Playing Stackelberg security games in perfect formulations

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Abstract

Protecting critical infrastructure from intentional damage requires foreseeing the strategies of possible attackers. The problem faced by the defender of such infrastructure can be formulated as a Stackelberg security game. A *defender* must decide what specific targets to protect with limited resources, maximizing their expected utility (e.g., minimizing damage value) and considering that a second player (or players), called *attacker*, responds in the best possible way.

Since Stackelberg security games are generally NP-Hard, the main challenge in finding optimal strategies in real applications is developing efficient methodologies for large instances.

We propose a general methodology to find a Strong Stackelberg Equilibrium for Stackelberg security games whose set of defender's mixed strategies can be represented as a perfect formulation. This methodology consists in two steps. First, we formulate the problem using variables representing the probabilities of each target being defended. The formulation must be either a polynomial-size MILP and/or a MILP with an exponential size of constraints that can be efficiently separated through branch-and-cut. In the second step, we recover the mixed strategies in the original space efficiently (in polynomial time) using column generation. This methodology has been applied in various security applications studied in the last decade. We generalize and propose new examples. Finally, we provide extensive computational study of different formulations based on marginal probabilities.

Keywords: OR in Defense, Bilevel Optimization, Polyhedral structure, Stackelberg Games

1. Introduction

Critical infrastructure ensures the proper functioning of a country [41] and includes, e.g., airports and railways. Damage to a country's critical infrastructure can cause significant economic loss and seriously affect its population [9, 10].

The growing threat of terrorist attacks on this infrastructure has increased the need for security systems in recent years [11, 38]. Security forces and terrorists act respectively as defenders and attackers of critical infrastructure. The defender has limited resources to protect the infrastructure's most sensitive components. At the same time, the attacker or attackers seek to cause maximum damage by targeting one or more critical infrastructure components. This problem falls in the category of a *Stackelberg security game* (SSG for short). These games are a particular case of the Stackelberg games (SG), which model sequential player interactions.

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A Stackelberg security game models a competitive and successive interaction between two agents, usually denoted as defender and attacker. In this interaction, the defender has to use limited resources to protect the set of targets that optimizes the expected protected value without knowing the strategy that the attacker will deploy but aware of the fact that the attacker will, at a later time, respond optimally [8]. The main characteristic of SSGs is that for both defender and attacker, there is a utility per target that only depends on whether the target being attacked was or was not protected by the defender at the time of the attack [37].

In this article, we deal with Bayesian Stackelberg security game models, in which the defender knows that there exists a set of potential attackers as well as the probability of each one of them being the actual attacker, but only faces one of them [8]. Because of this uncertainty regarding the attackers, the game belongs to the category of games with incomplete information. Additionally, the game has perfect information, as each player is aware of all the previous events [18].

In our game, the defender deploys an optimal mixed (randomized) defense strategy. The mixed strategy is a probability distribution over all the possible actions in the defender space. The defender's strategy is observed by the attackers, who respond optimally. Due to its importance in security applications, and being the dominant modeling choice in the literature [24, 39, 19, 5, 6, 25], we find a Strong Stackelberg Equilibrium (SSE), i.e., in case of a tie between different possible responses of the follower, the most convenient response for the leader is chosen.

One of the challenges in solving Stackelberg security games in real applications is the size of the problem. In fact, finding optimal strategies for both Bayesian SG and Bayesian SSG is NP-Hard [13, 30]. The difficulty of computing an SSE in a Stackelberg security game is directly related to the defender strategy space [40]. Even in the simplest case, when the defender's strategy is to assign resources to protect targets, the enumeration of all possible actions is intractable.

Different formulations of the problem and methods have been devised to find mixed strategies for the defender [8]. One approach is to formulate and solve the problem using variables associated with the defender and attackers' pure strategies, where the decision variables represent the probabilities with which each action is played. The optimal solution of these non-compact formulations provides the SSE directly. Another approach is to reformulate the problem as a compact formulation using as variables the frequencies of the targets being defended, or *marginal probabilities* (see Figure 1). This reformulation reduces the number of variables, but it requires retrieving a mixed strategy in the original defender's strategy space, to implement the solution. This is not always possible or efficient, and it depends on the representability of the defender strategy domain as a set with good *polyhedral properties.*

We focus on the class of SSG problems for which the space of defender mixed strategies satisfies two properties: it can be represented as a perfect formulation, and the number of constraints is either polynomial or it can be separated efficiently. Perfect formulations enjoy the property that their linear relaxation coincides with the convex hull of the (integer) feasible points of the original problem [12]. Typical examples of such problems are the shortest path or bipartite matching problems. There also exist problems whose feasible set can be represented by a polyhedron with exponentially many linear inequalities that can be separated efficiently, such as matching problems in general graphs. In this article, we take advantage of such problems and propose a general methodology to find an SSE when the strategies of the defender constitute such combinatorial objects.

Our contributions are the following: First, we propose a general methodology to find an SSE for SSGs for problems in which the set of defender strategies can be represented as a perfect formulation, and the set of constraints has either polynomial size or, having an exponential size, can be separated in polynomial time. In this last case, the problem is solved via branch-and-cut. This methodology (see Figure 1) consists in first formulating the problem of computing an SSE as a MILP using a polynomial number of variables associated with the marginal probabilities (compact formulation), and solving it (solution phase). In a second stage, we obtain feasible strategies in the original space using a column generation approach where the pricing subproblem is polynomial (implementation phase). This two-stage method outperforms the traditional branch-and-price based on the non-compact formulation D2 of Jain et al. [23]. We use some problems already studied in the literature as examples: protecting targets with fairness constraints and combined resources.

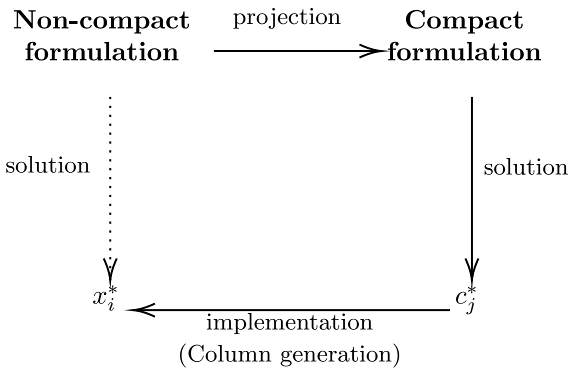

Figure 1: Graphical representation of our approach.

Second, we propose a new formulation and solve the case where the goal is to protect one or two adjacent targets with each resource. Third, we study for the first time a SSG where the defender has to protect a spanning tree on a graph, and the attacker attacks one edge. For this game, we study several formulations and four ways to represent spanning tree strategies, and we provide an algorithmic discussion. These games are new and fall in the category of a Stackelberg security game. Fourth, we provide experiments that show that our method outperforms branch-and-price over non-compact formulations.

The paper is structured as follows. In Section 2, we provide a literature review. Section 3 introduces the base model and formulations utilized throughout this article. Section 4 outlines a general methodology for finding an SSE in SSGs when it is possible to represent the defender's strategies with a perfect formulation efficiently. In Section 5, we apply the proposed methodology to various known and new problems. We present a computational study in Section 6, and our conclusions are presented in Section 7.

2. Literature Review

In recent years, many articles have studied the complexity of solving security games in different contexts. Conitzer and Sandholm [13] discuss polynomial-time algorithms to find an SSE in Stackelberg Games, considering one type of follower. These authors also establish that in the Bayesian case, with more than two followers, the problem of computing an SSE is NP-hard. For Bayesian security games, calculating Stackelberg strategies is also NP-Hard, even if each attacker type has only a single resource [30].

Korzhyk et al. [26] studied Stackelberg security games with one type of attacker, where each security resource can be allocated to protect a subset of targets. They provided polynomial-time algorithms for games with heterogeneous resources and defense strategies that protect single targets, also for games with homogeneous resources and defense strategies that protect at most 2 targets. They also showed that games with an underlying bipartite graph, heterogeneous resources, and defense strategies that protect at most 2 targets are NP-hard, even when the game is zero-sum. Furthermore, the research showed the NP-hardness of games with homogeneous resources that can protect up to three targets per resource.

Letchford and Conitzer [29] explored security games that involve one type of attacker, where targets are nodes on an underlying graph and security resources patrol various substructures of the graph. They provided polynomial-time algorithms for solving Stackelberg games under certain conditions. For example, suppose the graph is a rooted set of trees and resources are heterogeneous, and the defense strategies are paths starting from a specific node known as the root. In that case, it is possible to use a polynomial-time algorithm. Similarly, if the graph is a set of paths, resources are homogeneous, and each security resource protects subpaths of the graph. They also find structures that result in NP-hardness. For instance, if defender resources are heterogeneous, the graph is a path, and defense strategies protect edges, the problem becomes NP-hard. Similarly, if resources are homogeneous, the graph is general, and the defense strategies are paths from a specific node called the root. Also, if the resources are heterogeneous, the graph is a tree, and the defense strategies protect paths that pass through the root.

Several mixed-integer linear programming (MILP) formulations have been developed to address the NP-hardness of computing an SSE. For instance, Paruchuri et al. [35] suggested a mixed-integer formulation called DOBSS that can solve Bayesian SGs. This approach replaces the follower's best response with linear inequalities while linearizing the leader's objective function. Another formulation for SSGs using variables related to marginal probabilities is ERASER, which was introduced by Kiekintveld et al. [25]. Furthermore, Casorrán et al. [8] studied multiple formulations for Bayesian SG and SSGs, examining their relationship. They introduced a new SSG formulation called Mip-p-S. They also proved that for the case of 1 attacker, this is a perfect formulation.

In the article of Budish et al. [7], the authors discussed relevant work and introduced the idea of implementability for a random mixed strategy. A set of constraints is implementable if any random mixed strategy under that set can be decomposed as a convex combination of pure strategies that satisfy the constraints. The authors derived sufficient and necessary conditions for implementability based on the bi-hierarchy structure of constraints. Examples that fulfill this condition include One-to-One Assignments, the Birkhoff-von Neumann Theorem, Endogenous Capacities, Group-Specific Quotas, and Course Allocation.

Finally, Xu [40] demonstrated that for any given set I of pure defender strategies, the following problems are reducible to each other in polynomial time: a) Combinatorial optimization over I, b) Computing the minimax equilibrium for zero-sum security games over I, c) Computing the strong Stackelberg equilibrium for security games over I, and d) Computing the best or worst (for the defender) Nash equilibrium for security games over I. The author's fundamental concept is that the complexity of finding an equilibrium is directly linked to the computation of the best response function for the leader. The leader's best response is the strategy that returns the maximum utility for a fixed vector of weights over the targets. Additionally, the author stated that the set of pure strategies primarily determines the complexity of a security game.

We extend the literature, addressing for the first time a class of problems characterized by defender's mixed strategies with marginal probability variables that can be efficiently represented using perfect formulations. We use column generation to obtain feasible defender mixed strategies and demonstrate how the proposed methodology can be applied to different problems. Additionally, for the first time, we solve a SSG problem that focuses on protecting a tree structure over a graph. This scenario arises where preserving a connection between a supplier and demand points is mandatory, as in product distribution and warfare. Furthermore, we propose a novel approach for protecting one or two adjacent targets within a network.

Our results are also relevant to the definition of implementability introduced by Budish et al. [7]. We exploit the fact that when working with defender strategies whose marginal probabilities are modeled as perfect formulations, decomposition into feasible mixed strategies is always possible.

3. Notation and MILP formulations

This section reviews equivalent MILP formulations to solve Stackelberg security games.

3.1. Stackelberg security game

Consider a Bayesian Stackelberg security game where players aim to maximize their payoff in a sequential, one-off encounter. In this game, the defender faces a set of K attackers, each with a probability p^k of acting or appearing.

We denote by I the set of the defender's pure strategies and by J the set of the attacker's pure strategies. Note that each attacker attacks one target, so each pure strategy $j \in J$ is associated with target j being attacked. From the defender's side, each pure strategy $i \in I$ is associated with a subset of targets to be defended. A mixed strategy for the leader means that every pure strategy i is chosen with a probability x_i. Analogously, a mixed strategy for the attacker k involves selecting each pure strategy j with a probability q_j^k. Without loss of generality, we can assume that $\mathbf{q}$ determines a pure strategy since it is the best response to $\mathbf{x}$: it maximizes the attacker expected payoff given $\mathbf{x}$. In other words, $q_j^k \in \{0,1\}$ (we refer the reader to Casorrán et al. [8] for a proof).

Determining the optimal solution in this domain depends on how we define acting optimally, or equivalently, the type of equilibrium we choose. Leitmann [28] pointed out the need to generalize the standard Stackelberg equilibrium in static games, introducing a conservative version of it. This concept is formalized by Breton et al. [3] as the weak Stackelberg Equilibrium, and the strong Stackelberg Equilibrium [16, 2]. In all cases, the leader selects a strategy to maximize its utility, considering that the follower will respond optimally. If the follower has multiple optimal strategies and is indifferent between them, the strong Stackelberg Equilibrium assumes that the follower will choose the strategy that benefits the leader the most. In contrast, the weak Stackelberg Equilibrium assumes the opposite: that the follower chooses the worst strategy for the leader when there are multiple best responses.

It is worth noting that a strong Stackelberg equilibrium exists in all Stackelberg games, but a weak Stackelberg equilibrium may not exist [31, 14]. Furthermore, the leader can often induce the attacker to choose a preferred action by making an infinitesimal adjustment to her strategy [21]. The strong Stackelberg Equilibrium has received considerable attention and is the most studied case to date [24, 39, 19, 25]. Therefore, in this paper we use the strong Stackelberg equilibrium.

Formally, our setting is represented by an underlying graph $G = (V, E)$. Within this graph, the set J represents targets or objectives that can be attacked by the follower or attacker $k \in K$. These objectives $j \in J$ can be nodes, edges, or subgraphs of G.

The leader or defender possesses a limited number m of security resources to protect the targets. If all resources can protect the same set of targets, the resources are *homogeneous*. Otherwise, they are *heterogeneous*. Recall that each attacker attacks one target $j \in J$.

One of the main characteristics of SSG is that the payoffs for both the defender and attacker depend solely on the targets being attacked [37]. For each target $j \in J$ facing an attacker type $k \in K$, the possible payoffs of the defender are $D^k(j|p)$ and $D^k(j|u)$ if target $j \in J$ is protected or unprotected, respectively. Analogously, $A^k(j|p)$ and $A^k(j|u)$ are the payoffs for the attacker type k when target j is defended or not, respectively. We assume that $D(j|p) \geq D(j|u)$ and $A(j|p) \leq A(j|u)$.

In a SSG, the defender's decisions will affect the attacker's decisions. This hierarchical structure can be modeled as a bilevel optimization problem as follows:

(Bilevel-SSG)

$$\max_{x,q,r} \sum_{j\in J} \sum_{k\in K} p^k q_j^k \left(\sum_{i\in I: j\in i} x_i D^k(j|p) + \left(1 - \sum_{i\in I: j\in i} x_i\right) D^k(j|u) \right) \tag{3.1}$$

$$\text{s.t} \sum_{i\in I} x_i = 1, \mathbf{x} \in [0,1]^{|I|} \tag{3.2}$$

$$q^k = \arg\max_{r^k} \left\{ \sum_{j\in J} r_j^k \left(\sum_{i\in I: j\in i} x_i A^k(j|p) + \left(1 - \sum_{i\in I: j\in i} x_i\right) A^k(j|u) \right) \right\} \quad \forall k \in K \tag{3.3}$$

$$\sum_{j\in J} r_j^k = 1, \boldsymbol{r}^k \in \{0,1\}^{|J|} \qquad \forall k \in K \tag{3.4}$$

The objective function (3.1) maximizes the defender's expected reward. Constraints (3.2) require that the sum of probabilities of the pure strategies used by the defender to form a mixed strategy must add up to 1. The second level is modeled by (3.3)-(3.4), and it states that a) each follower k

will attack one target j, and b) the followers respond to the leader's decision optimizing their payoff. If multiple optimal strategies exist for the follower, the strategy that favors the leader is chosen. The auxiliary variables r_j^k allow us to define this second level.

From this bilevel formulation, three single-level, MILP SSG reformulations are presented in the literature: D2$_{x,q,f,s}$ [8], Mip-p-G$_{h,q}$ [8] and DOBSS$_{h,q,s}$ [35]. The defender mixed strategies are obtained directly by solving these formulations, which we call *non-compact*. In these formulations, the defender generally has exponentially many pure strategies. If there are m security resources and n targets to defend, the number of possible pure defender strategies is $\binom{n}{m}$. As a result, MILP and LP formulations that enumerate strategies are exponential in size [13].

We now consider formulations specific to Stackelberg security games by exploiting the payoff structure of these games, proposed by Kiekintveld et al. [25]. These *compact formulations* are cast by introducing variables c defined as

$$c_j = \sum_{i \in I : j \in i} x_i, \quad j \in J \tag{3.5}$$

and dropping out variables x. Here, $j \in i$ denotes that defender strategy i includes protecting target j. If we represent strategy i as a vector of size J, in which protected targets are represented by a 1 and unprotected targets by a 0, the j^{th} position in vector i is equal to 1. Variables c_j are interpreted as the probability that each target j is covered (or protected) under the mixed strategy x. This can be done because of the payoff structure: c_j can replace expressions involving x variables in (3.1)-(3.3), reducing the number of variables of each formulation. A vector c is said to be implementable if we can retrieve a vector x satisfying (3.5). The solution of compact formulations is not always implementable; to be so, the model has to fulfill specific requirements that we will detail in the next section.

We review three different compact formulations. The first model is a generalization of the game proposed by Kiekintveld et al. [25], and it has the following formulation:

(ERASER$_{c,q,f,s}$)

$$\max_{c,f,s,q} \sum_{k \in K} p^k f^k \tag{3.6}$$

$$\text{s.t } f^k \leq D^k(j|p)c_j + D^k(j|u)(1-c_j) + (1-q_j^k)M \qquad \forall j \in J, k \in K \tag{3.7}$$

$$0 \leq s^k - A^k(j|p)c_j - A^k(j|u)(1-c_j) \leq (1-q_j^k)M \qquad \forall j \in J, k \in K \tag{3.8}$$

$$\sum_{j \in J} q_j^k = 1 \qquad \forall k \in K \tag{3.9}$$

$$q_j^k \in \{0,1\} \qquad \forall j \in J, k \in K \tag{3.10}$$

$$s^k, f^k \in \mathbb{R} \qquad \forall k \in K \tag{3.11}$$

$$\boldsymbol{c} \in \text{conv}(\mathcal{P}). \tag{3.12}$$

The objective function (3.6) maximizes the expected utility of the defender. Constraints (3.7) and (3.8) ensure that the leader and followers choose strategies that maximize their respective expected payoffs. Constraints (3.9) indicate that the attacker $k \in K$ attacks a single target $j \in J$. Constraint (3.10) and (3.11) state the nature of variables. The expression (3.12) says that the marginal probability vector $\boldsymbol{c} = [c_1, c_2, ...c_j, ..., c_{|J|}]$ is a convex combination of points in set $\mathcal{P}$. This set $\mathcal{P}$ contains all binary vectors $\boldsymbol{c}^i \in \mathcal{P}$ that encode defender pure strategies, i.e., $c_j^i = 1$ if $j \in i$ and $c_j^i = 0$ otherwise. For a given marginal probability vector $\boldsymbol{c}$, the coefficients in the convex combination represent the probabilities associated with the corresponding pure strategy and constitute the mixed strategy. Remark that expression (3.12) is equivalent to expression (3.5) and $\text{conv}(\mathcal{P}) \subseteq [0,1]^{|J|}$.

Another equivalent formulation is Mip-p-S$_{y,q}$ [8], which can be obtained by using the transformation of variables $y^k_{lj} = c_l q^k_j$ $\forall l, j \in J$ and $k \in K$.

$$
\begin{aligned}
&(\text{Mip-p-S}_{y,q}) \\
&\max_{y,q} \sum_{j\in J}\sum_{k\in K} p^k(D^k(j|p)y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) && && (3.13)\\
&\text{s.t} \sum_{j\in J} y^k_{lj} = \sum_{j\in J} y^1_{lj} && \forall l \in J, k \in K && (3.14)\\
&0 \le y^k_{lj} \le q^k_j && \forall l, j \in J, k \in K && (3.15)\\
&A^k(j|p)y^k_{jj} + A^k(j|u)(q^k_j - y^k_{jj}) - \\
&\quad A^k(l|p)y^k_{lj} - A^k(l|u)(q^k_j - y^k_{lj}) \ge 0 && \forall j \in J, k \in K && (3.16)\\
&\sum_{j\in J} q^k_j = 1 && \forall k \in K && (3.17)\\
&q^k_j \in \{0,1\} && \forall j \in J, k \in K && (3.18)\\
&Proy_c(y^k_{lj}) \in \text{conv}(\mathcal{P}) && \forall l, j \in J, k \in K && (3.19)
\end{aligned}
$$

The objective function (3.13) maximizes the expected utility of the defender. Constraints (3.14) express that every follower sees the same strategy of the defender. Since y^k_{lj} represents a joint probability of a target being attacked and being protected, expressions (3.15) specify an upper and lower bound of variables y^k_{lj}. Constraints (3.16) ensure that each follower responds optimally. Constraints (3.17) indicate that the attacker $k \in K$ attacks a single target $j \in J$. Constraints (3.18) state the nature of variables $\boldsymbol{q}$. The expression (3.19) says that the total marginal probability belongs to the convex hull of the polytope of defender strategies $\mathcal{P}$.

A further equivalent MILP formulation for the SSG is SDOBSS$_{y,q,s}$ [8], and it can be constructed from Mip-p-S$_{y,q}$ by replacing constraints (3.16) by:

$$0 \le s^k - A^k(j|p)\sum_{l\in J} y^k_{jl} - A^k(j|u)(1 - \sum_{l\in J} y^k_{jl}) \le (1 - q^k_j)M \qquad \forall j \in J, k \in K \qquad (3.20)$$

Once Mip-p-S$_{y,q}$ and SDOBSS$_{y,q,s}$ are solved, and variables y^k_{jl} and q^k_j are known, variables c_j can be directly recovered.

Formulations ERASER$_{c,q,f,s}$ and SDOBSS$_{y,q,s}$ use big-M to model optimal responses. In the article of Casorrán et al. [8] it is shown that the tightest correct M values are:

- In (3.7), $M = \max_{l\in J}\{D^k(l|p), D^k(l|u)\} - \min\{D^k(j|p), D^k(j|u)\} \forall j \in J, k \in K$.
- In (3.8) and (3.20), $M = \max_{l\in J}\{A^k(l|p), A^k(l|u)\} - \min\{A^k(j|p), A^k(j|u)\} \forall j \in J, k \in K$.

ERASER$_{c,q,f,s}$, Mip-p-S$_{y,q}$ or SDOBSS$_{y,q,s}$ are all compact formulations. In general, they are more efficient in terms of computational solving times compared to non-compact formulations such as D2$_{x,q,f,s}$ [8], Mip-p-G$_{h,q}$ [8] and DOBSS$_{h,q,s}$ [35] However, compact formulations require post-processing to obtain the mixed strategy $\mathbf{x}$, and their solutions are not always implementable strategies.

In terms of formulation size, and excluding the defender strategy space, ERASER$_{c,q,s,f}$ has the fewest number of constraints and variables, with a complexity of $O(|J||K|)$ [8]. On the other hand, Mip-p-S$_{q,y}$ and SDOBSS$_{q,y,s}$ have complexities of $O(|J|^2|K|)$ [8], resulting in LP relaxations that are more complex and time-consuming to solve compared to ERASER$_{c,q,s,f}$. We do not consider the complexity of the defender space since it remains the same for all three formulations and depends on the game type.

An important theoretical result of Casorrán et al. [8] is that the LP relaxation of Mip-p-S$_{y,q}$ is tighter in comparison to ERASER$_{c,q,f,s}$ and SDOBSS$_{y,q,s}$ for the case of protecting m single targets. In Section 6, we will observe the quantitative difference between formulations' relaxations through computational study. For simplicity, we do not use subscripts when mentioning these models from now on.

In the following section, we prove that if the set of defender strategies can be represented using a perfect formulation, the resulting mixed strategy will always be implementable. Additionally, the post-processing required to obtain the mixed strategy from an optimal solution of a compact formulation can be accomplished in polynomial time under certain conditions.

4. General Approach

Our general approach is applicable to Stackelberg security games whose defender mixed strategies can be efficiently represented with a perfect formulation. This methodology consists in first formulating the corresponding MILP using marginal probability variables, a so-called compact formulation. This representation can have either a polynomial-size MILP or a MILP with an exponential size of constraints that can be separated in polynomial time through branch-and-cut.

With this purpose, in this section we first define and analyze perfect formulations and their relationship to Stackelberg security games (subsection 4.1). We prove that, for the class of problems whose set of defender mixed strategies can be represented by using a perfect formulation, the solution of a compact formulation of the problem always corresponds to an associated mixed strategy, and that once a solution to the compact formulation has been obtained, the associated mixed strategy can be found in polynomial time.

Some compact formulations have a polynomial number of constraints (e.g., combined resources, and m targets with fairness constraints). These cases can be solved straightforwardly, for example, using a solver. However, there are cases in which the number of constraints is exponential. In subsections 4.1 and 4.2, we describe how to solve a compact Stackelberg game formulation, using branch-and-cut (when needed) and to use a column generation method to recover a mixed strategy corresponding to an optimal marginal vector, even in the case of an exponential number of constraints, provided that they are efficiently separable.

4.1. Perfect formulations

A perfect formulation of a set $Q = \{x \in \mathbb{Z}^n : Ax \leq b\}$ is a linear system of inequalities $A'x \leq b'$, such that $\text{conv}(Q) = \{x \in \mathbb{R}^n : A'x \leq b'\}$ [12]. In other words, a linear formulation of an integer optimization problem is perfect if that problem can be solved as a continuous linear problem on that perfect formulation. In particular, if the matrix A used to define set Q is totally unimodular, or if the linear system $Ax \leq b$ is totally dual integral, then it defines a perfect formulation for every $b \in \mathbb{Z}^m$ [12].

In Stackelberg security games, several types of defender mixed strategies can be modeled using perfect formulations. In fact, every mixed integer linear set has a perfect formulation when the data are rational [34]. Still, only some have a useful perfect formulation, i.e., with a polynomial number of constraints or with constraints that can be separated in polynomial time. It is important to note that finding a perfect and efficient formulation of an NP-hard problem, except if NP = P, is a task that cannot be performed easily. Therefore, this methodology should not be applied to cases where the defender strategy set has as its underlying combinatorial structure of an NP-hard problem.

Our general result concerns SSG for which the defender marginal probability vectors can be efficiently modeled with a perfect formulation. With this objective in mind, consider the set $\mathcal{P}$ whose vertices describe the defender's pure strategies. The definition of the set $\mathcal{P}$ will depend on the scenario being addressed, i.e., what is being protected by the defender. The defender's problem can be viewed as an optimization problem on $conv(\mathcal{P})$, as $\mathcal{P} = conv(\mathcal{P})$ [40] and the set $conv(\mathcal{P})$ contains all marginal probability vectors. If $conv(\mathcal{P})$ can be represented with a perfect formulation, then the problem is easier to solve. We not only show this for problems that have been previously solved, but we also show this characteristic for new problems.

The next proposition is proved for ERASER because, for the other two compact formulations (Mip-p-S and SDOBSS), we can retrieve the same variables c_j.

Proposition 1. *It is true that: (i) For any vector of marginal probabilities* $\boldsymbol{c} = \{c_1, c_2, ...c_{|J|}\}$ *feasible to ERASER, a mixed strategy* $\{x_1, x_2, ...x_{|I|}\}$ *can be found that implements the desired marginal probabilities, provided that the set of defense strategies* $\mathcal{P}$ *can be modeled as a perfect formulation. This mixed strategy involves, at most,* $|J| + 1$ *pure strategies with positive probability.*
(ii) Furthermore, if a perfect formulation is known involving only constraints that can be separated in polynomial time, then the mixed strategy can be determined in polynomial time.

Proof. (i) From Carathéodory's theorem, any point in $\boldsymbol{c} \in Conv(\mathcal{P})$ is the convex combination of at most $|J| + 1$ points of $\mathcal{P}$. Recall that $\mathcal{P}$ can be modeled as a perfect formulation, so $Conv(\mathcal{P}) = \mathcal{P}$. The coefficients of this convex combination constitute a mixed strategy that implements the marginal probability vector c.
(ii) Next, given a marginal probability vector $\boldsymbol{c}$ feasible to ERASER, a mixed strategy x that implements $\boldsymbol{c}$ is obtained by solving the following linear system with a possibly exponential number of variables:

$$\sum_{i=1}^{|I|} c_j^i x_i = c_j \qquad \forall j \in J \tag{4.1}$$

$$\sum_{i=1}^{|I|} x_i = 1 \tag{4.2}$$

$$x_i \geq 0 \qquad \forall i \in I, \tag{4.3}$$

where c_j^i is 1 if target j is protected in strategy i. A target j is protected in strategy i if and only if the j-th component of c^i equals to 1 (i.e. $j \in i$), and it equals to 0 otherwise. However, this feasibility problem can be stated as an LP whose dual contains $|J| + 1$ variables and one constraint for each pure strategy $i \in I$. The separation problem for this set of constraints can be solved in polynomial time since it amounts to optimize on $Conv(\mathcal{P})$ (or equivalently, over $\mathcal{P}$). Hence, the dual, and thus the primal of the original problem, can be solved in polynomial time. □

We remark that a marginal probability vector can be induced by different mixed strategies, but each mixed strategy has a unique marginal probability vector.

4.2. Column generation

By Proposition 1, the marginal probabilities vector is always implementable when defense strategies can be modeled as perfect formulations. In other words, a corresponding vector of probabilities over the pure strategies of the defender can be found. In this section, we describe how to use the optimal solutions of ERASER, Mip-p-S, and SDOBSS formulations to retrieve defender strategies $i \in I$ and their corresponding probabilities x_i, in other words, how to transform marginal probabilities into an implementable mixed strategy.

Given a vector $\boldsymbol{c} \in \mathrm{Conv}(\mathcal{P})$, we want to obtain a set I^* of optimal defender pure strategies and a set of weights x_i satisfying (4.1)-(4.3). This can be done by solving the following linear optimization problem:

$$\text{(MP)} \ \min \sum_{j \in J} \gamma_j + \eta \tag{4.4}$$

$$\text{s.t.} \sum_{i \in I} x_i c_j^i + \gamma_j = c_j \qquad \forall j \in J \ (\alpha_j) \tag{4.5}$$

$$\sum_{i \in I} x_i + \eta = 1 \qquad (\beta) \tag{4.6}$$

$$x_i \geq 0 \qquad \forall i \in I \tag{4.7}$$

$$\eta \geq 0 \tag{4.8}$$

$$\gamma_i \geq 0 \qquad \forall i \in I \tag{4.9}$$

where $\boldsymbol{c}^i \in \mathcal{P}$ are the binary encoding of strategies I. Variables $\boldsymbol{\gamma}$ and η represent slack variables associated to (4.1) and (4.2), respectively, and $\boldsymbol{\alpha}$ and β are dual variables of constraints (4.5) and (4.6). An optimal mixed strategy has been found when the objective function 4.4 is zero, which is always true for our case by Proposition 1. Note that it is impossible to retrieve an optimal mixed strategy when the value of the objective (4.4) is not zero.

To solve the problem above and find an optimal mixed strategy $\mathbf{x}$, we use column generation. The reduced cost of variable x_i is given by $-\boldsymbol{\alpha}^\intercal \boldsymbol{c}^i - \beta$. So, to obtain new strategies, we must solve the following pricing problem:

$$\max \boldsymbol{\alpha}^\intercal \boldsymbol{c} \tag{4.10}$$

$$\text{s.t } \boldsymbol{c} \in \mathcal{P}. \tag{4.11}$$

Its solution $\overline{\boldsymbol{c}}$ provides the column of a new variable with minimum reduced cost, given the current set of variables in the restricted master problem.

If $\boldsymbol{\alpha}^\intercal \overline{\boldsymbol{c}} + \beta \leq 0$, then $(\overline{\boldsymbol{\gamma}}, \overline{\eta}, \overline{x})$ is the current solution and is optimal, i.e., there is no need to add new variables, and $\boldsymbol{c}$ is implementable by the pure strategies such that $\bar{x}_i > 0$. Otherwise, we must include this new variable, which column is given by the current solution of the pricing problem.

Solving the problem (4.10)-(4.11) allows us to find new columns $\boldsymbol{c}^i$. If we work with perfect formulations that have a polynomial number of constraints, or which constraints are separable in polynomial time, we can solve these problems in polynomial time through different algorithms.

In the next section, we analyze some existing examples in the literature of SSG under the lens of this methodology. In particular, we provide a new mathematical formulation for one of the examples.

5. Applications

The examples we provide in this section are problems addressed in the existing literature, as well as new games solved here for the first time. We will use all these examples to test our methodology later in Section 6. Some of these games, such as combined resources (Section 5.2), and protecting one or two adjacent targets (Section 5.3), have an exponential number of constraints that can be efficiently separated. For all the applications, we model the space of defender strategies $\mathcal{P}$ with a perfect formulation in the space of marginal probabilities.

5.1. Single targets and fairness constraints with labels

Bucarey and Labbé [6] studied the problem of ensuring fairness in police patrolling to prevent discrimination when implementing surveillance. A set of targets J is protected, and the defender has m homogeneous security resources.

To ensure that the protection of targets is fair and non-discriminatory, we partition the set of targets J into subsets, denoted by $\{J_l\}_{l\in\mathcal{L}}$, based on the type of population represented by each target. The objective is to allocate security resources to each partition J_l in proportion to the percentage of the population they represent. We define that the total number of resources allocated to each partition J_l has to lie within the range defined by the parameters d_l^L (minimum) and d_l^U (maximum):

$$d_l^L = \left\lfloor (1-\Delta) m \frac{|J_l|}{|J|} \right\rfloor, \quad d_l^U = \left\lceil (1+\Delta) m \frac{|J_l|}{|J|} \right\rceil \tag{5.1}$$

where Δ represents the highest acceptable percentage by which the number of security resources used in J_l may deviate. This limit is determined based on the population proportion that J_l represents.

Now the set of possible defender pure strategies is in the form:

$$I = \{i \subseteq J : |i| \leq m, d_l^L \leq |i \cap J_l| \leq d_l^U \forall l \in \mathcal{L}\} \tag{5.2}$$

where the notation $i \subseteq J$ means that strategy i represents a subset of m targets in J being defended. In adition, the condition must hold that every partition must be protected among the limits d_l^L and d_l^U defined previously.

To apply our methodology, we rewrite the set $\mathcal{P}$ of binary vectors corresponding to all possible pure strategies as follows:

$$\mathcal{P} = \{c \in \{0,1\}^{|J|} : \sum_{j \in J} c_j \leq m, d_l^L \leq \sum_{j \in J_l} c_j \leq d_l^U l \in \mathcal{L}\} \tag{5.3}$$

The set $conv(\mathcal{P})$ is obtained by replacing $c \in \{0,1\}^{|J|}$ in (5.3) by $c \in [0,1]^{|J|}$. It should be noted that the scenario studied by Kiekintveld et al. [25], which involves protecting m targets with homogeneous resources and without taking fairness into account, can be expressed by assigning $d_l^L = 0$ and $d_l^U \geq m$ for all $l \in \mathcal{L}$ in equations (5.2) and (5.3).

After solving the compact formulation and obtaining the vector of marginal probabilities, we decompose it into feasible defender strategies. As was shown by Bucarey and Labbé [6], the defense marginal probability vector can be implemented universally.

The pricing model is defined by constraints (4.10)- (4.11), where $\mathcal{P}$ is determined by (5.3). We follow the poly-time algorithm proposed by Budish et al. [7] to retrieve feasible strategies. For the case of protecting m targets, the pricing model can be solved by finding the j positions of the m biggest values of vector α. In both cases, this problem can be solved in polynomial time.

5.2. Combined resources and matching strategies

The second example is the game described by Bucarey et al. [4, 5], which focuses on patrolling borders. The area to be protected is divided into precincts, with one security resource available per precinct. Each precinct has various targets. Due to the limited resources in each precinct, some security resources must be paired to conduct m patrol camps at night. Only resources from geographically adjacent precincts can be paired, and this pairing can defend one target in either precinct.

Figure 2 illustrates an example scenario in which precincts are labeled from 1 to 5. The edges represent feasible precinct pairings, while the targets in each precinct are denoted by letters [a] through [i]. For example, when $m = 2$, a viable defense strategy could combine precincts 1 and 3 to protect [a], and precincts 2 and 5 to protect [i]. The defender's set of all feasible actions comprises all pairings and the possible $m = 2$ targets covered by these pairings.

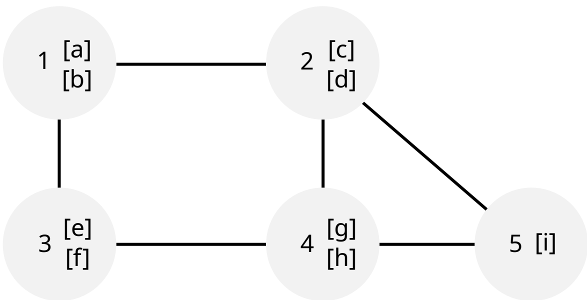

Figure 2: Graph representation of an instance of border patrolling.

Formally, let $G = (V, E)$ be a graph where V represents the set of police precincts and E the possible pairings that can be performed. We use $\delta(v) \subseteq E$ to indicate the set of edges incident to precinct $v \in V$. Similarly, for any $U \subseteq V$, $\delta(U) \subseteq E$ denotes the edges between U and $V \setminus U$, and $E(U) \subseteq E$ represents the edges between precincts in U.

Let $\mathcal{M}_m$ be the set of all matchings of size m in graph G. Also, let J_v be the set containing all targets in precinct v. Therefore, the entire set of feasible targets to protect is $J = \cup_{v \in V} J_v$, and a precinct pairing (or edge) $e = \{u, v\} \in E$ can only defend a target $j \in J_e = J_u \cup J_v$.

A pure defender strategy involves a configuration of m paired precincts, where each pairing protects a specific target associated with one of the precincts in the pair. We can formally define the set of the defender pure strategy as follows:

$$I = \{i \subseteq J : |i| \leq m, \exists F \in \mathcal{M}_m \quad \text{s.t.} |i \cap J_e| = 1, \forall e \in F\}. \tag{5.4}$$

Where F is a matching that belongs to set $\mathcal{M}_m$ such that for each $e \in F$, J_e contains exactly one target to protect.

Let $\mathcal{Q}$ represent the set of feasible solution (c, z, g) to the following constraint set:

$$\sum_{j \in J} c_j = m, \tag{5.5}$$

$$\sum_{e \in E} z_e = m, \tag{5.6}$$

$$\sum_{e \in \delta(v)} z_e \leq 1, \qquad v \in V \tag{5.7}$$

$$\sum_{e \in E : j \in J_e} g_{e,j} = c_j, \qquad j \in J \tag{5.8}$$

$$\sum_{j \in J_e} g_{e,j} = z_e, \qquad e \in E \tag{5.9}$$

$$z_e \in \{0, 1\} \qquad e \in E \tag{5.10}$$

$$g_{e,j} \in \{0, 1\} \qquad e \in E, j \in J_e \tag{5.11}$$

$$c_j \in \{0, 1\} \qquad j \in J \tag{5.12}$$

The set $\mathcal{P}$ of binary vectors corresponding to a defender feasible pure strategies is given by:

$$\mathcal{P} = \text{Proj}_c(Q) = \{c \in \{0, 1\}^{|J|} : \exists \quad z, g \quad \text{s.t.} (5.5) - (5.11)\} \tag{5.13}$$

The set $conv(\mathcal{Q})$ is obtained by adding:

$$\sum_{e \in E(U)} z_e \leq \frac{|U| - 1}{2} \qquad \forall U \subseteq V, |U| \geq 3, |U| \text{ odd} \tag{5.14}$$

and relaxing the domain of variables z_e, c_j and $g_{e,j}$ to be $[0, 1]$. This is a product of the perfect formulation given by Edmonds [17] for the convex hull of the binary vectors defining matchings.

Consequently, this Stackelberg game can be solved using ERASER in which constraint (3.12) is replaced by the system of constraints defining $conv(Q)$. The resulting compact formulation can be solved with a branch-and-cut algorithm. Constraints (5.14) are known to be separable in polynomial time. The detection process is reduced to a global min-cut set problem with odd cardinality in a related network, which can be solved in polynomial time with the Gomory-Hu algorithm [20]. We implement a simple version of the Gomory-Hu algorithm, given by Gusfield [22].

Finally, a polynomial algorithm is presented by Bucarey et al. [5] to recover a set of feasible pure strategies corresponding to a feasible marginal probability vector.

5.3. One or two adjacent targets

A SSG with homogeneous resources where the objective is to protect one or two adjacent nodes in a general graph has been proposed but not solved [29]. Now we study a mathematical formulation for the setting where each of the m resources can cover at most two nodes, and the attacker chooses a node to attack. Not all the possible pairings are feasible to defend. We represent this situation through an undirected graph $G = (V, E)$ where $J = V$ is the set of targets, and E the set of edges linking nodes that can be paired. Then, the set of defender and attacker pure strategies can be defined as:

$$I = \{i \subseteq V : i = R \cup V(T), R \subseteq V, T \subseteq E, |R| + |T| \leq m\} \tag{5.15}$$
$$J = V, \tag{5.16}$$

where $V(T)$ denotes the set of nodes of V incident to edges in T.

To derive a mathematical formulation, we reformulate this game as follows. Define $G' = (V', E')$ from G by duplicating the set of nodes and adding "antennas" for each node $v \in V$, see Figure 3. Formally,

$$\begin{aligned} V' &= V \cup \{u_v \text{ for each } v \in V\} \\ E' &= E \cup \{\{v, u_v\} \text{ for each } v \in V\}. \end{aligned} \tag{5.17}$$

With this, we define a new, equivalent game. If one resource is allocated to a single node in the original game, in this new game, it selects an edge in the form $\{v, u_v\}$.

Let $\mathcal{M}'_{\leq m}$ be the set of all matchings of size at most m in G', and define:

$$I' = \{i' \in V : i = V'(M') \cap V, M' \in \mathcal{M}'_{\leq m}\} \tag{5.18}$$
$$J' = J. \tag{5.19}$$

where $V'(M')$ denotes the set of nodes of V' incident to edges in a matching M'

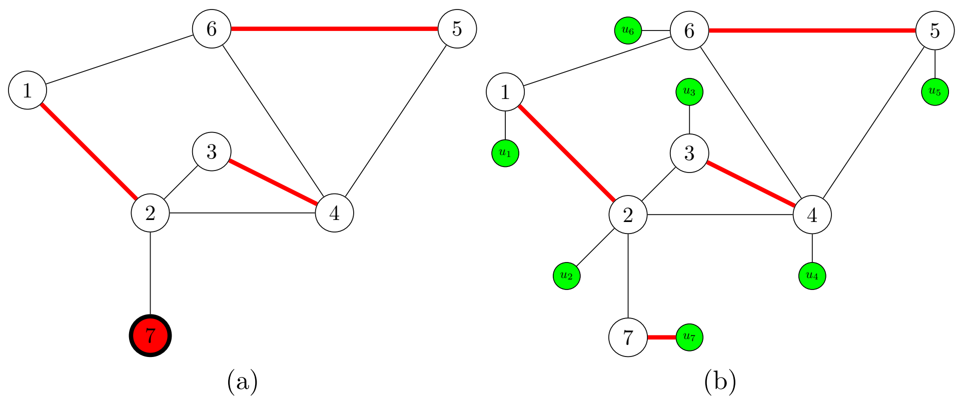

Figure 3: Example of a strategy of the original game (a) and its reformulation (b).

Proposition 2. *The set of the defender's pure strategies coincides: $I = I'$.*

Proof. Let $i \in I$. There exist $R \subseteq V$ and $T \subseteq E$ such that $i = R \cup V(T)$ and $|R| + |T| \leq m$. The set $N' = T \cup \{\{v, u_v\} : v \in R\}$ contains m edges of G' and covers all the nodes of i.

It is possible to transform N' into a matching with at most m edges while covering the same set of nodes. Indeed, if for $v, w \in V, \{v, u_v\}$ and $\{v, w\}$ belong to N', then we can remove $\{v, u_v\}$ from N'. If for $v, w, s \in V, \{v, w\}$ and $\{w, s\}$ belong to N', then we can replace $\{w, s\}$ by $\{s, u_s\}$. As a consequence, $i \in I'$.

Conversely, it is easy to see that a pure strategy $i' \in I'$ also belongs to I. Let M' be a matching of size at most m that covers all nodes belonging to i'. Sets $T = M' \cap E$ and $R = \{v \in V : \{v, u_v\} \in M'\}$ satisfy condition (5.16) for that pure strategy i'.

$\square$

This 'equivalence' relies on the fact that we can assign at most m resources and that the number of resources protecting one-single target is irrelevant in the payoff structure.

Let $\mathcal{Q}$ represent the set of the feasible solution (c, z) to the following constraint set:

$$\sum_{e \in E'} z_e \leq m \tag{5.20}$$

$$\sum_{e \in \delta(j)} z_e \leq 1 \qquad \forall j \in V' \tag{5.21}$$

$$z_e \in \{0, 1\} \qquad e \in E' \tag{5.22}$$

$$\sum_{e \in \delta(v)} z_e = c_v \qquad \forall v \in V, \tag{5.23}$$

where (5.20) states that there are only m resources to protect m edges or pairs of nodes. Constraints (5.21) state that the selected edges must constitute a matching. Constraints (5.23) imply that a node is protected if and only if it is incident to one selected edge.

On the one hand, it is easy to see that $\mathcal{P} = Proj_c(\mathcal{Q})$. On the other hand, from Edmond's result [17], the convex hull of binary vectors satisfying (5.20)-(5.23) is obtained by replacing (5.22) by non-negativity constraints and adding the blossom inequalities (5.14).

As Blossom inequalities are the same as in Section 5.2, we use this branch-and-cut implementation. To retrieve an implementable set of strategies from the compact formulation, the pricing problem (4.10)-(4.11) is reduced to find a maximum cost matching of size at most m where the costs are variables α, duals of constraint (4.5). To do so, we use the algorithm of Plesnik [36], where this problem is transformed to a maximum cost matching without any budget constraint. Afterwards, we find a maximum-weighted matching of maximum-cardinality over $\hat{G}$ [17].

The obtained solution is over $\hat{G}$. Then, to get a solution for the original graph, we only keep the original edges of the optimal solution over $\hat{G}$, resulting in a solution over G.

The optimal value of the pricing problem will be the value of the maximum-weighted matching of maximum cardinality.

5.4. Spanning trees

In this problem, a defender deploys limited security resources to preserve connectivity in a region. This problem is relevant in distribution systems, e.g., when resources are required to be delivered safely to any point within a region, including military applications. To the best of our knowledge, this has not been studied in the literature in the context of Stackelberg games. Baïou and Barahona [1] study a similar scenario focused on Nash-Equilibrium mixed strategies.

Given a graph $G = (V, E)$ that represents the region, one attacker aims to attack one of the E edges. The set I of pure strategies of the defender includes all the spanning trees that could be protected. The defender has enough security resources to cover up to $m = n - 1$ edges. We aim to find the Strong Stackelberg Equilibrium for the game.

Let $\mathcal{T}$ denote the set of all spanning trees in graph G. Now the set of possible pure strategies is in the form:

$$I = \{i \subseteq E : \exists T \in \mathcal{T} s.t. \quad i = E(T)\}. \tag{5.24}$$

where $E(T)$ represents the edges in tree T.

In this case, the set $\mathcal{P}$ in (3.12) represents the binary encoding of the spanning trees. These trees can be represented using different **perfect** formulations, including the *directed multicommodity flow model* ($dflo$), *extended multicommodity flow model* ($mc'flo$), and *subtour* model (sub) with subtour constraints (see Magnanti and Wolsey [32] for a detailed discussion). Additionally, Martin [33] proposed the *reformulated minimum spanning tree formulation* (RMST). We study this game under these four perfect formulations of the spanning trees. Furthermore, we conducted preliminary experiments using several formulations considering these tree representations. In the instances we have solved, formulations RMST and dflo are the top-performing approaches. When paired with Mip-p-S or ERASER, they demonstrate similar competitiveness, while dflo exhibits clear advantages when used with SDOBSS, which is the slowest among the formulations. In Section 6, we select the best-performing tree formulation for each experiment to showcase optimal performance.

In the following subsection we present the *RMST* formulation. For the sake of shortness the other formulations are presented in Appendix A.

5.4.1. Reformulated minimum spanning tree (RMST) formulation

This formulation was proposed by Martin [33] and is $O(n^3)$ in both variables and constraints. The formulation comes from the fact that spanning trees do not contain cycles. Let u^h_{ij} be the flow of commodity h in arc (i,j). One unit of commodity h must be delivered to node h. Variables c_e represent the marginal probability of defending edge e.

Let $\mathcal{Q}$ represent the set of feasible solutions (c,u) to the following constraint set:

$$\sum_{s\in V: s>i} u^h_{is} + \sum_{j\in V: j<i} u^h_{ij} \leq 1 \qquad \forall h, i \in V, i \neq h \tag{5.25}$$

$$\sum_{s\in V: s>h} u^h_{hs} + \sum_{j\in V: j<h} u^h_{hj} \leq 0 \qquad \forall h \in V \tag{5.26}$$

$$\sum_{e\in E} c_e = n-1 \tag{5.27}$$

$$u^h_{ij} + u^h_{ji} = c_e \qquad \forall e = (i,j) \in E, h \in V \tag{5.28}$$

$$u^h_{ij} \geq 0 \qquad \forall i,j,h \in V \tag{5.29}$$

$$c_e \in \{0,1\} \qquad \forall e \in E. \tag{5.30}$$

First, note that that each target is assigned a numerical label. Thus, when we refer to the notation $s > i$, it means that the label of target s is greater than the label of target i. Constraint (5.25) states that every node has at most one flow of commodity h directed out of it, except for the destination node of commodity h. This prohibits undirected cycles. Constraint (5.26) enforces that node h is the final destination of commodity h. This constraint also prohibits directed cycles. Constraint (5.27) states that a n-node spanning tree should have $n-1$ edges. Constraint (5.28) forces the flow of commodity h over arc (i,j) and (j,i) to be equal to the marginal probability of defending edge e. Constraints (5.29) and (5.30) state the nature of variables.

The set $\mathcal{P}$ of binary vectors corresponding to a defender feasible pure strategies is given by:

$$\mathcal{P} = \text{Proj}_c(\mathcal{Q}) = \{c \in \{0,1\}^{|J|} : \exists \quad c, u \quad \text{s.t.}(5.25) - (5.29)\}. \tag{5.31}$$

Since the formulation (5.25) - (5.29) is a perfect formulation, $conv(\mathcal{P})$ is obtained by replacing $c \in \{0,1\}^{|J|}$ in (5.30) by $c \in [0,1]^{|J|}$.

Once we obtain a solution c, we perform the column generation approach of Section 4.2. We use Kruskal's algorithm [27] to solve the pricing problem. The optimal value of the pricing problem will be the minimum spanning tree weight value.

6. Computational Experiments

We first compare our general methodology (using compact formulations and column generation) with branch-and-price on D2 (non-compact formulation) using the methodology of Jain et al. [23]. We then run different tests using our approach, comparing different formulations and instances.

We assume that the game is played over an underlying graph $G = (V,E)$. The graph is connected and partially complete (70% of the edges of a complete graph). As before, J refers to the set of targets, with $|J| = n$, K denotes the set of attackers, with $|K| = k$, and m represents the number of security resources.

Following Casorrán et al. [8], we use two ways of generating reward and penalty matrices. First, we create penalty matrices in which the entries are randomly generated between 0 and 5, and reward matrices whose entries are randomly generated between 5 and 10. We refer to these as matrices with **no variability**. Second, we build penalty and reward matrices in which 90% of the values are randomly generated as before, that is between 0 and 5 (penalty) and between 5 and 10 (reward), while 10% of the values for the penalty matrices are randomly generated between 0 and 50, and 10% of the values of the reward matrices are between 50 and 100. We refer to these as matrices with **variability**. The reason for using payoff matrices with and without variability is that the value of the big-M parameter in both compact and non-compact formulation is directly affected by these matrices.

We performed our experiments on an 11th Gen Intel Core i9-11900, 2.50GHz, equipped with 32 gigabytes of RAM, 16 cores, 2 threads per core, and running the Ubuntu operating system release 20.04.6 LTS. The experiments were coded in Python v.3.8.10 and SCIP v.8.0.0 as the optimization solver, considering a 3-hour solution time limit for Section 6.1, and 1-hour for the rest of the cases.

We present the results in Figures and Tables. Figures present solution times in log scale and LP gap in linear scale. Figures showing LP gap only show results for instances solved within the specified time limit.

6.1. Compact vs. Non-compact formulations

We compare our approach with branch-and-price on D2 (non-compact formulation), as described by Jain et al. [23].

From the point of view of problem size, when using branch-and-price, the problem has a combinatorial number of variables x_i. These variables represent all/some possible combinations of defense strategies, and their number depends on the problem being solved. In contrast, by projecting variables x onto c variables (or an equivalent), only $|J|$ variables c are required. Even though branch-and-price may not necessarily need to enumerate all variables, the upper bound is still combinatorial. Solving the compact formulation requires retrieving a strategy x from marginal probabilities c. However, we show that retrieving such a strategy is done in polynomial time for this article's broad class of problems. This fact and the experimental results in this section show that the approach presented in this article is significantly more efficient.

We conducted experiments comparing the two approaches (ours versus branch-and-price) to have an idea of this efficiency gain from an empirical point of view. We used compact formulations Mip-p-S, ERASER, and SDOBSS for our approach. For the method using non-compact formulations, we chose D2, as it has an exponential number of variables and a polynomial number of constraints, which is ideal for column generation. We denote D2_BP to this implementation. We compare the methods in two games: i) protecting m targets, which is a simple case, and ii) protecting spanning trees, a more elaborate case. For each game, we considered five instances with different numbers of targets, defense resources, and no variability. In all cases, we limited solution times to three hours. We present the results graphically in Figures 4 and 5, and numerically in Tables 1 and 2.

For the case of protecting m targets, the instances considered targets $n \in \{10, 20, 30, 40, 50\}$, attackers $k \in \{2, 4, 8\}$, and security resources m of a 25%, 50%, and 75% of the total number of targets, rounded to the nearest integer number. For each instance size, we generate 5 random instances.

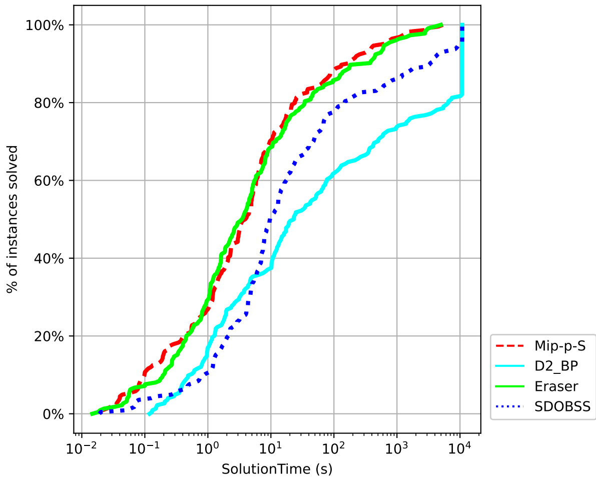

Figure 4: Percentage of instances (ordinate axis) solved in the time shown in the abscissa. Our method (compact formulation plus column generation) vs. D2_BP. Protecting m targets. Instances without variability.

Figure 4 shows that the compact formulations Mip-p-S and ERASER take less than 100 seconds to solve 90% of the instances, while D2 plus branch-and-price (D2_BP) takes more than 3 hours to solve the same percentage of instances. It should be noted that D2_BP is initially competitive, particularly with smaller instances, but becomes inefficient as the instances grow larger. Among the compact formulations examined, SDOBSS exhibits the slowest performance. Note that our method using ERASER can solve all the instances in less than 20 minutes while D2_BP can only solve 78% of the instances within the same time.

Table 1 displays a fragment of detailed results for the case of protecting m targets, for both methods. The columns show the numbers of targets (n), attackers (k) and security resources (m). For our method, the columns show the time required to solve the compact formulation **t**, the solution time of the column generation process **tCG**, and the total time for the method **tTotal**. The optimality gap is always zero. For the branch-and-price on non-compact formulation (D2_BP), the Table displays the optimality gap **Gap**, and the Total time **tTotal**.

			Mip-p-S			ERASER			SDOBSS			D2_BP	
n	**m**	**k**	**t**	**tCG**	**tTotal**	**t**	**tCG**	**tTotal**	**t**	**tCG**	**tTotal**	**Gap**	**tTotal**
10	2	2	0,02	0,00	**0,03**	0,17	0,00	0,17	0,47	0,00	0,48	0,00	0,21
10	5	2	0,03	0,00	**0,04**	0,21	0,00	0,21	1,20	0,00	1,20	0,00	0,23
10	8	2	0,04	0,00	0,04	0,59	0,00	0,59	0,02	0,00	**0,02**	0,00	0,13
10	2	4	0,13	0,00	**0,13**	1,07	0,00	1,07	1,90	0,00	1,90	0,00	1,87
10	5	4	2,52	0,00	2,52	1,60	0,00	**1,60**	4,09	0,00	4,10	0,00	2,38
10	8	4	0,04	0,00	**0,04**	0,45	0,00	0,45	0,07	0,00	0,07	0,00	0,61
10	2	8	1,40	0,00	**1,40**	8,20	0,00	8,20	1,96	0,00	1,96	0,00	42,22
10	5	8	3,91	0,00	**3,91**	14,45	0,00	14,45	7,25	0,00	7,26	0,00	86,00
10	8	8	5,53	0,00	**5,53**	5,80	0,00	5,80	6,53	0,00	6,54	0,00	52,62
20	5	2	0,07	0,00	**0,07**	1,88	0,00	1,88	6,99	0,00	6,99	0,00	0,81
20	10	2	0,34	0,00	**0,34**	1,23	0,00	1,24	2,26	0,01	2,26	0,00	0,97
20	15	2	0,10	0,00	**0,10**	0,60	0,00	0,60	0,12	0,00	0,13	0,00	0,96
20	5	4	3,29	0,01	**3,30**	7,29	0,00	7,29	5,39	0,01	5,39	0,00	23,03
20	10	4	8,16	0,01	8,17	1,62	0,01	**1,62**	9,66	0,01	9,67	0,00	11,21
20	15	4	6,33	0,00	6,34	1,54	0,00	**1,55**	10,61	0,00	10,61	0,00	2,00
20	5	8	3,41	0,00	**3,41**	45,57	0,00	45,58	35,71	0,00	35,71	0,00	507,64
20	10	8	6,72	0,01	**6,73**	34,46	0,01	34,47	47,56	0,01	47,57	0,00	1419,06
20	15	8	14,38	0,01	14,39	5,00	0,01	**5,00**	20,30	0,01	20,31	0,00	416,51
30	8	2	3,05	0,01	3,05	1,10	0,01	**1,11**	2,74	0,01	2,75	0,00	4,36
30	15	2	0,52	0,01	**0,53**	1,03	0,01	1,04	7,44	0,01	7,45	0,00	2,55
30	22	2	0,23	0,01	**0,24**	0,68	0,01	0,68	0,40	0,01	0,40	0,00	0,60
30	8	4	4,43	0,01	**4,44**	8,09	0,01	8,10	20,74	0,01	20,75	0,00	77,45
30	15	4	9,00	0,01	9,01	5,16	0,01	**5,17**	24,13	0,01	24,14	0,00	68,21
30	22	4	5,04	0,01	5,05	3,45	0,01	**3,46**	13,11	0,01	13,12	0,00	35,69
30	8	8	20,88	0,01	**20,89**	616,58	0,01	616,58	1462,89	0,01	1462,90	0,28	10882,76
30	15	8	63,31	0,01	**63,32**	123,82	0,01	123,83	1010,62	0,01	1010,63	0,18	10866,25
30	22	8	197,33	0,01	197,34	14,56	0,01	**14,57**	210,29	0,01	210,30	0,08	10848,02
40	10	2	2,32	0,02	2,34	1,04	0,01	**1,05**	7,55	0,02	7,57	0,00	10,39
40	20	2	2,00	0,02	2,02	1,59	0,02	**1,61**	7,46	0,02	7,48	0,00	17,35
40	30	2	0,39	0,01	0,40	0,06	0,01	**0,07**	0,73	0,01	0,74	0,00	1,83
40	10	4	9,67	0,01	9,68	5,40	0,01	**5,41**	40,07	0,01	40,08	0,00	369,93
40	20	4	15,76	0,02	15,78	4,26	0,01	**4,27**	22,20	0,02	22,22	0,00	30,36
40	30	4	2,27	0,01	2,28	1,05	0,01	**1,06**	3,95	0,02	3,96	0,00	11,37
40	10	8	86,37	0,01	**86,38**	701,14	0,01	701,16	3391,81	0,01	3391,82	0,46	10853,06
40	20	8	831,75	0,02	831,77	455,88	0,02	**455,89**	8952,75	0,02	8952,77	0,34	10848,88
40	30	8	226,65	0,02	226,66	28,43	0,02	**28,45**	1794,44	0,02	1794,46	0,17	10822,93

Table 1: Detailed performance results for our method and the branch-and-price method. Protecting m targets

The running time of compact and non-compact models escalates notably as the game size expands, especially when there is an increase in the number of targets and attackers. However, the non-compact formulation method exhibits a much faster rate of growth. The optimality gap in our method in this fragment of results, is consistently zero, indicating that it always finds the optimal solution within the time limit. The time required by the column generation process (**tCG**) is always marginal. On the other hand, the optimality gap of D2_BP is sometimes non-zero, indicating that it could not reach an optimal solution within three hours.

For the case of protecting spanning tree structures over a graph, we consider $n \in \{10, 20, 30, 40\}$ and $k \in \{2, 4, 6\}$. The targets protected are arcs of a spanning tree, and the number of security resources is $m = n - 1$. For each combination of values (n, m, k), and always maintaining the graph density of 70%, we generate 5 instances ($i = \{0, 1, 2, 3, 4\}$) with random payoff matrices, using the values for matrices without variability. Fig. 5 compares our method with D2 plus branch-and-price (D2_BP), for the game of protecting spanning trees.

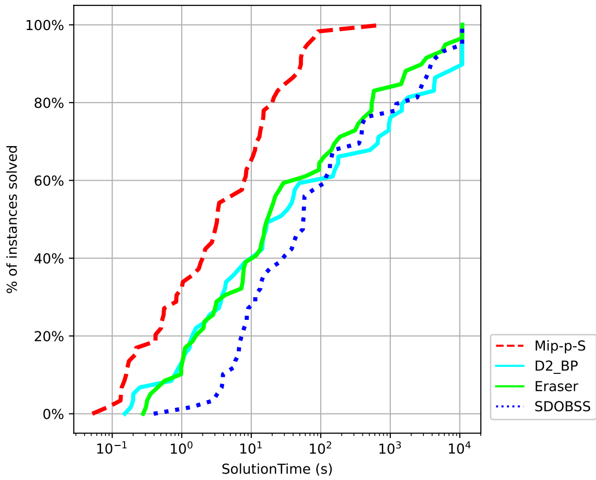

Figure 5: Percentage of instances (ordinate axis) solved in the time shown in the abscissa. Our method (compact formulation plus column generation) vs. D2 plus branch-and-price (D2_BP). Protecting spanning trees. Instances without variability.

Fig. 5 shows that, for the game of protecting tree structures, our method using Mip-p-S takes 20 seconds to solve 80% of the cases, while D2 plus branch-and-price takes more than one hour. All of the instances were solved by Mip-p-S in less than 17 minutes, while branch-and-price could not solve 10% of the cases within the three hours limit. Once again, it should be emphasized that D2_BP is a competitive approach in smaller instances, even outperforming a compact formulation such as SDOBSS. Additionally, D2_BP performs similarly to ERASER in 60% of the instances.

Table 2 shows a fragment of the detailed results of all the instances for the case of protecting spanning trees. The first three columns indicate the number of targets, attackers and the identification (i) of the random payoff matrices used. The next columns report the different metrics related to the computational performance of the two methods, with the same notation as in Table 1.

			Mip-p-S			ERASER			SDOBSS			D2_BP	
n	**k**	**i**	**t**	**tCG**	**tTotal**	**t**	**tCG**	**tTotal**	**tBC**	**tCG**	**tTotal**	**Gap**	**tTotal**
10	2	0	0,17	0,01	**0,17**	0,30	0,01	0,31	3,23	0,01	3,24	0,00	0,25
10	4	0	0,14	0,01	**0,15**	2,10	0,02	2,12	9,03	0,01	9,04	0,00	2,21
10	6	0	0,41	0,01	**0,42**	11,77	0,01	11,78	18,47	0,01	18,48	0,00	38,28
20	2	0	0,81	0,02	**0,83**	0,96	0,02	0,98	7,87	0,02	7,89	0,00	1,44
20	4	0	2,06	0,02	**2,08**	28,85	0,02	28,87	55,31	0,02	55,33	0,00	178,75
20	6	0	3,31	0,02	**3,33**	111,56	0,02	111,58	143,16	0,03	143,19	0,00	676,49
30	2	0	1,38	0,03	1,41	3,12	0,03	3,16	7,34	0,03	7,37	0,00	**1,02**
30	4	0	7,63	0,03	**7,67**	14,03	0,04	14,07	56,95	0,05	57,00	0,00	40,54
30	6	0	19,68	0,04	**19,72**	563,73	0,04	563,77	1200,68	0,04	1200,72	0,04	10861,36
40	2	0	13,46	0,10	13,55	15,33	0,11	15,45	56,96	0,10	57,07	0,00	**4,19**
40	4	0	39,66	0,11	**39,77**	539,69	0,09	539,78	2887,60	0,11	2887,71	0,00	508,72
40	6	0	776,87	0,08	**776,95**	5515,08	0,10	5515,18	5827,05	0,11	5827,16	0,16	10834,75
10	2	1	0,13	0,01	**0,13**	0,31	0,01	0,32	1,36	0,01	1,36	0,00	0,19
10	4	1	0,08	0,01	**0,09**	1,63	0,01	1,64	5,87	0,01	5,88	0,00	1,60
10	6	1	0,13	0,01	**0,14**	7,80	0,01	7,81	11,43	0,01	11,44	0,00	16,77
20	2	1	0,49	0,02	**0,52**	1,05	0,02	1,07	6,66	0,02	6,68	0,00	0,80
20	4	1	1,74	0,02	**1,76**	19,17	0,02	19,19	44,27	0,02	44,29	0,00	49,22
20	6	1	3,02	0,02	**3,03**	580,98	0,02	581,00	150,94	0,02	150,96	0,00	1473,43
30	2	1	3,22	0,04	3,26	2,96	0,05	**3,01**	14,98	0,06	15,04	0,00	3,81
30	4	1	8,45	0,06	**8,51**	94,28	0,05	94,33	134,93	0,07	134,99	0,00	42,24
30	6	1	13,05	0,06	**13,10**	421,44	0,05	421,49	1219,87	0,05	1219,92	0,00	953,79
40	2	1	10,39	0,12	**10,51**	16,22	0,15	16,37	112,04	0,14	112,18	0,00	14,43
40	4	1	48,53	0,12	**48,65**	2793,03	0,09	2793,12	2639,69	0,11	2639,80	0,00	4419,78
40	6	1	51,66	0,10	**51,76**	3313,28	0,10	3313,38	3606,00	0,10	3606,10	0,00	7011,83

Table 2: Extract of results of Mip-p-S and D2_BP for the case of protecting spanning trees.

The solution time strongly increases with the number of targets and the number of attackers. As before, the column generation phase takes a negligible time (**tCG**). The time that D2_BP takes is at least an order of magnitude larger than Mip-p-S when the number of attackers is 4, and several orders of magnitude for 6 attackers. Specifically, for the first instance of 6 attackers and 40 targets or more, D2_BP cannot find an optimal solution within the time limit.

In conclusion, for problems whose space of strategies in the marginal probabilities space can be represented as perfect formulation, and have either a polynomial number of constraints, or an exponential number but separable efficiently, our method clearly outperforms branch-and-price, the standard approach for addressing SSG problems.

Having compared our general approach with D2_BP, we now evaluate and contrast the efficiency of the general approach using three compact SSG formulations (ERASER, Mip-p-S, and SDOBSS) on randomly generated instances for four different games, as follows: i) protecting m single targets with fairness constraints with labels, ii) protecting single targets or 2 adjacent targets with m security resources, iii) using m combined security resources, and iv) protecting a tree structure over a graph. Note that, for brevity reasons, in Section 6 when we refer to ERASER, Mip-p-S, and SDOBSS, we mean using the respective compact model including the column generation process of Section 4.2, that retrieves the mixed strategy. We show the results in graphs showing the LP Gap, i.e., the percentage difference between the optimal solution and the LP relaxation value, and graphs displaying the Solution Time, which includes the entire procedure, including solving the compact formulation and retrieving a mixed strategy.

6.2. Single targets and fairness constraints with labels

We analyzed the protection of m targets (single nodes) with fairness constraints using payoff matrices both with and without variability. Our experiments were conducted for $n \in \{20, 30, 40\}$, $k \in \{2, 3\}$, $m \in \{5, 10\}$, $nL \in \{3, 5\}$ partitions and $\Delta \in \{0.1, 0.25, 0.5\}$ deviation. We created five instances for each configuration.

In cases without variability, as depicted in Figure 6, Mip-p-S outperforms both SDOBSS and ERASER regarding speed. Additionally, ERASER is always faster than SDOBSS. When variability is introduced, as shown in Figure 7, Mip-p-S remains the fastest formulation in most cases. There is no clear dominance of ERASER over SDOBSS in terms of solution time.

Regarding the LP formulations' gap percentage (Figures 6 and 7), Mip-p-S's relaxation is consistently tighter, supporting the theoretical findings by Casorrán et al. [8], who proved this for the case of protecting m single targets. For instances without variability (Figure 6), Mip-p-S's average LP gap is 36.08% lower than ERASER's and 30.5% lower than SDOBSS's. SDOBSS's relaxation is 7.6% tighter on average than ERASER's. For instances with variability (Figure 7)), Mip-p-S's average LP gap is 52.01% lower than ERASER's and 49.4% lower than SDOBSS's. SDOBSS's relaxation is 5.2% tighter on average than ERASER.

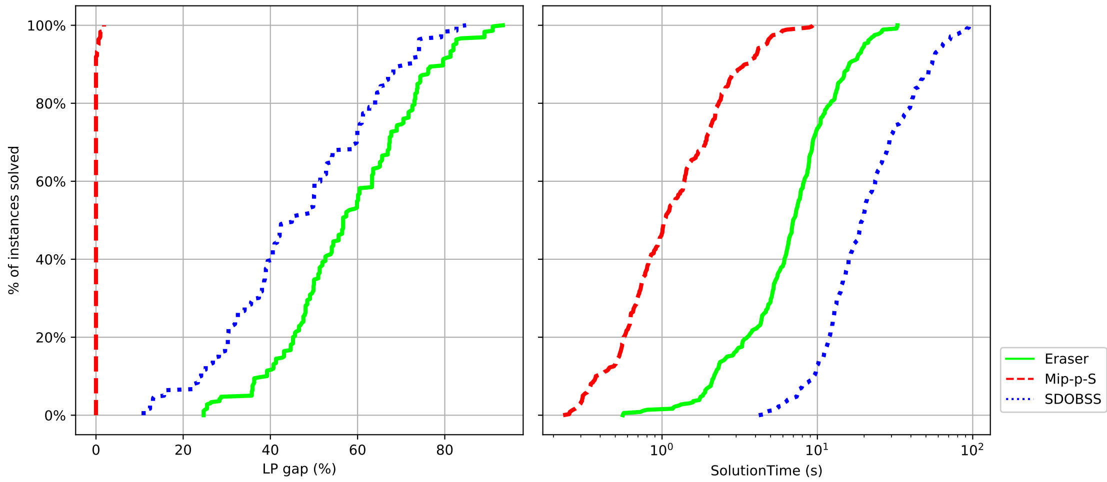

Figure 6: Time to solve the integer problem and %LP gap. Single targets and fairness constraints with labels. Instances without variability.

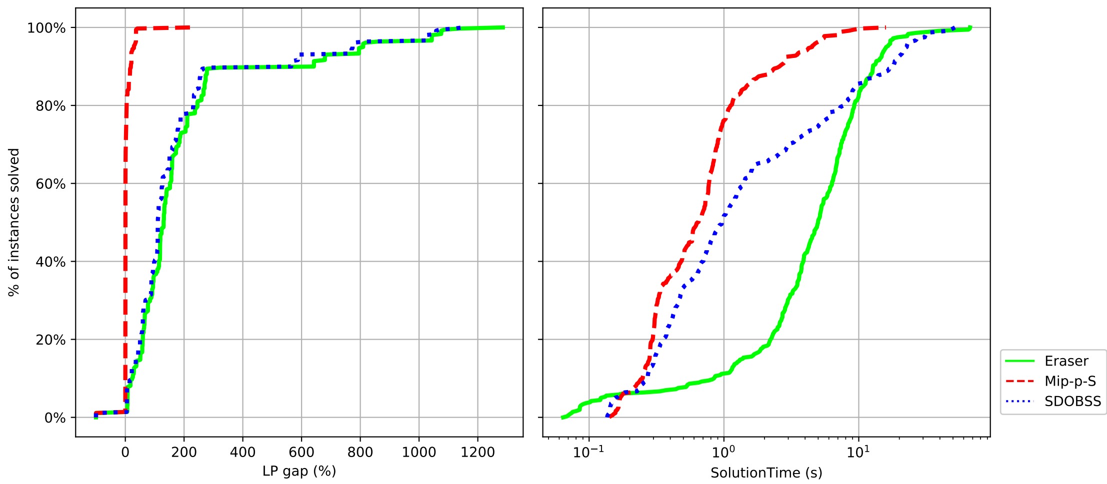

Figure 7: Time to solve the integer problem and %LP gap. Single targets and fairness constraints with labels. Instances with variability.

6.3. Combined resources and matching strategies

When the objective is to protect targets with a combination of m resources, we evaluated scenarios with $\{15, 20, 25\}$ precints, $k \in \{4, 6\}$, $m \in \{2, 3, 4\}$. We consider three targets per precinct ($t = \{3\}$), therefore, the targets are represented by $n \in \{45, 60, 75\}$. Our analysis included instances with and without variability, and we considered five instances for each configuration.

Regardless of variability (refer to Figure 8 and 9), Mip-p-S outperforms both SDOBSS and ERASER in solution time in all of cases. Additionally, SDOBSS cannot solve close to 20% of instances within the given time limit in instances without variability.

Mip-p-S's relaxation consistently produces a tighter percentage of gap in the LP formulations (refer to Figures 8 and 9). In instances without variability (Figure 8), the average LP gap for Mip-p-S is 0.2%. This indicates that the solutions generated by the relaxation of Mip-p-S are close to the optimal solution. In contrast, ERASER shows an average LP gap of 100.2%, while SDOBSS has a slightly lower average LP gap of 97.6%. In instances with variability (Figure 9), Mip-p-S maintains its superiority but experiences a higher average LP gap of 5.01%. This increase can be attributed to the larger values of the payoff matrices, introduced by variability. ERASER has a significantly higher average LP gap of 389.5%. Similarly, SDOBSS exhibits an average LP gap of 261.5%.

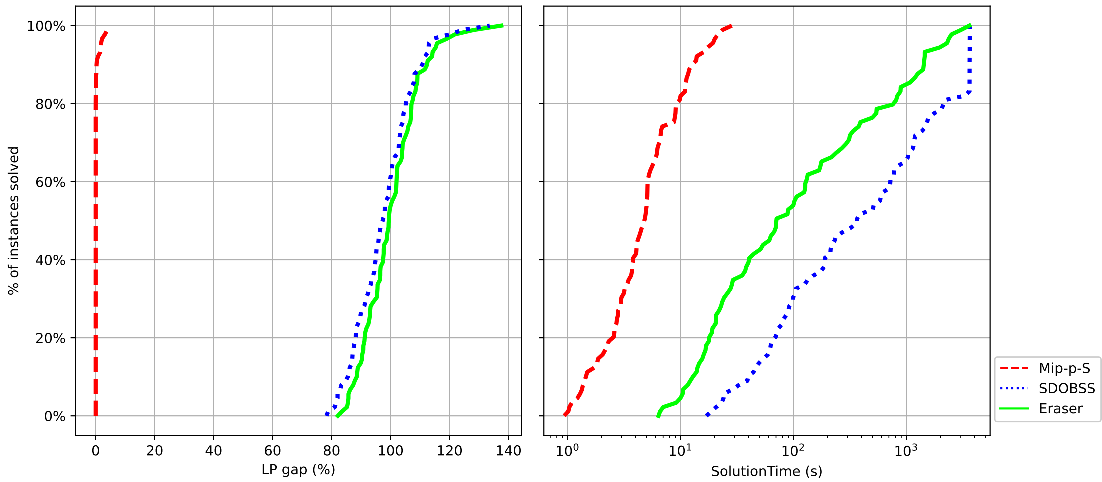

Figure 8: Time to solve the integer problem and %LP gap. Combined resources and matching strategies. Instances without variability.

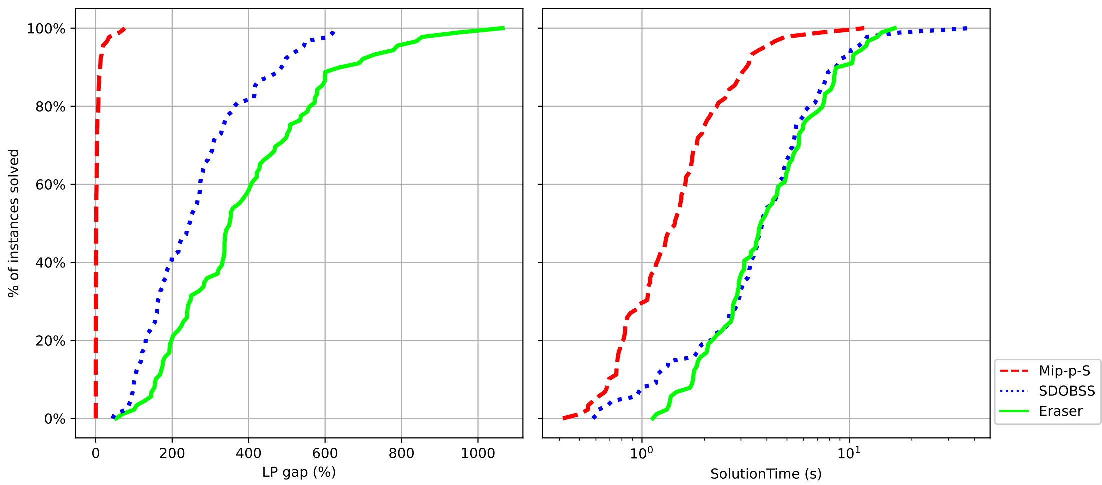

Figure 9: Time to solve the integer problem and %LP gap. Combined resources and matching strategies. Instances with variability.

6.4. One or two adjacent targets

In the next example, the aim is to protect one or two adjacent targets, for instances with and without variability. The parameters we used are as follows: $n \in \{30, 40\}$, $k \in \{2, 4, 6\}$, and m is $5\%, 10\%$ or 25% of the number of targets (Figures 10, 11). We solve five different instances in each case.

Regarding solution time (as shown in Figure 10), the Mip-p-S formulation is faster than the other formulations, with ERASER being faster than SDOBSS. Related to the LP gap, in instances with and without variability (Figure 10 and 11), again, Mip-p-S has the tightest LP gap [8]. In instances without variability (Figure 10), Mip-p-S's average LP gap is 0.06%, while ERASER's 62.3%, and SDOBSS's 52.3%. For instances with variability (Figure 11), Mip-p-S's average LP gap is 17.5%, ERASER's 300.4%, and SDOBSS's 247.3%.

Mip-p-S formulation not only achieves faster solution times but also consistently delivers solutions with a tighter LP gap, making it a favorable choice over ERASER and SDOBSS in both instances with and without variability.

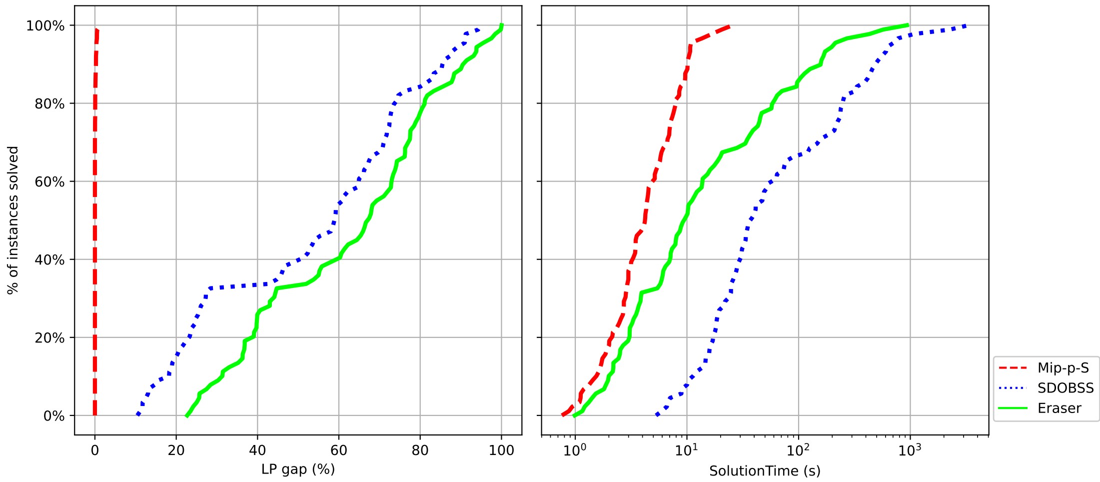

Figure 10: Time to solve the integer problem, and %LP gap. One or two adjacent targets. Instances without variability.

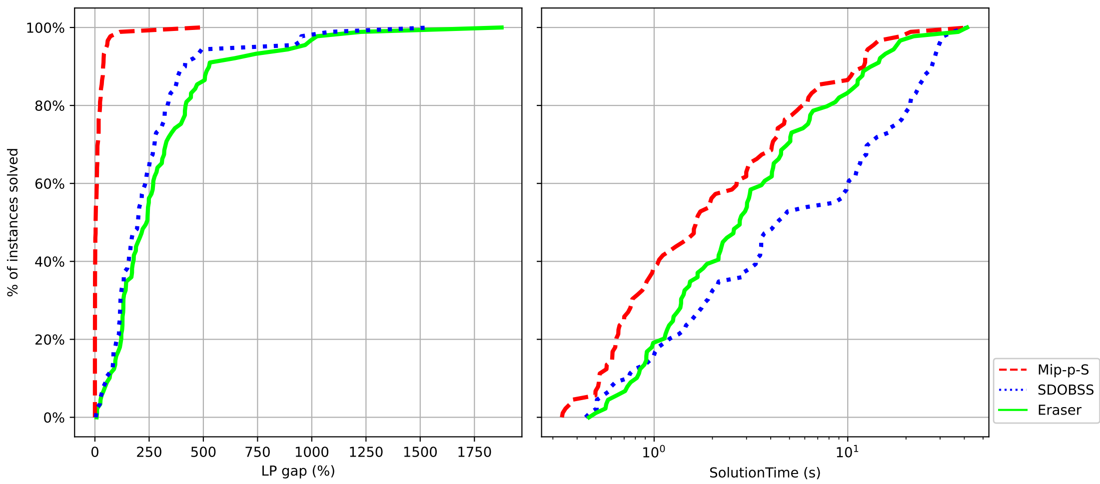

Figure 11: Time to solve the integer problem, and %LP gap. One or two adjacent targets. Instances with variability.

6.5. Spanning trees

When protecting a tree structure over a graph, for instances with and without variability, we considered $n \in \{10, 20, 30, 40\}$, $k \in \{2, 4, 6\}$. For every configuration, we solve five different instances per setting.

Regardless of the variability considered, Mip-p-S is the fastest formulation. In instances without variability, ERASER's solution time is better than SDOBSS, while in cases with variability, SDOBSS outperforms ERASER.

In instances with and without variability (Figure 12 and 13), the Mip-p-S LP relaxation is the tightest. In instances without variability (Figure 12), Mip-p-S's average LP gap is 27.7% smaller than ERASER's and 19.05% smaller than SDOBSS's. SDOBSS's relaxation is 10.7% tighter on average than ERASER. Concerning instances with variability (Figure 13), Mip-p-S's average LP gap is 55.2% smaller than ERASER's and 53.06% smaller than SDOBSS's. SDOBSS's relaxation is 4.6% tighter on average than ERASER.

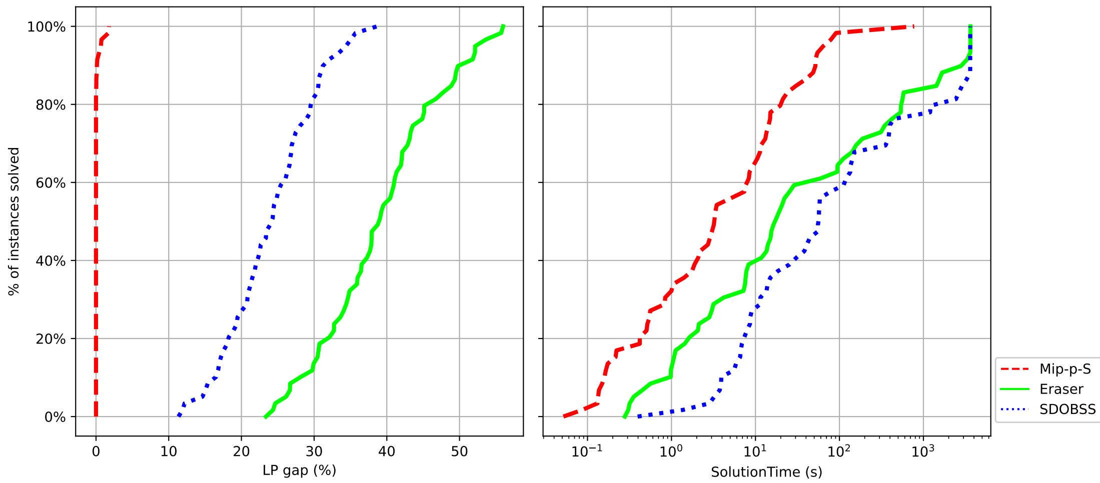

Figure 12: Time to solve the integer problem, and %LP gap. Spanning tree. Instances without variability.

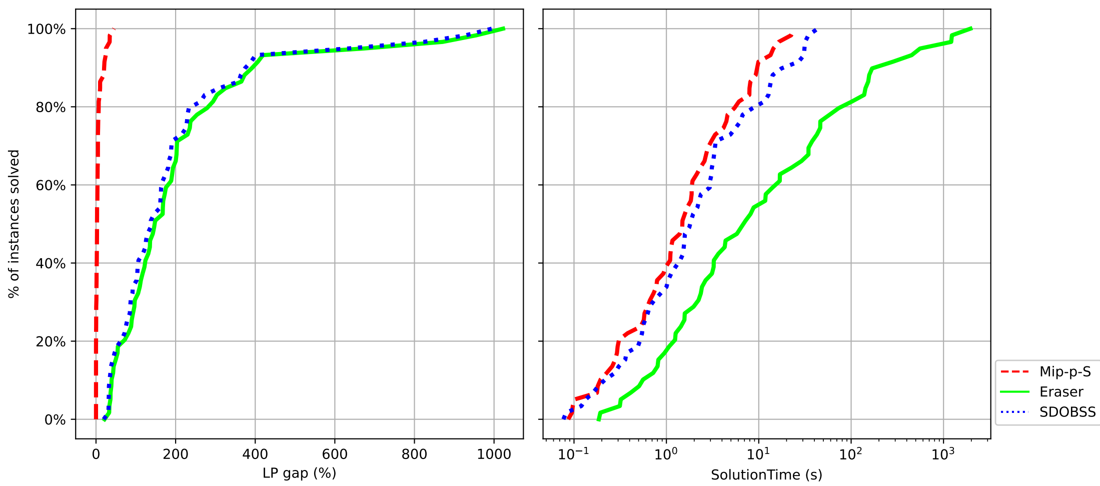

Figure 13: Time to solve the integer problem, and %LP gap. Spanning tree. Instances with variability.

6.6. Discussion

Our approach outperforms branch-and-price over D2 non-compact formulation [23]. We discuss the formulation-related reasons as well as show this fact empirically through two examples: protecting m targets, and spanning trees. Our experiments show that as the instances' size increase, D2_BP takes noticeably longer than Mip-p-S and ERASER to find a solution. The difference in solution time between D2_BP and Mip-p-S (or ERASER) can be several orders of magnitude. Furthermore, it is worth noting that SDOBSS exhibits slower performance in smaller instances than D2_BP. While compact formulations can be highly efficient, not all are equally effective in terms of time.

Compact formulations such as Mip-p-S are more efficient than non-compact ones, even though they require post-processing to obtain mixed strategies. Generally, solutions for the compact formulations may not always result in implementable strategies. However, we show that, when defender strategy spaces are modeled as perfect formulations, it is always possible to retrieve feasible strategies. Thanks to this property, we use compact formulations as they are more efficient, and use a column generation method to retrieve strategies, which is polynomial.

Regarding the different compact formulations studied, we could observe that Mip-p-S has a significantly tighter percentage of LP gap compared to the other two formulations, resulting in a better quality of the upper bound. This usually leads to a much smaller node usage in the branch & bound tree [8]. In many cases, this results in achieving optimality of the integer problem faster. Experimentally, we conclude that Mip-p-S is the fastest formulation for the instances whenever the number of targets and/or attackers are large enough.

Regarding formulation size, and excluding the defender strategy space, ERASER is the least complex, with the fewest number of constraints and variables, at a complexity of $O(|J||K|)$ [8]. On the other hand, Mip-p-S and SDOBSS have complexities of $O(|J|^2|K|)$ [8], resulting in LP relaxations that are more complex and time-consuming to solve than ERASER.

7. Conclusions

We address the class of problems whose defender strategies can be efficiently represented by perfect formulations. We prove that the solution of a compact representation of this kind of problem can always be described as a feasible mixed strategy. Once a solution to a compact formulation has been obtained, the associated strategy can be found in polynomial time using column generation if the defender strategy space has a polynomial number of constraints, or an exponential number of poly-time separable constraints.

For the studied setting, our approach using Mip-p-S and ERASER outperforms branch-and-price over non-compact formulations. We showed this through two cases: protecting m targets, and spanning trees. Additionally, our findings highlight the fact that while compact formulations can be highly efficient, not all are equally effective in terms of time.

Based on these results, we propose a general methodology to find a Strong Stackelberg Equilibrium for Stackelberg security games for cases where perfect formulations can efficiently represent the set of defender strategies. This process consists in initially representing through marginal probabilities a polynomial-size MILP or a MILP with an exponential size of constraints that may be efficiently separated through branch-and-cut. Once the problem is described, we obtain feasible defender strategies (in polynomial time) through column generation. We use three different formulations in the space of marginal probabilities, and we develop branch-and-cut schemes to manage large instances.

We consider different defense strategies as applications: single targets with fairness constraints, one or two adjacent targets using a novel solution approach, combined resources, and for the first time, protecting spanning trees in Stackelberg Security Games. We test our methodology for these examples, and we analyze the results.

We observe that Mip-p-S has a significantly tighter percentage of LP gap compared to the other two formulations, resulting in a better quality of the upper bound and a much smaller node usage in the branch & bound tree. In many cases, this results in achieving optimality of the integer problem faster. Experimentally, we conclude that Mip-p-S is the fastest formulation for the instances with a large number of targets and/or attackers.

In the same line of the results of Casorrán et al. [8], we show experimentally that Mip-p-S has the tightest LP gap for the cases studied, followed by SDOBSS and ERASER, in that order. In instances without variability, Mip-p-S's average LP gap ranges from 27.7% to 100% smaller than ERASER's and ranges from 19.1% to 96% smaller than SDOBSS's. In instances with variability, Mip-p-S's average LP gap ranges from 52% to 384% smaller than ERASER's, and ranges from 4.6% to 256% smaller than SDOBSS's. Therefore, instances without variability show less LP gap difference between formulations than instances with variability.

Overall, these results can be useful for understanding the performance of different algorithms for solving Bayesian Stackelberg Security Games and for selecting an appropriate formulation for a given instance of the game. We recommend using compact formulations when the defender strategy set is modeled as a perfect formulation. Among the three studied formulations, we recommend using Mip-p-S when the number of targets is high.

8. Acknowledgments

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Appendix A SSG Formulations: Defending spanning trees

A.1 ERASER

Given a graph $G = (V, E)$ that represents a region, an attacker aims to attack one of the E edges. In this scenario, the set I of pure strategies of the defender includes all the spanning trees that could be protected. The defender possesses sufficient security resources to cover up to $m = n - 1$ edges.

Spanning trees can be represented using different formulations, including the *directed multicommodity flow model* (*dflo*) [32], *extended multicommodity flow model* (*mc′flo*) [32], *subtour* model (*sub*) [32], and *reformulated minimum spanning tree formulation* (RMST) [33]. In this section, we will present various formulations for defending a spanning tree structure, considering these four different approaches to represent trees.

The first formulation we present is $ERASER_{sub}$, based on the *subtour model* [32]:

$$(\text{ERASER}_{sub})$$

$$\max \sum_{k \in K} \pi^k f^k \tag{A.1}$$

$$\text{s.t } (3.7) - (3.11)$$

$$\sum_{e \in E} c_e = n - 1 \tag{A.2}$$

$$\sum_{e \in E(T)} c_e \leq |T| - 1 \qquad T \subseteq V \tag{A.3}$$

In this formulation, the variable c_e represents the marginal probability of defending edge e. Constraint (A.2) is a cardinality constraint that ensures exactly that $n-1$ edges are chosen to be protected. The exponential number of subtour constraints (A.3) guarantee that the selected edges do not form any cycles. Note that $E(T) \subseteq E$ represents the edges between targets in $T \subseteq V$. We generated subtour equations as lazy constraints following Cunningham's method for forest matroids [15].

The next formulation we propose is $ERASER_{dflo}$, based on the *directed multicommodity flow model* [32]. Consider an arbitrary node $r \in V$ as the root node for any possible spanning tree, and consider a digraph $D = (V, A)$ formed by replacing each edge $\{i, j\}$ in E by arcs (i, j) and (j, i) in A. In this model, every node $k \in V, k \neq r$ defines a commodity: one unit of commodity k originates at the root node r and must be delivered to node k. Let u^k_{ij} be the flow of commodity k in arc (i, j), and the variable w_{ij} be the capacity for the flow of each commodity k in arc (i, j). Let $w_{ij} = 1$ if the tree contains arc (i, j) when we root it at node r. We formulate this model as follows:

$$
\begin{aligned}
&(\text{ERASER}_{dflo})\\
&\max \sum_{k \in K} \pi^k f^k && && \text{(A.4)}\\
&\text{s.t } (3.7) - (3.11)\\
&\sum_{e \in \delta^-(r)} u_e^h - \sum_{e \in \delta^+(r)} u_e^h = -1 && \forall h \in V, h \neq r && \text{(A.5)}\\
&\sum_{e \in \delta^-(v)} u_e^h - \sum_{e \in \delta^+(v)} u_e^h = 0 && \forall v, h \in V, v \neq r, v \neq h && \text{(A.6)}\\
&\sum_{e \in \delta^-(h)} u_e^h - \sum_{e \in \delta^+(h)} u_e^h = 1 && \forall h \in V, h \neq r && \text{(A.7)}\\
&u_e^h \geq 0 && \forall e \in A && \text{(A.8)}\\
&u_{ij}^h \leq w_{ij} && \forall (i,j) \in E, h \neq r && \text{(A.9)}\\
&w_e \geq 0 && \forall e \in A && \text{(A.10)}\\
&\sum_{e \in A} w_e = n - 1 && && \text{(A.11)}\\
&w_{ij} + w_{ji} = c_e && \forall\ e = (i,j) \in E && \text{(A.12)}
\end{aligned}
$$

Note that $\delta^+(v)$ and $\delta^-(v)$ represent the sets of outward and inward arcs of a node $v \in V$, respectively. Constraints (A.5), (A.6) and (A.7) enforce flow balance. Expressions (A.5) state that the root node r has an outgoing flow of 1 of every commodity $h \in V \setminus \{r\}$. Constraints (A.6) ensure that for every node $v \in V \setminus \{r\}$, except for the destination node of commodity h and the root node, the sum of the incoming flow of commodity $h \in V$ should be equal to the sum of the outgoing flow of the same commodity. Equations (A.7) state that every node h has an incoming flow of 1 of the same commodity. The constraint (A.9) implies that we can send flow of each commodity on arc (i, j) only if that arc is a member of the directed spanning tree defined by the variables w. Constraint (A.11) states that the network defined by any solution contains $n-1$ edges, and every feasible solution must be a spanning tree. Expressions (A.12) state that the total capacity of an edge e is the marginal probability c_e. Constraints (A.8) and (A.10) express the nature of variables.

We consider a closely related formulation by eliminating the w_{ij} variables. This way of representing trees is named *extended multicommodity flow model* [32]. The resulting formulation is:

$$
\begin{aligned}
&(\text{ERASER}_{mc'flo})\\
&\max \sum_{k \in K} \pi^k f^k && && \text{(A.13)}\\
&\text{s.t } (3.7) - (3.11), (\text{A.2}), (\text{A.5}) - (\text{A.8})\\
&u_{ij}^h + u_{ji}^{h'} \leq c_e && \forall h, h' \in V,\ e = (i,j) \in E && \text{(A.14)}
\end{aligned}
$$

This model considers an undirected graph $G = (V, E)$. Although, we allow bidirectional flows u for each commodity k on edge e = $\{i, j\}$. The bidirectional flow inequalities (A.14) link the flow of different commodities flowing in different directions on the edge $\{i, j\}$.

To understand these constraints intuitively, let's consider a feasible spanning tree. If we remove the edge $\{i, j\}$, the graph nodes will be split into two separate components. In this scenario, any commodity associated with a node located in the same component as the root node will not flow on the edge $\{i, j\}$. However, if two commodities have associated nodes in the component without the root, they will flow on the edge $\{i, j\}$ in the same direction. Consequently, when we have two commodities h and h', flowing on the same edge $\{i, j\}$, they will both flow in the same direction. This implies that either u_{ij}^h or $u_{ji}^{h'}$ will be zero.

The next formulation is based on the tree representation $RMST$ by Martin [33]. This formulation stems from the fact that spanning trees do not contain cycles. Let u_{ij}^h be the flow of commodity h in arc (i, j). One unit of commodity h must be delivered to node h. Variables c_e represent the marginal probability of defending edge e.

$$
\begin{aligned}
&(\text{ERASER}_{RMST}) \\
&\max \sum_{k \in K} \pi^k f^k && && \text{(A.15)} \\
&\text{s.t } (3.7) - (3.11) \\
&\sum_{s \in V: s > i} u_{is}^h + \sum_{j \in V: j < i} u_{ij}^h \leq 1 && \forall h, i \in V, i \neq h && \text{(A.16)} \\
&\sum_{s \in V: s > h} u_{hs}^h + \sum_{j \in V: j < h} u_{hj}^h \leq 0 && \forall h \in V && \text{(A.17)} \\
&u_{ij}^h \geq 0 && \forall i, j, h \in V && \text{(A.18)} \\
&\sum_{e \in E} c_e = n - 1 && && \text{(A.19)} \\
&u_{ij}^h + u_{ji}^h = c_e && \forall e = (i, j) \in E, h \in V && \text{(A.20)}
\end{aligned}
$$

First, note that that each target is assigned a numerical label. Thus, when we refer to the notation $s > i$, it means that the label of target s is greater than the label of target i. Constraint (A.16) states that every node has at most one flow of commodity h directed out of it, except for the destination node of commodity h. This prohibits undirected cycles. Constraint (A.17) enforces that node h is the final destination of commodity h. This constraint also prohibits directed cycles. Constraint (A.19) states that a n-node spanning tree should have $n - 1$ edges. Constraint (A.20) forces the flow of commodity h over arc (i, j) and (j, i) to be equal to the marginal probability of defending edge e. Constraint (A.18) states the nature of variables.

A.2 Mip-p-S

We will now present formulation Mip-p-S [8] considering the four previously defined approaches to representing trees: $(dflo)$, $(mc'flo)$, (sub), $(RMST)$. The main difference with ERASER is using the transformation of variables $y_{lj}^k = c_l q_j^k$ $\forall l, j \in J$ and $k \in K$. Note that $c_l = \sum_{j \in J} y_{lj}^0$.

$$
\begin{aligned}
&(\text{Mip-p-S}_{sub}) \\
&\max_{y,q} \sum_{j \in J} \sum_{k \in K} p^k (D^k(j|p) y_{jj}^k + D^k(j|u)(q_j^k - y_{jj}^k)) && && \text{(A.21)} \\
&\text{s.t } (3.14) - (3.17) \\
&\sum_{e \in E} \sum_{l \in j} y_{el}^0 = n - 1 && && \text{(A.22)} \\
&\sum_{e \in E(T)} \sum_{l \in J} y_{el}^0 \leq |T| - 1 && T \subseteq V && \text{(A.23)}
\end{aligned}
$$

$$
\begin{aligned}
&(\text{Mip-p-S}_{dflo}) \\
&\max_{y,q} \sum_{j \in J} \sum_{k \in K} p^k (D^k(j|p) y_{jj}^k + D^k(j|u)(q_j^k - y_{jj}^k)) && && \text{(A.24)} \\
&\text{s.t } (3.14) - (3.17), (\text{A.5}) - (\text{A.11}) \\
&w_{ij} + w_{ji} = \sum_{l \in J} y_{el}^0 && \forall\ e = (i, j) \in E && \text{(A.25)}
\end{aligned}
$$

(Mip-p-S$_{mc'flo}$)

$$\max_{y,q} \sum_{j\in J}\sum_{k\in K} p^k(D^k(j|p)y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) \tag{A.26}$$

$$\text{s.t } (3.14)-(3.17), (\text{A}.22), (\text{A}.5)-(\text{A}.8)$$

$$u^h_{ij} + u^{h'}_{ji} \leq \sum_{l\in J} y^0_{el} \qquad \forall e=(i,j)\in E, h\in V \tag{A.27}$$

(Mip-p-S$_{RMST}$)

$$\max_{y,q} \sum_{j\in J}\sum_{k\in K} p^k(D^k(j|p)y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) \tag{A.28}$$

$$\text{s.t } (3.14)-(3.17), (\text{A}.16)-(\text{A}.18)$$

$$\sum_{e\in E}\sum_{l\in E} y^0_{el} = n-1 \tag{A.29}$$

$$u^h_{ij} + u^h_{ji} = \sum_{l\in J} y^0_{el} \qquad \forall e=(i,j)\in E, h\in V \tag{A.30}$$

8.3. SDOBSS

Recall that the SSG formulation SDOBSS [8], can be constructed from Mip-p-S by replacing constraints (3.16) by:

$$0 \leq s^k - A^k(j|p)\sum_{l\in J} y^k_{jl} - A^k(j|u)(1-\sum_{l\in J} y^k_{jl}) \leq (1-q^k_j)M \qquad \forall j\in J, k\in K \tag{8.31}$$

By replacing this constraint on formulations Mip-p-S$_{sub}$, Mip-p-S$_{dflo}$, Mip-p-S$_{mc'flo}$, and Mip-p-S$_{RMST}$, we can derive their corresponding versions in the SDOBSS framework.