A New Single-Layer Inverse-Free Fixed-Time Dynamical System for Absolute Value Equations

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***Abstract*—In this technical note, a novel single-layer inverse-free fixed-time dynamical system (SIFDS) is proposed to address absolute value equations. The proposed SIFDS directly employs coefficient matrix and absolute value equation function that aims at circumventing matrix inverse operation and achieving fixed-time convergence. The equilibria of the proposed SIFDS is proved to be the unique solution of absolute value equation under the mild condition. In contrast to most existing dynamical systems, the salient feature of the proposed SIFDS is its concise structure and tighter upper bound of convergence time. Moreover, theoretical analysis shows that our SIFDS possesses fixed-time convergence which is independent of the initial values. To further improve the upper bound of convergence time of SIFDS, we establish a new global error bound for absolute value equation. Finally, numerical simulation results are presented to validate the effectiveness of the proposed SIFDS.**

***Index Terms*—Absolute value equations, fixed-time convergence, global error bound, single-layer inverse-free dynamical system.**

I. Introduction

Various equation systems have received widespread attention in recent decades [1]–[4]. As a class of extremely important equations, absolute value equations (AVEs) [5] have been investigated and applied widely in many fields, such as quadratic programming problem, variational inequality (VI), linear complementarity problem (LCP), mixed integer programming problem, bimatrix game and economic equilibrium problem [6]–[10]. The problem of AVE is cast as

$$\boldsymbol{A}\boldsymbol{z} - |\boldsymbol{z}| - \boldsymbol{b} = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, $\boldsymbol{z} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^m$, and $|\boldsymbol{z}| \in \mathbb{R}^m$ represents the component-wise absolute value of $\boldsymbol{z}$ whose the k-th component is z_k if $z_k \in \mathbb{R}_+ \cup \{0\}$ and $-z_k$ otherwise. It is well known that AVE (1) is NP-hard [5] owing to its non-differentiability and nonlinearity.

To effectively address the AVEs, many numerical computation methods have been investigated, such as Levenberg-Marquardt method [11], and fixed point iteration method [12] and its improved form [13]. These methods are all discrete, and they usually need to coordinate the iteration step length and search direction to achieve the fast computation goal.

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The numerical methods [11]–[13] are easily implemented on digital computers. However, they cannot efficiently obtain real-time solutions, and often encounter difficulties in coordinating the iteration step length and search direction. To tackle this issue, this paper considers the dynamical systems. In contrast to discrete methods, dynamical systems have the advantages of parallel processing of information, acquisition of real-time solutions, convenience of convergence analysis via Lyapunov theory, and induction of some possible discrete methods [14]–[18]. In the last decades, dynamical systems for addressing the AVEs have attracted much attention in [19]–[25]. For instance, to find the exact solution, the globally convergent double-projection dynamical system was presented in [19] and the asymptotically stable projection dynamical system was constructed in [20]. In addition, the authors in [21] reformulated AVE (1) as a smooth unconstrained problem by the smooth approximation technique, and proposed a unified smoothing dynamical system based on eight systematically generated smoothing functions to tackle this smooth unconstrained problem. To overcome the dependence of the upper bound for convergence time upon the initial states, the authors in [22] reformulated AVE (1) as an LCP, and proposed a fixed-time convergent projection dynamical system (FCPDS) for solving this LCP to obtain the solution of AVE (1). It is worth pointing out that these dynamical systems cannot avoid matrix inversion, which may increase computation load, especially for the large scale AVEs problems. To overcome this limitation, several inverse-free dynamical systems were designed for tackling the AVEs in the literatures. Recently, the asymptotically convergent inverse-free dynamical system (ACIDS) was proposed in [23]. In addition, the authors in [24] extended the ACIDS, and proposed an inertial inverse-free dynamical system (IIDS) by introducing an inertial term. From the perspective of structure, we know that the ACIDS is one-layer while the IIDS is two-layer and involves more neurons than the ACIDS. Considering fixed-time stability and the ACIDS with simple structure, more recently, the authors in [25] proposed a fixed-time convergent inverse-free dynamical system (FCIDS) with one-layer structure.

To inherit the characteristics of matrix inverse-free operation and single-layer structure of the ACIDS [23], and the fixed-time stability of the FCIDS [25], we aim to design an inverse-free dynamical system with simple structure and accelerated convergence to solve AVE (1). To put it briefly, our motivation is to develop a new inverse-free dynamical system to improve the convergence of the FCIDS [25]. The main contributions of this technical note are highlighted as follows.

- A novel single-layer inverse-free fixed-time dynamical system (SIFDS) is proposed to address AVE (1). Based on the fixed-time stability theory in [26], SIFDS is rigorously proved to be convergent in fixed time, and its convergence point is just the unique solution of AVE (1). In contrast to the FCIDS [25], the proposed SIFDS has a smaller upper bound for convergence time to achieve the solution of AVE (1). Moreover, for AVE (1), a new error bound which is more compact than that in [23] is established to further improve the convergence of SIFDS.

- Compared with the two-layer FCPDS [22], the proposed SIFDS possesses less neurons and the feature of matrix inverse-free operation, and only involves Assumption 1 which indicates that we have more relaxed restriction on AVE (1).
- Detailed comparisons of SIFDS with the FCIDS [25] are given by a numerical example. It can be observed from the numerical simulation results that the proposed SIFDS is effective and enjoys faster convergence.

This paper is outlined as below. Section II introduces some preliminaries on AVE (1) and fixed-time stability theory. Section III designs a novel single-layer inverse-free fixed-time dynamical system, and provides some convergence results. In Section IV, numerical simulation results are listed. The conclusion is presented in Section V.

II. PRELIMINARIES

A. Notations

$\mathbb{R}_+$ represents the set of positive real numbers. Let $\boldsymbol{0}$ denote a column vector or square matrix with all entries equal to $0 \in \mathbb{R}$ (its size is to be understood from this brief). Let $\boldsymbol{E} \in \mathbb{R}^{m\times m}$ stand for the identity matrix. $(\cdot)^\top$ represents the transpose of some vector or matrix. $\langle \cdot, \cdot \rangle$ denotes the inner product. For $\boldsymbol{B} \in \mathbb{R}^{m\times m}$, the spectral norm and spectral radius will be represented by $\|\boldsymbol{B}\|_2$ and $\rho(\boldsymbol{B})$, respectively. For $\boldsymbol{x} \in \mathbb{R}^m$, the ℓ_1-norm will be denoted by $\|\boldsymbol{x}\|_1$ and the ℓ_2-norm by $\|\boldsymbol{x}\|$. Denote $|\cdot|$ as the absolute value vectors defined by $|\boldsymbol{b}| = (|b_1|, \cdots, |b_m|)^\top$. $\mathrm{evs}(\boldsymbol{B})$ denotes the eigenvectors space of $\boldsymbol{B} \in \mathbb{R}^{m\times m}$ For $\boldsymbol{B} \in \mathbb{R}^{m\times m}$, $\lambda_{\min}(\boldsymbol{B})$ denotes $\boldsymbol{B}$'s smallest eigenvalue. Denote $\boldsymbol{tridiag}(p, q, r)$ as a matrix whose the sub-diagonal, the main diagonal, the super-diagonal and other entries are $p \in \mathbb{R}$, $q \in \mathbb{R}$, $r \in \mathbb{R}$ and $0 \in \mathbb{R}$, respectively. Set $\boldsymbol{C} > 0$ if $\boldsymbol{C} \in \mathbb{R}^{m\times m}$ is positive definite.

B. Absolute Value Equation (AVE)

Let $\Theta(\boldsymbol{z}) := (\boldsymbol{A} + \boldsymbol{E})\boldsymbol{z} - \boldsymbol{b}$, $\Xi(\boldsymbol{z}) := (\boldsymbol{A} - \boldsymbol{E})\boldsymbol{z} - \boldsymbol{b}$ and $\Omega := \{\boldsymbol{z} \in \mathbb{R}^m : \boldsymbol{z} \geq \boldsymbol{0}\}$. According to the analysis of [5], [23], it can be known that AVE (1) is equivalent to the following general LCP: find a $\boldsymbol{z} \in \mathbb{R}^m$ such that

$$\Theta(\boldsymbol{z}) \geq 0,\ \Xi(\boldsymbol{z}) \geq 0 \text{ and } \langle \Theta(\boldsymbol{z}), \Xi(\boldsymbol{z}) \rangle = 0. \tag{2}$$

Notice that (2) is equivalent to the following VI: find a $\boldsymbol{z} \in \mathbb{R}^m$ such that $\Theta(\boldsymbol{z}) \in \Omega$ and

$$\langle \boldsymbol{y} - \Theta(\boldsymbol{z}), \Xi(\boldsymbol{z}) \rangle \geq 0,\ \forall \boldsymbol{y} \in \Omega, \tag{3}$$

which can be equivalently transformed into the following projection equation [29]:

$$\Theta(\boldsymbol{z}) = P_\Omega[\Theta(\boldsymbol{z}) - \Xi(\boldsymbol{z})], \tag{4}$$

where $P_\Omega(\boldsymbol{v})$ represents a projection operator of the vector $\boldsymbol{v} \in \mathbb{R}^m$ on the set Ω, and $P_\Omega(\boldsymbol{v}) := \arg\min_{\boldsymbol{x}\in\Omega} \|\boldsymbol{x} - \boldsymbol{v}\|$.

Before proceeding, the following assumption is adopted for our analysis.

Assumption 1. *For AVE* (1)*, $\sigma_{min}(\boldsymbol{A}) \in (1, +\infty)$, where $\sigma_{min}(\boldsymbol{A})$ represents the smallest singular value of $\boldsymbol{A}$.*

Remark 1. Assumption 1 has received extensive attention (see [5], [22], [23], [25]). From [5, Proposition 3], it can be seen that Assumption 1 is given to ensure that AVE (1) enjoys a unique solution, which means that it makes sense to design some computing approaches to address AVE (1).

C. Some Necessary Definitions and Lemmas

Consider the following system:

$$\begin{cases} \dot{\boldsymbol{z}}(t) = \mathcal{F}(\boldsymbol{z}(t)), \\ \boldsymbol{z}(t_0) = \boldsymbol{z}_0 \in \mathbb{R}^m, \end{cases} \tag{5}$$

where $\mathcal{F} : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function.

The following lemma results show the global fixed-time stability characteristic of system (5).

Lemma 1. *[26] The vector $\boldsymbol{z}^\star \in \mathbb{R}^m$ is called as an equilibria of system* (5) *if $\mathcal{F}(\boldsymbol{z}^\star) = \boldsymbol{0}$. There exists a radially unbounded and continuously differentiable function $\mathcal{W} : \mathbb{R}^m \to \mathbb{R}_+ \cup \{0\}$ such that $\mathcal{W}(\boldsymbol{z}(t)) = 0$ iff $\boldsymbol{z}(t) = \boldsymbol{0}$. Moreover, for each solution $\boldsymbol{z}(t) \in \mathbb{R}^m$ with $t \geq 0$, if the following inequality*

$$\dot{\mathcal{W}}(\boldsymbol{z}(t)) \leq -\left[\hat{\alpha}\mathcal{W}(\boldsymbol{z}(t)) + \hat{\beta}\mathcal{W}^c(\boldsymbol{z}(t)) + \hat{\gamma}\mathcal{W}^d(\boldsymbol{z}(t))\right]$$

is satisified, where $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \mathbb{R}_+, c \in (0, 1)$ and $d \in (1, +\infty)$, then the equilibria $\boldsymbol{z}^\star \in \mathbb{R}^m$ of system (5) *is globally fixed-time stable, and the estimate $T_{\sup}$ of the settling time $T(\boldsymbol{z}(t_0))$ is presented as follows:*

$$T(\boldsymbol{z}(t_0)) \leq \frac{\ln\left(1 + \frac{\hat{\alpha}}{\hat{\beta}}\right)}{\hat{\alpha}(1-c)} + \frac{\ln\left(1 + \frac{\hat{\alpha}}{\hat{\gamma}}\right)}{\hat{\alpha}(d-1)} = T_{\sup},\quad \forall \boldsymbol{z}(t_0) \in \mathbb{R}^m.$$

Remark 2. Utilizing the fact that the inequality $0 < \ln(z + 1) < z$ always holds for all $z \in \mathbb{R}_+$, it is not difficult to deduce that the supremum $T_{\sup}$ of the settling time $T(\boldsymbol{z}(t_0))$ in Lemma 1 is slightly tighter than that in [28].

In the following, the properties for AVE (1) are listed.

Lemma 2. *[23] Let $\varepsilon(\boldsymbol{z}) := \Theta(\boldsymbol{z}) - P_\Omega[\Theta(\boldsymbol{z}) - \Xi(\boldsymbol{z})]$ in* (4)*. Then, $\boldsymbol{z}^\star$ is a solution to AVE* (1) *iff $\varepsilon(\boldsymbol{z}^\star) = \boldsymbol{0}$. Moreover, $\varepsilon(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} - |\boldsymbol{z}| - \boldsymbol{b}$ can be obtained directly.*

Lemma 3. *[23] If $\boldsymbol{z}^\star \in \mathbb{R}^m$ is a solution to AVE* (1) *and Assumption 1 holds, then*

$$\left\langle \boldsymbol{z} - \boldsymbol{z}^\star, \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{z} - |\boldsymbol{z}| - \boldsymbol{b}) \right\rangle \geq \frac{1}{2}\left\|\boldsymbol{A}\boldsymbol{z} - |\boldsymbol{z}| - \boldsymbol{b}\right\|^2,\ \forall\, \boldsymbol{z} \in \mathbb{R}^m.$$

Lemma 4. *[23] Let $\Omega(\boldsymbol{z}) := \boldsymbol{A}\boldsymbol{z} - |\boldsymbol{z}| - \boldsymbol{b}$. Under Assumption 1, AVE* (1) *possesses a unique solution, say $\boldsymbol{z}^\star \in \mathbb{R}^m$, and*

$$\|\boldsymbol{z} - \boldsymbol{z}^\star\| \in \left[\frac{1}{\tau_1 + \tau_2}\|\Omega(\boldsymbol{z})\|, \frac{\tau_1 + \tau_2}{\eta}\|\Omega(\boldsymbol{z})\|\right] \tag{6}$$

TABLE I: COMPARISON OF RELATED DYNAMICAL SYSTEMS FOR AVE (1)

Algorithms	Neurons	Layers	Matrix inversion	Assumption 1 only
FCPDS [22]	$2m$	2	yes	no
FCIDS [25]	m	1	no	yes
IIDS [24]	$2m$	2	no	no
SIFDS	$\boldsymbol{m}$	**1**	**no**	**yes**

holds for each $z \in \mathbb{R}^m$*, where* $\tau_1 = \|A+E\|_2$, $\tau_2 = \|A-E\|_2$*, and* $\eta = \lambda_{\min}(A^T A) - 1$.

III. SIFDS and Its Convergence Analysis[1]

For tackling AVE (1), the following SIFDS is proposed:

$$\dot{z} = -\hbar\Big[\gamma + \alpha\big\|\Omega(z)\big\|^{u-1} + \beta\big\|\Omega(z)\big\|^{v-1}\Big]\Lambda(z), \quad (7)$$

where $\Omega(z) := Az - |z| - b$, $\Lambda(z) := A^\top \Omega(z)$, and $\hbar \in \mathbb{R}_+$, $\gamma \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+$, $u \in (-1, 1)$ and $v \in (1, +\infty)$ are six design parameters. The dynamical system described in (7) is easy to be implemented by a one-layer recurrent neural network.

Remark 3. The proposed SIFDS (7) is inspired from the inverse-free dynamical systems in [23], [25]. It is worth noticing that if $\gamma = 1$ and $\alpha = \beta = 0$, then SIFDS (7) degenerates to the ACIDS [23]. If $\gamma = 0$ and $u \in (0, 1)$, then SIFDS (7) degenerates to the FCIDS [25]. Meanwhile, it can be observed from Table I that our SIFDS inherits the advantages of single-layer structure, inverse-free calculation and less restrictions on AVE (1) in [25]. However, different from the FCIDS [25], our algorithm enjoys a smaller upper bound for convergence time to achieve the unique solution of AVE (1) (see Theorem 1 in this work for details). Moreover, compared with the FCPDS [22] and IIDS [24], our algorithm has fewer restrictions on AVE (1) and fewer neurons. In contrast to the FCPDS [22], our algorithm can avoid matrix inversion which effectively reduces the hardware consumption.

The following lemma states the equivalence between the solution to AVE (1) and the equilibria of SIFDS (7).

Lemma 5. *Under Assumption 1,* $z^\star \in \mathbb{R}^m$ *is an equilibria of SIFDS* (7) *iff it is the unique solution to AVE* (1).

Proof: It is pointed out that if $z^\star \in \mathbb{R}^m$ is an equilibrium point of SIFDS (7), then

$$\Big[\alpha\big\|\Omega(z)\big\|^{u-1} + \beta\big\|\Omega(z)\big\|^{v-1} + \gamma\Big]\Lambda(z) = \boldsymbol{0},$$

that is,

$$\Lambda(z^\star) = A^\top \Omega(z^\star) = \boldsymbol{0}. \quad (8)$$

Note that it can be directly deduced from Assumption 1 that square matrix A is invertible. It thus follows from (8) that

$$\Omega(z^\star) = Az^\star - |z^\star| - b = \boldsymbol{0}, \quad (9)$$

which means that vector $z^\star$ is the solution to AVE (1). Conversely, if vector $z^\star$ is the solution to AVE (1), it is not difficult to find that it is the equilibria of SIFDS (7). Hence, the conclusion is established. ■

We now show Lipschitz continuity of $\Lambda(z)$ in (7).

Lemma 6. *The mapping* $\Lambda(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ *involved in SIFDS* (7) *is Lipschitz continuous in* $\mathbb{R}^m$.

Proof: By (7), the triangle inequality and the compatibility [30] of the matrix spectral norm and vector ℓ_2-norm, we deduce:

$$\big\|\Lambda(x) - \Lambda(y)\big\| \le \big\|A^\top A(x - y)\big\| + \big\|A^\top(|x| - |y|)\big\| \quad (10)$$

for each $x, y \in \mathbb{R}^m$. From [30, Remark 1], it follows that

$$\big\|A^\top A(x - y)\big\| \le l_1\big\|x - y\big\|, \quad (11)$$

and

$$\Big\|A^\top(|x| - |y|)\Big\| \le l_2\Big\||x| - |y|\Big\|, \quad (12)$$

where $l_1 = \|A^\top A\|_2$ and $l_2 = \|A\|_2$. Then, combining (10), (11) and (12), one has

$$\big\|\Lambda(x) - \Lambda(y)\big\| \le l_1\big\|x - y\big\| + l_2\Big\||x| - |y|\Big\|. \quad (13)$$

In light of (13) and the fact that $\Big\||x| - |y|\Big\| \le \|x - y\|$, it implies that

$$\big\|\Lambda(x) - \Lambda(y)\big\| \le (l_1 + l_2)\big\|x - y\big\|,$$

which means that the mapping $\Lambda(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is globally Lipschitz continuous with the Lipschitz constant $\mathcal{L} = l_1 + l_2$. Therefore, this completes the proof. ■

Remark 4. From [27, Lemma 3], it can be seen that $\Lambda(\cdot)$ is a Lipschitz continuous vector field. By virtue of [27, Lemma 3] and Lemma 6, one deduces that SIFDS (7) has a unique equilibrium point.

In what follows, the fixed-time convergence analysis of SIFDS (7) is presented.

Theorem 1. *Under Assumption 1, SIFDS* (7) *is globally fixed-time convergent with the settling time given as*

$$T\big(z(t_0)\big) \le \frac{\ln\big(1 + \frac{\hat{\alpha}}{\hat{\beta}}\big)}{\hat{\alpha}(1 - c)} + \frac{\ln\big(1 + \frac{\hat{\alpha}}{\hat{\gamma}}\big)}{\hat{\alpha}(d - 1)} = T_{\sup} \quad (14)$$

for any initial vector $z(t_0) \in \mathbb{R}^m$*, where*

$$\hat{\alpha} = \hbar\gamma\zeta^2, \quad \hat{\beta} = \hbar\alpha\zeta^{u+1}, \quad \hat{\gamma} = \hbar\beta\zeta^{v+1}, \quad (15a)$$

$$\zeta = \frac{\eta}{\tau_1 + \tau_2}, \quad c = \frac{u+1}{2} \in (0, 1), \quad d = \frac{v+1}{2} \in (1, +\infty) \quad (15b)$$

are all positive constants.

Proof: The Lyapunov function candidate is defined as

$$\mathcal{W}\big(z(t)\big) := \big\|z(t) - z^\star\big\|^2. \quad (16)$$

For the sake of simplification discusssed, $\mathcal{W}$ is used to stand for $\mathcal{W}\big(z(t)\big)$ in the remainder of this brief, unless necessary.

[1]For notational simplicity, (t) will be omitted for all variables (like $z(t)$) containing (t) in the remainder of this work, unless necessary.

From (16), it follows that $\mathcal{W} > 0$ for each $z \in \mathbb{R}^m \setminus \{z^\star\}$, and $\mathcal{W} = 0$ iff $z = z^\star$, and

$$\mathcal{W} \geq \frac{1}{2}\|z - z^\star\|^2,$$

which means that $\mathcal{W} \to +\infty$ when $\|z\| \to +\infty$. Taking $\mathcal{W}$'s time derivative yields $\dot{\mathcal{W}} = \frac{d\mathcal{W}}{dz} \cdot \frac{dz}{dt}$, it then follow from (7), (16), and Lemma 3 that

$$\begin{aligned}\dot{\mathcal{W}} =& 2\langle z - z^\star, \dot{z}\rangle \\ =& -2\hbar\Big\langle z - z^\star, \Big[\gamma + \alpha\|\Omega(z)\|^{u-1} + \beta\|\Omega(z)\|^{v-1}\Big]\Lambda(z)\Big\rangle \\ \leq& -\Big[\hbar\alpha\|\Omega(z)\|^{u-1} + \hbar\beta\|\Omega(z)\|^{v-1} + \hbar\gamma\Big]\|\Omega(z)\|^2 \\ \leq& 0,\end{aligned} \tag{17}$$

which implies that $\mathcal{W}$ is non-increasing. It is worth pointing out that if $\dot{\mathcal{W}} = 0$, then $\Omega(z) = \boldsymbol{0}$ is satisfied by utilizing (17) and Lemma 5, which means that $\dot{z} = \boldsymbol{0}$ is established. The above indicates that $\mathcal{W}$ is a Lyapunov function.

By virtue of (6) in Lemma 4, it implies that

$$\|\Omega(z)\| \geq \frac{\eta}{\tau_1 + \tau_2}\|z - z^\star\|. \tag{18}$$

Based on the definition of $\mathcal{W}$, one has $\mathcal{W}^{\frac{1}{2}} = \|z - z^\star\|$. Further, by (18), one derives that

$$\|\Omega(z)\|^2 \geq \Big(\frac{\eta}{\tau_1 + \tau_2}\Big)^2\mathcal{W}. \tag{19}$$

It then follows from (19) that

$$\begin{cases}\|\Omega(z)\|^{u+1} \geq \Big(\frac{\eta}{\tau_1+\tau_2}\Big)^{u+1}\mathcal{W}^{\frac{u+1}{2}}, \\ \|\Omega(z)\|^{v+1} \geq \Big(\frac{\eta}{\tau_1+\tau_2}\Big)^{v+1}\mathcal{W}^{\frac{u+1}{2}}.\end{cases} \tag{20}$$

By (17), (19) and (20), one deduces:

$$\begin{aligned}\dot{\mathcal{W}} \leq& -\hbar\gamma\Big(\frac{\eta}{\tau_1 + \tau_2}\Big)^2\mathcal{W} - \hbar\alpha\Big(\frac{\eta}{\tau_1 + \tau_2}\Big)^{u+1}\mathcal{W}^{\frac{u+1}{2}} \\ &- \hbar\beta\Big(\frac{\eta}{\tau_1 + \tau_2}\Big)^{v+1}\mathcal{W}^{\frac{v+1}{2}} \\ =& -\Big(\hat{\alpha}\mathcal{W} + \hat{\beta}\mathcal{W}^c + \hat{\gamma}\mathcal{W}^d\Big),\end{aligned} \tag{21}$$

where $\hat{\alpha} \in \mathbb{R}_+$, $\hat{\beta} \in \mathbb{R}_+$, $\hat{\gamma} \in \mathbb{R}_+$, $c \in (0, 1)$, and $d \in (1, +\infty)$ are five constants defined as in (15). Consequently, it follows from (21) and Lemma 1 that SIFDS (7) is fixed-time stable, and the supremum T_{sup} for the time $T(z(t_0))$ is listed as in (14). Hence, this proof is completed. ■

Remark 5. Theorem 1 indicates that SIFDS (7) can effectively deal with AVE (1) within fixed time for any initial conditions. In other words, AVE (1) is not necessarily satisfied at the initial time and will hold after a settling time. It is worth pointing out that if $\gamma = 0$ in (7), then the proposed SIFDS is also globally fixed-time convergent (see Theorem 2). However, the supremum of the settling time for our SIFDS with $\gamma = 0$ is larger than one for our SIFDS as noted in Remark 2, which means that $\tilde{T}_{\text{sup}}$ in (25) is greater than T_{sup} in (14). In addition, from Theorem 1, it is not difficult to obtain that if $\gamma = 0$ and $\beta = 0$ in (7), then the proposed SIFDS is globally finite-time convergent.

Theorem 2. *If Assumption 1 is satisfied, then SIFDS* (7) *with $\gamma = 0$ is globally fixed-time convergent with the settling time given as*

$$T(z(t_0)) \leq \frac{1}{\tilde{\alpha}(1 - c)} + \frac{1}{\tilde{\beta}(d - 1)} = \tilde{T}_{\text{sup}} \tag{22}$$

for any initial point $z(t_0) \in \mathbb{R}^m$, where $\tilde{\alpha} = \hbar\alpha\zeta^{u+1} \in \mathbb{R}_+$, $\tilde{\beta} = \hbar\beta\zeta^{v+1} \in \mathbb{R}_+$, and the designed parameters ζ, c and d are defined as in (15b).

Proof: This proof follows from Theorem 1, and thus it is omitted for brevity. ■

Remark 6. It is worth noticing that the proposed SIFDS possesses the same fixed-time convergence characteristics as the dynamical systems in [22], [25]. However, from Remark 3 and Theorem 2, one deduces that the upper bound of convergence time of our SIFDS is smaller than that of the FCPDS [22] and FCIDS [25]. Specifically, their fixed-time convergence are established for the case that $\lambda_1 \in (0, 1)$, excluding the case that $\lambda_1 \in (-1, 0]$, where λ_1 plays the same role as u in this work. Meanwhile, notice that the other parameters in [22], [25] correspond completely to the roles of the parameters α, β and v in SIFDS (7) with $\gamma = 0$, and these parameters can correspond to equality. Thus, it can be seen from Theorems 1-2 that we fill the gap. In addition, from Table I, one can observe that our SIFDS not only circumvents the matrix inversion operations involved in the FCPDS [22], but also inherits the conciseness of the FCIDS [25].

To further improve the convergence time upper bound of the proposed SIFDS, we present the following new global error bound for AVE (1), which is more compact than that in [25].

Lemma 7. *The function $\Omega(z)$ is defined as in* (7). *If Assumption 1 holds, then AVE* (1) *possesses the unique solution $z^\star \in \mathbb{R}^m$, and*

$$\|z - z^\star\| \in \Big[\frac{1}{\tau_1^{\text{new}} + \tau_2^{\text{new}}}\|\Omega(z)\|, \frac{\tau_1^{\text{new}} + \tau_2^{\text{new}}}{\eta}\|\Omega(z)\|\Big] \tag{23}$$

holds for every $z \in \mathbb{R}^m$, where

$$\begin{cases}\tau_1^{\text{new}} = \begin{cases}\rho(\boldsymbol{A} + \boldsymbol{E}), & z - z^\star \in \text{evs}(\boldsymbol{A} + \boldsymbol{E}), \\ \|\boldsymbol{A} + \boldsymbol{E}\|_2, & \text{otherwise},\end{cases} \\ \tau_2^{\text{new}} = \begin{cases}\rho(\boldsymbol{A} - \boldsymbol{E}), & z - z^\star \in \text{evs}(\boldsymbol{A} - \boldsymbol{E}), \\ \|\boldsymbol{A} - \boldsymbol{E}\|_2, & \text{otherwise},\end{cases}\end{cases} \tag{24}$$

and $\eta = \lambda_{\min}(\boldsymbol{A}^\top\boldsymbol{A}) - 1$.

Proof: See Appendix for detailed proof. ■

Remark 7. Notice that it can be deduced from Lemma 7 that $\tau_1^{\text{new}} + \tau_2^{\text{new}}$ in (23) is less than or equal to $\tau_1 + \tau_2$ in (6) for Lemma 4 (i.e., [23, Theorem 4.1]) by virtue of the property (see [31, Theorem 5.6.9.]) of the spectral radius and compatibility (see [30, Remark 1]) of the matrix spectral norm and the vector ℓ_2-norm. Therefore, theoretically, the upper and lower bounds of the error involved in Lemma 7 are more compact than those in Lemma 4, which means that the results in [23] are improved.

Through Lemma 7 and the proof of and Theorem 1, we can directly deduce two other fixed-time convergence conclusions of SIFDS, so the proof of Theorems 3-4 is omitted.

Theorem 3. *If Assumption 1 holds, SIFDS* (7) *is globally fixed-time convergent with the settling time given as*

$$T\big(\mathbf{z}(t_0)\big) \leq \frac{\ln\left(1+\frac{\hat{\alpha}^{\text{new}}}{\hat{\beta}^{\text{new}}}\right)}{\hat{\alpha}^{\text{new}}(1-c)} + \frac{\ln\left(1+\frac{\hat{\alpha}^{\text{new}}}{\hat{\gamma}^{\text{new}}}\right)}{\hat{\alpha}^{\text{new}}(d-1)} = T_{\text{sup}}^{\text{new}}$$

for any initial point $\mathbf{z}(t_0) \in \mathbb{R}^m$, where $\hat{\alpha}^{\text{new}}$, $\hat{\beta}^{\text{new}}$ and $\hat{\gamma}^{\text{new}}$ are similar to the definition of (15)*, except that $\zeta = \eta/(\tau_1 + \tau_2)$ in* (15) *is replaced by $\zeta^{\text{new}} = \eta/(\tau_1^{\text{new}} + \tau_2^{\text{new}})$.*

Theorem 4. *Under Assumption 1, SIFDS* (7) *with $\gamma = 0$ is globally fixed-time convergent with the settling time given as*

$$T\big(\mathbf{z}(t_0)\big) \leq \frac{1}{\tilde{\alpha}^{\text{new}}(1-c)} + \frac{1}{\tilde{\beta}^{\text{new}}(d-1)} = \tilde{T}_{\text{sup}}^{\text{new}} \tag{25}$$

for any initial vector $\mathbf{z}(t_0) \in \mathbb{R}^m$, where $\tilde{\alpha}^{\text{new}}$ and $\tilde{\beta}^{\text{new}}$ are similar to the definition of Theorem 3, except that $\zeta = \eta/(\tau_1+\tau_2)$ is replaced by $\zeta^{\text{new}} = \eta/(\tau_1^{\text{new}} + \tau_2^{\text{new}})$.

Remark 8. It is not difficult to derive $T_{\text{sup}}^{new} \leq T_{\text{sup}}$ and $\tilde{T}_{\text{sup}}^{new} \leq \tilde{T}_{\text{sup}}$ from Lemma 7 and Theorems 3-4. This indicates that theoretically, the upper bound of the convergence time of the proposed SIFDS is theoretically improved with the aid of the new global error bound formula in Lemma 7.

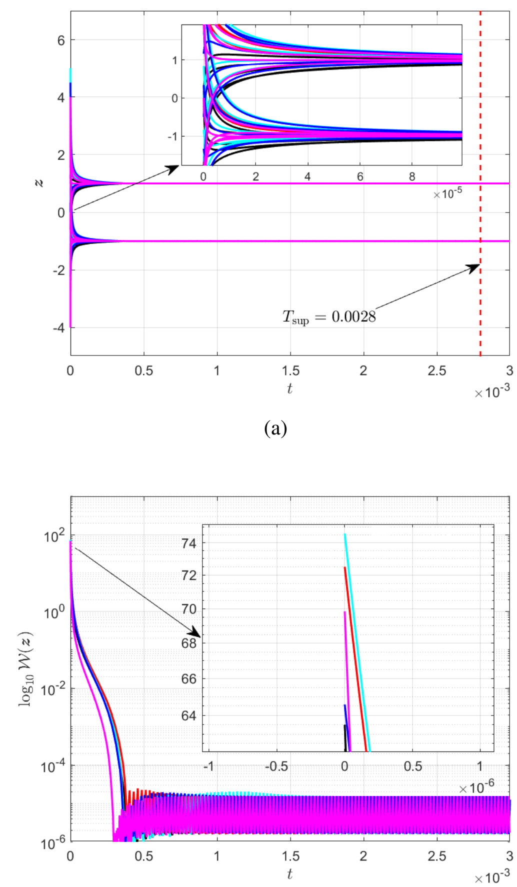
IV. EXPERIMENTAL RESULTS

This section provides a numerical example to illustrate the computation performance and theoretical results of the proposed SIFDS. The ODE45 solver in MATLAB R2019b is used to conduct the simulation on the FCIDS [25] and the proposed SIFDS. In this simulation, we choose the same initial value for different dynamical systems. For the sake of convenience, the parameters $\hbar$, α, β, u, and v are used to replace parameters γ, ρ_1, ρ_2, λ_1, and λ_2 in the FCIDS [25], respectively. In addition, we set the design parameters as $\hbar = 100$ and $\alpha = \beta = 1$. We take $\gamma = 0.5$, $u = -0.8$, and $v = 1.2$ in our SIFDS. Consequently, in our SIFDS, the upper bound of the estimation for the settling time is $T_{\text{sup}} = 0.0028$s. To evaluate the performance of all comparison algorithms, we regard (16) as the tracking error.

Example 1 ([32])*:* Consider AVE (1) with $\boldsymbol{A} = \boldsymbol{tridiag}(-1, 8, -1) \in \mathbb{R}^m$, and $b = \boldsymbol{A}\boldsymbol{y} - |\boldsymbol{y}| \in \mathbb{R}^m$, where

$$\begin{cases} \boldsymbol{tridiag}(-1,8,-1) = \begin{bmatrix} 8 & -1 & & & & \\ -1 & 8 & -1 & & & \\ & -1 & 8 & & & \\ & & & \ddots & -1 & \\ & & & -1 & 8 & -1 \\ & & & & -1 & 8 \end{bmatrix}, \\ \boldsymbol{y} = \big(-1, 1, -1, 1, \cdots, -1, 1\big)^\top. \end{cases}$$

As pointed out in [32], $\lambda_{\min}(\boldsymbol{A}^\top\boldsymbol{A}) > 1$, which means that Assumption 1 holds. Let $m = 10$ in Example 1. All simulation results are presented in Figs. 1-2.



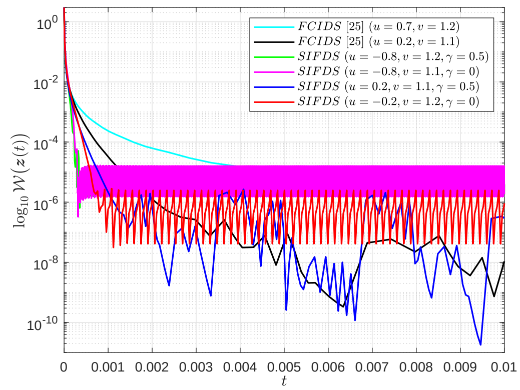
(a)

(b)

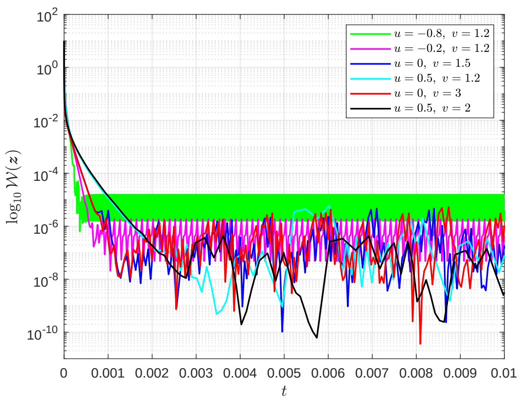
Fig. 1: (a) Transient responses for the SIFDS (7); (b) Tracking error convergence responses for the SIFDS (7).

Fig. 1 reports the transient behaviors of $\mathbf{z}(t)$ and the tracking errors of SIFDS with five different initial points. From Fig. 1(a)-(b), one observes that SIFDS can converge to the solution $\mathbf{z}^\star = \boldsymbol{y}$ under different initial conditions, and its convergence time is upper bounded and far less than 0.0028s. Moreover, from Fig. 1(b), it can be seen that the tracking error trajectories of SIFDS with five different initial values reach stability before 0.0028s, and their final error values hover around 10^{-5}.

Fig. 2(a) depicts the tracking error responses under the FCIDS [25] and SIFDS. Meanwhile, Fig. 2(b) reports several sample results on the tracking error responses for SIFDS with $\gamma = 0.5$ under different values of u and v. It can be seen from Fig. 2(a) that these algorithms ultimately achieve the solution of Example 1. It is worth noticing that the settling time for the FCIDS [25] is strictly larger than 0.0028s, while the settling time for SIFDS less than 0.0028s. Furthermore, one can observe from Fig. 2(a) that when u and v are almost unchanged, the settling time for SIFDS becomes smaller as the value of γ increases. When $u = 0.2$ and $v = 1.1$ are fixed, the error trajectory of SIFDS with $\gamma = 0.5$ reaches stability faster than that of FCIDS [25]. If $\gamma = 0.5$ and v changes slightly, the convergence time for SIFDS becomes shorter as the value of u decreases. We also notice that when v is almost equal and $\gamma = 0$, the convergence time for all algorithms becomes shorter as u tends to -1, and the convergence time of our SIFDS is naturally shorter than that of the FCIDS [25]. As can be observed from Fig. 2(b), the convergence time of SIFDS is improved as the value of u decreases, but the value of v has almost no effect on it. Therefore, it can be found from the above analysis that the proposed SIFDS enjoys faster convergence than the FCIDS [25].



(a)



(b)

Fig. 2: (a) Tracking error convergence responses for the FCIDS [25] and SIFDS (7); (b) For the SIFDS (7), effect of the design parameters ($\hbar = 100$, $\alpha = \beta = 1$, $\gamma = 0.5$).

V. Conclusions

In this paper, we proposes a SIFDS for tackling AVE (1), and rigorously prove its fixed-time convergence. The solution of AVE (1) is achieved in fixed time which does not depend on the initial states of the proposed SIFDS, and an upper bound for the settling time is also explicitly obtained by virtue of Lemma 1. In addition, for AVE (1), we establish a more compact global error bound to further improve the convergence. Numerical simulation results are finally presented to demonstrate the effectiveness and advantages of our SIFDS. Potential future work includes designing distributed algorithms to address AVE (1) and its general form.

Appendix
Proof of Lemma 7

Proof: For simplification, we replace $\varepsilon(z)$, $\Theta(z)$, $\Theta(z^\star)$, $\Xi(z)$ and $\Xi(z^\star)$ with ε, Θ, $\Theta^\star$, Ξ and $\Xi^\star$, respectively. From the VI (3) and the proof of [23, Theorem 3.5], it follows that

$$\Big\langle \varepsilon, \big(\Theta - \Theta^\star\big) + \big(\Xi - \Xi^\star\big) \Big\rangle \geq \big\|\varepsilon\big\|^2 + \big\langle \Theta - \Theta^\star, \Xi - \Xi^\star \big\rangle. \tag{26}$$

Let $\boldsymbol{G} = \boldsymbol{A}^\top \boldsymbol{A} - \boldsymbol{E}$. Using Assumption 1 and the fact that $\boldsymbol{G} = \boldsymbol{G}^\top$, we obtain $\boldsymbol{G} > 0$. It then follows that

$$\big\langle \Theta - \Theta^\star, \Xi - \Xi^\star \big\rangle = \big\langle z - z^\star, \boldsymbol{G}\big(z - z^\star\big) \big\rangle \geq 0. \tag{27}$$

In addition, by [33, Lemma 6], for $0 < \boldsymbol{G} \in \mathbb{R}^{m\times m}$, there is an orthogonal matrix $\boldsymbol{P} \in \mathbb{R}^{m\times m}$, such that

$$\big\langle \boldsymbol{x}, \boldsymbol{G}\boldsymbol{x} \big\rangle = \sum_i^m \lambda_i \big(\boldsymbol{P}\boldsymbol{x}\big)_i^2 \geq \eta \sum_i^m \big(\boldsymbol{P}\boldsymbol{x}\big)_i^2 = \eta \big\|\boldsymbol{x}\big\|^2 \tag{28}$$

always holds for any vector $\boldsymbol{x} \in \mathbb{R}^m$ (which may not be the eigenvector of $\boldsymbol{G}$), where $\lambda_i \in \mathbb{R}_+\big(i \in \{1, \cdots, m\}\big)$ is the eigenvalue of $\boldsymbol{G}$ and $\eta = \lambda_{\min}(\boldsymbol{A}^\top \boldsymbol{A}) - 1 = \lambda_{\min}\big(\boldsymbol{G}\big)$. It then follows from (27) and (28) that

$$\big\langle \Theta - \Theta^\star, \Xi - \Xi^\star \big\rangle \geq \eta \big\|z - z^\star\big\|^2. \tag{29}$$

Further, by (26) and (29), one has

$$\eta \big\|z - z^\star\big\|^2 \leq \big\langle \varepsilon, \big(\Theta - \Theta^\star\big) + \big(\Xi - \Xi^\star\big) \big\rangle. \tag{30}$$

Applying the Cauchy-Schwarz inequality and the triangle inequality, (30) translates to

$$\big\|z - z^\star\big\|^2 \leq \frac{1}{\eta} \big\|\varepsilon\big\| \big(\big\|\Theta - \Theta^\star\big\| + \big\|\Xi - \Xi^\star\big\|\big). \tag{31}$$

If $z - z^\star \in \operatorname{evs}\big(\boldsymbol{A}+\boldsymbol{E}\big)$ $\big(z - z^\star \in \operatorname{evs}\big(\boldsymbol{A}-\boldsymbol{E}\big)\big)$, then $\big\|(\boldsymbol{A}+\boldsymbol{E})\big(z - z^\star\big)\big\| \leq \rho\big(\boldsymbol{A}+\boldsymbol{E}\big)\big\|z - z^\star\big\|$ $\big(\big\|(\boldsymbol{A}-\boldsymbol{E})\big(z - z^\star\big)\big\| \leq \rho\big(\boldsymbol{A}-\boldsymbol{E}\big)\big\|z - z^\star\big\|\big)$ by utilizing [31, Theorem 5.6.9] and the homogeneity of the matrix norm. Otherwise, by the compatibility (see [30, Remark 1]) of the matrix spectral norm and the vector ℓ_2-norm, we obtain that $\big\|(\boldsymbol{A}+\boldsymbol{E})\big(z - z^\star\big)\big\| \leq \big\|\boldsymbol{A}+\boldsymbol{E}\big\|_2 \big\|z - z^\star\big\|$ and $\big\|(\boldsymbol{A}-\boldsymbol{E})\big(z - z^\star\big)\big\| \leq \big\|\boldsymbol{A}-\boldsymbol{E}\big\|_2 \big\|z - z^\star\big\|$.

Thus, by the definitions of $\Theta(\cdot)$ and $\Xi(\cdot)$, (31) can be rewritten as

$$\begin{aligned}\big\|z - z^\star\big\|^2 \leq& \frac{1}{\eta}\big\|\varepsilon\big\| \Big[\big\|(\boldsymbol{A}+\boldsymbol{E})(z - z^\star)\big\| + \big\|(\boldsymbol{A}-\boldsymbol{E})(z - z^\star)\big\|\Big] \\ \leq& \frac{\tau_1^{\text{new}} + \tau_2^{\text{new}}}{\eta} \big\|\varepsilon\big\| \big\|z - z^\star\big\|,\end{aligned}$$

namely,

$$\big\|z - z^\star\big\| \leq \frac{\tau_1^{\text{new}} + \tau_2^{\text{new}}}{\eta} \big\|\Omega(z)\big\|, \tag{32}$$

where τ_1^{new} and τ_2^{new} are defined by (24). Note that $\tau_1^{\text{new}} \geq 0$ and $\tau_2^{\text{new}} \geq 0$ always hold, and $\tau_1^{\text{new}} = 0$ $\big(\tau_2^{\text{new}} = 0\big)$ iff $\boldsymbol{A} = -\boldsymbol{E}$ ($\boldsymbol{A} = \boldsymbol{E}$), so $\tau_1^{\text{new}} + \tau_2^{\text{new}} \in \mathbb{R}_+$ must be true. From (32), it then follows that the right-hand inequality of (23) holds.

Besides, for any $\boldsymbol{y} \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$, if $\boldsymbol{y} - z \in \operatorname{evs}\big(\boldsymbol{A} + \boldsymbol{E}\big)$, then $\big\|(\boldsymbol{A}+\boldsymbol{E})\big(\boldsymbol{y} - z\big)\big\| \leq \rho\big(\boldsymbol{A}+\boldsymbol{E}\big)\big\|\boldsymbol{y} - z\big\|$ by using [31, Theorem 5.6.9] and the homogeneity of the matrix norm;

otherwise, it follows from [30, Remark 1] that $\big\|(\boldsymbol{A}+\boldsymbol{E})(\boldsymbol{y}-\boldsymbol{z})\big\| \leq \big\|\boldsymbol{A}+\boldsymbol{E}\big\|_2\big\|\boldsymbol{y}-\boldsymbol{z}\big\|$. Thus, by the definitions of $\Theta(\cdot)$ and τ_1^{new}, we conclude that

$$\big\|\Theta(\boldsymbol{y})-\Theta(\boldsymbol{z})\big\| \leq \tau_1^{\mathrm{new}}\big\|\boldsymbol{y}-\boldsymbol{z}\big\| \tag{33}$$

for any $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{z} \in \mathbb{R}^m$. It then follows from (33) that $\Theta(\cdot)$ is globally Lipschitz continuous with the Lipschitz constant $\mathcal{L} = \tau_1^{\mathrm{new}}$. By (26) and (27), one has

$$\big\|\varepsilon\big\|^2 \leq \Big\langle \varepsilon, \big(\Theta-\Theta^\star\big)+\big(\Xi-\Xi^\star\big)\Big\rangle. \tag{34}$$

By (24), (34), the Cauchy-Schwarz inequality, the triangle inequality, [31, Theorem 5.6.9] and [30, Remark 1], one has

$$\begin{aligned}\big\|\varepsilon\big\|^2 \leq & \big\|\varepsilon\big\|\Big(\big\|\Theta-\Theta^\star\big\|+\big\|\Xi-\Xi^\star\big\|\Big)\\ \leq & \big(\tau_1^{\mathrm{new}}+\tau_2^{\mathrm{new}}\big)\big\|\varepsilon\big\|\big\|\boldsymbol{z}-\boldsymbol{z}^\star\big\|,\end{aligned}$$

that is

$$\big\|\boldsymbol{z}-\boldsymbol{z}^\star\big\| \geq \frac{1}{\tau_1^{\mathrm{new}}+\tau_2^{\mathrm{new}}}\big\|\Omega(\boldsymbol{z})\big\|. \tag{35}$$

It thus follows from (35) that the left-hand inequality of (23) is true. This proof is completed. ■

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