Unboundedness and Infeasibility in Linear Bilevel Optimization: How to Overcome Unbounded Relaxations

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Abstract

Bilevel optimization problems are known to be challenging to solve in practice. In particular, the feasible set of a bilevel problem is, in general, non-convex, even for linear bilevel problems. In this work, we aim to develop a better understanding of the feasible set of linear bilevel problems. Specifically, we develop means by which to identify when a bilevel problem is unbounded or infeasible. We show that extending the well-known high point relaxation with lower-level dual feasibility constraints is relevant to detecting when a bilevel problem is infeasible due to its lower-level problem being unbounded. Moreover, we present a new linear model to detect that a bilevel problem is unbounded when that unboundedness originates from the upper-level variables alone. Furthermore, we derive two sets of sufficient conditions to guarantee bilevel boundedness. Finally, we highlight that constraints that are implied by others are not necessarily redundant for bilevel problems.

1 Introduction

Bilevel optimization is a hierarchical modelling framework involving two nested optimization problems: the upper-level and the lower-level problems. On the one hand, the upper-level decision-maker must anticipate the reaction of the lower level to determine its set of optimal solutions. On the other hand, given an upper-level decision, the lower-level decision-maker acts optimally according to its own interests.

This decision-making process can be represented mathematically in its as follows:

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$$
\begin{aligned}
\min_{x,\tilde{y}} \quad & c^\top x + d_1^\top \tilde{y} && \text{(B.1)}\\
\text{s.t.} \quad & A_1 x + B_1 \tilde{y} \leq b_1 && \text{(B.2)}\\
& x \geq 0 && \text{(B.3)}\\
& \tilde{y} \in \arg\min_{y} \quad d_2^\top y && \text{(B.4)}\\
& \qquad \text{s.t.} \quad A_2 x + B_2 y \leq b_2 && \text{(B.5)}\\
& \qquad\qquad\quad y \geq 0 && \text{(B.6)}
\end{aligned}
$$

where $x \in \mathbb{R}^{n_x}$, $y, \tilde{y} \in \mathbb{R}^{n_y}$, $A_1 \in \mathbb{R}^{n_1 \times n_x}$, $A_2 \in \mathbb{R}^{n_2 \times n_x}$, $B_1 \in \mathbb{R}^{n_1 \times n_y}$, $B_2 \in \mathbb{R}^{n_2 \times n_y}$, $b_1 \in \mathbb{R}^{n_1}$, and $b_2 \in \mathbb{R}^{n_2}$. In this formulation, (B.1)-(B.3) corresponds to the upper-level problem, while (B.4)-(B.6) corresponds to the lower-level problem. This formulation is said to be optimistic because when there are multiple optimal solutions for the lower level, the best one with respect to the upper-level objective value is selected. We consider this optimistic formulation throughout this paper. Furthermore, we assume that the lower-level objective is not constant (i.e., that $d_2 \neq 0$). This assumption on non-trivial lower-level problems is not restrictive, as, without it, the bilevel problem can be reduced to a single-level linear problem (see Remark 1).

There has been a growing interest in bilevel optimization in recent years [9], greatly due to the various real-life applications for which this framework provides an ideal modelling paradigm. A few examples of bilevel optimization applications are network design and energy market operation [2, 18]. For an introduction to bilevel optimization refer to [4], and for a review of current methods and applications for bilevel problems, as well as an extensive bibliography, consult [9].

Despite the numerous potential applications in the field, bilevel problems are known to be challenging to solve in practice. In fact, even in the linear case, the feasible sets of bilevel problems are, in general, non-convex, and bilevel problems are strongly NP-hard [11].

In this paper, we are concerned with gaining a better understanding of the feasible set of linear bilevel problems, often denoted as the inducible region. We seek to develop means to identify when a bilevel problem is unbounded or infeasible. In particular, we emphasise the study of bilevel problems with unbounded single-level relaxations.

The main contributions of our work are as follows: We show that extending the high point relaxation (formally defined in Section 3) with lower-level dual constraints is relevant to detecting when a bilevel is infeasible due to the unboundedness of its lower-level problem. Moreover, we present a new linear model to detect that the bilevel is unbounded when that unboundedness originates from the upper-level variables alone. Furthermore, we derive sufficient conditions to guarantee bilevel boundedness and further strengthen these conditions by taking advantage of lower-level dual information. Finally, we highlight issues arising from redundancy in bilevel problems and discuss how it differs from redundancy in its single-level relaxations.

Below, we denote some standard notation that is used throughout this paper. For any feasible upper-level variable x, the lower-level optimal-value function is defined as: $\varphi(x) = \min_{y \geq 0} \left\{ d_2^\top y : A_2 x + B_2 y \leq b_2 \right\}$. Similarly, the optimal-value function of the dual lower-level problem is $\varphi_{\mathrm{D}}(x) = \max_{\lambda \geq 0} \left\{ (A_2 x - b_2)^\top \lambda : B_2^\top \lambda \geq -d_2^\top \right\}$. Following [10], we say that these functions are well-defined if the problems have a finite optimal value and are ill-defined otherwise. We denote the inducible region, which is the feasible set of the bilevel problem, as $\mathcal{F}_{\mathrm{B}} = \{(x, y) \in \mathbb{R}^{n_x \times n_y} : (\mathrm{B.2})\text{-}(\mathrm{B.6})\}$. In addition, let the extended inducible region, including the lower-level dual variables, be denoted as $\mathcal{F}_{\mathrm{B}}^+ = \{(x, y, \lambda) \in \mathbb{R}^{n_x \times n_y \times n_2} : (x, y) \in \mathcal{F}_{\mathrm{B}};\ B_2^\top \lambda \geq -d_2^\top;\ \lambda \geq 0;\ (A_2 x - b_2)^\top \lambda \geq \varphi_{\mathrm{D}}(x)\}$.

This paper is organised as follows. In Section 2, we gather relevant background results on the inducible region and its properties, as well as on handling unbounded single-level relaxations. In Section 3, we introduce the high point relaxation and extend it with lower-level dual constraints. In addition, in Section 3.1, we show that the latter is particularly useful for detecting infeasible bilevel problems due to an unbounded lower-level problem. Section 4 is divided into two parts. In Section 4.1, we present two results that help describe the inducible region, and in Section 4.2, we discuss redundancy in the bilevel setting. In Section 5, we investigate bilevel problems with unbounded single-level relaxations. First, in Section 5.1, we consider the case when the relaxation model is unbounded, but the lower-level variables are bounded and derive conditions that lead to bilevel unboundedness. Second, in Section 5.2, we consider the general case when neither upper- nor lower-level variables are known to be bounded and derive two sets of sufficient conditions for guaranteeing the boundedness of a bilevel problem. Finally, we finish with some concluding remarks in Section 6.

2 Background

In this section, we provide motivation for our work and relevant background literature. This section is divided into two parts: In Section 2.1, we discuss the inducible region and some of its properties, while in Section 2.2, we discuss unbounded single-level relaxations.

2.1 Inducible Region

In this section, we focus on the feasible region of the bilevel problem, denoted as the inducible region, and summarise some important results regarding the characterisations and properties of this set. This is not an overview of algorithmic approaches to solving bilevel problems. For more information on these approaches, the reader is redirected to [12].

Bilevel problems are challenging to solve, and that is reflected in the complex structure of their feasible set. In fact, the inducible region of bilevel problems is generally non-convex, even in the linear case. In addition, this feasible set can be disconnected or empty in the presence of coupling constraints, which are upper-level constraints that depend on lower-level variables [7].

One interesting theorem lends greater geometric insight into what the inducible region looks like. This result states that the inducible region of a linear bilevel problem is comprised of faces of the feasible shared constraint set of upper- and lower-level constraints, and it is obtained under the assumption that this shared constraint set is non-empty and compact [2]. These faces are associated with optimal vertices of the dual of the lower-level problem and their objective value. Furthermore, if the bilevel does not have coupling upper-level constraints, then these faces are connected [1].

To the best of our knowledge, there is currently no way to directly feed a bilevel problem into a ready-to-use optimization solver. Therefore, most approaches reformulate the bilevel problem as a single-level optimization model. This reformulation of the inducible region into a single-level feasible set can be achieved by replacing the lower level with optimality conditions. Some of the most commonly studied reformulations include the optimal-value function, the Karush-Kuhn-Tucker (KKT), and the strong-duality reformulations.

In the optimal-value function reformulation [17], the inducible region is reformulated into the shared constraint set in addition to the constraint ($d_2^\top y \leq \varphi(x)$), ensuring the lower-level value is optimal. This reformulation is often difficult to obtain in analytical form. In the KKT reformulation, the lower-level problem is replaced with its equivalent KKT optimality conditions [14], thereby introducing linear complementarity constraints to the feasible set. This reformulation is valid when the lower-level problem is convex and satisfies a constraint qualification. Lastly, in the strong-duality reformulation, the lower-level problem is replaced with the lower-level primal and dual constraints as well as the strong-duality condition ($d_2^\top y \leq (A_2 x - b_2)^\top \lambda$). This strong-duality constraint introduces bilinearities to the feasible set and works for linear lower-level problems.

Moreover, various representability results concerning the inducible region have been introduced to study what types of sets can be modelled as feasible regions of linear bilevel problems [3]. It is shown that the inducible region of a continuous linear bilevel problem can be equivalently modelled as the feasible region of a linear complementarity problem or as a finite union of polyhedra. Conversely, any finite union of polyhedra (or linear complementarity feasible set) can be represented using the feasible set of a linear bilevel problem.

Finally, we note that most research assumes that the inducible region is bounded either directly or by assuming that the feasible set of its relaxation is bounded. To the best of our knowledge, the complexity of the decision problem regarding whether there is a direction of unboundedness for the bilevel problem has never been directly answered. We suspect that solving this problem is a challenging task. As we previously detailed, linear bilevel problems can be equivalently reformulated as linear complementarity problems. In addition, finding a direction of unboundedness for a linear complementarity problem corresponds to finding a non-zero solution to the corresponding homogeneous linear complementarity problem [8], and linear complementarity problems are NP-hard [6]. This suggests that deciding whether a direction of unboundedness of a linear bilevel problem exists might also be a difficult task.

2.2 Unbounded Relaxations

As noted in Section 2.1, reformulating the bilevel problem as a single-level optimization model comes at the cost of introducing nonlinearities to the problem. Consequently, many bilevel solution approaches start by relaxing these complicating constraints and solving a simpler single-level relaxation.

The most common relaxation of a bilevel problem is the high point relaxation (HPR) which is obtained by simply optimizing the upper-level objective over the shared constraint set of upper- and lower-level constraints (formal definition in Section 3). It is known that an optimal solution of the bilevel can be found at a vertex of this relaxation [2], which hints at the relevance of the HPR model in bilevel optimization.

When this relaxation is infeasible, we can conclude that the corresponding bilevel problem is also infeasible. If this relaxation is finite optimal, then the bilevel problem can either be finite optimal or infeasible, and each algorithm proceeds differently to try to find an optimal solution when one exists. However, if this relaxation is unbounded, nothing can be concluded about the optimality status of the corresponding bilevel. The examples in [12, 19] show that when the HPR model is unbounded, the corresponding bilevel can be finite optimal, unbounded, or infeasible.

Due to this inconclusiveness, most bilevel solution approaches assume that the feasible set of the HPR is bounded. Consequently, there is little existing research regarding what happens to the bilevel problem when its relaxation is unbounded.

The majority of progress in the study of unbounded HPR models is made under the assumption that this unboundedness originates in the lower-level problem alone. In fact, if there is an upper-level variable that is feasible for the HPR and its corresponding lower-level problem is unbounded, then the bilevel problem is infeasible (Lemma 2 in [19]). This key theorem has driven most of the results in this field.

In [19], a mixed-integer linear problem is designed to track the reason for the unboundedness of the HPR. Depending on whether an optimal solution of this mixed-integer linear problem has a positive, zero, or negative objective value, we can conclude that the bilevel is infeasible, unbounded, or finite optimal, respectively (Lemmas 6-8 in [19]). These results are derived for mixed-integer linear bilevel problems under the assumption that all upper-level variables are integer and bounded, and they can be easily adapted to linear bilevel problems with bounded upper-level variables.

Furthermore, it is shown in [10] that, when the HPR is unbounded, one can detect upfront whether the lower-level problem is unbounded by solving a linear problem that does not depend on upper-level variables. Depending on whether the optimal value of this linear model is negative or not, one can conclude that either the bilevel is infeasible or the lower-level problem is well-defined for every feasible point of the HPR. Nevertheless, when the HPR is unbounded, but the lower-level problem is not unbounded, solving this linear model will not allow us to determine the status of the original bilevel.

To sum up, studying which conclusions can be drawn about the bilevel problem when its relaxation is unbounded is a relevant topic that is often overlooked. In this paper, we propose some results to help close this gap by studying the inducible region in relation to the feasible set of bilevel relaxations.

3 Single-Level Relaxations

Linear single-level relaxations are very important in (linear) bilevel optimization, especially since solving these relaxations is an essential part of many bilevel solution approaches [12].

As previously mentioned, the most common single-level relaxation of a bilevel problem is obtained by relaxing the lower-level optimality constraint (B.4). This relaxation is called the high point relaxation and it is defined as:

$$\min_{x,y} \left\{ c^\top x + d_1^\top y : \ (x,y) \in \mathcal{F}_{\text{HPR}} \right\} \tag{HPR}$$

where $\mathcal{F}_{\text{HPR}} = \{(x,y) : (B.2), (B.3), (B.5), (B.6)\}$ denotes the feasible set. Note that this model is indeed a relaxation of the corresponding bilevel problem [2].

Recall that we assume that the lower-level objective is not constant, as $d_2 \neq 0$. In Remark 1, we show that without this assumption, the bilevel problem is equivalent to its HPR.

Remark 1 (Non-constant Lower-level Objective Assumption)**.** *The single-level lower-level optimal value reformulation of the bilevel problem (B) can be written as:*

$$\begin{aligned} \min_{x,y} \quad & c^\top x + d_1^\top y \\ s.t. \quad & A_1 x + B_1 y \leq b_1 \\ & A_2 x + B_2 y \leq b_2 \\ & d_2^\top y \leq \varphi(x) \\ & x, y \geq 0 \end{aligned}$$

In the case where the lower-level objective is constant and $d_2 = 0$, it is easy to observe that the optimal value constraint $d_2^\top y \leq \varphi(x)$ becomes trivial, because $d_2^\top y = 0$ for any y, and $\varphi(x) = 0$ for any feasible x. Removing this constraint leads to the HPR. Hence, when $d_2 = 0$, the bilevel feasible set is equivalent to the feasible set of its HPR. Thus, the original problem did not require a bilevel optimization framework.

Consequently, we assume $d_2 \neq 0$ throughout the rest of the paper.

Another valid single-level linear relaxation is the HPR with lower-level dual feasibility, which can be defined as:

$$\begin{aligned} \min_{x,y,\lambda} \quad & c^\top x + d_1^\top y \\ \text{s.t.} \quad & (x,y) \in \mathcal{F}_{\text{HPR}} \\ & B_2^\top \lambda \geq -d_2^\top \\ & \lambda \geq 0 \end{aligned} \tag{HPR+DF}$$

where the two constraints added correspond to the feasibility conditions of the dual lower-level problem $\varphi_{\text{D}}(x)$.

Note that this is indeed a relaxation of the original bilevel problem. In fact, the (HPR+DF) problem can be obtained by relaxing the complementarity conditions of the single-level KKT reformulation of a linear bilevel problem. Alternatively, (HPR+DF) can be obtained by relaxing the strong-duality condition from the strong-duality single-level reformulation.

3.1 Advantages of (HPR+DF)

In the HPR+DF relaxation, the dual variables λ are not explicitly connected with the other variables (x, y) and do not appear in the objective. Therefore, the question as to whether this relaxation brings any added benefit when compared to the classic HPR formulation naturally arises. Theorem 1 partly answers this question. We show that even though the dual information is not explicitly connected with the primal information, there may be an advantage in using this model for bilevel algorithms that require solving a linear relaxation.

Theorem 1. *If (HPR+DF) is feasible, then the lower-level optimal-value function $\varphi(x)$ is well-defined for all points (x, y, λ) that are feasible for (HPR+DF).*

Proof. Assume that (HPR+DF) is feasible, and let (x, y, λ) be a feasible solution. Then, we know that y is feasible for the lower-level problem $\varphi(x)$. Therefore, the lower-level problem is feasible (either finite optimal or unbounded).

Similarly, we know that λ is feasible for the dual lower-level problem $\varphi_{\mathrm{D}}(x)$. Hence, the dual lower-level problem is feasible (either finite optimal or unbounded). Consequentially, the lower-level problem $\varphi(x)$ (dual of the dual lower-level problem) is either finite optimal or infeasible.

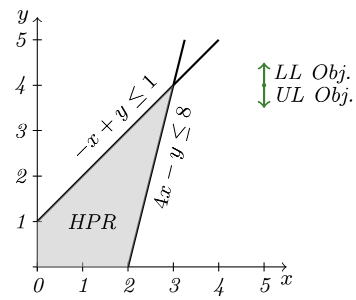
From the two statements above, we can conclude that the lower-level problem is finite optimal for x. From the arbitrariness of (x, y, λ), we conclude that the lower-level optimal-value function $\varphi(x)$ is well-defined for every feasible point of the (HPR+DF). □

Note that Theorem 1 does not hold in general for the HPR problem, as we illustrate in Example 1.

Example 1 (HPR Feasible and $\varphi(x)$ Ill-defined)**.** *Consider the following linear bilevel problem illustrated in Figure 1:*

$$
\begin{aligned}
\min_{x,\tilde{y}} \quad & y \\
s.t. \quad & x - \tilde{y} \leq -1 \\
& x \geq 0 \\
& \tilde{y} \in \arg\min_{y \geq 0} \{ \ -y : \\
& \qquad\qquad 4x - y \leq 8 \}
\end{aligned}
$$

Figure 1



In this example, the HPR model is feasible, as $(x, y) = (1, 1) \in \mathcal{F}_{HPR}$*. However, for* $x = 1$*, the lower-level problem is unbounded; accordingly,* $\varphi(1) = -\infty$ *is ill-defined.*

Theorem 1 indicates that this alternative HPR+DF relaxation is particularly relevant for instances where the lower-level problem is unbounded, and the bilevel problem is, therefore, infeasible. In these instances, the HPR model might not help us detect that the bilevel problem is infeasible (see Example 1). However, since the HPR+DF problem would be infeasible, we could conclude that the original bilevel problem is also infeasible.

4 Descriptions of the Inducible Region

The aim of this section is to better understand and describe the inducible region. In Section 4.1, we present two descriptions of the inducible region using faces generated by lower-level constraints, while in Section 4.2, we highlight how redundancy differs in bilevel optimization compared with single-level optimization.

4.1 Structure of Lower-Level Faces

In this section, we derive some theorems to describe and approximate the inducible region. First, we show that the inducible region is contained in a union of a subset of faces defined by lower-level constraints. Second, we describe the finite union of polyhedra, which is equivalent to the extended inducible region following the work in [3]. These results are the premises for deriving sufficient conditions for the unboundedness of a bilevel problem in Section 5.2.

We start by defining the lower-level candidate faces of the HPR feasible set as:

$$
\begin{aligned}
\mathcal{H}_i &= \{(x, y) \in \mathcal{F}_{\text{HPR}} : (A_2 x + B_2 y)_i = (b_2)_i\}, && \text{for } i \in \{1, ..., n_2\} \\
\mathcal{H}_{i+n_2} &= \{(x, y) \in \mathcal{F}_{\text{HPR}} : y_i = 0\}, && \text{for } i \in \{1, ..., n_y\}
\end{aligned}
$$

Note that some of these sets $\mathcal{H}_i$ might be empty, some might contain a single point, and others might be a facet of $\mathcal{F}_{\text{HPR}}$.

We introduce the first main result of this section in Theorem 2.

Theorem 2. *There exists a subset $I \subseteq \{1, \dots, n_2 + n_y\}$ such that*

$$\mathcal{F}_B \subseteq \bigcup_{i \in I} \mathcal{H}_i$$

where $\mathcal{F}_B$ is the inducible region.

Proof. If $\mathcal{F}_B = \emptyset$, the result is trivial. If $\mathcal{F}_B \neq \emptyset$, then let $(x, y) \in \mathcal{F}_B$. Since $y \in \varphi(x)$ and $d_2 \neq 0$, we know that there exists (at least) one lower-level constraint active at (x, y). Equivalently, there exists $i \in \{1, \dots, n_2 + n_y\}$ such that $(x, y) \in \mathcal{H}_i$. By building I such that $i \in I$, we conclude that $(x, y) \in \bigcup_{i \in I} \mathcal{H}_i$. We can repeat this process for every $(x, y) \in \mathcal{F}_B$ to obtain a subset I such that $\mathcal{F}_B \subseteq \bigcup_{i \in I} \mathcal{H}_i$. □

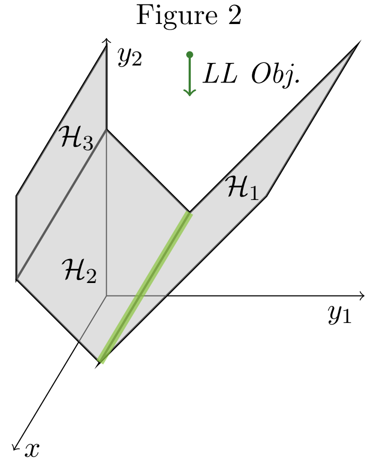
Note that we have built the subset I such that for all $i \in I$, $\mathcal{H}_i \neq \emptyset$. Therefore, $\{\mathcal{H}_i\}_{i \in I}$ are faces of the HPR feasible set, and we refer to them as such throughout the rest of the paper.

Note also that there might be no subset I such that the opposite inclusion $\bigcup_{i \in I} \mathcal{H}_i \subseteq \mathcal{F}_B$ is true, as depicted in Example 2.

Example 2 (Converse Inclusion of Theorem 2 Does Not Hold)**.** *Consider the following linear bilevel problem illustrated in Figure 2:*

$$\begin{aligned} \min_{x, \tilde{y}} \quad & x + y_1 + y_2 \\ s.t. \quad & x \geq 0 \\ & (\tilde{y_1}, \tilde{y_2}) \in \arg\min_{y_1, y_2 \geq 0} \{ y_2 : \\ & \qquad y_1 - y_2 \leq 0 \\ & \qquad -y_1 - y_2 \leq -2 \\ & \qquad y_1, y_2 \geq 0 \} \end{aligned}$$

Figure 2



In this example, the lower level faces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are defined by constraints $(y_1 - y_2 \leq 0)$, $(-y_1 - y_2 \leq -2)$, and $(y_1 \geq 0)$, respectively, and the inducible region is shaded in green. The candidate face associated with $y_2 \geq 0$ is empty.

It is clear that $\mathcal{H}_3$ does not intersect the inducible region. Moreover, $\mathcal{H}_1$ and $\mathcal{H}_2$ both intersect the inducible region but are not contained in it. Thus, there is no subset I of lower-level faces whose union is contained in the inducible region.

In addition, we can define candidate faces of (HPR+DF) associated with lower-level dual constraints as:

$$\begin{aligned} \mathcal{D}_i &= \{(x,y,\lambda) \in \mathcal{F}_{\text{HPR+DF}} : \lambda_i = 0\}, && \text{for } i \in \{1,...,n_2\} \\ \mathcal{D}_{i+n_2} &= \{(x,y,\lambda) \in \mathcal{F}_{\text{HPR+DF}} : (B_2^\top \lambda)_i = (-d_2)_i\}, && \text{for } i \in \{1,...,n_y\} \end{aligned}$$

Once again, some of these $\mathcal{D}_i$ sets might be empty, some might contain a single point, and others might be a facet of $\mathcal{F}_{\text{HPR+DF}}$.

Finally, we present a characterisation of the inducible region as a finite union of polyhedra based on the results in [3], which show that a continuous linear bilevel representable set is precisely a finite union of polyhedra.

Lemma 1. *The extended inducible region is the finite union of polyhedra:*

$$\mathcal{F}_B^+ = \bigcup_{\omega \in \{1,2\}^{n_2+n_y}} \mathcal{P}_\omega$$

where, for $\omega \in \{1,2\}^{n_2+n_y}$*, the polyhedron* $\mathcal{P}_\omega$ *is defined as:*

$$\begin{aligned} \mathcal{P}_\omega = \{(x,y,\lambda) \in \mathcal{F}_{HPR+DF} : & \ (x,y) \in \mathcal{H}_i && \forall i : \omega_i = 1; \\ & \ \lambda \in \mathcal{D}_i && \forall i : \omega_i = 2\} \end{aligned}$$

Proof. This proof uses Lemmas 29, 30, and 27 from [3] in the first, second, and third equalities, respectively.

$$\begin{aligned} \mathcal{F}_B^+ &= \{(x,y,\lambda) \in \mathcal{F}_{\text{HPR+DF}} : (b_2 - A_2x - B_2y) \cdot \lambda = 0;\ y \cdot (B_2^\top \lambda + d_2) = 0\} \\ &= \big\{(x,y,\lambda) \in \mathcal{F}_{\text{HPR+DF}} : \max\{-(b_2 - A_2x + B_2y); -\lambda\} \geq 0; \\ &\qquad \max\{-y; -(B_2^\top \lambda + d_2)\} \geq 0\}\big\} \\ &= \bigcup_{\omega \in \{1,2\}^{n_2}} \mathcal{P}_\omega \end{aligned}$$

□

4.2 Redundancy in the Bilevel-Setting

In this section, we present some observations and results about redundancy in bilevel optimization. We discuss redundancy in the bilevel setting as a mechanism to remove constraints, as well as some potential pitfalls in extending this concept from single-level optimization.

A redundant constraint is often defined as a constraint that is implied by other constraints and, consequently, is unnecessary to describe the feasible region. In addition to unnecessarily increasing the size of an optimization problem, redundant constraints can also lead to degeneracy. In single-level optimization, when detected, these constraints can be removed without altering the problem.

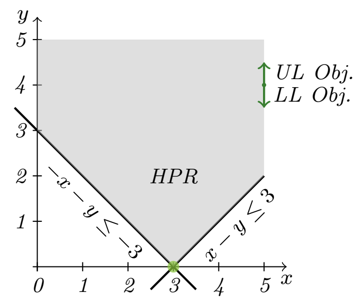
Detecting redundant constraints in bilevel optimization, in particular lower-level constraints, could be relevant to reducing the subset I of lower-level faces containing the inducible region (see Theorem 2). Nevertheless, in bilevel optimization, degeneracy cannot always be avoided due to the parametric nature of the lower-level problem [15].

Although redundant constraints can be safely removed when solving the HPR, this removal is not generally valid for the bilevel problem as depicted in Example 3.

Example 3 (Difficulty with Removing Redundant Constraints in Bilevel Optimization)**.** *Consider the following linear bilevel problem illustrated in Figure 3:*

$$\begin{aligned} \min_{x,\tilde{y}} \quad & -y \\ s.t. \quad & -x-y \leq -3 \\ & x-y \leq 3 \\ & x \geq 0 \\ & \tilde{y} \in \arg\min_{y}\{y : y \geq 0\} \end{aligned}$$

Figure 3



In this example, the lower-level constraint $y \geq 0$ is redundant as it is implied by the upper-level constraints ($-x-y \leq -3$) and ($x-y \leq 3$). However, deleting it changes the inducible region and the problem's optimality status by making the only bilevel feasible solution $(x, y) = (3, 0)$ infeasible.

The main reason for this issue is that the redundant constraint is a lower-level inequality that is implied by a set of upper-level constraints. A more detailed discussion of this can be found later on in this section.

In fact, the same problem can prevail even if the redundant constraint is not active at the bilevel optimal solution. This issue arises because the independence of irrelevant constraints (IIC) property does not hold for bilevel problems in general [16]. In other words, adding a lower-level constraint that is inactive at the optimal solution of the bilevel problem might change the inducible region and, consequently, the optimal solution of that bilevel problem [13]. Despite being valid in single-level optimization, this property is shown to only hold in bilevel optimization if the HPR optimal solution is also bilevel optimal [16]. In practice, this condition is seldom verified for bilevel problems.

We define that a constraint is bilevel-redundant for the feasible set $\mathcal{F}_{\text{B}}$ (or $\mathcal{F}_{\text{B}}^{+}$) if it can be removed from the feasible set without changing it.

As depicted in Example 3, some constraints might be implied by others without being bilevel-redundant. In the rest of this section, we discuss the cases in which these redundant constraints that are implied by others are also bilevel-redundant and can therefore be removed. We consider upper- and lower-level redundant constraints separately.

Upper-level redundant constraints can always be safely removed, because they do not affect the lower-level optimal value function. Therefore, removing an upper-level redundant constraint corresponds to removing a redundant constraint from the single-level optimal-value function reformulation. In other words, upper-level redundant constraints are also bilevel-redundant. Note that this conclusion is not contradictory with the results in [16] because, in that work, the IIC property is defined with respect to lower-level constraints.

By contrast, lower-level redundant constraints require a more detailed analysis, as Example 3 indicates that they are not always bilevel-redundant. We distinguish two types of lower-level redundant constraints, depending on the type of constraints implying them.

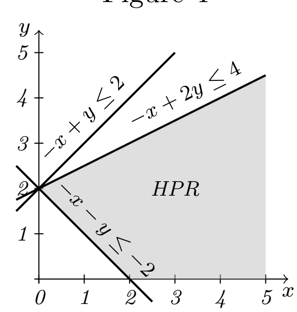
The first type of lower-level redundant constraints are implied only by other lower-level constraints. In this case, the constraint is redundant for both the shared constraint set and the lower-level feasible set for any fixed feasible upper-level variable x. This redundancy at both levels implies that removing the redundant constraint will not affect the shared constraint set, nor will it change the lower-level optimal-value function. Therefore, lower-level redundant constraints that are implied only by lower-level constraints are always bilevel-redundant.

The second type of lower-level redundant constraints cannot be implied only by other lower-level constraints. As illustrated in Example 4, this type of redundant constraint can, in some cases, be bilevel-redundant and, in others, not. Hence, each case needs to be accessed individually.

Example 4 (Lower-Level Redundant Constraint Implied by Upper Level)**.** *Consider the following linear bilevel problem with parameter* $\alpha \in \{-1, 1\}$ *illustrated in Figure 4:*

$$\begin{aligned} \min_{x,\tilde{y}} \quad & -x + \tilde{y} \\ s.t. \quad & -x + 2\tilde{y} \leq 4 \\ & x \geq 0 \\ & \tilde{y} \in \quad \arg\min_{y \geq 0} \{ \alpha y : \\ & \qquad\qquad -x + y \leq 2, \\ & \qquad\qquad -x - y \leq -2 \} \end{aligned}$$

Figure 4



The lower-level constraint $(-x + y \leq 2)$ *is redundant because it is implied by the upper-level constraint* $(-x + 2y \leq 4)$ *and the lower-level constraint* $(-x - y \leq -2)$*. Depending on the value of the parameter* α*, this constraint can either be bilevel-redundant or not. We detail these two possibilities below.*

Case 1: $\alpha = 1 \Rightarrow$ ***Bilevel-redundant***

In this case, the inducible region is defined by the union of the line segments $\{(x, y) : -x - y = -2;\ x \in [0, 2]\}$ *and* $\{(x, 0) : x \geq 2\}$*. If we remove the redundant constraint, the inducible region remains unchanged. Consequently, the constraint is bilevel-redundant.*

Case 2: $\alpha = -1 \Rightarrow$ ***Not bilevel-redundant***

In this case, the inducible region is the singleton $\{(0,2)\}$*. If we remove the redundant constraint, the lower-level problem at* $x = 0$ *becomes unbounded. Hence, the bilevel problem becomes infeasible. This clearly changes the inducible region, and consequently, the constraint is not bilevel-redundant.*

When lower-level redundant constraints cannot be implied only by lower-level constraints, further conclusions can be drawn if the redundant constraint is parallel to another constraint. In fact, in [13], the authors present a complete presolve method for removing parallel rows, including cases where both upper- and lower-level constraints are simultaneously involved.

Finally, the results in [5] regarding moving lower-level constraints to the upper-level problem can also be relevant to the study of redundant constraints in the bilevel setting. Here, the authors show that, under certain assumptions, lower-level constraints that only involve upper-level variables can be moved to the upper level. This would allow us to handle such constraints as belonging to the upper-level problem, where we know that redundant constraints are bilevel-redundant.

5 bilevel problems with Unbounded Relaxations

As established in Section 2.2, when (HPR) is unbounded, nothing can be concluded about the optimality status of the corresponding bilevel problem. The same holds true when (HPR+DF) is unbounded. Example 5 depicts this inconclusiveness by depicting three cases where the bilevel is infeasible, unbounded, and finite optimal.

Example 5 ((HPR+DF) Unbounded and Bilevel Inconclusive)**.** *Consider the following linear bilevel problem with parameters* $\alpha \in \{-1, 1\}$ *and* $\beta \in \{1, 2\}$*:*

$$\begin{aligned}
\min_{x,\tilde{y}} \quad & -x - \tilde{y} \\
s.t. \quad & -x - \tilde{y} \leq -2 \\
& \alpha(x - \tilde{y}) \leq \beta \\
& x \geq 0 \\
& \tilde{y} \in \arg\min_{y} \{y : -x + y \leq 2;\ y \geq 0\}
\end{aligned}$$

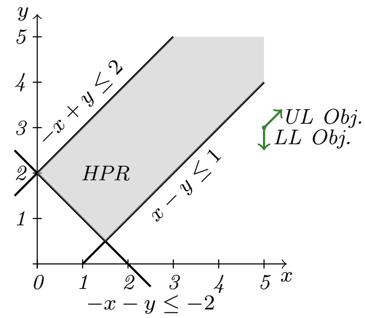
The dual lower-level feasibility constraints are $\lambda \geq 0$ *and* $\lambda \geq -1$*. The HPR+DF problem is unbounded since it admits a feasible solution* $(1, 1, 0)$ *and a direction of unboundedness* $(\Delta x, \Delta y, \Delta\lambda) = (1, 1, 0)$*.*

However, depending on the values of α *and* β*, the bilevel can be infeasible, unbounded, or finite optimal. We detail these three cases below.*

Case 1: $(\alpha, \beta) = (1, 1) \Rightarrow$ ***Bilevel Infeasible***

This case is illustrated in Figure 5. The bilevel is infeasible because no solution of the form $(x, y) = (x, 0)$ verifies the upper-level constraints.

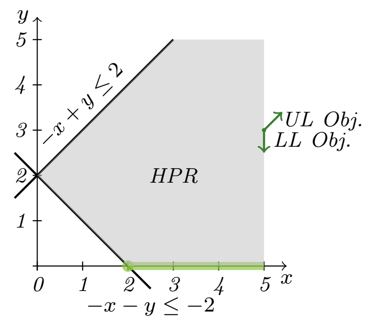
Figure 5



Case 2: $(\alpha, \beta) = (-1, 2) \Rightarrow$ ***Bilevel Unbounded***

This case is illustrated in Figure 6. The bilevel is unbounded because the sequence of bilevel-feasible points $(x, y) = (k, 0)$ for $k \in \mathbb{Z}^+ : k \geq 2$ has an increasing upper-level objective value.

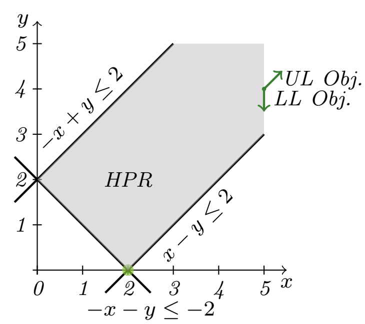
Figure 6



Case 3: $(\alpha, \beta) = (1, 2) \Rightarrow$ ***Bilevel Finite Optimal***

This case is illustrated in Figure 7. The bilevel is finite optimal because it has a single feasible solution $(x, y) = (2, 0)$, which is, consequently, optimal.

Figure 7



The rest of this section is divided as follows. In Section 5.1, we study the special case when the relaxation model is unbounded, but the lower-level variables are bounded. In this case, we introduce a linear model and derive a condition on its optimal value, which allows us to conclude that the bilevel is unbounded. In Section 5.2, we derive two sets of sufficient conditions to guarantee that a bilevel problem is bounded, despite its relaxation being unbounded.

5.1 bilevel problems with Bounded Lower Levels

A first natural extension to the work in the literature presented in Section 2.2 is to consider the opposite case to the one studied so far, wherein the HPR model is unbounded, but the lower-level variables are bounded. In this case, the unboundedness of the HPR model originates in the upper-level problem.

Analogously to the model presented in [10], in this section, we introduce a linear model that does not depend on lower-level variables. Moreover, we derive Theorem 3, which allows us to extract some conclusions about the optimality status of the bilevel problem when this model's optimal value is negative.

Consider the following linear problem that does not depend on lower-level variables:

$$\begin{aligned} \min_{\Delta x} \quad & c^\top \Delta x \\ \text{s.t.} \quad & A_1 \Delta x \leq 0 \\ & A_2 \Delta x = 0 \\ & 0 \leq \Delta x \leq 1 \end{aligned} \tag{M}$$

Note that (M) is neither unbounded (because $0 \leq \Delta x \leq 1$) nor infeasible (because $\Delta x = 0$ is a solution). Therefore, it must have a finite optimal solution with a non-positive optimal objective value. Theorem 3 allows us to draw some conclusions about the optimality status of the bilevel problem (B) from the optimal value of this linear problem (M).

Theorem 3. *Assume (B) is feasible. If (M) has a strictly negative optimal value, then (B) is unbounded.*

Proof. Let (x, y) be a feasible solution of (B), and let Δx^* be an optimal solution of (M) with $c^\top \Delta x^* < 0$. We will show that $(\Delta x, \Delta y) = (\Delta x^*, 0)$ is a direction of unboundedness of the bilevel problem. Let $\epsilon \in \mathbb{R}^+$.

First, note that $(x + \epsilon \Delta x^*, y)$ is a bilevel feasible point because it verifies all upper- and lower-level constraints:

$$\begin{aligned} & A_1(x + \epsilon \Delta x^*) + B_1 y \leq b_1 && (\text{as } A_1 \Delta x^* \leq 0;\ \epsilon \geq 0) \\ & x + \epsilon \Delta x^* \geq 0 && (\text{as } \Delta x^* \geq 0;\ \epsilon \geq 0) \\ & A_2(x + \epsilon \Delta x^*) + B_2 y \leq b_2 && (\text{as } A_2 \Delta x^* \leq 0;\ \epsilon \geq 0) \\ & y \geq 0 \end{aligned}$$

and it is lower-level optimal:

$$d_2^\top y \leq \varphi(x + \epsilon \Delta x^*) = \varphi(x) \qquad (\text{as } A_2 \Delta x^* = 0)$$

Second, since $c^\top \Delta x^* < 0$, the upper-level objective value at $(x + \epsilon \Delta x^*, y)$ decreases with increasing $\epsilon \in \mathbb{R}^+$.

Therefore, $(\Delta x^*, 0)$ is a direction of unboundedness of the bilevel problem, and it follows that the bilevel problem is unbounded. □

5.2 Guaranteeing Bilevel Boundedness

In this section, we consider the general case when neither upper- nor lower-level variables are known to be bounded. In Section 5.2.1, we derive sufficient conditions to guarantee that a bilevel problem is bounded. These conditions involve solving up to $(n_2 + n_y)$ linear problems. In Section 5.2.2, we extend these conditions to use lower-level dual information. If the correct constraint index is provided, this extension could allow us to guarantee bilevel boundedness by solving two linear problems.

5.2.1 Sufficient Conditions for Bilevel Boundedness

We introduce a linear optimization problem to help detect whether the unboundedness of the HPR originates in the unboundedness of the corresponding bilevel problem. Consider the following linear problem (U_i) for $i \in \{1, ..., n_2 + n_y\}$:

$$
\begin{aligned}
\xi_i = \min_{\Delta x, \Delta y} \quad & c^\top \Delta x + d_1^\top \Delta y \\
\text{s.t.} \quad & A_1 \Delta x + B_1 \Delta y \leq 0 \\
& A_2 \Delta x + B_2 \Delta y \leq 0 \\
& 0 \leq \Delta x, \Delta y \leq 1 \\
& (\Delta x, \Delta y) \in \mathcal{H}_i^\Delta
\end{aligned}
\tag{U_i}
$$

where $\mathcal{H}_i^\Delta$ ensures that once active, the lower-level constraint i remains active along the direction $(\Delta x, \Delta y)$:

$$
\begin{aligned}
\mathcal{H}_i^\Delta &= \{(\Delta x, \Delta y) : (A_2 \Delta x + B_2 \Delta y)_i = 0\}, && \text{for } i \in \{1, ..., n_2\} \\
\mathcal{H}_{i+n_2}^\Delta &= \{(\Delta x, \Delta y) : \Delta y_i = 0\}, && \text{for } i \in \{1, ..., n_y\}
\end{aligned}
$$

Note that (U_i) is neither unbounded (because $0 \leq \Delta x, \Delta y \leq 1$) nor infeasible (because $(\Delta x, \Delta y) = (0, 0)$ is a solution). Therefore, it is finite optimal and has a nonpositive optimal value. The following theorem shows how (U_i) can be used to derive sufficient conditions for bilevel boundedness.

Theorem 4. *Assume (B) is feasible. If ξ_i=0 for all $i \in I$, then (B) is bounded.*

Proof. We prove the theorem by proving its contrapositive: If the bilevel problem is unbounded, then there exists $k \in I$, such that (U_k) has a strictly negative objective value: $\xi_k < 0$.

Assume the bilevel problem is unbounded. From Theorem 2, we know that $\mathcal{F}_{\mathrm{B}} \subseteq \bigcup_{i \in I} \mathcal{H}_i$. Hence, there exists $k \in I$ such that $\mathcal{H}_k$ is an unbounded face of the feasible set of (HPR), and there exists $(\Delta x, \Delta y)$ a direction of unboundedness for the bilevel problem along $\mathcal{H}_k$. Without loss of generality, we assume that $\Delta x, \Delta y \leq 1$. By the definition of a direction of unboundedness, we know that:

$$\begin{aligned}
&c^\top \Delta x + d_1^\top \Delta y < 0,\\
&A_1 \Delta x + B_1 \Delta y \leq 0,\\
&A_2 \Delta x + B_2 \Delta y \leq 0,\\
&\Delta x, \Delta y \geq 0.
\end{aligned}$$

Since along this direction one remains on $\mathcal{H}_k$, then we also know that $(\Delta x, \Delta y) \in \mathcal{H}_k^\Delta$. Therefore, $(\Delta x, \Delta y)$ is a feasible solution of (U_k) and it has a strictly negative objective value. Hence, $\exists k \in I : \xi_i < 0$. □

Furthermore, to obtain this guarantee on bilevel boundedness, we need only check the subset $I \subseteq \{1, ..., n_2 + n_y\}$. The task of reducing subset I by excluding lower-level faces that are unnecessary for the inducible region was briefly discussed in Section 4.2. Finally, note that the converse of Theorem 4 is not true, as shown by Example 6.

Example 6 (Counterexample of the Converse of Theorem 4)**.** *Consider the linear bilevel problem in Case 3 of Example 5 illustrated in Figure 7. In this example, the bilevel has an optimal solution $(x, y) = (2, 0)$. Hence, it is bounded. However, the (U_i) model associated with the lower-level constraint $(-x + y \leq 2)$ has a strictly negative optimal value since the corresponding face is unbounded in a direction of increasing upper-level objective.*

5.2.2 Extending Sufficient Conditions with Dual Information

In this section, we will extend the sufficient condition for bilevel boundedness presented in Section 5.2.1. Analogously to (U_i), we introduce another linear problem using lower-level dual information defined for $i \in \{1, ..., n_2 + n_y\}$:

$$\begin{aligned}
\xi_i^{\mathrm{D}} = \min_{\Delta x, \Delta y, \Delta\lambda} \quad & c^\top \Delta x + d_1^\top \Delta y \\
\text{s.t.} \quad & A_1 \Delta x + B_1 \Delta y \leq 0\\
& A_2 \Delta x + B_2 \Delta y \leq 0\\
& B_2^\top \Delta\lambda \geq 0\\
& 0 \leq \Delta x, \Delta y, \Delta\lambda \leq 1\\
& \Delta\lambda \in \mathcal{D}_i^\Delta
\end{aligned} \tag{$\mathrm{U}_i^{\mathrm{D}}$}$$

where $\mathcal{D}_i^\Delta$ ensures that once active, the lower-level dual constraint i remains active along the direction $(\Delta x, \Delta y, \Delta\lambda)$:

$$\begin{aligned}
\mathcal{D}_i^\Delta &= \{\Delta\lambda : \Delta\lambda_i = 0\}, && \text{for } i \in \{1, ..., n_2\}\\
\mathcal{D}_{i+n_2}^\Delta &= \{\Delta\lambda : (B_2^\top \Delta\lambda)_i = 0\}, && \text{for } i \in \{1, ..., n_y\}
\end{aligned}$$

The problems (U_i) and ($\mathrm{U}_i^{\mathrm{D}}$), along with Lemma 1, allow us to strengthen the previous sufficient conditions for bilevel boundedness in Theorem 4.

Theorem 5. *Assume (B) is feasible. If* $\exists i \in \{1, \ldots, n_2 + n_y\} : (\xi_i = 0)$ *and* $(\xi_i^D = 0)$*, then (B) is bounded.*

Proof. Assume (B) is feasible. We prove this result by proving its contrapositive: If (B) is unbounded, then $\forall i \in \{1, \ldots, n_2 + n_y\} : (\xi_i < 0)$ or $(\xi_i^{\mathrm{D}} < 0)$.

Assuming (B) is unbounded, then there exists (x, y, λ) a bilevel feasible point and $(\Delta x, \Delta y, \Delta\lambda)$ a direction of unboundedness at (x, y, λ).

Given Lemma 1, we know there exists $\omega \in \{1, 2\}^{n_2+n_y}$, such that $(x, y, \lambda) \in \mathcal{P}_\omega$, and $(\Delta x, \Delta y, \Delta\lambda)$ is a direction along $\mathcal{P}_\omega$. This is:

$$(\Delta x, \Delta y) \in \mathcal{H}_i^\Delta \quad \forall i : \omega_i = 1$$
$$(\Delta\lambda) \in \mathcal{D}_i^\Delta \quad \forall i : \omega_i = 2$$

Assume without loss of generality that $(\Delta x, \Delta y, \Delta\lambda) \leq 1$.

Let $k \in \{1, \ldots, n_2 + n_y\}$. If $\omega_k = 1$, then $(\Delta x, \Delta y)$ is a feasible solution of (U_k) with a negative objective value. Therefore, $\xi_k < 0$. If $\omega_k = 2$, then $(\Delta x, \Delta y, \Delta\lambda)$ is a feasible solution of $(\mathrm{U}_k^{\mathrm{D}})$ with a negative objective value. Therefore, $\xi_k^{\mathrm{D}} < 0$.

Hence, either (U_k) or $(\mathrm{U}_k^{\mathrm{D}})$ have a strictly negative objective. By arbitrariness of the choice of k, we conclude that $\forall i \in \{1, \ldots, n_2 + n_y\} : (\xi_i < 0)$ or $(\xi_i^{\mathrm{D}} < 0)$. □

The worst-case scenario for this theorem involves solving $2(n_2 + n_y)$ linear problems, which is worse than the worse-case scenario in Theorem 4 requiring the solution of $(n_2 + n_y)$ linear problems. However, provided the correct constraint index i, Theorem 5 allows us to guarantee bilevel boundedness by solving only two linear optimization problems. While in Theorem 4 we have to check every constraint index, in Theorem 5 we merely need to find one that verifies the assumptions.

Finally, note that the converse of Theorem 5 does not hold in general, as we illustrate in Example 7.

Example 7 (Counterexample for Converse of Theorem 5)**.** *Consider the linear bilevel problem in Case 3 of Example 5 illustrated in Figure 7. In that example, the constraint* $(\Delta\lambda \in \mathcal{D}_i^\Delta)$ *of* (U_i^D) *is* $\Delta\lambda = 0$ *for both indexes* $i \in \{1, 2\}$*. Furthermore,* $(\Delta x, \Delta y, \Delta\lambda) = (0, 1, 0)$ *is a feasible solution of* (U_i^D) *for* $i \in \{1, 2\}$*, and it has a strictly negative objective value.*

Therefore, (U_i^D) *has a strictly negative optimal value for* $i \in \{1, 2\}$*, representing the dual constraints* $(\lambda \geq -1)$ *and* $(\lambda \geq 0)$*. Consequently, the assumption of Theorem 5 does not hold. However, (B) is bounded as* $\mathcal{F}_B = \{(x, y) = (2, 0)\}$*.*

6 Conclusion

In this paper, we investigated linear bilevel problems and, in particular, their feasible set in order to better understand when bilevel problems are unbounded or infeasible, especially those with unbounded relaxations.

We began by showing that the HPR+DF relaxation with lower-level dual constraints helps to detect when a bilevel is infeasible due to its lower-level problem being unbounded. In fact, we showed that the lower-level value function is well-defined for every feasible point of (HPR+DF).

Moreover, we studied bilevel problems whose relaxations are unbounded, since these problems can be finite optimal, unbounded, or infeasible. In this context, we derived ways to conclude about the optimality status of a bilevel when its relaxation is unbounded. Namely, we designed a mechanism to determine whether a bilevel problem with bounded lower-level variables is unbounded. For the general case when neither upper- nor lower-level variables are known to be bounded, we derived two sets of sufficient conditions for bilevel boundedness. These conditions use information about the structure of the inducible region and the importance of lower-level faces in its representation and require solving linear models.

Finally, we also highlighted a key difference between redundancy in the single-level and the bilevel settings. We concluded that constraints implied by others are not always redundant for the bilevel problem. In other words, removing such constraints can alter the inducible region and even the optimality status of the bilevel problem.

Future research could focus on determining the complexity of deciding whether a linear bilevel problem admits a direction of unboundedness. Additionally, it would be interesting to investigate ways to refine the subset I of lower-level faces containing the inducible region (see Theorem 2). Reducing this subset would strengthen the sufficient conditions for bilevel boundedness presented in Theorem 4. Another possible next step is to further study lower-level redundant constraints, which cannot be implied only by lower-level constraints and have properties that might make them bilevel-redundant. Lastly, when bilevel boundedness has been guaranteed, the next step should be to develop mechanisms to generate cuts that bound the feasible set of the relaxation without excluding bilevel-feasible solutions.

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