Robust optimization: from the uncertainty set to the solution and back

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Abstract

So far, robust optimization have focused on computing solutions resilient to data uncertainty, given an uncertainty set representing the possible realizations of this uncertainty. Here, instead, we are interested in answering the following question: once a solution of a problem is given, which is the largest uncertainty set that this solution can support? We address this question for a popular uncertainty set used in robust optimization, the cardinality constrained one, proving that an answer can be provided in polynomial time.

keywords: robust optimization, cardinality constrained uncertainty, computing the largest uncertainty set.

1 Introduction

Consider the optimization problem DP below, where I and J are the sets of the indices of constraints and variables, respectively, and $N \subseteq J$ contains the indices of the variables that are required to be integer.

$$\begin{aligned} DP \quad & \max \mathbf{c}^T\mathbf{x} \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & x_j \in \mathbb{Z} \quad j \in N \end{aligned}$$

Traditionally, a method for solving optimization problem DR assumes that the problem data, that is, constraint matrix $\mathbf{A}$, right-hand side vector $\mathbf{b}$ and objective coefficients $\mathbf{c}$, are known when the problem is solved and that they are constant over time. If this is not true, as it is the case in most problems coming from real-life applications, the computed solution may degrade its performances or even become feasible, if some modifications in the problem data occur at some point. For this reason, methodologies for handling data uncertainty, such as simulation [1, 2], stochastic programming [3, 4] and robust optimization [5, 6], have received an increasing attention in the literature and have been massively used for solving applied problems (see, e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). These techniques acknowledge the possibility that the problem data may be affected by some level of uncertainty and their aim is to produce a solution that is resilient to data uncertainty.

Among the available approaches to handle data uncertainty, in the present paper we focus on robust optimization [5]. Under this framework, it is assumed that there exists a given set of values, or *realizations*, of the problem data that are of interest, whereas the others are supposed to be unlikely to happen and just ignored. The set of the considered realizations is known as the *uncertainty set* and several ways to model it have been proposed in the literature, including both general purpose frameworks [19, 20] and application-driven sets [17, 21]. The real probability distribution is unknown and the aim of robust optimization methods is to compute the best solution among the ones that are feasible for any realization of the uncertainty in the given uncertainty set.

We consider here the robust optimization framework defined in [20]. That is: the uncertainty only affects matrix $\mathbf{A}$; we must use the same solution $\mathbf{x}$, independently of the realization of the uncertainty; the uncertainty is expressed using the *cardinality constrained* model. In the cardinality constrained model, each uncertain entry a_{ij} of matrix $\mathbf{A}$ can be regarded as a symmetric bounded random variable [22], whose probability distribution is unknown, but for which a nominal value $\bar{a}_{ij}$ and a maximum deviation $\delta_{ij} \geq 0$ from the nominal value are available. Hence, a_{ij} takes values in the interval $[\bar{a}_{ij} - \delta_{ij}, \bar{a}_{ij} + \delta_{ij}]$. At most Γ_i parameters can deviate from the nominal value at the same time in constraint $i \in I$. In a worst case realization, if parameter a_{ij} deviates from the nominal value, the deviation is the largest possible, that is, either $a_{ij} = \bar{a}_{ij} + \delta_{ij}$ or $a_{ij} = \bar{a}_{ij} - \delta_{ij}$. If a solution is resilient to these worst case realizations, then it is resilient to any realization in the set. We assume, without loss of generality, that Γ_i is integer for any $i \in I$.

Definition 1.1. *Let z^i_j be a binary variable that takes value 1 if parameter a_{ij} deviates from the nominal value and 0 otherwise. Any worst case realization of the uncertainty for constraint $i \in I$ corresponds to a vector $\mathbf{z}^i$ and, therefore, with a little abuse of notation, we regard $\mathcal{U}_i(\Gamma_i)$ as the uncertainty set.*

$$\mathcal{U}_i(\Gamma_i) = \left\{ \mathbf{z}^i \in \{0,1\}^{|J|} : \sum_{j \in J} z^i_j \leq \Gamma_i \right\}$$

Under this framework, computing a robust solution $\mathbf{x}$ means solving the optimization problem RP below.

$$RP \quad \max \mathbf{c}^T \mathbf{x}$$

$$\sum_{j \in J} \bar{a}_{ij} x_j + \max_{\mathbf{z}^i \in \mathcal{U}_i(\Gamma_i)} \sum_{j \in J} \delta_{ij} |x_j| z^i_j \leq b_i \quad i \in I$$

$$x_j \in \mathbb{Z} \qquad j \in N$$

The inner max computes the worst case realization in $\mathcal{U}_i(\Gamma_i)$ for solution $\mathbf{x}$. If $z^i_j = 1$ and $x_j > 0$, then the worst value of parameter a_{ij} corresponds to the end of the interval, that is, it takes value $a_{ij} + \delta_{ij}$. If $z^i_j = 1$ and $x_j < 0$, then the worst case value is the beginning of the interval, that is, $a_{ij} - \delta_{ij}$. It follows that the contribution due to the deviation that must be considered for parameter a_{ij} when $z^i_j = 1$ is $\delta_{ij}|x_j|$.

Value Γ_i represents the level of robustness of the computed solution with respect to constraint $i \in I$: when Γ_i increases, the computed solution becomes more robust, because it is resilient to more realizations of the uncertainty, but it is also more expensive; when Γ_i decreases, the solution protects against less realizations of the uncertainty, but it is cheaper. This is the so-called *price of robustness* [20]. Values δ_{ij} represent, instead, the level of uncertainty of parameter a_{ij} and the confidence in the nominal value $\bar{a}_{ij}$: if $\delta_{ij} = 0$, then parameter a_{ij} is not uncertain, as it always assumes the nominal value $\bar{a}_{ij}$; the larger δ_{ij}, the larger the level of uncertainty of a_{ij} and the smaller the confidence in nominal value $\bar{a}_{ij}$. As it is easy to see, the larger Γ_i and δ_{ij}, $j \in J$, the larger the uncertainty set.

While it is known which is the probability that a constraint $i \in I$ is violated by solution $\bar{\mathbf{x}}$ when more than Γ_i parameters change at the same time [20], no robust optimization study investigates which is the largest uncertainty set that a given solution $\bar{\mathbf{x}}$ can handle. This is not only an interesting theoretical question, but also has some relevance in practical applications of robust optimization. Indeed, although a solution might have been computed using the best available approach to handle data uncertainty, no method considers the possibility that the information used to determine the uncertainty set (or the probability distribution, for stochastic methods) can become, at a certain point, unreliable. In real-life problems, it may happen that the lifetime of the implemented solution may be very long, making any estimation outdated after some time, or some unexpected phenomena can arise, deeply changing the whole system, beyond any prediction.

Consider, for example, the problem of designing a telecommunication network: once realized, the network is supposed be used for a very long time (decades), given the cost and the time effort required to build it. Although sophisticated prediction methods may have been used to forecast future traffic (uncertainty set), after some time any prediction becomes naturally obsolete. Moreover, some technological innovations that could not be predicted when the network was realized (e.g., smartphones), contribute to delineate a completely different scenario with respect to the initial assumptions. Similarly, if one has to locate some health services over a territory, it may happen that these services will no longer correspond to the needs of the population of that territory at some point, because the population may increase, decrease or modify its age structure after some (long) time. In addition, an unexpected pandemic may occur, changing the perspective completely, and possibly permanently, on health services and population needs. As a consequence, there are many systems (solutions) that either were not planned in a robust way or were planned using predictions that are now obsolete. Those solutions may not be immediately modifiable and, then, it would be interesting to know how long they can still be used, despite the changes that occurred in the meantime, before major modifications are required.

To fill this gap, the focus of this paper is not on how to compute robust solutions, but on how to find the largest possible uncertainty set that a given solution can support, that is, on computing the level of robustness and the increase in the level of uncertainty that a given solution $\bar{\mathbf{x}}$ can handle, without becoming infeasible. Formally, we want to answer the following questions.

1. If new nominal and/or deviation values replace the old ones (or deviations are introduced for the first time), which is the maximum level of robustness that solution $\bar{\mathbf{x}}$ can guarantee?

2. If we want to ensure a given level of robustness, which is the maximum increase in the uncertainty level (that is, in the deviation values) that we can accept, before $\mathbf{x}$ becoming infeasible?

We will show that it is possible to answer to both questions in polynomial time. In §2 we address the first question. In §3 the second one is considered. Conclusions are given in §4.

2 The Maximum Robustness problem

The first question corresponds to the Maximum Robustness (MR) problem below.

Problem 2.1. *Given a solution $\bar{\mathbf{x}}$ and values $\bar{a}_{ij}$ and δ_{ij} for $i \in I, j \in J$, the MR problem consists of determining the largest Γ for which $\bar{\mathbf{x}}$ remains feasible for any realization of the uncertainty, that is, value $\Gamma^* = \min_{i \in I}\{\Gamma_i^*\}$, where Γ_i^* is the largest value for which constraint i is still satisfied.*

Assume that $\bar{\mathbf{x}}$ is feasible for the DR problem, that is, when no parameter deviates from the nominal value ($a_{ij} = \bar{a}_{ij}$ for any $j \in J$). Γ_i^* can be computed solving the problem below.

$$
\begin{aligned}
MR^i_{\bar{\mathbf{x}}} \quad & \min \Gamma_i \\
& \sum_{j \in J} \bar{a}_{ij}\bar{x}_j + \max_{\mathbf{z}^i \in \mathcal{U}_i(\Gamma_i + 1)} \sum_{j \in J} \delta_{ij} |\bar{x}_j| z^i_j > b_i \\
& \Gamma_i \in \mathbb{Z}_+
\end{aligned}
$$

Recall that $\bar{\mathbf{x}}$ is given, the only variable of the outer problem is Γ_i, while in the inner max, when both Γ_i and $\bar{\mathbf{x}}$ are constant, the variables are represented by vector $\mathbf{z}^i$. Problem $MR^i_{\bar{\mathbf{x}}}$ computes the smallest value Γ_i for which $\bar{\mathbf{x}}$ satisfies constraint i for any realization in uncertainty set $\mathcal{U}_i(\Gamma_i)$, while this is no longer true for $\mathcal{U}_i(\Gamma_i + 1)$. $MR^i_{\bar{\mathbf{x}}}$ always admits feasible solution $\Gamma_i = 0$, since $\bar{\mathbf{x}}$ is a feasible solution of DR.

We show below that Γ_i^* can be computed in polynomial time by finding the minimum cardinality cover of a *binary knapsack*. Denote by d_i the amount $b_i - \sum_{j \in J} \bar{a}_{ij}\bar{x}_j$ and let w^i_j be equal to $\delta_{ij}|\bar{x}_j|$.

Theorem 2.1. *Solving $MR^i_{\bar{\mathbf{x}}}$ amounts to compute a minimum cardinality cover of the knapsack below.*

$$K = \left\{ \mathbf{z}^i \in \{0,1\}^{|J|} : \sum_{j \in J} w^i_j z^i_j \leq d_i \right\}$$

Proof. Recall that $\mathbf{z}^i$ is the incident vector of a set of parameters that deviate from the nominal value at the same time. The vectors corresponding to deviations supported by $\bar{\mathbf{x}}$, that is, to deviations that do not lead to the violation of constraint i, correspond to the following set.

$$\left\{ \mathbf{z}^i \in \{0,1\}^{|J|} : \sum_{j \in J} \bar{a}_{ij}\bar{x}_j + \sum_{j \in J} \delta_{ij}|\bar{x}_j| z^i_j \leq b_i \right\}$$

Using d_i and $\mathbf{w}^i$ defined above, it becomes set K. Note that, since we assumed that $\bar{\mathbf{x}}$ is feasible when all the parameters are at the nominal value, then $\sum_{j \in J} \bar{a}_{ij}\bar{x}_j \leq b_i$ and, hence, $d_i \geq 0$. Moreover, since $\delta_{ij} \geq 0$, then $w^i_j = \delta_{ij}|\bar{x}_j| \geq 0$ for any $j \in J$. A cover of K is a set $C \subseteq J$ such that $\sum_{i \in C} w^i_j > d_i$. It corresponds to a set of variables z^i_j that cannot take value 1 at the same time, without violating the considered inequality, that is, to a vector $\mathbf{z}^i$ not supported by $\bar{\mathbf{x}}$. Let C^* be a minimum cardinality cover of K, that is, $C^* \in \arg\min\{|C| : C \text{ is a cover of } K\}$. Since C^* is a cover of K, then $\Gamma_i^* \leq |C^*| - 1$, as C^* represents a vector $\mathbf{z}^i$ not supported by $\bar{\mathbf{x}}$. Since C^* is a minimum cardinality cover, then no set with less than $|C^*|$ elements violates constraint i. It follows that $\Gamma_i^* = |C^*| - 1$. □

A polynomial time algorithm for computing a minimum cardinality cover of K is illustrated below.

Theorem 2.2. *A minimum cardinality cover of K can be computed in polynomial time in $O(|J| \log |J|)$.*

Proof. Order the variables $z^i_j, j \in J$ in non decreasing order with respect to values $\mathbf{w}^i$, with ties broken arbitrarily. Ordering vector $\mathbf{w}^i$ can be done in $O(|J| \log |J|)$. Start adding to set C, initially empty, an index j at a time, according to the order, and stop as soon as $\sum_{j\in C} w_j^i > d_i$ or $C = J$. Assume that, when the algorithm terminates, $C \subset J$. Then, C is a minimum cardinality cover of K. In fact, by construction, no $D \subset C$ is a cover. Moreover, for any $j \in C$ and $h \notin C$, $w_j^i \geq w_h^i$ and, hence, we cannot reduce the number of elements in C by replacing them with the ones outside. If $C = J$, then no cover exists and, hence, $\bar{\mathbf{x}}$ supports any realization of the uncertainty in $\mathcal{U}_i(|J|)$. □

3 The Maximum Uncertainty Level problem

The second question corresponds to the Maximum Level of Uncertainty (MUL) problem that follows. Given to modified external conditions, deviation values $\boldsymbol{\delta}$ could become no longer reliable, because the level of uncertainty in the system could increase. Since the real probability distribution is unknown, we assume here that, if the uncertainty in the system increases, all the parameters are affected in the same way, that is, the increasing in the deviation values is the same for all of them.

Problem 3.1. *Given $\bar{\mathbf{x}}$ and values $\Gamma_i > 0$, $\bar{a}_{ij}$ and δ_{ij} for $i \in I, j \in J$ ($\boldsymbol{\delta} = \mathbf{0}$ if $\bar{\mathbf{x}}$ was computed without considering any uncertainty), the MUL problem consists of determining the largest increase λ for values δ_{ij}, such that $\bar{\mathbf{x}}$ remains feasible for any realization, that is, value $\lambda^* = \min_{i\in I}\{\lambda_i^*\}$, where λ_i^* is the maximum increase in the deviation values for constraint $i \in I$.*

Assume that $\bar{\mathbf{x}}$ is a feasible solution of RP for the given $\bar{\mathbf{a}}$, $\boldsymbol{\delta}$ and Γ_i, although it is not necessarily optimal for the considered Γ_i. Value λ_i^* corresponds to the optimal solution of the problem below.

$$
\begin{aligned}
MUL_{\bar{\mathbf{x}}}^i \quad & \max \lambda_i \\
& \sum_{j\in J} \bar{a}_{ij}\bar{x}_j + \max_{\mathbf{z}^i\in\mathcal{U}_i(\Gamma_i)} \sum_{j\in J} (\delta_{ij} + \lambda_i)|\bar{x}_j| z_j^i \leq b_i \\
& \lambda_i \geq 0
\end{aligned}
$$

Recall that Γ_i is a constant value here. The only variable of the outer problem is λ_i, while in the inner max, where also λ_i is regarded as a constant, the variables are represented by vector $\mathbf{z}^i$. $MUL_{\bar{\mathbf{x}}}^i$ always admits feasible solution $\lambda_i = 0$, since $\bar{\mathbf{x}}$ is a feasible solution of RP for the given $\bar{\mathbf{a}}$, $\boldsymbol{\delta}$ and Γ_i. Recall that $d_i = b_i + \sum_{j\in J} \bar{a}_{ij}\bar{x}_j \geq 0$ and $w_j^i = \delta_{ij}|\bar{x}_i|$. We show below how to obtain a problem without the inner max by linear duality.

Theorem 3.1. *The $MUL_{\bar{\mathbf{x}}}^i$ problem can be solved by solving the problem below.*

$$
\begin{aligned}
MULsl_{\bar{\mathbf{x}}}^i \quad & \max \lambda_i \\
& \Gamma_i\alpha_i + \sum_{j\in J} \beta_j^i \leq d_i \\
& \alpha_i + \beta_j^i - |\bar{x}_j|\lambda_i \geq w_j^i \quad j \in J
\end{aligned}
$$

$$\lambda_i \geq 0, \alpha_i \geq 0$$
$$\beta^i_j \geq 0 \qquad j \in J$$

Proof. Following what done in [20], we can exploit linear duality to deal with the inner max of problem $MUL^i_{\bar{\mathbf{x}}}$. The inner problem and its dual are reported below.

$$
\begin{array}{llll}
P & \max \sum_{j \in J} (\delta_{ij} + \lambda_i)|\bar{x}_j| z^i_j & D & \min \Gamma_i \alpha_i + \sum_{j \in J} \beta^i_j \\
(\alpha_i) & \sum_{j \in J} z^i_j \leq \Gamma_i & (z^i_j) & \alpha_i + \beta^i_j \leq (\delta_{ij} + \lambda_i)|\bar{x}_j| \quad j \in J \\
(\beta^i_j) & 0 \leq z^i_j \leq 1 \qquad j \in J & & \alpha_i \geq 0, \boldsymbol{\beta}^i \in \mathbb{R}^{|J|}_+
\end{array}
$$

By strong duality, since the primal problem P is feasible and bounded for any λ_i, so is the dual and the two problems produce the same optimal value. Hence, one can replace the primal by the dual in formulation $MUL^i_{\bar{\mathbf{x}}}$. While the max is not redundant if we use P, the min is redundant if we use D, since we have a less than or equal to constraint. Therefore, the min can be eliminated and, setting $d_i = b_i - \sum_{j \in J} \bar{a}_{ij}|\bar{x}_i|$ and $w^i_j = \delta_{ij}|\bar{x}_i|$, we obtain problem $MULsl^i_{\bar{\mathbf{x}}}$. □

Given the above transformation, we can prove that the problem of computing the maximum increase λ^*_i of uncertainty that $\bar{\mathbf{x}}$ can handle is polynomially solvable.

Theorem 3.2. *Value λ^*_i can be computed in polynomial time.*

Proof. Value λ^*_i is the optimal value of the compact linear programming formulation $MULsl^i_{\bar{\mathbf{x}}}$ and linear programming problems are polynomially solvable [23]. □

It is easy to see that we can use the same arguments to prove that the above results also holds when: $\boldsymbol{\delta} > \mathbf{0}$ and the deviations change from δ_{ij} to $(1+\lambda_i)\delta_{ij}$; the deviations are independent from one another and they change from δ_{ij} to $\delta_{ij}+\tau_{ij}$ and $\lambda_i = \sum_{j \in J} \tau_{ij}$; $\boldsymbol{\delta} > \mathbf{0}$, the deviations are independent of one another and they change from δ_{ij} to $(1-\tau_{ij})\delta_{ij}$ and $\lambda_i = \min_{j \in J}\{\tau_{ij}\}$.

4 Conclusions

Differently from the traditional robust optimization perspective, here we were not interested in computing a robust solution, but the focus is on determining the largest uncertainty set that a given solution can handle. We considered the cardinality constrained uncertainty set, which is characterized by two parameters: the deviations of the uncertain parameters from their nominal values, that measure the level of uncertainty in the system; the number of parameters that can deviate from the nominal value at the same time, that gives a measure of the desired robustness. The larger the values of these parameters, the larger the uncertainty set. We proved that, for a given solution, the corresponding robustness and the maximum increase in the uncertainty level, that is, the largest uncertainty set that it can support, can be computed in polynomial time.

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