of LCPs.** In [20], Mangasarian introduced a class of matrices $M \in \mathbb{R}^{n\times n}$ for which the LCP, which we denote by the pair (q, M): $0 \leq z \perp w = q + Mz \geq 0$, with $q \in \mathbb{R}^n$, where $\perp$ is the perpendicularity notation which in this context denotes the complementarity relation between the two vectors z and w, can be solved by a single LP. Some important properties of this class of matrices are obtained in [5]. Coined a hidden Z-matrix in [23, 24], a matrix M belongs to this class if there exist Z-matrices X and Y and positive vectors $\bar{r}$ and $\bar{s}$ such that (a) $MX = Y$ and (b) $\bar{r}^\top X + \bar{s}^\top Y > 0$. We highlight two important properties of such a matrix M: (i) X must be nonsingular; and (ii) the matrix $A \triangleq \left[\begin{array}{cc} X^\top & Y^\top \end{array}\right] \in \mathbb{R}^{n \times 2n}$ is pre-Leontief; see [5]. Most importantly, a solution of the LCP (q, M), which must exist if the problem is feasible, can be obtained by letting $\bar{z} = X\bar{v}$, where $\bar{v}$ is the unique solution of the LP:

$$\begin{array}{ll} \underset{v}{\textbf{minimize}} & p^\top v \\ \textbf{subject to} & q + Yv \geq 0 \text{ and } Xv \geq 0, \end{array}$$

for any vector $p \in \mathbb{R}^n$ such that $p^\top X > 0$; such a vector p must necessarily be positive if X is Minkowski. The dual LP is

$$\begin{array}{ll} \underset{(r,s) \geq 0}{\textbf{minimize}} & q^\top s \\ \textbf{subject to} & X^\top r + Y^\top s = p. \end{array} \tag{2}$$

We can apply Proposition 1 to the latter dual LP by assuming that the matrix $\overline{A}$ with entries defined below is Minkowski:

$$\bar{a}_{ij} = \left\{ \begin{array}{cl} \min(X_{ii}, Y_{ii}) & \text{if } i = j \\ -\max(|X_{ji}|, |Y_{ji}|) & \text{if } i \neq j \end{array} \right\} \quad \forall\, i, j = 1, \cdots, n. \tag{3}$$

We recall that a real square matrix is a P-matrix if all its principal minors are positive. It is a fundamental result in LCP theory that if $M \in \mathbb{R}^{n \times n}$ is a P-matrix, then the LCP (q, M) has a unique solution for all vectors $q \in \mathbb{R}^n$. Without assuming nondegeneracy, the following result gives an upper bound on the number of pivots for the simplex method to compute such a solution when M is additionally hidden Z.

Proposition 2. Let M be a hidden Z-matrix with the two defining Z-matrices X and Y. Suppose that the matrix $\overline{A}$ with entries defined by (3) is Minkowski. Then both M and X must be P-matrices and the following two statements hold.

• For every vector $q \in \mathbb{R}^n$, the LCP (q, M) has a unique solution; moreover, such a solution can be obtained via LP duality by solving the dual LP (2) for any vector p satisfying $p^\top X > 0$.

• For every vector $q \in \mathbb{R}^n$ and any vector p satisfying $p^\top X > 0$, an optimal solution to the dual LP (2) can be obtained by the simplex method with the least reduced cost rule in no more than $\mathrm{O}(\,n\,\lceil n\,C\,\log(n\,C)\rceil\,)$ pivots, where $C \triangleq \left[\max\limits_{1\le i\le n}\,\left[\,(\,\overline{A}\,)^{-1}p\,\right]_i\right] \times \left[\max\limits_{1\le i\le n}\dfrac{\max(\,X_{ii},\,Y_{ii}\,)}{p_i}\right]$.

Proof. By [6, Theorem 3.11.19], being hidden Z, the matrix M is P if and only if there exists a vector $v > 0$ such that for any index set $\alpha \subseteq [n] \triangleq \{1,\cdots,n\}$, we have $Wv > 0$, where $W \triangleq \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar\alpha} \\ & \\ Y_{\bar\alpha\alpha} & Y_{\bar\alpha\bar\alpha}\end{bmatrix}$, where $\bar\alpha$ is the complement of α, and each of the four blocks in W is a submatrix of X and Y, respectively, whose rows and columns are indexed by the pair $(\alpha,\bar\alpha)$, correspondingly. For any nonnegative vector a, it is easy to show that $Wa \ge \overline{A}^{\top} a$, provided that X and Y have nonnegative diagonal elements. Consequently, if $\overline{A}$ is Minkowski, then so is its transpose; hence a positive vector v exists such that $\overline{A}^{\top} v > 0$. This is enough to show that M is P. By taking $\alpha = [n]$, it follows that $W = X$ and thus $Xv > 0$, which implies X is a Minkowski matrix and thus a P-matrix. Finally, the claim about the number of pivots of the simplex method for solving the dual LP (2) follows readily from part B(iii) in Proposition 1. □

3. Nonnegatively Constrained Convex Quadratic Programs

In this section, we derive an upper bound for the DvPW pivoting algorithm for solving the convex quadratic program (QP):

$$\underset{z\in\mathbb{R}^n_+}{\textbf{minimize}}\; v(z) \triangleq \tfrac{1}{2}\,z^{\top}Mz + q^{\top}z, \tag{4}$$

where the matrix M is symmetric positive semidefinite and q is arbitrary. This QP is the dual of the strictly convex QP:

$$\left\{\underset{x\in\mathbb{R}^m}{\textbf{minimize}}\;\frac{1}{2}\,x^{\top}Qx + p^{\top}x \textbf{ subject to } Ax \le b\right\}$$

with a positive definite $Q \in \mathbb{R}^{m\times m}$ via the identifications: $M = AQ^{-1}A^{\top}$ and $q = b + AQ^{-1}p$. Thus a solution of (4) will yield an optimal solution of the latter QP. The Karush-Kuhn-Tucker (KKT) conditions for (4) are given by the LCP (q,M). In terms of the vector $w = q + Mz$ we have $v(z) = \frac{1}{2}\,[q^{\top}z + w^{\top}z]$ for an arbitrary vector z. Note that if A has linearly independent rows, then the matrix M is positive definite.

For any index subsets α and γ of $[n] \triangleq \{1,\cdots,n\}$, $M_{\alpha\gamma}$ is the submatrix of M with rows indexed by α and columns indexed by γ; in particular, $M_{\alpha\alpha}$ is a principal submatrix of M; q_α is similarly defined. Associated with an index set α with $M_{\alpha\alpha}$ nonsingular is the *basic solution* $(z_\alpha, 0)$, where $z_\alpha = -(M_{\alpha\alpha})^{-1}q_\alpha$. We call the index set α *feasible* if $z_\alpha \ge 0$. The full vector $(z_\alpha, 0)$ is a solution of the LCP (q,M) if the vector of *reduced costs*

$w_{\bar{\alpha}} = q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}(M_{\alpha\alpha})^{-1}q_\alpha$ is nonnegative, where $\bar{\alpha}$ is the complement of α in $[n]$. We note that by the symmetry and positive semidefiniteness of M, the nonsingularity of the principal submatrix $M_{\alpha\alpha}$ is equivalent to its positive definiteness; moreover, the Schur complement, denoted $(M/M_\alpha) \triangleq M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}(M_{\alpha\alpha})^{-1}M_{\alpha\bar{\alpha}}$, remains symmetric positive semidefinite.

Throughout we make a blanket assumption that the optimal value of the QP (4) is finite. Two constants play an important role in the following analysis: $\lambda_{\max}(M)$, which is the largest eigenvalue of M, and $\rho_{\min}(M) \triangleq \min\limits_{\alpha:M_{\alpha\alpha}\succ 0} \lambda_{\min}(M_{\alpha\alpha})$, where $M_{\alpha\alpha} \succ 0$ means that $M_{\alpha\alpha}$ is positive definite and $\lambda_{\min}(\bullet)$ denotes the smallest eigenvalue of a positive definite matrix. We make some comments on $\rho_{\min}(M)$. If M itself is (symmetric) positive definite, then $\rho_{\min}(M) = \lambda_{\min}(M)$. In general, it can be shown that $\rho_{\min}(M) \leq \lambda^+_{\min}(M)$, where $\lambda^+_{\min}(M) > 0$ is the smallest positive eigenvalues of M. Indeed, consider the eigen-decomposition $M = \sum_{i=1}^{k} \lambda_i u^i(u^i)^\top$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ are the positive eigenvalues of M and the family $\{u^i\}_{i=1}^k$ are the associated normalized eigenvectors (thus $\|u^i\| = 1$). Take any index set α such that $|\alpha| = k$ and $\{u^i_\alpha\}_{i=1}^k$ are linearly independent, and let $\widehat{M} \triangleq M_{\alpha\alpha} = \sum_{i=1}^{k} \lambda_i\, \widehat{u}^{\,i}(\widehat{u}^{\,i})^\top \succ 0$, where $\widehat{u}^{\,i} \triangleq u^i_\alpha$. Take a vector $v \in \mathbb{R}^k$ such that $\|v\| = 1$ and v is orthogonal to $\widehat{u}^{\,i}$ for all $i = 1, \ldots, k-1$. It follows that $\rho_{\min}(M) \leq \rho_{\min}(\widehat{M}) \leq v^\top \widehat{M}\, v = \lambda_k \left(v^\top \widehat{u}^{\,k}\right)^2 \leq \lambda_k \|v\|^2 \|u^k\|^2 \leq \lambda_k = \lambda^+_{\min}(M)$. The following upper bound on the norm of any basic feasible solution of (4) can easily be obtained.

Lemma 1. For any basic feasible solution z of the LCP (q, M) with M being symmetric positive semidefinite, one has $\| z\|_1 \leq \dfrac{\sqrt{n}\, \| q_- \|_2}{\rho_{\min}(M)}$, where $q_- \triangleq \max\{0, -q\}$.

Proof. For any basic feasible solution $z = (z_\alpha, 0)$, we have $\rho_{\min}(M) \| z \|_2^2 \leq z^\top M z = -z^\top q \leq z^\top q_-$ because z is nonnegative. Hence $\| z \|_1 \leq \sqrt{n}\, \| z \|_2 \leq \dfrac{\sqrt{n}\, \| q_- \|_2}{\rho_{\min}(M)}$, which is the desired bound. $\square$

For a matrix $F \in \mathbb{R}^{n\times m}$, we denote its largest singular value by $\lambda_{\max}(F) = \sqrt{\lambda_{\max}(FF^\top)} = \lambda_{\max}(F^\top)$. For a subset α of $[n]$ with cardinality $|\alpha|$, we denote by $F_{\alpha\bullet}$ the rows of F indexed by α and the transpose of $F_{\alpha\bullet}$ by $F^\top_{\alpha\bullet} \in \mathbb{R}^{m\times|\alpha|}$. Note that $\lambda_{\max}(F_{\alpha\bullet}) = \lambda_{\max}(F^\top_{\alpha\bullet}) \leq \lambda_{\max}(F)$. Similar definitions apply to $F_{\bullet\beta}$ for any subset β of $[m]$.

In contrast to many other pivoting-type algorithms for solving QPs, a significant feature of the DvPw algorithm is that it can always terminate in finite steps even when the problem (4) is degenerate. Referring to [6, Algorithm 4.2.11] for details, we sketch the DvPW algorithm for solving the LCP (q, M) as follows. It is convenient to use the tableau form to represent $w = q + Mz$;

$$\begin{array}{c|c|c|}
 & 1 & z \\ \hline
q^\top z & 0 & q^\top \\ \hline
w & q & M \\ \hline
\end{array} \tag{5}$$

After each principle pivot in which the basic variables z_α and w_β (where β is the complement of α), and the product $q^\top z$ are expressed in terms of the nonbasic variables w_α and z_β, we obtain the following tableau:

$$\begin{array}{c|c|c|c|}
 & 1 & w_\alpha & z_\beta \\ \hline
q^\top z & -q_\alpha^\top (M_{\alpha\alpha})^{-1} q_\alpha & q_\alpha^\top (M_{\alpha\alpha})^{-1} & q_\beta^\top - q_\alpha^\top (M_{\alpha\alpha})^{-1} M_{\alpha\beta} \\ \hline
z_\alpha & -(M_{\alpha\alpha})^{-1} q_\alpha & (M_{\alpha\alpha})^{-1} & -(M_{\alpha\alpha})^{-1} M_{\alpha\beta} \\ \hline
w_\beta & q_\beta - M_{\beta\alpha}(M_{\alpha\alpha})^{-1} q_\alpha & M_{\beta\alpha}(M_{\alpha\alpha})^{-1} & M_{\beta\beta} - M_{\beta\alpha}(M_{\alpha\alpha})^{-1} M_{\alpha\beta} \\ \hline
\end{array} \tag{6}$$

Consisting of major and minor cycles, the algorithm starts with an index set α of the basic z-variables for which the following two conditions hold: (a) $M_{\alpha\alpha}$ is nonsingular (thus positive definite), and (b) $z_\alpha = -(M_{\alpha\alpha})^{-1} q_\alpha \geq 0$. With $w_\beta \not\geq 0$ where β is the complement of α, a *most negative* component w_r with $r \in \min_{i \in \beta} w_i$ is identified and the algorithm attempts to increase the value of z_r by pivoting while keeping the basic z-components nonnegative; this is determined by a ratio test as in the standard simplex method in linear programming. There are two types of pivots, depending on whether (a) the increase of z_r is *blocked* by a basic z-component becoming zero, or (b) w_r is the blocking variable, i.e., its value reaches zero. In the former case, there is one less basic z-component after the pivot, and z_r continues its increase; this is termed a *minor cycle*. In the latter case, z_r becomes basic while its complement w_r becomes nonbasic; this completes the current *major cycle*. Notice there are at most $n - 1$ minor cycles within each major cycle because there are at most $n - 1$ basic z-variables to be (potentially) pivoted out during each major cycle. Thus, in order to analyze the total number of pivots in the algorithm, it suffices to upper bound the number of major cycles. The derivation of such bounds is the principal task in the following analysis. Note that at the beginning of each major cycle t, the pair (z^t, w^t) is primal feasible and satisfies complementarity; i.e., $0 \leq z^t \perp w^t$; for such a pair, we have $v_t = v(z^t) = \frac{1}{2}\, q^\top z^t$.

The derivation proceeds in several lemmas, the first of which gives an amount of decrease of the objective value from the beginning of one major cycle to the end of the same cycle.

Lemma 2 (Strict decrease for each major cycle)**.** Let $\bar{w}_r$ be the value of w_r at the beginning of t-th major cycle. Then $v_t - v_{t+1} \geq \dfrac{\bar{w}_r^2}{4\lambda_{\max}(M)}$.

Proof. Assume that the t-th major cycle consists of K minor cycles and we denote the increment of z_r during each minor cycle as Δ_k. Let $\bar{w}_r^k$ be the value of w_r at the beginning of the k-th minor cycle so that $\bar{w}_r^1 = \bar{w}_r \leq \cdots \leq \bar{w}_r^{K+1} = 0$. We have $|\bar{w}_r^k| - |\bar{w}_r^{k+1}| = \Delta_k\, s_k$, where $s_k \triangleq m_{rr} - M_{r\alpha_k}(M_{\alpha_k\alpha_k})^{-1}M_{\alpha_k r}$ with α_k being the corresponding α-index set in k-th minor cycle within the major cycle t. Then we can see that $0 \leq s_k \leq m_{rr} \leq \lambda_{\max}(M)$, which follows from the positive definiteness of $M_{\alpha_k\alpha_k}$ and the Courant-Fischer Theorem. Hence $\Delta_k \geq \dfrac{|\bar{w}_r^k| - |\bar{w}_r^{k+1}|}{\lambda_{\max}(M)}$. Letting $\bar{z}^{t,k}$ and $\bar{w}^{t,k}$ denote the values of the vectors z and w, respectively, at the beginning of the k-th minor cycle within the t-th major cycle, we then have

$$
\begin{aligned}
& v(\bar{z}^{t,k}) - v(\bar{z}^{t,k+1}) = \tfrac{1}{2}\, q^\top (\, \bar{z}^{t,k} - \bar{z}^{t,k+1}\,) + \tfrac{1}{2}\, \Big[\, (\, \bar{z}^{t,k})^\top \bar{w}^{t,k} - (\, \bar{z}^{t,k+1})^\top \bar{w}^{t,k+1} \Big] \\
& = -\frac{\Delta_k}{2}\, \Big[\, q_r - q_{\alpha_k}^\top (M_{\alpha_k\alpha_k})^{-1} M_{\alpha_k,r} \Big] + \tfrac{1}{2}\, \Big[\, (\, \bar{z}^{t,k})^\top \bar{w}^{t,k} - (\, \bar{z}^{t,k+1})^\top \bar{w}^{t,k+1} \Big] \\
& = \ \frac{\Delta_k}{2}\, |\, \bar{w}_r^k\, | + \tfrac{1}{2}\, \Big[\, (\, \bar{z}^{t,k})^\top \bar{w}^{t,k} - (\, \bar{z}^{t,k+1})^\top \bar{w}^{t,k+1} \Big],
\end{aligned}
$$

where the last equality is due to $\bar{z}_{\alpha_k} = -(M_{\alpha_k\alpha_k})^{-1} q_{\alpha_k}$ and $\bar{w}_r^k = q_r + M_{r\alpha_k}\bar{z}_{\alpha_k}$. Hence, with $v_t = v(\bar{z}^{t,1})$ and $v_{t+1} = v(\bar{z}^{t,K+1})$ and since $(\, \bar{z}^{t,1})^\top \bar{w}^{t,1} - (\, \bar{z}^{t,K+1})^\top \bar{w}^{t,K+1} = 0$, we deduce

$$
\begin{aligned}
2(v_t - v_{t+1}) =& \sum_{k=1}^{K} |\bar{w}_r^k|\, \Delta_k \geq \sum_{k=1}^{K} \frac{|\bar{w}_r^k| + |\bar{w}_r^{k+1}|}{2}\, \Delta_k \quad (\text{since } |\, \bar{w}_r^k\, | \geq |\, \bar{w}_r^{k+1}\, |) \\
\geq & \sum_{k=1}^{K} \frac{\big(|\bar{w}_r^k| + |\bar{w}_r^{k+1}|\big)}{2}\, \frac{\big(|\bar{w}_r^k| - |\bar{w}_r^{k+1}|\big)}{\lambda_{\max}(M)} \\
=& \frac{1}{2\lambda_{\max}(M)} \sum_{k=1}^{K} \Big(\, \bar{w}_r^k\, \Big)^2 - \Big(\, \bar{w}_r^{k+1}\, \Big)^2 = \frac{\bar{w}_r^2}{2\lambda_{\max}(M)}.
\end{aligned}
$$

This completes the proof. ∎

Remark 1. From the proof, we see that we can replace $\lambda_{\max}(M)$ in the denominator of the bound by $\max\limits_i M_{ii}$. We retain $\lambda_{\max}(M)$ in the bound as it will be combined with $\lambda_{\min}(M)$ to obtain the condition number of M in Theorem 2.

By the bound in Lemma 1, we obtain the next lemma that gives the deviation from optimality of the iterates during the DvPW algorithm.

Lemma 3 (Distance to v_*)**.** For each major cycle, $v_t - v_* \leq \dfrac{\sqrt{n}\,\|\, q_- \,\|_2}{\rho_{\min}(M)} \,|\, \bar{w}_r \,|$, where v_* is the optimal value of (4).

Proof. With $(\bar{z}^t, \bar{w}^t)$ denoting the value of the pair (z, w) at the beginning of the t-th major cycle, we have the following string of inequalities:

$$\begin{aligned} v_* - v_t &\geq (\,\bar{w}^t\,)^\top (\,\bar{z}^* - \bar{z}^t\,) && \text{(by the gradient inquality of the objective)} \\ &= (\,\bar{w}^t_\beta\,)^\top \bar{z}^*_\beta && \text{(since } \bar{z}^t \perp \bar{w}^t \text{ and } \bar{w}^t_\alpha = 0) \\ &\geq \bar{w}_r \sum_{i=1}^n \bar{z}^*_i = -|\bar{w}_r| \|\bar{z}^*\|_1, && \text{(by the definition of the negative } \bar{w}_r = \min_{i\in\beta} w^t_i) \end{aligned}$$

from which the desired bound of $v_t - v_*$ follows readily. $\square$

We can now combine Lemmas 2 and 3 to yield an upper bound on the suboptimality of any basic feasible solution obtained during the DvPW algorithm for solving the QP (4).

Theorem 1. Let $\kappa \triangleq \dfrac{4\lambda_{\max}(M)\,\|\, q_- \,\|_2^2}{\rho_{\min}(M)^2}$. Then $v_t - v_* \leq \dfrac{\kappa\, n}{t-1}$.

Proof. Define a sequence of deviations: $\{e_t\}$ with $e_t \triangleq v_t - v_*$. Then by Lemmas 2 and Lemma 3, we have

$$\frac{e_t - e_{t+1}}{e_t^2} = \frac{v_t - v_{t+1}}{(v_t - v_*)^2} \geq \frac{\bar{w}_r^2}{4\lambda_{\max}(M)} \, \frac{\rho_{\min}(M)^2}{n\,\|\, q_- \,\|_2^2\, \bar{w}_r^2} = \frac{1}{\kappa n}\,.$$

Since $e_t \geq e_{t+1}$, which implies $e_t^2 \geq e_t\, e_{t+1}$, hence, it follows that $\dfrac{1}{e_{t+1}} - \dfrac{1}{e_t} \geq \dfrac{1}{\kappa\, n}$, which yields $\dfrac{1}{e_t} \geq \dfrac{1}{e_1} + \dfrac{1}{\kappa\, n}(t-1) \geq \dfrac{1}{\kappa\, n}(t-1)$ from which the claimed bound of e_t follows readily. $\square$

While the DvPW algorithm is a finite algorithm on the pivots (see e.g. [6]), this finite termination is not captured by Theorem 1. In order to complete this finite-termination analysis, we need to introduce the two constants $\overline{\gamma}$ and $\underline{\delta}$ for the QP (4). In the following, we restrict our definition to a positive definite matrix M. Specifically, let

$$\delta_{\text{qp}} \triangleq \min_{\text{feasible }\alpha} \min_{i\in\alpha} \left\{ z^\alpha_i \mid z^\alpha_i \triangleq \left[-(M_{\alpha\alpha})^{-1} q_\alpha \right]_i > 0 \right\} \text{ and } \gamma_{\text{qp}} \triangleq \max_{\text{feasible }\alpha} \max_{i\in\alpha} z^\alpha_i.$$

We first restate Lemma 3 in terms of $\gamma_{\rm qp}$ when M is positive definite.

Lemma 3′. If $M \succ 0$, then for each major cycle, $v_t - v_* \leq n\,\gamma_{\rm qp}\,|\,\bar{w}_r\,|$.

Proof. This follows from $v_t - v_* \leq |\bar{w}_r|\,\|\,z^*\,\|_1$ proved in Lemma 3 and from $\|\,z^*\,\|_1 \leq n\,\gamma_{\rm qp}$. □

We can now prove the following finite termination of the DvPW algorithm when M is positive definite. A noteworthy point about this result is that the condition number of the matrix M appears in the bound, This seems to be the first time that the condition number of a matrix appears in bounding the number of pivots in a simplex-type method for solving quadratic programs.

Theorem 2. If $M \succ 0$, then for any vector q, the DvPW algorithm with the least reduced cost rule computes the unique optimal solution of the QP (4) in no more than $1 + 8\left(\dfrac{n\,\gamma_{\rm qp}}{\delta_{\rm qp}}\right)^2 \operatorname{cond}(M)$ iterations, where $\operatorname{cond}(M) \triangleq \dfrac{\lambda_{\max}(M)}{\lambda_{\min}(M)}$ is the condition number of M.

Proof. Similar to the proof of Theorem 1 and using Lemma 3′ instead, we can derive

$$\frac{e_t - e_{t+1}}{e_t^2} \geq \frac{\bar{w}_r^2}{4\,\lambda_{\max}(M)}\,\frac{1}{n^2\,\bar{w}_r^2\,\gamma_{\rm qp}^2} = \frac{1}{4\,\lambda_{\max}(M)\,n^2\,\gamma_{\rm qp}^2},$$

which implies

$$v_t - v_* \leq \frac{4\,\lambda_{\max}(M)\,n^2\,\gamma_{\rm qp}^2}{t-1}. \tag{7}$$

Define $I_1 \triangleq \operatorname{supp}(z^t)\backslash\operatorname{supp}(z^*)$ and $I_2 \triangleq \operatorname{supp}(z^*)\backslash\operatorname{supp}(z^t)$, where $\operatorname{supp}(\bullet)$ denotes the support of a vector, i.e., the index set of the nonzero components of the vector. We claim that if v^t is not optimal, then $|I_1 \cup I_2|$ is not empty, i.e. $|I_1 \cup I_2| \geq 1$. Indeed if $I_1 \cup I_2 = \emptyset$, then $\operatorname{supp}(z^t) = \operatorname{supp}(z^*)$; this would then imply that $z^t = z^*$ by the nonsingularity of the principal submatrix of M induced by $\operatorname{supp}(z^*)$. We can now establish the desired upper bound on the iteration count t by the following string of derivations:

$$\begin{aligned}
v_t - v_* &= (Mz^* + q)^\top(z^t - z^*) + \tfrac{1}{2}(z^t - z^*)^\top M(z^t - z^*) \quad \text{(Taylor expansion)}\\
&\geq \tfrac{1}{2}(z^t - z^*)^\top M(z^t - z^*) \quad \text{(by the optimality of } z^*)\\
&\geq \frac{\lambda_{\min}(M)}{2}\|z^t - z^*\|_2^2 \geq \frac{\lambda_{\min}(M)}{2}\left(\sum_{i\in I_1}(z_i^t)^2 + \sum_{i\in I_2}(z_i^*)^2\right) \geq \frac{\lambda_{\min}(M)}{2}\delta_{\rm qp}^2.
\end{aligned}$$

Combining the last inequality with (7), we have $t \leq \dfrac{8n^2\lambda_{\max}(M)\gamma_{\rm qp}^2}{\lambda_{\min}(M)\delta_{\rm qp}^2} + 1$. □

If M is a Stieltjes matrix (i.e., a symmetric Minkowski matrix), the DvPW algorithm for solving (4) essentially reduces to Chandrasekaran's algorithm [4]. It is known that in this case, the algorithm can terminate in n steps. In essence, the proof is based on the observation that for any index set α, we must have $(M_{\alpha\alpha})^{-1}M_{\alpha\beta} \leq 0$; this then implies that a z-variable once becomes basic can only increase in value, and in particular, will not become nonbasic. Thus, every pivot is the exchange of the nonbasic z_r-variable with the basic w_r-variable, and the method terminates in n steps. In the recent paper [12], it is shown that similar results can be established for submodular objectives, which include the Stieltjes quadratic function as a special case. This indicates that the iteration count for solving (4) with a Stieltjes matrix is drastically lower than the general case for an arbitrary positive definite M. Whether the bound in Theorem 2 can be improved remains a subject to be further studied.

4. Some Applications

In this section, we give two applications of the theoretical results developed in Section 3. The first application pertains to Theorem 1 and is a variant of network flow least squares problems. The second application provides a class of matrices Q to illustrate the estimation of the two key constants γ_{qp} and δ_{qp} in Theorem 2; specifically, we consider Q that is the sum of a "simple" matrix and a low-rank matrix.

4.1. **Least squares in network flow problems.** Consider a directed graph $G = (V, E)$ without self-loops, where $V = [n]$ is the set of vertices and $E \subseteq V \times V$ is the set of arcs. We assume $n \geq 2$ in this section. Let $A \in \mathbb{R}^{|V|} \times \mathbb{R}^{|E|}$ denote the vertex-arc incidence matrix of G, i.e., for all $v \in V$ and $e \in E$, $A_{ve} = 1$ if $e = (v, u)$ for some $u \in V$, -1 if $e = (u, v)$ for some $u \in V$, and 0 otherwise. We aim to solve

$$\underset{x \geq 0}{\textbf{minimize}} \ \|Ax - b\|_2^2 + c^\top x, \tag{8}$$

where $b \in \mathbb{R}^n$ and $c \in \mathbb{R}_+^{|E|}$. In this context, $M = A^\top A$ is called the *edge-Laplacian matrix* of G, in contrast to the standard Laplacian matrix defined as $L = AA^\top$. Note that since $\mathbf{1}^\top A = 0$, one can see that $\lambda_{\min}(L) = \lambda_n(L) = 0$ and unless $\mathbf{1}^\top b = 0$, the equation system $Ax = b$ has no solution. Thus, the quadratic term in the problem (8) can be interpreted as follows: given a node-vector $b \in \mathbb{R}^{|V|}$ that is "unbalanced", i.e., for which the network system $Ax = b$, $x \geq 0$ is not feasible (due perhaps to corrupted or historical data), compute a "small" correction (measured in the Euclidean norm) δb of b so that the corrected system $Ax = b + \delta b$, $x \geq 0$ is feasible. Moreover, the cost of traversing arc $e \in E$ is denoted by c_e. If $c = 0$, the problem (8) can be efficiently solved by combinatorial algorithms; see [13]. Nevertheless, With $c \neq 0$, the complexity for solving (8) does not seem to have been studied in the literature, especially when c is not in the range of $A^\top$. For any arc subset $\alpha \subseteq E$, the undirected counterpart of the subgraph induced by α is represented as G_α. A symmetric matrix Q is *reducible* if it is block diagonal and each block is a proper submatrix of Q. Matrix Q is called *irreducible* if it is not reducible. Lemma 4 states some basic results from spectral graph theory (see e.g., [2]), which are needed to estimate the iteration count of the DvPW algorithm on (8).

Lemma 4. Given a submatrix $M_{\alpha\alpha}$ of M with $\alpha \subseteq E$, the following statements hold true.

- $M_{\alpha\alpha}$ is nonsingular if and only if G_α is a forest;
- $M_{\alpha\alpha}$ is irreducible if and only if G_α is connected;
- If $M_{\alpha\alpha}$ is irreducible and nonsingular, then $\det(M_{\alpha\alpha}) = |\alpha| + 1$;
- $\lambda_{\max}(M_{\alpha\alpha}) \leq 2|\alpha|$, and if $M_{\alpha\alpha}$ is nonsingular, then $\lambda_{\min}(M_{\alpha\alpha}) \geq \dfrac{2\pi}{n^2}$.

Proof. Write $M_{\alpha\alpha} = A_{\bullet\alpha}^\top A_{\bullet\alpha}$. Since $A_{\bullet\alpha}$ is the incidence matrix of the subgraph induced by the arc set α, it suffices to prove the results for $\alpha = E$. For this reason, we drop the subscript α throughout the proof. The first inequality in the last conclusion follows from $\lambda_{\max}(M) = \lambda_{\max}(L) \leq 2|\alpha|$ by Corollary 4.14 [2]. Since M is nonsingular if and only if A has full column rank and the latter is further equivalent to G containing no (undirected) circle, the first conclusion holds true. The second conclusion is trivial. Due to the second conclusion, we can assume without loss of generality that G is a tree in the rest of the proof. Because the eigenvalue spectrum of $A^\top A$ coincides with the one of $AA^\top$, $\det(M) = \prod_{i=1}^{n-1} \lambda_i(L)$ which is exactly n times the number of spanning tree of G by the renowned Matrix-Tree Theorem; see Theorem 4.11 [2] for details. Since G itself is a tree, this implies that $\det(M) = n = |\alpha| + 1$. It remains to prove the second inequality in the last conclusion. We first observe that $\lambda_{\min}(M) = \lambda_{n-1}(M) = \lambda_{n-1}(L)$, where $\lambda_{n-1}(L)$ is called the *algebraic connectivity* of G. Since G is connected, according to Proposition 4.3 of [11], $\lambda_{n-1}(L) \geq 2(1 - \cos(\pi/n)) \geq \dfrac{2\pi}{n^2}$, where the last inequality is due to $1 - \cos(t) \geq t^2/\pi \ \forall t \in [0, \pi/2]$. This finishes the proof. □

Combining Lemma 4 and Theorem 1, one can immediately deduce Proposition 3.

Proposition 3. When applying the DvPW algorithm to (8), one has $v_t - v_* \leq \dfrac{4n^6 \|b\|_2^2}{\pi^2(t-1)}$.

Proof. By Lemma 4, $\lambda_{\max}(M) \leq 2n$ and $\rho_{\min}(M) \geq \dfrac{2\pi}{n^2}$. Now we estimate

$$\begin{aligned} \|q_-\|_2^2 &\leq \left\| \left[A^\top b + c\right]_- \right\|_2^2 \leq \left\| \left[A^\top b\right]_- \right\|_2^2 \\ &\leq \lambda_{\max}(L) \|b\|_2^2 = \lambda_{\max}(M) \|b\|_2^2 \\ &\leq 2n \|b\|_2^2, \end{aligned}$$

where the second inequality is due to $c \geq 0$. To get a shaper complexity result, we note from Remark 1 that one can use $\max\limits_{i \in E} M_{ii} = \max\limits_{i \in E} \|A_{\bullet i}\|_2^2 \leq 2$ in place of $\lambda_{\max}(M)$ in the definition of κ. The conclusion follows from Theorem 1 by plugging the bound of $\rho_{\min}(M)$ and $\|q_-\|_2^2$ in the formula of κ. $\square$

Proposition 4. Assume b and c are integer data. Then the DvPW algorithm with the least reduced cost rule computes the unique optimal solution of the QP (8) in no more than $n^4 \|b\|_2^2 / 2$ iterations.

Proof. Since the matrix $M = A^\top A$ may not be positive definite unless G is a forest, Theorem 2 is not directly applicable. For this reason, we proceed after Lemma 2 and use notations consistent with those in this lemma. Since $M_{\alpha\alpha}^{-1}$ is nonsingular, by Lemma 4, there exists a partition $\alpha = \cup_{i=1}^k \alpha_i$ such that each G_{α_i} is a connected component of G_α. Note that G_{α_i} is a tree. Since $M_{\alpha\alpha}$ is block diagonal, by definition we have $\bar{w}_r = q_r - M_{r\alpha} M_{\alpha\alpha}^{-1} q_\alpha = q_r - \sum_{i=1}^k (A_{\bullet r})^\top A_{\bullet \alpha_i} M_{\alpha_i \alpha_i}^{-1} q_{\alpha_i}$. Observe that $r \notin \alpha$, and $(A_{\bullet r})^\top A_{\bullet \alpha_i} \neq 0$ if and only if the subgraph G_{α_i} is incident with edge r in G. Hence there are at most two indices in $[k]$, say i_1 and i_2, such that $(A_{\bullet r})^\top A_{\bullet \alpha_i} \neq 0$. Consequently, $\bar{w}_r = q_r - (A_{\bullet r})^\top A_{\bullet \alpha_{i_1}} \tilde{q}^1 - (A_{\bullet r})^\top A_{\bullet \alpha_{i_2}} \tilde{q}^2$, where $\tilde{q}^j \triangleq M_{\alpha_{i_j} \alpha_{i_j}}^{-1} q_{\alpha_{i_j}}$ is an integral multiple of $1/\det(M_{\alpha_{i_j} \alpha_{i_j}})$ by Cramer's rule for $j = 1, 2$. Moreover, by our assumption $q = A^\top b + c$ is integral. This implies that $|\bar{w}_r| = \dfrac{W}{\det(M_{\alpha_{i_1} \alpha_{i_1}}) \det(M_{\alpha_{i_2} \alpha_{i_2}})}$ for some integer $W > 0$. Therefore, one can deduce from Remark 1 that

$$\begin{aligned} v_t - v_{t+1} &\geq \frac{\bar{w}_r^2}{4 \max\limits_i M_{ii}} \geq \frac{1}{4 \left(\max\limits_i M_{ii} \right) \left[\det(M_{\alpha_{i_1} \alpha_{i_1}}) \det(M_{\alpha_{i_2} \alpha_{i_2}}) \right]^2} \\ &\geq \frac{1}{8 \left[(|\alpha_{i_1}| + 1)(|\alpha_{i_2}| + 1) \right]^2} \geq \frac{2}{(|\alpha_{i_1} + 1| + |\alpha_{i_2} + 1|)^4} \geq \frac{2}{n^4}, \end{aligned}$$

where the third inequality is due to Lemma 4 and the fact that $\max_i M_{ii} \leq 2$, and the fourth inequality is due to the mean-value inequality $a_1 a_2 \leq (a_1 + a_2)^2/4$. The desired conclusion now follows from the fact that the objective of 8 is nonnegative and has a trivial upper bound $\|b\|_2^2$. □

Note that any basic solution has the form $M_{\alpha\alpha}^{-1} q_\alpha$ which is homogeneous in q for fixed M. Hence, if b and c are generic rational numbers, one can scale b and c to make them integral and get an equivalent problem for which Proposition 4 is applicable. It is important to note the significant difference between the two bounds in the above two propositions. There are two key factors that contribute to this difference; one is the integrality assumption of the vectors b and c Proposition 4, which enables us to derive a constant amount of decrease in each iteration, whereas no such constant decrease is possible in Proposition 3. Another key background result is Lemma 4 that enables the bound of $\det(M_{\alpha\alpha})$. These two properties: the integrality assumption and a bound of $\det(M_{\alpha\alpha})$, persist in the next class of problems for which the general results of the last section can be sharpened.

4.2. **A special class: $M = K\Xi + FF^\top$.** In this subsection, we consider a structured version of (4)

$$\underset{z \geq 0}{\textbf{minimize}}\ z^\top \left(K\Xi + FF^\top\right) z + q^\top z, \tag{9}$$

where $K > 0$ is an integer, $q \in \mathbb{R}^n$, Ξ is a positive definite matrix, and $F \in \mathbb{R}^{n \times r}$ is a low-rank matrix. We assume all data of (9) are rational numbers. Moreover, by properly scaling, one can further assume that the entries of Ξ, F and q are integers. We are interested in the case where the determinant of all $\Xi_{\alpha\alpha}$, denoted by $\det(\Xi_{\alpha\alpha})$, and the rank of F are small. For instance, in factor models of portfolio risk analysis, $\Xi = I$ captures the idiosyncratic variance of the portfolio in question, and the rank r represents the number of economic factors and is usually a small number [1, 3, 17]. The case where a certain index of the stock market serves as the solely economic factor ($r = 1$) is studied in [26]. Based on Theorem 2, the next proposition offers an estimate for the number of iterations incurred during the implementation of the DvPW algorithm.

Proposition 5. Assume that K, Ξ, F and, q are integer data. Define $D \triangleq \max_{\alpha \subseteq [n]} \det(\Xi_{\alpha\alpha})$. The DvPW algorithm with the least reduced cost rule computes the unique optimal solution of the QP (9) in no more than

$$1 + \frac{8n^2 D^{2r+2} \left[\, K\lambda_{\min}(\Xi) + \lambda_{\max}(F)^2 \,\right]^{2r} \left[\, K\lambda_{\max}(\Xi) + \lambda_{\max}(F)^2 \,\right] \|q\|_2^2}{K\lambda_{\min}(\Xi)^{2r+3}} \tag{10}$$

iterations. In particular, with $\Xi = I$, the iteration bound reduces to $1 + \dfrac{8n^2\left[K+\lambda_{\max}(F)^2\right]^{2r+1}\|q\|_2^2}{K}$.

Proof. We first estimate δ. By the Morrison-Woodbury formula, $M_{\alpha\alpha}^{-1} = \dfrac{1}{K}\Xi_{\alpha\alpha}^{-1} - \dfrac{1}{K}\Xi_{\alpha\alpha}^{-1}F_{\alpha\bullet}R^{-1}F_{\alpha\bullet}^{\top}\Xi_{\alpha\alpha}^{-1}$, where $R = KI + F_{\alpha\bullet}^{\top}\Xi_{\alpha\alpha}^{-1}F_{\alpha\bullet} \in \mathbb{R}^{r\times r}$. Moreover, one has $\Xi_{\alpha\alpha}^{-1} = \dfrac{1}{\det(\Xi_{\alpha\alpha})}\widetilde{\Xi}$, where $\widetilde{\Xi}$ is a certain integer matrix. As a result, $M_{\alpha\alpha}^{-1} = \dfrac{1}{K\det(\Xi_{\alpha\alpha})}\widetilde{\Xi} - \dfrac{1}{K\det(\Xi_{\alpha\alpha})}\widetilde{\Xi}F_{\alpha\bullet}(\det(\Xi_{\alpha\alpha})R)^{-1}F_{\alpha\bullet}^{\top}\widetilde{\Xi}$. Observe that $\det(\Xi_{\alpha\alpha})R$ is an integral matrix. Since q is an integer vector, one can deduce from Cramer's rule that any component of the solution z is a nonnegative integral multiple of $\dfrac{1}{K\det(\Xi_{\alpha\alpha})\det(\det(\Xi_{\alpha\alpha})R)}$. Thus,

$$
\begin{aligned}
\frac{1}{\delta} &\le K\det(\Xi_{\alpha\alpha})\det(\det(\Xi_{\alpha\alpha})R)\\
&\le K\det(\Xi_{\alpha\alpha})^{r+1}\lambda_{\max}(R)^r = K\det(\Xi_{\alpha\alpha})^{r+1}\left[K+\lambda_{\max}\left(F_{\alpha\bullet}^{\top}\Xi_{\alpha\alpha}^{-1}F_{\alpha\bullet}\right)\right]^r\\
&\le K\det(\Xi_{\alpha\alpha})^{r+1}\left[K+\lambda_{\max}(F_{\alpha\bullet}^{\top}F_{\alpha\bullet})/\lambda_{\min}(\Xi_{\alpha\alpha})\right]^r\\
&\le KD^{r+1}\left[K+\lambda_{\max}(F)^2/\lambda_{\min}(\Xi)\right]^r.
\end{aligned}
$$

Furthermore, we have $\lambda_{\max}(M) = \lambda_{\max}(K\Xi + FF^{\top}) \le K\lambda_{\max}(\Xi) + \lambda_{\max}(FF^{\top}) = K\lambda_{\max}(\Xi) + \lambda_{\max}(F)^2$, $\lambda_{\min}(M) \ge K\lambda_{\min}(\Xi)$, and $\gamma \le \max_{\alpha}\left\|M_{\alpha\alpha}^{-1}q_\alpha\right\|_2 \le \dfrac{\|q\|_2}{K\lambda_{\min}(\Xi)}$. Plugging all these bounds in Theorem 2, we can easily deduce the claimed bound (10). □

5. Conclusions

In this note, we have studied the iteration count of Dantzig's Simplex Methods for solving linear and convex quadratic programs. The complexity bounds rely on the condition number of the matrix in question or/and the magnitude of the basic solutions. These results supplement those in the existing literature and are particularly useful when some key constants can be estimated under certain circumstances. Whether the iteration bounds can be improved is an open question to be investigated in the future.

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