From the uncertainty set to the solution and back: the two stage case

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Abstract

Robust optimization approaches compute solutions resilient to data uncertainty, represented by a given uncertainty set. Instead, the problem of computing the largest uncertainty set that a given solution can support was, so far, quite neglected and the only results refer to the single stage framework. For that setting, it was proved that this problem can be solved in polynomial time, when the uncertainty is modeled using the cardinality constrained set. Here we study what happens if we consider a two stage framework.

keywords: robust optimization, cardinality constrained uncertainty set, two stage

1 Introduction

?⟨sec:intro⟩? Data uncertainty in the parameters of mathematical programming formulations has been recognized as a major issue in computing solutions of problems coming from real-life applications. Therefore, several methods have been studied in the literature to compute solutions that are resilient to data uncertainty. The most popular ones are simulation [3, 28], stochastic programming [10, 14] and robust optimization [4, 15]. Applications of these techniques to real-life problems are very common, ranging from healthcare [1] to personnel scheduling [24] and network design [19]. They differ from one another on the statistical information that is required to derive the corresponding models and on the computational effort needed to solve them. In the present paper we focus on robust optimization methods. These approaches assume that a limited statistical information can be exploited, that is, the probability distribution associated with the uncertain parameters is unknown, but it is possible to partition the values that the uncertain parameters can take (*realizations*) into realizations of interest (*the uncertainty set*) and realizations that are unlikely to occur. The underlying assumption is that all the realizations in the uncertainty set have the same probability to occur, whereas the others can be neglected. The scope is to compute the best solution among the ones that are feasible for any realization of the uncertainty in the given uncertainty set. The uncertainty set must be detailed enough to represent the variability of the parameters, but simple enough to be tractable from a computational point of view. A popular way to represent uncertainty sets is to use ellipsoids or polytopes: some examples can be found in [7, 8, 12, 24].

In mixed-integer programming, the problem without uncertainty is commonly referred to as the *deterministic* optimization (DET) problem and it can be mathematically formalized, in a general way, as follows.

$$
\begin{aligned}
fDET \quad & \min \mathbf{c}^T \mathbf{x} \\
& \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& x_j \in \mathbb{Z} \quad j \in N
\end{aligned}
$$

We denote by I and J the indices of constraints and variables and by $N \subseteq J$ the variables that are required to take integer values.

The problem that takes data uncertainty into account, instead, is known as the *robust* problem and its mathematical model depends on the flexibility that the robust solution must have. If one has to use the same solution $\mathbf{x}$ for all the realizations of the uncertainty, we speak of *single stage* approaches. If, instead, one can (at least partially) change the computed solution according to the actual realization, we speak of *multi stage* or *adjustable* models. Single stage approaches lead to models that are computationally easier to solve, but multi stage frameworks correspond to less conservative (cheaper) solutions. We consider here a two stage problem, where the $\mathbf{x}$ variables are partitioned into first stage variables $\mathbf{x}^1$ and second stage variables $\mathbf{x}^2$. The value of the first stage variables must be the same for any realization of the uncertainty (*here and now* decisions), while the second stage ones can be changed, depending on the actual realization (*wait and see* decisions). If $\mathbf{x}^2$ can be changed completely when the realization changes, we speak of *unrestricted recourse*. If, instead, the modifications that can be done are limited, we speak of *restricted policies*. The unrestricted recourse provides the maximum flexibility at the expenses of solving a computationally hard problem. The scope of the restrictions in the policy is to guarantee some flexibility, while keeping the robust problem computationally tractable [5]. A comparison of several policies that can be used for the second stage variables in network design problems can be found in [23]. We suppose that the uncertainty only affects the coefficients of the first stage variables and/or the right-hand sides of the constraints, whereas the coefficients of the second stage variables are not uncertain (*fixed recourse*).

Denote by U the uncertainty set and by p the policy that must be used in computing the value of the second stage variables. Let $[\mathbf{A}^1, \mathbf{A}^2]$ be the partition of the constraint matrix $\mathbf{A}$, where $\mathbf{A}^1$ represents the (uncertain) coefficients of the first stage variables and $\mathbf{A}^2$ the (not uncertain) ones of the second stage variables. Similarly, objective coefficients $\mathbf{c}$ are partitioned into $\mathbf{c}^1$ and $\mathbf{c}^2$. A two stage robust problem (2ROB) can be formulated as follows, where the value of the second stage variables depends on the policy p and the realization u. J^1 denotes the set of the indices of the first stage variables and J^2 refers to the second stage ones. Set N is partitioned into N^1 and N^2 accordingly. The goal is to find the first stage solution $\mathbf{x}^1$ that can be completed by a suitable second stage solution under policy p for any $u \in U$ and that has the most favorable worst case.

$$
\begin{array}{lll}
f2ROB & \min q & \\
& q \geq \mathbf{c}^{1T}\mathbf{x}^1 + \max\limits_{u \in U} \mathbf{c}^{2T}\mathbf{x}^2(u,p) & \\
& \mathbf{A}^{1u}\mathbf{x}^1 + \mathbf{A}^2\mathbf{x}^2(u,p) \geq \mathbf{b}^u & u \in U \\
& x_j^1 \in \mathbb{Z} & j \in N^1 \\
& x_j^2(u,p) \in \mathbb{Z} & j \in N^2, u \in U
\end{array}
$$

As already noted in [16], in many practical applications the validity of the estimations used to define the uncertainty set and to compute a robust solution, may become unreliable long before the solution lifetime has reached its end, leading to the necessity of reassessing, under more accurate and up-to-date values, the already computed solutions. This is also the case when unexpected technological innovations or unpredictable external events have a disruptive effect on the considered system.

The aim of this paper is to study what happens to a given first stage solution $\bar{\mathbf{x}}^1$, when the predictions that were used to define the uncertainty set U are not longer reliable and one has to determine which is the largest uncertainty set that $\bar{\mathbf{x}}^1$ can handle, under the modified conditions. In [16] the single stage case was considered and it is proved that, computing the largest uncertainty set that a solution can handle, is a problem that can be solved in polynomial time, when the uncertainty set is defined using the *cardinality constrained* model [8]. To our knowledge, no other paper in the literature considers the problem of computing the largest uncertainty set that a solution can support. Here we focus on the two stage framework and address what happens in this case, showing that the problem can be polynomially solvable or not, depending on the recourse policy. We will use network design problems to provide examples.

In §2 we formalize the problems of computing the largest uncertainty set in a two stage framework, under the cardinality constrained uncertainty model. In §3 we define the *restricted* problem and show how it is related to the considered problems. In §4 we study the complexity of the restricted problem, considering both the unrestricted recourse and the *affinely adjustable* one [5]. In §5 conclusions are given.

2 The problem

⟨sec:prob⟩ The cardinality constrained model assumes that each uncertain parameter corresponds to a symmetric bounded random variable [6]. The probability distribution is not given, but a nominal value and a maximum deviation from the nominal value are known. Given uncertain parameter a^1_{ij}, $i \in I, j \in J^1$, let $\bar{a}^1_{ij}$ and δ_{ij} be its nominal value and maximum deviation. The actual value of parameter a^1_{ij} lies in the interval $[\bar{a}^1_{ij} - \delta_{ij}, \bar{a}^1_{ij} + \delta_{ij}]$. Similarly, uncertain parameter b_i, $i \in I$ takes values in $[\bar{b}_i - \sigma_i, \bar{b}_i + \sigma_i]$, where $\bar{b}_i$ is its nominal value and σ_i the maximum deviation. It is supposed that at most Γ parameters can deviate from the nominal value at the same time. Let Γ be an integer number, we denote by $\mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\sigma})$ the uncertainty set obtained by applying the cardinality constrained framework.

?⟨def:ui⟩? **Definition 2.1.** *Denote by z^+_{ij} (z^-_{ij}) the positive (negative) percentage deviation of parameter a^1_{ij} from its nominal value and let y^+_i (y^-_i) be variables that play the same role for b_i. The uncertainty set $\mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\sigma})$ can be defined as below.*

$$
\mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\sigma}) = \Big\{ \begin{array}{ll}
a^1_{ij} = \bar{a}_{ij} + \delta_{ij}(z^+_{ij} - z^-_{ij}) & i \in I, j \in J^1 \\
b_i = \bar{b}_i + \sigma_i(y^+_i - y^-_i) & i \in I \\
\sum_{i \in I} \sum_{j \in J^1} (z^+_{ij} + z^-_{ij}) + \sum_{i \in I} (y^+_i + y^-_i) \leq \Gamma & \\
0 \leq z^+_{ij} \leq 1, 0 \leq z^-_{ij} \leq 1 & i \in I, j \in J^1 \\
0 \leq y^+_i \leq 1, 0 \leq y^-_i \leq 1 & i \in I
\end{array} \Big\}
$$

The uncertainty set depends on two parameters: the level of robustness to be ensured, represented by Γ; the level of uncertainty to be handled, represented by $\boldsymbol{\delta}$ and $\boldsymbol{\sigma}$. When the values of these parameters increase, the uncertainty set becomes larger. We want to answer to the following questions.

Q1 If new nominal and/or deviation values replace the old ones, which is the maximum level of robustness Γ that solution $\bar{\mathbf{x}}^1$ can guarantee?

Q2 If new nominal and/or deviation values replace the old ones, which is the maximum level of robustness Γ that solution $\bar{\mathbf{x}}^1$ can guarantee, without exceeding a given budget for the second stage costs?

Q3 If we want to ensure a given level of robustness Γ, which is the maximum percentage increase λ in the uncertainty level (deviation values) that we can accept?

Q4 If we want to ensure a given level of robustness Γ, which is the maximum percentage increase λ in the uncertainty level (deviation values $\boldsymbol{\delta}$ and $\boldsymbol{\sigma}$) that we can accept, without exceeding a given budget for the second stage costs?

Q5 Given new nominal and/or deviation values and a robustness level Γ to ensure, which is the minimum budget that we need for the second stage solution?

Questions Q1 and Q3 are related to the maximum robustness that first stage solution $\bar{\mathbf{x}}^1$ can ensure and they correspond to the two stage maximum robustness (2MR) problem below.

⟨pr:2mr⟩ **Problem 2.1.** *Given a solution $\bar{\mathbf{x}}^1$, a budget B (possibly infinite), values $\bar{a}^1_{ij}$, δ_{ij} for $i \in I$, $j \in J^1$ and $\bar{b}_i$, σ_i for $i \in I$, the 2MR problem consists of determining the largest Γ for which $\bar{\mathbf{x}}^1$ can be completed by a suitable second stage solution $\mathbf{x}^2(u, p)$ respecting the policy p, for any realization of the uncertainty u in $\mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\sigma})$, without exceeding B.*

To avoid trivial cases, we suppose that $\bar{\mathbf{x}}^1$ cannot support the realization where $|I| \times (1 + |J^1|)$ parameters (that is, all the uncertain parameters) change simultaneously. Mathematically, the problem can be formalized as follows, where we look for the minimum Γ such that there exists a realization $\eta \in \mathcal{U}(\Gamma + 1, \boldsymbol{\delta}, \boldsymbol{\sigma})$ with the property that: either (i) there is a constraint $i \in I$ that cannot be satisfied by any second stage solution for realization η; or (ii) any solution that is feasible for all constraints $i \in I$ and realizations $u \in \mathcal{U}(\Gamma + 1, \boldsymbol{\delta}, \boldsymbol{\sigma})$, exceeds budget B for realization η.

$$
\begin{aligned}
f2MR \quad & \min \Gamma \\
& \text{such that } \exists \eta \in \mathcal{U}(\Gamma + 1, \boldsymbol{\delta}, \boldsymbol{\sigma}) \text{ with the property that} \\
& \text{either} \\
& \quad \exists i \in I \text{ such that} \\
& \quad \max_{\mathbf{x}^2} \sum_{j \in J^1} a^{1\eta}_{ij} \bar{x}^1_j + \sum_{j \in J^2} a^2_{ij} x^2_j(\eta, p) < b^\eta_i \\
& \text{or} \\
& \quad \mathbf{c}^{2T} \mathbf{x}^2(\eta, p) > B \text{ for all } \mathbf{x}^2 \text{ such that} \\
& \quad \mathbf{A}^{1u} \bar{\mathbf{x}}^1 + \mathbf{A}^2 \mathbf{x}^2(u, p) \leq \mathbf{b}^u \qquad u \in \mathcal{U}(\Gamma + 1, \boldsymbol{\delta}, \boldsymbol{\sigma})
\end{aligned}
$$

Let us now address questions Q2 and Q4, which are formalized into the two stage maximum uncertainty (2MU) problem that follows.

⟨pr:2mu⟩ **Problem 2.2.** *Given a solution $\bar{\mathbf{x}}^1$, a budget B (possibly infinite), and values Γ, $\bar{a}^1_{ij}$, δ_{ij} for $i \in I$, $j \in J^1$ and $\bar{b}_i$, σ_i for $i \in I$, the 2MU problem consists of determining the largest percentage increase λ in the deviation values for which $\bar{\mathbf{x}}^1$ can still be completed by a suitable second stage solution $\mathbf{x}^2(u, p)$ respecting the policy p, for any realization of the uncertainty $u \in \mathcal{U}(\Gamma, (1 + \lambda)\boldsymbol{\delta}, (1 + \lambda)\boldsymbol{\sigma})$, without exceeding B.*

The 2MU problem can be formulated as below and, to avoid trivial cases, we assume that the problem is feasible when $\lambda = 0$.

$$f2MU \quad \max \lambda$$

$$
\begin{array}{ll}
B \leq \mathbf{c}^{2T}\mathbf{x}^2(u,p) & u \in \mathcal{U}(\Gamma,(1+\lambda)\boldsymbol{\delta},(1+\lambda)\boldsymbol{\sigma}) \\
\mathbf{A}^{1u}\bar{\mathbf{x}}^1 + \mathbf{A}^2\mathbf{x}^2(u,p) \geq \mathbf{b}^u & u \in \mathcal{U}(\Gamma,(1+\lambda)\boldsymbol{\delta},(1+\lambda)\boldsymbol{\sigma}) \\
\lambda \geq 0 & \\
x_j^2(u,p) \in \mathbb{Z} & j \in N^2, u \in \mathcal{U}(\Gamma,(1+\lambda)\boldsymbol{\delta},(1+\lambda)\boldsymbol{\sigma})
\end{array}
$$

Consider now question Q4, which correspond to the minimum budget (2MB) problem below.

⟨pr:2mb⟩ **Problem 2.3.** *Given a solution $\bar{\mathbf{x}}^1$, values Γ, $\bar{a}_{ij}^1$, δ_{ij} for $i \in I$, $j \in J^1$ and $\bar{b}_i$, σ_i for $i \in I$, the 2MB problem consists of determining the minimum value B, if any, for which $\bar{\mathbf{x}}^1$ can still be completed by a suitable second stage solution $\mathbf{x}^2(u,p)$ respecting the policy p, for any realization of the uncertainty $u \in \mathcal{U}(\Gamma,\boldsymbol{\delta},\boldsymbol{\sigma})$.*

This problem can be expressed by the formulation that follows. To avoid trivial cases, we assume that f2MU is feasible.

$$
\begin{array}{lll}
f2MB & \min B & \\
& B \leq \mathbf{c}^{2T}\mathbf{x}^2(u,p) & \mathbf{b} \in \mathcal{U}(\Gamma,\boldsymbol{\delta}) \\
& \mathbf{A}^{1u}\bar{\mathbf{x}}^1 + \mathbf{A}^2\mathbf{x}^2(u,p) \geq \mathbf{b}^u & u \in \mathcal{U}(\Gamma,\boldsymbol{\delta},\boldsymbol{\sigma}) \\
& x_j^2(u,p) \in \mathbb{Z} & j \in N^2, u \in \mathcal{U}(\Gamma,\boldsymbol{\delta},\boldsymbol{\sigma})
\end{array}
$$

3 The restricted problem

⟨sec:ristretto⟩ We now show that all the problems defined above are related to a unique problem, that we call the *restricted* (RES) problem.

?⟨pr:res⟩? **Problem 3.1.** *Let $\bar{\mathbf{x}}^1$ be given values for the first stage variables, the RES problem consists of solving the 2ROBC problem with $\mathbf{x}^1$ fixed to $\bar{\mathbf{x}}^1$.*

Mathematically, the restricted problem can be formulated as follows

$$
\begin{array}{lll}
fRES & \min q & \\
& q \geq \max\limits_{u \in \mathcal{U}(\Gamma,\boldsymbol{\delta},\boldsymbol{\sigma})} \mathbf{c}^{2T}\mathbf{x}^2(u,p) & \\
& \mathbf{A}^{1u}\bar{\mathbf{x}}^1 + \mathbf{A}^2\mathbf{x}^2(u,p) \geq \mathbf{b}^u & u \in \mathcal{U}(\Gamma,\boldsymbol{\delta},\boldsymbol{\sigma}) \\
& x_j^2(u,p) \in \mathbb{Z} & j \in N^2, u \in \mathcal{U}(\Gamma,\boldsymbol{\delta},\boldsymbol{\sigma})
\end{array}
$$

and it is easy to see that it coincides with the 2MB problem (Problem 2.3). The relation between the 2MR problem (Problem 2.1) and the RES problem is summarized in the result below.

Theorem 3.1. *The 2MR problem is polynomially solvable if and only if the RES problem is polynomially solvable and it can be solved solving at most $\log_2(|I| \times (1+|J^1|))$ instances of the RES problem.*

Proof. If $\mathbf{x}^2$ respects the budget, the pair $(\Gamma,\mathbf{x}^2)$ is feasible for the 2MR problem if and only if $\mathbf{x}^2$ is feasible for the RES problem with the given Γ. By the equivalence between optimization and separation [13], the 2MR problem is polynomially solvable if and only if the RES problem is polynomially solvable. The 2MR problem can be solved by performing a binary search on Γ, where, at each step, a RES problem is solved. This search has complexity $\log_2(|I| \times (1+|J^1|))$. Since the solution computed by solving the RES problem, if any, is the cheapest possible, if the optimal solution of the RES problem does not respect the finite budget B (if any) no solution within the budget exists for the considered Γ value. □

A similar argument can be used for problem 2MU (Problem 2.2).

Theorem 3.2. *The 2MU problem is polynomially solvable if and only if the RES problem is polynomially solvable.*

Proof. If $\mathbf{x}^2$ respects the budget, the pair $(\lambda,\mathbf{x}^2)$ is feasible for the 2MU problem if and only if $\mathbf{x}^2$ is feasible for the RES problem with deviations $(1+\lambda)\boldsymbol{\delta}$ and $(1+\lambda)\boldsymbol{\sigma}$. By the equivalence between optimization and separation [13], the 2MU problem is polynomial if and only if the RES problem is polynomially solvable. □

In the rest of the paper we investigate the complexity of the RES problem and, hence, of answering the questions in §2, depending on the chosen flexibility policy p.

4 Complexity results

⟨sec:complessita⟩ The RES problem include, as special case, the DET problem with fixed $\mathbf{x}^1$, which corresponds to situations where either both $\boldsymbol{\delta} = \mathbf{0}$ and $\boldsymbol{\sigma} = \mathbf{0}$ or $\Gamma = 0$. If the DET problem with fixed $\mathbf{x}^1$ is already NP-hard, then the RES problem is NP-hard as well, independently of the policy p. Therefore, we assume in what follows that the DET problem is polynomially solvable for fixed $\mathbf{x}^1$ values. Note that this setting is quite common in the applications, where the decision process is usually decomposed into (more) difficult decisions to be made at the strategic level (first stage variables) and somehow easier decisions to be made at the operational level (second stage variables) [24]. Below, we investigate the complexity of the RES problem, depending on the policy p. We consider two policies: the unrestricted recourse and the affinely adjustable one [5].

Note that we did not make any assumptions on the first stage variables and, hence, the polynomiality of the RES problem does not directly imply the polynomiality of the 2ROB problem. On the other hand, if the RES problem is NP-hard, the 2ROB problem is also NP-hard by the equivalence between optimization and separation [13]. We will present examples where the RES problem is polynomial, but the 2ROB problem is not. To keep the presentation simple, without loss of generality we assume that no costs are associated with the second stage variables and, hence, the RES problem becomes a feasibility problem, where one has to check if there exist solutions $\mathbf{x}^2(u,p)$ for $u \in \mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\sigma})$ satisfying the constraints below.

$$\mathbf{A}^{1u}\bar{\mathbf{x}}^1 + \mathbf{A}^2\mathbf{x}^2(u,p) \geq \mathbf{b}^u \quad u \in \mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\sigma})$$

4.1 The unrestricted recourse policy

?⟨sec:ur⟩? The unrestricted recourse policy assumes that the values of the second stage variables can be arbitrarily changed depending on the current value of the uncertain parameters. This ensures the maximum possible flexibility, which means the minimum costs and the minimum conservatism, but it has the drawback that the corresponding problems are generally difficult to solve. This is true also for the RES problem, even when the uncertainty affects only the right-hand sides and, to show it, we will refer to network design applications.

Let us consider two network design problems, the Capacitated Edge Activation (CEA) problem [20, 21] and the Network Loading (NL) problem [2, 17]. The CEA problem is the following.

⟨pr:cea⟩ **Problem 4.1.** *Given a capacitated graph $G(V,E)$ (network), where the capacity of each edge becomes available only if the edge is activated, and a set of demands between node pairs (commodities), to be routed on the network, the CEA problem consists of selecting a minimum cost set of edges to be activated, to ensure a feasible routing of the commodities.*

The NL problem can be defined as follows.

⟨pr:nl⟩ **Problem 4.2.** *Given an uncapacitated graph $G(V,E)$ (network) and a set of demands between node pairs (commodities) to be routed on the network, the NL corresponds to determining minimum cost integer capacities to be installed on the edges to route the commodities.*

The main difference between the CEA problem (Problem 4.1) and the NL problem (Problem 4.2) is that in the former the capacities are given, whereas in the latter they must be computed. Assume now that the demands are uncertain and that the uncertainty set is modeled using the cardinality constrained approach [22, 23].

When we solve a network design problem using a two stage framework, the decisions on the capacities (both to install and to activate) are regarded as first stage variables, as they correspond to strategic choices, whereas the actual routing of the demands corresponds to second stage variables because, in practical applications, the related decisions are adopted in a second time, at the operational level. The unrestricted recourse policy is known as *dynamic* routing, the single stage approach is called *static* routing and there exist other policies, for example the *affine* routing [27], corresponding to the affinely adjustable policy in [5]. The problem of checking if given capacities (either activated or installed) can support all the realizations of the demands in the uncertainty set by a given routing policy, is known as the *separation* problem and it corresponds to the RES problem.

The separation problem for the robust CEA and the robust NL problem with dynamic routing is known to be NP-hard by the results in [25]. Hence, the same holds for the general RES problem.

Theorem 4.1. *The RES problem with unrestricted recourse and right-hand side uncertainty is NP-hard.*

Proof. The problem of checking if given capacities can support uncertain demands by dynamic routing under the cardinality constrained uncertainty model is NP-hard [25]. □

Also note that the separation problem for the network design problems mentioned above is not polynomial also when the cardinality constrained uncertainty set is replaced by the so-called *hose* polyhedron [12], by the results in [11, 18]. This also confirms that, in general, the 2ROB problem with unrestricted recourse is difficult, even when the deterministic problem is polynomial, the uncertainty set is computationally tractable and the uncertainty only affects the right-hand sides [26].

4.2 Adjustable policies

?⟨sec:adj⟩? An adjustable policy assumes that the value of the second stage variables can be changed within some restriction. Typically, it is assumed that they change according to some function of the uncertain parameters. We discuss the most popular of such functions, the affinely adjustable (*aff*) one [5]. This policy was introduced to ensure some flexibility, while guaranteeing, at the same time, that the 2ROB problem remains computationally tractable. The affinely adjustable policy forces the values of the second stage variables to be affine functions of the uncertainty. This means that the second stage variables must follow the rule below, which limits $\mathbf{x}^2$ to be an affine function of $\mathbf{A}^1$ and $\mathbf{b}$.

$$x_j^2(u, aff) = w_j + \sum_{h \in I} \sum_{h \in J^1} v_j^{hk} a_{hk}^{1u} + \sum_{h \in I} t_j^h b_h^u \quad j \in J \tag{1}$$ eq:adj

In this case, the RES problem can be solved in polynomial time.

Theorem 4.2. *The RES problem is polynomially solvable under the affinely adjustable policy.*

Proof. Using equations (1) in the RES problem, we have that $\bar{\mathbf{x}}^1$ can be completed by suitable second stage values if the system below is feasible.

$$\sum_{j \in J^1} a_{ij}^{1u} \bar{x}_j^1 + \sum_{j \in J^2} a_{ij}^2 (w_j + \sum_{h \in I} \sum_{k \in J^1} v_j^{hk} a_{hk}^{1u} + \sum_{h \in I} t_j^h b_h^u) \geq b_i^u \quad i \in I, u \in \mathcal{U}(\Gamma, \boldsymbol{\delta}, \boldsymbol{\gamma})$$

Recall that $a^{1u}_{ij} = \bar{a}^1_{ij} + \delta_{ij}(z^+_{ij} - z^-_{ij})$ and $b^u_i = \bar{b}_i + \sigma_i(y^+_i - y^-_i)$. The above system is feasible if the optimization problem below, whose variables are $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{t}$, has optimal value $\alpha \leq 0$.

$$
\begin{aligned}
&\min \alpha \\
&\alpha + \sum_{j\in J^2} a^2_{ij}(w_j + \sum_{h\in I}\sum_{k\in J^1} \bar{a}^1_{hk} v^{hk}_j) + \\
&+ \min_{\mathbf{z}^+,\mathbf{z}^-,\mathbf{y}^+,\mathbf{y}^-} \Bigg\{ \sum_{j\in J^1} \delta_{ij}\bar{x}^1_j (z^+_{ij} - z^-_{ij}) + \sum_{j\in J^2}\sum_{h\in I}\sum_{k\in J^1} a^2_{ij}\delta_{hk} v^{hk}_j (z^+_{hk} - z^-_{hk}) \\
&+ \sum_{j\in J^2}\sum_{h\in I} t^h_j \sigma_h (y^+_h - y^-_h) - \sigma_i(y^+_i - y^-_i) \Bigg\} \geq \bar{b}_i - \sum_{j\in J^1} \bar{a}^1_{ij} x^1_j \qquad i \in I \\
&\mathbf{t} \in \mathbb{R}^{|I|\times|J^1|}, \mathbf{w} \in \mathbb{R}^{|J^1|}, \mathbf{v} \in \mathbb{R}^{|I|\times|J^1|\times|J^2|}
\end{aligned}
$$

The inner min assumes that $\mathbf{t}, \mathbf{v}$ and $\mathbf{w}$ are given and it is a linear programming problem in the variables $\mathbf{z}^+, \mathbf{z}^-, \mathbf{y}^+, \mathbf{y}^-$. We report below the primal and the dual problem for constraint i and fixed $\bar{\mathbf{w}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{t}}$.

$$
\begin{aligned}
P_i(\bar{\mathbf{w}}, \bar{\mathbf{t}}, \bar{\mathbf{v}}) \quad & \min \sum_{j\in J^1} \delta_{ij}\bar{x}^1_j (z^{+i}_{ij} - z^{-i}_{ij}) \\
& + \sum_{j\in J^2}\sum_{h\in I}\sum_{k\in J^1} a^2_{ij}\delta_{hk}\bar{v}^{hk}_j (z^{+i}_{hk} - z^{-i}_{hk}) \\
& + \sum_{j\in J^2}\sum_{h\in I} \bar{t}^h_j \sigma_h (y^{+i}_h - y^{-i}_h) - \sigma_i (y^{+i}_i - y^{-i}_i) \\
(\gamma^i) \quad & \sum_{i\in I} (y^{+i}_i + y^{-i}_i) + \sum_{i\in I}\sum_{j\in J^1} (z^{+i}_i + z^{-i}_i) \leq \Gamma \\
(\rho^{+i}_h) \quad & 0 \leq y^{+i}_h \leq 1 \qquad h \in I \\
(\rho^{-i}_h) \quad & 0 \leq y^{-i}_h \leq 1 \qquad h \in I \\
(\beta^{+i}_{hj}) \quad & 0 \leq z^{+i}_{hk} \leq 1 \qquad h \in I, k \in J^1 \\
(\beta^{-i}_{hj}) \quad & 0 \leq z^{-i}_{hk} \leq 1 \qquad h \in I, k \in J^1
\end{aligned}
$$

$$
\begin{aligned}
D_i(\bar{\mathbf{w}}, \bar{\mathbf{t}}, \bar{\mathbf{v}}) \quad & \max -\Gamma\gamma^i - \sum_{h\in I} (\rho^{+i}_h + \rho^{-i}_h) - \sum_{h\in I}\sum_{j\in j^1} (\beta^{+i}_{hj} + \beta^{-i}_{hj}) \\
(y^{+i}_h) \quad & -\gamma^i - \rho^+_h \leq \sum_{j\in J^2} \bar{t}^h_j \sigma_h \qquad h \in I \setminus \{i\} \\
(y^{-i}_h) \quad & -\gamma^i - \rho^-_h \leq -\sum_{j\in J^2} \bar{t}^h_j \sigma_h \qquad h \in I \setminus \{i\}
\end{aligned}
$$

$$
\begin{aligned}
&(y_i^{+i}) && -\gamma^i - \rho_i^+ \le \sum_{j\in J^2} \bar{t}_j^i \sigma_i + \sigma_i && \\
&(y_i^{-i}) && -\gamma^i - \rho_i^- \le -\sum_{j\in J^2} \bar{t}_j^i \sigma_i - \sigma_i && \\
&(z_{hk}^{+i}) && -\gamma^i - \beta_{hk}^+ \le \delta_{hk}\bar{x}_k^1 + \sum_{j\in J^2} a_{ij}^2 \delta_{hk} \bar{v}_j^{hk} && h\in I, k\in J^1 \\
&(z_{hk}^{-i}) && -\gamma^i - \beta_{hk}^+ \le -\delta_{hk}\bar{x}_k^1 - \sum_{j\in J^2} a_{ij}^2 \delta_{hk} \bar{v}_j^{hk} && h\in I, k\in J^1 \\
& && \gamma^i \ge 0, \boldsymbol{\beta}^{+i}, \boldsymbol{\beta}^{-i} \le \mathbf{0}, \boldsymbol{\rho}^{+i}, \boldsymbol{\rho}^{-i} \ge \mathbf{0} &&
\end{aligned}
$$

Replacing the inner min by its dual and noting that the max is redundant, we get the formulation below.

$$
\begin{aligned}
&\min \alpha && \\
&\alpha + \sum_{j\in J^2} a_{ij}^2 (w_j + \sum_{h\in I}\sum_{k\in J^1} \bar{a}_{hk}^1 v_j^{hk}) + && \\
&-\Gamma\gamma^i - \sum_{h\in I}(\rho_h^{+i} + \rho_h^{-i}) - \sum_{h\in I}\sum_{j\in j^1}(\beta_{hj}^{+i} + \beta_{hj}^{-i}) \ge \bar{b}_i - \sum_{j\in J^1} \bar{a}_{ij}^1 x_j^1 && i\in I \\
&-\gamma^i - \rho_h^+ \le \sum_{j\in J^2} t_j^h \sigma_h && h\in I\setminus\{i\}, i\in I \\
&-\gamma^i - \rho_h^- \le -\sum_{j\in J^2} t_j^h \sigma_h && h\in I\setminus\{i\}, i\in I \\
&-\gamma^i - \rho_i^+ \le \sum_{j\in J^2} t_j^i \sigma_i + \sigma_i && i\in I \\
&-\gamma^i - \rho_i^- \le -\sum_{j\in J^2} t_j^i \sigma_i - \sigma_i && i\in I \\
&-\gamma^i - \beta_{hk}^+ \le \delta_{hk}\bar{x}_k^1 + \sum_{j\in J^2} a_{ij}^2 \delta_{hk} v_j^{hk} && h\in I, k\in J^1, i\in I \\
&-\gamma^i - \beta_{hk}^+ \le -\delta_{hk}\bar{x}_k^1 - \sum_{j\in J^2} a_{ij}^2 \delta_{hk} v_j^{hk} && h\in I, k\in J^1, i\in I \\
&\mathbf{t}\in\mathbb{R}^{|I|\times|J^1|}, \mathbf{w}\in\mathbb{R}^{|J^1|}, \mathbf{v}\in\mathbb{R}^{|I|\times|J^1|\times|J^2|} && \\
&\gamma^i \ge 0, \boldsymbol{\beta}^{+i}, \boldsymbol{\beta}^{-i} \le \mathbf{0}, \boldsymbol{\rho}^{+i}, \boldsymbol{\rho}^{-} \ge \mathbf{0} && i\in I
\end{aligned}
$$

Hence, the RES problem corresponds to a (polynomially solvable) linear programming problem.

□

It is easy to see that the same argument can be used to prove that, for the robust NL and CEA problem with affine routing and/or hose uncertainty, the RES problem is also polynomial. In this way, we get two examples of problems where the RES problem is polynomial, but the 2ROB problem is not. In fact, independently of the routing policy and of the uncertainty set describing the demands, the robust NL and CEA problem include the problems without uncertainty and, hence, the Steiner Tree problem, as special case and are, therefore, NP-hard [9, 20].

5 Conclusions

⟨sec:conclusioni⟩ We considered a two stage robust optimization problem with cardinality constrained uncertainty, where the computed first stage solution must be re-assessed. The goal was to compute the largest uncertainty set that a given first stage solution can handle. We proved that the polynomiality of this problem depends on the recourse policy. In particular, if the values of the second stage variables can arbitrarily be changed to adapt them to the current values of the parameters, then the problem is NP-hard. If the affinely adjustable policy is adopted, then the problem can be solved in polynomial time.

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