Randomized Robust Price Optimization

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The robust multi-product pricing problem is to determine the prices of a collection of products so as to maximize the worst-case revenue, where the worst case is taken over an uncertainty set of demand models that the firm expects could be realized in practice. A tacit assumption in this approach is that the pricing decision is a deterministic decision: the prices of the products are fixed and do not vary. In this paper, we consider a *randomized* approach to robust pricing, where a decision maker specifies a distribution over potential price vectors so as to maximize its worst-case revenue over an uncertainty set of demand models. We formally define this problem – the *randomized robust price optimization* problem – and analyze when a randomized price scheme performs as well as a deterministic price vector, and identify cases in which it can yield a benefit. We also propose two solution methods for obtaining an optimal randomization scheme over a discrete set of candidate price vectors based on constraint generation and double column generation, respectively, and show how these methods are applicable for common demand models, such as the linear, semi-log and log-log demand models. We numerically compare the randomized approach against the deterministic approach on a variety of synthetic and real problem instances; on synthetic instances, we show that the improvement in worst-case revenue can be as much as 1300%, while on real data instances derived from a grocery retail scanner dataset, the improvement can be as high as 92%.

Key words: pricing, randomization, robust optimization

1. Introduction

Price optimization is a key problem in modern business. The price optimization problem can be stated as follows: we are given a collection of products. We are given a demand model which tells us, for each product, what the expected demand for that product will be as a function of the price of that product as well as the price of the other products. Given this demand model, the price optimization problem is to decide a price vector – i.e., what price to set for each product – so as to maximize the total expected revenue arising from the collection of products.

The primary input to a price optimization approach is a demand model, which maps the price vector to the vector of expected demands of a product. However, in practice, the demand model is never known exactly, and must be estimated from data. This poses a challenge because data is typically limited, and thus a firm often faces uncertainty as to what the demand model is. This is problematic because a mismatch between the demand model used for price optimization – the nominal demand model – and the demand model that materializes in reality can lead to suboptimal revenues.

As a result, there has been much research in how to address demand model uncertainty in pricing. In the operations research community, a general framework for dealing with uncertainty is robust optimization. The idea of robust optimization is to select an uncertainty set, which is a set of values for the uncertain parameter that we believe could plausibly occur, and to optimize the worst-case value of the objective function, where the worst-case is taken over the uncertainty set. In the price optimization context, one would construct an uncertainty set of potential demand models and determine the prices that maximize the worst-case expected revenue, where the worst-case is the minimum revenue over all of the demand models in the uncertainty set. In applying such a procedure, one can ensure that the performance of the chosen price vector is good under a multitude of demand models, and that one does not experience the deterioration of a price vector optimized from a single nominal demand model.

Typically in robust optimization, the robust optimization problem is to find the *single* best decision that optimizes the worst-case value of the objective function. Stated in a slightly different way, one *deterministically* implements a single decision. However, a recent line of research (Delage et al. 2019) has revealed that with regard to the worst-case objective, it is possible to obtain better performance than the traditional deterministic robust optimization approach by *randomizing* over multiple solutions. Specifically, instead of optimizing over a single decision in some feasible set that optimizes the worst-case objective, one optimizes over a *distribution* supported on the feasible set that informs the decision maker how to randomize.

In this work, we propose a methodology for robust price optimization that is based on randomization. In particular, we propose solving a randomized robust price optimization (RRPO) problem, which outputs a probability distribution that specifies the frequency with which the firm should use different price vectors. From a practical perspective, such a randomization scheme has the potential to be implemented in modern retailing as a strategy for mitigating demand uncertainty. In particular, in an ecommerce setting, randomization is already used for A/B testing, which involves randomly assigning some customers to one experimental condition and other customers to a different experimental condition. Thus, it is plausible that the same form of randomization could be used to display different price vectors with certain frequencies. In the brick-and-mortar setting, one can potentially implement randomization by varying prices geographically or temporally. For example, if the RRPO solution is the three price vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ with probabilities 0.2, 0.3, 0.5, then for a set of 50 regions, we would choose 10 ($= 50 \times 0.2$) to assign to price vector $\mathbf{p}_1$, 15

($= 50 \times 0.3$) to assign to $\mathbf{p}_2$, and 25 ($= 50 \times 0.5$) to assign to $\mathbf{p}_3$. Similarly, if one were to implement the same randomization scheme temporally, then for a selling horizon of 20 weeks, one would use the price vector $\mathbf{p}_1$ for 4 ($= 20 \times 0.2$) weeks, $\mathbf{p}_2$ for 6 ($= 20 \times 0.3$) weeks and $\mathbf{p}_3$ for 10 ($= 20 \times 0.5$) weeks.

We make the following specific contributions:

1. **Benefits of randomization**. We formally define the RRPO problem and analyze it under different conditions to determine when the underlying robust price optimization problem is *randomization-receptive* – there is a benefit from implementing a randomized decision over a deterministic decision – versus when it is *randomization-proof* – the optimal randomized and deterministic decisions perform equally well. We show that the robust price optimization problem is randomization-proof in several interesting settings, which can be described roughly as follows: (1) when the set of feasible price vectors is convex and the set of uncertain revenue functions is concave; (2) when the set of feasible price vectors and the set of uncertain demand parameters are convex, and the revenue function obeys certain quasiconvexity and quasiconcavity properties with respect to the price vector and the uncertain parameter; and (3) when the set of feasible price vectors is finite, and a certain minimax property holds. We showcase a number of examples of special cases that satisfy the hypotheses of these results and consequently are randomization-proof. We also present several examples showing how these results can fail to hold when certain assumptions are relaxed and the problem thus becomes randomization-receptive.
2. **Tractable solution algorithms**. We propose algorithms for solving the RRPO problem in two different settings:
   (a) In the first setting, we assume that the set of possible price vectors is a finite set and that the uncertainty set of demand function parameters is a convex set. In this setting, when the revenue function is quasiconvex in the uncertain parameters, we show that the RRPO problem can be solved via delayed constraint generation. The separation problem that is solved to determine which constraint to add is exactly the nominal pricing problem for a fixed uncertain parameter vector of the demand function. For the log-log and semi-log demand models, we show that this nominal pricing problem can be reformulated and solved to global optimality as a mixed-integer exponential cone program. We believe these reformulations are of independent interest as to the best of our knowledge, these are the first exact mixed-integer convex formulations for these problems in either the marketing or operations literatures, and they leverage recent advances in solution technology for mixed-integer conic programs (as exemplified in the conic solver Mosek).

(b) In the second setting, we assume that both the price set and the uncertainty set are finite sets. In this setting, we show how the RRPO problem can be solved using a double column generation method, which involves iteratively generating new uncertainty realizations and price vectors by solving primal and dual separation problems, respectively. We show how the primal and dual separation problems can be solved exactly for the linear, semi-log and log-log demand models.

3. **Numerical evaluation with synthetic and real data**. We evaluate the effectiveness of randomized pricing on different problem instances generated synthetically and problem instances calibrated with real data. Using synthetic data instances, we show that randomized pricing can improve worst-case revenues by as much as 1300% over deterministic pricing, while in our real data instances, the benefit can be as high as 92%. Additionally, we show that for instances of realistic size (up to 20 products), our algorithm can solve the RRPO problem in a reasonable amount of time (no more than four minutes on average).

The rest of this paper is organized as follows. Section 2 reviews the related literature on pricing, robust optimization and randomized robust optimization. Section 3 formally defines the nominal price optimization problem, the deterministic robust price optimization problem and the randomized robust price optimization problem. Section 4 analyzes the robust price optimization problem and provides conditions under which the price optimization problem is randomization-receptive and randomization-proof. Section 5 presents our constraint generation approach for solving the RRPO problem when the price set is a finite set and the uncertainty set is a convex set, and discusses how this approach can be adapted for different families of demand models. Section 6 provides a brief overview of our methodology for solving the RRPO problem when the price set and uncertainty sets are finite sets, with the details provided in Section EC.3 of the companion. Section 7 presents our numerical experiments. Lastly, in Section 8, we conclude and highlight some directions for future research.

2. Literature review

Our paper is closely related to three streams of research: pricing, robust optimization and the use of randomized strategies in optimization. We discuss each of these three streams below.

Pricing optimization and demand models. Optimal pricing has been extensively studied in many fields such as revenue management and marketing research; for a general overview of this research area, we refer readers to Soon (2011) and Gallego and Topaloglu (2019). An important stream of pricing literature is on static pricing, which involves setting a fixed price for a product. The most commonly considered demand models in the static pricing literature are the linear and log-log models. For example, the papers of Zenor (1994) and Bernstein and Federgruen (2003) assume linear demand functions in the study of pricing strategies. The papers of Reibstein and Gatignon (1984), and Montgomery and Bradlow (1999) use log-log (multiplicative) demand functions to represent aggregate demand. The paper of Kalyanam (1996) considers a semi-log demand model. Besides linear, semi-log and log-log demand functions, another type of demand form that is extensively discussed in pricing literature is based on an underlying discrete choice model. For example, Hanson and Martin (1996), Aydin and Ryan (2000), and Hopp and Xu (2005) consider the product line pricing problem under the multinomial logit (MNL) model. Keller et al. (2014) consider attraction demand models which subsume MNL models. Li and Huh (2011) study the pricing problem with the nested logit (NL) models, and show the concavity of the profit function with respect to market share holds. Gallego and Wang (2014) characterize the optimal pricing structure under the general nested logit model with product-differentiated price sensitivities and arbitrary nest coefficients. The papers of Keller (2013) and Zhang et al. (2018) study the multiproduct pricing problem under the family of generalized extreme value (GEV) models which includes MNL and NL models as special cases. Our work differs from this prior work on multiproduct pricing in that the demand model is not assumed to be known, and that there is an uncertainty set of plausible demand models that could actually materialize. Correspondingly, the firm is concerned not with expected revenue under a single, nominal demand model, but with the worst-case revenue with respect to this uncertainty set of demand models.

Another significant stream of pricing literature is to consider multiple-period pricing decisions where the prices of products change over time and there is a fixed inventory of each product; we refer readers to McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003), and Talluri and van Ryzin (2004) for a comprehensive review on dynamic pricing strategies with inventory considerations. As in the static pricing literature, studies in dynamic pricing also vary in the types of demand models. Besbes and Zeevi (2015) assume linear demand in a multiperiod single product pricing problem, and show that the corresponding pricing policy can perform well even under model misspecification. Caro and Gallien (2012) consider multiplicative models where the demand rate and price discount have the logarithmic relationship, and consider a multiproduct clearance pricing optimization problem for the fast-fashion retailer Zara. Akçay et al. (2010) consider dynamic pricing under MNL models for horizontally differentiated products and show that the profit function is unimodal in prices, while Song et al. (2021) and Dong et al. (2009) reformulate the MNL profit as a concave function of its market share rather than prices. Our work focuses on the static, single-period setting where there is no inventory consideration, and is thus not directly related to this stream of the pricing literature.

Robust Optimization. Within the literature mentioned above, either the probability distributions of demand are assumed to be known exactly or the demand models are estimated from historical data. However, in practice, the decision maker often has no access to the complete information of demand distributions. Also, in many real applications, the lack of sales data makes it hard to obtain a good estimation of demand models, which leads to model misspecification and thus suboptimal pricing decisions. In operations research, this type of challenge is most commonly addressed using the framework of robust optimization, where uncertain parameters are assumed to belong to some uncertainty set and one optimizes for the worst-case objective of parameters within the set. We refer readers to Ben-Tal et al. (2009), Bertsimas et al. (2011), Gabrel et al. (2014) and Bertsimas and den Hertog (2022) for a detailed overview of this approach. Robust optimization has been widely applied in various problem settings such as assortment optimization (Rusmevichientong and Topaloglu 2012, Bertsimas and Mišić 2017, Sturt 2021b), inventory management (Bertsimas and Thiele 2006, Govindarajan et al. 2021) and financial option pricing (Bandi and Bertsimas 2014, Sturt 2021a).

Within this literature, our paper contributes to the substream that considers robust optimization for pricing. Thiele (2009) consider tractable robust counterparts to the deterministic multiproduct pricing problem with the budget-of-resource-consumption constraint in the case of additive demand uncertainty, and investigate the impact of uncertainty on the optimal prices of multiple products sharing capacitated resources. Mai and Jaillet (2019) consider robust multiproduct pricing optimization under the generalized extreme value (GEV) choice model and characterize the robust optimal solutions for unconstrained and constrained pricing problems. Hamzeei et al. (2021) study the robust pricing problem with interval uncertainty of the price sensitivity parameters under the multi-product linear demand model. For robust dynamic pricing problems, Lim and Shanthikumar (2007) and Lim et al. (2008) use relative entropy to represent uncertainty in the demand rate and thus the demand uncertainty can be expressed through a constraint on relative entropy. Perakis and Sood (2006), Adida and Perakis (2010) and Chen and Chen (2018) all study robust dynamic pricing problems with demand uncertainty modeled by intervals. Harsha et al. (2019) study robust dynamic price optimization on an omnichannel network with cross-channel interactions in demand and supply where demand uncertainty is modeled through budget constraints. From a different perspective, Cohen et al. (2018) develop a data-driven framework for solving the robust dynamic pricing problem by directly using samples in the optimization. Specifically, the paper considers three types of robust objective (max-min, min-max regret and max-min ratio), and uses the given sampled scenarios to approximate the uncertainty set by a finite number of constraints.

Our work differs from the majority of this body of work that takes a robust optimization approach to pricing in that the decision we seek to make is no longer a deterministic decision, but a randomized one. Within this body of work, the papers closest to our work are Allouah et al. (2021)

and Allouah et al. (2022). The paper of Allouah et al. (2021) considers a pricing problem under a valuation-based model of demand, where each customer has a valuation drawn from an unknown cumulative distribution function on the positive real line, and the firm only has one historical data point. The paper considers the pricing problem from a max-min ratio standpoint, where the firm seeks to find a pricing mechanism that maximizes the percentage attained of the true maximum revenue under the worst-case (minimum) valuation distribution consistent with the data point. The paper shows the approximation rate that is attainable given knowledge of different quantiles of the valuation distribution when the valuation distribution is a regular distribution or a monotone non-decreasing hazard rate distribution. The mechanisms that are proposed in the paper, which are mappings from the data point to a price, include deterministic ones that offer a fixed price, as well as randomized ones that offer different prices probabilistically. The paper of Allouah et al. (2022) considers a similar setting, where instead of knowing a point on the valuation CDF exactly or to within an interval, one has access to an IID sample of valuations drawn from the unknown valuation CDF, and similarly proposes deterministic and randomized mechanisms for this setting.

With regard to Allouah et al. (2021) and Allouah et al. (2022), our setup differs in a number of ways. First, our methodology focuses on a max-min revenue objective, as opposed to a max-min ratio objective that considers performance relative to oracle-optimal revenue. Our methodology also does not start from a valuation model, but instead starts from an aggregate demand model, and additionally considers the multi-product case in the general setting. Additionally, we do not take data as a starting point, but instead assume that the aggregate demand model is uncertain. Lastly, the overarching goals are different: while Allouah et al. (2021) and Allouah et al. (2022) seek to understand the value of data, and how well one can do with limited data, our goal is to demonstrate that from the perspective of worst-case revenue performance, a randomized pricing strategy can be preferable over a deterministic fixed price, and to develop tractable computational methods for computing such strategies under commonly used demand models in the multi-product setting.

Randomized strategies in optimization under uncertainty. The conventional robust optimization problems mentioned above only consider deterministic solutions. In recent years, the benefit of using randomized strategies has received increasing attention in the literature on decision making under uncertainty and robust optimization. Mastin et al. (2015) study randomized strategies for min-max regret combinatorial optimization problems in the cases of interval uncertainty and uncertainty representable by discrete scenarios, and provide bounds on the gains from randomization for these two cases. Bertsimas et al. (2016b) consider randomness in a network interdiction min-max problem where the interdictor can benefit from using a randomized strategy to select arcs to be removed.

The paper of Delage et al. (2019) considers the problem of making a decision whose payoff is uncertain and minimizing a risk measure of this payoff, and studies under what circumstances a randomized decision leads to lower risk than a deterministic decision. The paper characterizes the classes of randomization-receptive and randomization-proof risk measures in the absence of distributional ambiguity (i.e., classical stochastic programs), and discusses conditions under which problems with distributional ambiguity (i.e., distributionally robust problems) can benefit from randomized decisions.

Subsequently, the paper of Delage and Saif (2022) studies the value of randomized solutions for mixed-integer distributionally robust optimization problems. The paper develops bounds on the magnitude of improvement given by randomized solutions over deterministic solutions, and proposes a two-layer column generation method for solving single-stage and two-stage linear DRO problems with randomization. Our paper relates to Delage and Saif (2022) in that we apply a similar two-layer column generation approach for solving the randomized robust pricing problem when the price set and the uncertainty set are both finite; we discuss this connection in more detail in our discussion of the paper of Wang et al. (2020) below. The paper of Sadana and Delage (2023) develops a randomization approach for solving a distributionally robust maximum flow network interdiction problem with a conditional-value-at-risk objective, which is also solved using a column generation approach.

Our work is most closely related to the excellent paper of Wang et al. (2020). The paper of Wang et al. (2020) introduces randomization into the robust assortment optimization and characterizes the conditions under which a randomized strategy strictly improves worst-case expected revenues over a deterministic strategy. The paper proposes several different solution methods for finding an optimal distribution over assortments for the MNL, Markov chain and ranking-based models. For the MNL model in particular, the paper adapts the two-layer column generation method of Delage and Saif (2022) to solve the randomized robust assortment optimization problem when the uncertainty set is discrete.

Our paper shares a high-level viewpoint with the paper of Wang et al. (2020) in that revenue management decisions, such as assortment decisions and pricing decisions, are subject to uncertainty and from an operational point of view, have the potential to be randomized and to benefit from randomization. From a technical standpoint, several of our results on the benefit of randomization when the price set is discrete that are stated in Section 4.3 are generalizations of results in Wang et al. (2020) to the pricing setting that we study. In terms of methodology, the solution approach we apply when the price set and uncertainty set are discrete in Section 6 (described more fully in Section EC.3) is related to the approach in Wang et al. (2020) for the MNL model, as we also use the two-layer column generation scheme of Delage and Saif (2022). The main difference between the method in Wang et al. (2020) and our method lies in the nature of the subproblems. In the paper of Wang et al. (2020), the primal subproblem is a binary sum of linear fractional functions problem, and the dual subproblem is essentially a mixture of multinomial logits assortment problem that can be reformulated as a mixed-integer linear program. In our paper, the primal and dual subproblems that are used to generate new price vectors and uncertainty realizations comes from the underlying pricing problem and the structure of different demand models (linear, semi-log and log-log), which lead to different subproblems than in the assortment setting. In particular, in the semi-log and log-log cases, both the primal and dual subproblems can be formulated as mixed-integer exponential cone programs. In the semi-log and log-log cases specifically, the formulations of the dual subproblems, which are used to identify new price vectors to add, require a logarithmic transformation together with a biconjugate representation of the log-sum-exp function. This technique is also used to develop a constraint generation scheme for the randomized robust pricing problem when the price set is discrete and the uncertainty set is convex (Section 5); the subproblem in this case involves solving a nominal pricing problem under the semi-log or log-log model, which we are also able to reformulate exactly as a mixed-integer exponential cone program. As noted in the introduction, we believe these are the first exact formulations of these problems using mixed-integer conic programming.

Other work on randomization. Lastly, we comment on several streams of work that use randomization but are unrelated to our paper. Within the revenue management community, there are instances where randomization is an operational aspect of the algorithm. For example, in network revenue management, the heuristic of probabilistic allocation control involves using the primal variable values in the deterministic linear program (DLP) to decide how frequently requests should be accepted or rejected (Jasin and Kumar 2012). Here, randomization is used to ensure that the long run frequency with which different requests are accepted or rejected is as close as possible to the DLP solution, which corresponds to an idealized upper bound on expected revenue. As another example, randomization is often also a part of methods for problems that involve learning. For example, Ferreira et al. (2018) propose a method for network revenue management where there is uncertainty in demand rates based on Thompson sampling, which is a method from the bandit literature that involves taking a random sample from the posterior distribution of an uncertain parameter and taking the action that is optimal with respect to that sample. Here, randomization is a way of ensuring that the decision maker explores possibly suboptimal actions. In our work, the focus is not on using randomization to achieve better expected performance or using randomization to achieve a balance between exploration and exploitation, but rather to operationalize randomization to achieve better worst-case performance.

3. Problem definition

In this section, we begin by defining the nominal price optimization problem in Section 3.1. We subsequently define the deterministic robust price optimization problem in Section 3.2. Lastly, we define the randomized robust price optimization problem in Section 3.3.

3.1. Nominal price optimization problem

We assume that the firm offers I products, indexed from 1 to I. We let p_i denote the price of product $i \in [I]$, where we use the notation $[n] = \{1, \dots, n\}$ for any positive integer n. We use $\mathbf{p} = (p_1, \dots, p_I)$ to denote the vector of prices. We assume that the price vector $\mathbf{p}$ is constrained to lie in the set $\mathcal{P} \subseteq \mathbb{R}_+^I$, where $\mathbb{R}_+$ is the set of nonnegative real numbers.

We let d_i denote the demand function of product i, so that $d_i(\mathbf{p})$ denotes the demand of product i when the price vector $\mathbf{p}$ is chosen. The revenue function $R(\cdot)$ can then be written as $R(\mathbf{p}) = \sum_{i=1}^{I} p_i \cdot d_i(\mathbf{p})$.

The nominal price optimization (NPO) problem can be written simply as

$$\text{NPO}: \quad \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}).$$

There are numerous demand models that can be used in practice, which lead to different price optimization problems; we briefly review some of the more popular ones here.

1. *Linear demand model*: A linear demand model is defined by parameters $\boldsymbol{\alpha} \in \mathbb{R}^I$, $\boldsymbol{\beta} \in \mathbb{R}^I$, $\boldsymbol{\gamma} = (\gamma_{i,j})_{i,j \in [I], i \neq j} \in \mathbb{R}^{I \cdot (I-1)}$. The demand function $d_i(\cdot)$ of each product $i \in [I]$ has the form

$$d_i(\mathbf{p}) = \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j, \tag{1}$$

where $\beta_i \geq 0$ is the own-price elasticity parameter of product i, which indicates how much demand for product i is affected by the price of product i, whereas $\gamma_{i,j}$ is a cross-price elasticity parameter that describes how much demand for product i is affected by the price of a different product j. Note that $\gamma_{i,j}$ can be positive, which generally corresponds to products i and j being substitutes (i.e., when the price of product j increases, customers tend to switch to product i), or negative, which corresponds to products i and j being complements (i.e., products i and j tend to be purchased together, so when the price of product j increases, this causes a decrease in demand for product i). The corresponding revenue function $R(\cdot)$ is then

$$R(\mathbf{p}) = \sum_{i=1}^{I} p_i \cdot (\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j). \tag{2}$$

2. *Semi-log demand model*: A semi-log demand model is defined by parameters $\boldsymbol{\alpha} \in \mathbb{R}^I$, $\boldsymbol{\beta} \in \mathbb{R}^I$, $\boldsymbol{\gamma} = (\gamma_{i,j})_{i,j \in [I], i \neq j} \in \mathbb{R}^{I \cdot (I-1)}$. In a semi-log demand model, the logarithm of the demand function $d_i(\cdot)$ of each product $i \in [I]$ has a linear form in the prices $p_1, \ldots, p_I$:

$$\log d_i(\mathbf{p}) = \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j. \tag{3}$$

This implies that the demand function is

$$d_i(\mathbf{p}) = e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j}. \tag{4}$$

The corresponding revenue function $R(\cdot)$ is then

$$R(\mathbf{p}) = \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j}. \tag{5}$$

3. *Log-log demand model*: A log-log demand model is defined by parameters $\boldsymbol{\alpha} \in \mathbb{R}^I$, $\boldsymbol{\beta} \in \mathbb{R}^I$, $\boldsymbol{\gamma} = (\gamma_{i,j})_{i,j \in [I], i \neq j} \in \mathbb{R}^{I \cdot (I-1)}$. In a log-log demand model, the logarithm of the demand function $d_i(\cdot)$ of each product $i \in [I]$ has a linear form in the log-transformed prices $\log p_1, \ldots, \log p_I$:

$$\log d_i(\mathbf{p}) = \alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j. \tag{6}$$

This implies that the demand function for product i is

$$\begin{aligned} d_i(\mathbf{p}) &= e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} && (7) \\ &= e^{\alpha_i} p_i^{-\beta_i} \cdot \prod_{j \neq i} p_j^{\gamma_{i,j}}, && (8) \end{aligned}$$

and that the revenue function is therefore

$$\begin{aligned} R(\mathbf{p}) &= \sum_{i=1}^{I} p_i \cdot e^{\alpha_i} p_i^{-\beta_i} \cdot \prod_{j \neq i} p_j^{\gamma_{i,j}} && (9) \\ &= \sum_{i=1}^{I} e^{\alpha_i} \cdot p_i^{1-\beta_i} \cdot \prod_{j \neq i} p_j^{\gamma_{i,j}}. && (10) \end{aligned}$$

3.2. Deterministic robust price optimization problem

We now define the deterministic robust price optimization (DRPO) problem. To define this problem abstractly, we let $\mathcal{R}$ denote an uncertainty set of possible revenue functions. The DRPO problem is to then maximize the worst-case revenue, where the worst-case is the minimum revenue of a given price vector taken over all revenue functions in $\mathcal{R}$. Mathematically, this problem can be written as

$$\text{DRPO}: \quad \max_{\mathbf{p} \in \mathcal{P}} \min_{R \in \mathcal{R}} R(\mathbf{p}).$$

We use Z^*_{DR} to denote the optimal objective value of the DRPO problem.

Although $\mathcal{R}$ can be defined in many different ways, we will now focus on one general case that we will assume for most of our subsequent results in Sections 4, 5 and 6. Suppose that we fix the demand model to a specific parametric family, such as a log-log demand model. Let $\mathbf{u}$ denote the vector of demand model parameters. For example, for log-log, $\mathbf{u}$ would be the tuple $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Let $\mathcal{U}$ denote a set of possible values of $\mathbf{u}$; $\mathcal{U}$ is then an uncertainty set of model parameters. With a slight abuse of notation, let $d_i(\mathbf{p}, \mathbf{u})$ denote the demand for product i when the demand model parameters are specified by $\mathbf{u}$. Then $\mathcal{R}$ can be defined as:

$$\mathcal{R} = \{R(\cdot) \equiv \sum_{i=1}^{I} p_i \cdot d_i(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}, \tag{11}$$

i.e., it is all of the possible revenue functions spanned by the uncertain parameter vector $\mathbf{u}$ in $\mathcal{U}$.

To express the DRPO problem in this setting more conveniently, we will abuse our notation slightly and use $R(\mathbf{p}, \mathbf{u})$ to denote the revenue function evaluated at a price vector $\mathbf{p}$ with a particular parameter vector $\mathbf{u}$ specified. With this abuse of notation, the DRPO problem can be written as

$$\mathrm{DRPO}: \quad \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}).$$

3.3. Randomized robust price optimization problem

In the DRPO problem, we assume that the decision maker will deterministically implement a single price vector $\mathbf{p}$ in the face of uncertainty in the revenue function. In the RRPO problem, we instead assume that the decision maker will randomly select a price vector $\mathbf{p}$ according to some distribution F over the feasible price set $\mathcal{P}$. Under this assumption, we can write the RRPO problem as

$$\mathrm{RRPO}: \quad \max_{F \in \mathcal{F}} \min_{R \in \mathcal{R}} \int_{\mathcal{P}} R(\mathbf{p}) \, dF(\mathbf{p}),$$

where $\mathcal{F}$ is the set of all distributions supported on $\mathcal{P}$. We use Z^*_{RR} to denote the optimal objective value of the RRPO problem. Note that $Z^*_{\mathrm{RR}} \geq Z^*_{\mathrm{DR}}$. This is because for every $\mathbf{p}' \in \mathcal{P}$, the distribution $F(\cdot) = \delta_{\mathbf{p}'}(\cdot)$, where $\delta_{\mathbf{p}'}(\cdot)$ is the Dirac delta function at $\mathbf{p}'$, is contained in $\mathcal{F}$. For this distribution, $\min_{R \in \mathcal{R}} \int R(\mathbf{p}) \, dF(\mathbf{p}) = \min_{R \in \mathcal{R}} R(\mathbf{p}')$, which is exactly the worst-case revenue of deterministically selecting $\mathbf{p}'$.

A special instance of this problem arises when $\mathcal{P}$ is a discrete, finite set. In this case, F is a discrete probability distribution, and one can re-write the inner problem as an optimization problem over a discrete probability distribution $\boldsymbol{\pi} = (\pi_{\mathbf{p}})_{\mathbf{p} \in \mathcal{P}}$ over $\mathcal{P}$:

$$\mathrm{RRPO\text{-}D}: \quad \max_{\boldsymbol{\pi} \in \Delta_{\mathcal{P}}} \min_{R \in \mathcal{R}} \sum_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}) \pi_{\mathbf{p}},$$

where we use Δ_S to denote the $(|S|-1)$-dimensional unit simplex, i.e., $\Delta_S = \{\boldsymbol{\pi} \in \mathbb{R}^S \mid \sum_{i\in S} \pi_i = 1,\ \pi_i \geq 0\ \forall\ i \in S\}$.

Lastly, under the assumption that $\mathcal{R}$ is the set of revenue functions of a fixed demand model family whose parameter vector $\mathbf{u}$ belongs to a parameter uncertainty set $\mathcal{U}$, we can restate the RRPO problem when $\mathcal{P}$ is a generic set and when $\mathcal{P}$ is finite as

$$\text{RRPO}: \quad \max_{F\in\mathcal{F}} \min_{\mathbf{u}\in\mathcal{U}} \int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\, dF(\mathbf{p}), \tag{12}$$

$$\text{RRPO-D}: \quad \max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}} \min_{\mathbf{u}\in\mathcal{U}} \sum_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u})\pi_{\mathbf{p}}. \tag{13}$$

4. Benefits of randomization

In this section, we analyze when randomization can be beneficial. To aid us, we introduce some additional nomenclature in this section, which follows the terminology established in the prior literature on randomized robust optimization (Delage et al. 2019, Delage and Saif 2022, Wang et al. 2020). We say that a robust price optimization (RPO) problem is *randomization-receptive* if $Z^*_{\text{RR}} > Z^*_{\text{DR}}$, that is, randomizing over price vectors leads to a higher worst-case revenue than deterministically selecting a single price vector. Otherwise, we say that a RPO problem is *randomization-proof* if $Z^*_{\text{RR}} = Z^*_{\text{DR}}$, that is, there is no benefit from randomizing over price vectors.

In the following three sections, we derive three classes of results that establish when the RPO problem is randomization-proof. The first condition (Section 4.1) for randomization-proofness applies in the case when $\mathcal{P}$ is a convex set and $\mathcal{R}$ is an arbitrary set of revenue functions each of which is concave in $\mathbf{p}$. The second and third conditions apply to the case where $\mathcal{R}$ arises out of a single demand model, where the parameter vector $\mathbf{u}$ belongs to an uncertainty set. The second condition (Section 4.2) applies in the case when $\mathcal{P}$ and $\mathcal{U}$ are compact, convex sets and $R(\mathbf{p},\mathbf{u})$ obeys certain quasiconvexity and quasiconcavity properties. The third condition (Section 4.3) is for the case when $\mathcal{P}$ is finite and involves a certain minimax condition being met; as corollaries, we show that randomization-proofness occurs if the DRPO problem satisfies a strong duality property, and that randomization-receptiveness is essentially equivalent to the DRPO solution being different from the nominal price optimization problem solution at the worst-case $\mathbf{u}$. Along the way, we also give a number of examples where our results can be used to establish that a particular family of RPO problems is randomization-proof, and also highlight how the results fail to hold when certain hypotheses are relaxed.

The main takeaway from these results is that the set of RPO problems that are randomization-proof is small. As we will see, the conditions under which a RPO problem will be randomization-proof are delicate and quite restrictive, and are satisfied only for certain very special cases; in most other realistic cases, the RPO problem will be randomization-receptive. Consequently, in Sections 5 and 6, we will develop algorithms for solving the randomized robust when the candidate price vector $\mathcal{P}$ is finite, and in Sections 7 we will show a wide range of both synthetic and real data instances in which the RPO problem is randomization-receptive.

4.1. Concave revenue function uncertainty sets

Our first major result is for the case where $\mathcal{R}$ consists of concave revenue functions.

THEOREM 1. *Suppose that $\mathcal{P}$ is a convex set and that $\mathcal{R}$ is such that every $R \in \mathcal{R}$ is a concave function of* $\mathbf{p}$. *Then the RPO problem is randomization-proof, that is, $Z^*_{\mathrm{RR}} = Z^*_{\mathrm{DR}}$.*

The proof of this results (see Section EC.1.1 of the ecompanion) follows from a simple application of Jensen's inequality. We pause to make a few important comments about this result. First, one aspect of this result that is special is that $\mathcal{R}$ can be a very general set: it could be countable or uncountable, and it could consist of revenue functions corresponding to different families of demand models. This will not be the case for our later results in Section 4.2 and 4.3, which require that $\mathcal{R}$ is defined based on a single demand model family, and that the uncertainty set of parameter vectors for that family be a convex compact set.

Second, we remark that the condition that all functions in $\mathcal{R}$ be concave cannot be relaxed in general. We illustrate this in the following example, where $\mathcal{R}$ consists of two functions and one of the two is non-concave.

EXAMPLE 1. Consider a single-product RPO problem, and suppose that the revenue function uncertainty set $\mathcal{R} = \{R_1, R_2\}$, where $R_1(\cdot)$ and $R_2(\cdot)$ are defined as

$$\begin{aligned} R_1(p) &= p(10 - 2p), \\ R_2(p) &= p \cdot 10p^{-2}. \end{aligned}$$

Note that $R_1(p)$ is the revenue function corresponding to the linear demand function $d_1(p) = 10 - 2p$, while $R_2(p)$ is the revenue function corresponding to the log-log demand function $d_2(p) = \exp(\log(10) - 2\log(p)) = 10p^{-2}$. Note also that $R_1(\cdot)$ is concave, while $R_2(\cdot)$ is convex. Suppose additionally that $\mathcal{P} = [1, 4]$.

We first calculate the optimal value of the DRPO problem. Observe that in the interval $[1, 4]$, the only root of the equation $10 - 2p = 10p^{-2}$ is $p \approx p' = 1.137805\ldots$. For $p < p'$, $d_2(p) > d_1(p)$, and for $p > p'$, $d_1(p) > d_2(p)$. Therefore, the optimal value of the of the DRPO problem can be calculated as

$$\begin{aligned} &\max_{p \in [1,4]} \min_{R \in \mathcal{R}} R(p) \\ &= \max\{ \max_{p \in [1,p']} \min_{R \in \mathcal{R}} R(p), \max_{p \in [p',4]} \min_{R \in \mathcal{R}} R(p)\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{\max_{p \in [1,p']} p \cdot (10 - 2p), \max_{p \in [p',4]} p \cdot 10p^{-2}\} \\
&= 10p' - 2p'^2 \\
&= 8.78885
\end{aligned}
$$

In the above, the first step follows because the best value of the worst-case revenue over $[1,4]$ is equivalent to taking the higher of the best worst-case revenue over either $[1,p']$ or $[p',4]$. The second step follows because for every $p \in [1,p']$, $d_1(p) < d_2(p)$, and so $R_1(p) = p \cdot d_1(p) < p \cdot d_2(p) = R_2(p)$; similarly, for every $p \in [p',4]$, $d_1(p) > d_2(p)$, and so $R_2(p) < R_1(p)$. The third step follows by carrying out the maximization of each of the two functions from the prior step over its corresponding interval.

Now, let us lower bound the optimal value of the RRPO problem. Consider a distribution F that randomizes over prices in the following way:

$$
p = \begin{cases} 1 & \text{with probability } 17/21, \\ 2.5 & \text{with probability } 4/21. \end{cases} \tag{14}
$$

The worst-case revenue for this distribution is

$$
\begin{aligned}
&\min_{R \in \mathcal{R}} \int_1^4 R(p)\, dF(p) \\
&= \min\left\{\frac{17}{21} \cdot R_1(1) + \frac{4}{21} \cdot R_1(2.5), \frac{17}{21} \cdot R_2(1) + \frac{4}{21} \cdot R_2(2.5)\right\} \\
&= \min\left\{\frac{17}{21} \cdot 8 + \frac{4}{21} \cdot 12.5, \frac{17}{21} \cdot 10 + \frac{4}{21} \cdot 4\right\} \\
&= \min\left\{\frac{62}{7}, \frac{62}{7}\right\} \\
&= \frac{62}{7} \\
&= 8.857143.
\end{aligned}
$$

This implies that $Z^*_{\text{RR}} \geq 8.857143$, whereas $Z^*_{\text{DR}} = 8.78885$, and thus $Z^*_{\text{RR}} > Z^*_{\text{DR}}$. $\square$

Third, we note that the requirement that $\mathcal{P}$ be a convex set also cannot be relaxed in general. The following example illustrates how Theorem 1 can fail to hold when $\mathcal{P}$ is not a convex set.

Example 2. Consider again a single-product RPO problem. Suppose that $\mathcal{R} = \{R_1, R_2\}$, where $R_1(p) = p(10 - p)$, $R_2(p) = p(4 - 0.2p)$; R_1 and R_2 correspond to linear demand functions $d_1(p) = 10 - p$, $d_2(p) = 4 - 0.2p$. Suppose that $\mathcal{P} = \{p_1, p_2\}$, where $p_1 = 5$, $p_2 = 10$. From this data, observe that:

$$
\begin{aligned}
R_1(p_1) &= 5(10 - 5) = 25, \\
R_1(p_2) &= 10(10 - 10) = 0, \\
R_2(p_1) &= 5(4 - 0.2(5)) = 15, \\
R_2(p_2) &= 10(4 - 0.2(10)) = 20.
\end{aligned}
$$

We first calculate the optimal value of the DRPO problem:

$$\begin{aligned} Z^*_{\mathrm{DR}} &= \max_{p\in\{p_1,p_2\}} \min\{R_1(p), R_2(p)\} \\ &= \max\{\min\{25,15\}, \min\{0,20\}\} \\ &= \max\{15,0\} \\ &= 15. \end{aligned}$$

For the RRPO problem, the optimal value is given by the following LP:

$$\begin{aligned} \underset{\eta,\boldsymbol{\pi}}{\text{maximize}} \quad & \eta && \text{(15a)} \\ \text{subject to} \quad & \eta \leq \pi_{p_1}\cdot p_1 \cdot (10-p_1) + \pi_{p_2}\cdot p_2\cdot(10-p_2) && \text{(15b)} \\ & \eta \leq \pi_{p_1}\cdot p_1 \cdot (4-0.2p_1) + \pi_{p_2}\cdot p_2\cdot(4-0.2p_2) && \text{(15c)} \\ & \pi_{p_1} + \pi_{p_2} = 1 && \text{(15d)} \\ & \pi_{p_1}, \pi_{p_2} \geq 0. && \text{(15e)} \end{aligned}$$

The optimal distribution over $\mathcal{P} = \{p_1, p_2\}$ is given by $\pi_{p_1} = 2/3$, $\pi_{p_2} = 1/3$, which leads to $Z^*_{\mathrm{RR}} = 50/3 = 16.6667$. Since this is higher than Z^*_{DR}, we conclude that this particular instance is randomization-receptive. □

Lastly, Theorem 1 has a number of implications for different classes of demand models.

Example 3. (Single-product pricing under linear demand). Suppose that $I = 1$, which corresponds to a single-product pricing problem. Let $\mathbf{u} = (\alpha, \beta) \in \mathbb{R}^2$ denote the vector of linear demand model parameters, and let $\mathcal{U} \subseteq \mathbb{R}^2$ be an uncertainty set of possible values of (α, β). Let $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the set of revenue functions that arise from the uncertainty set $\mathcal{U}$. Note that each revenue function is of the form $R(p) = \alpha p - \beta p^2$. Therefore, the condition that each $R \in \mathcal{R}$ is concave implies that $R''(p) = -2\beta \leq 0$. Thus, if $\mathcal{U}$ is such that $\inf\{\beta \mid (\alpha,\beta) \in \mathcal{U}\} \geq 0$, then the robust price optimization problem is randomization-proof. □

Example 4. (Multi-product pricing under linear demand). In the more general multi-product pricing problem, let $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathbb{R}^I \times \mathbb{R}^I \times \mathbb{R}^{I(I-1)}$ denote the vector of linear demand model parameters, and let $\mathcal{U}$ be an arbitrary uncertainty set of these model parameter vectors. Let $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the set of revenue functions that arise from the uncertainty set $\mathcal{U}$. Observe that each revenue function $R(\cdot, \mathbf{u})$ is of the form

$$\begin{aligned} R(\mathbf{p}) &= \sum_{i=1}^{I} p_i(\alpha_i - \beta_i p_i + \sum_{j\neq i} \gamma_{i,j} p_j) \\ &= \boldsymbol{\alpha}^T \mathbf{p} - \mathbf{p}^T \mathbf{M}_{\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{p}, \end{aligned}$$

where $\mathbf{M}_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ is the matrix

$$\mathbf{M}_{\boldsymbol{\beta},\boldsymbol{\gamma}} = \begin{bmatrix} -\beta_1 & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,I-1} & \gamma_{1,I} \\ \gamma_{2,1} & -\beta_2 & \gamma_{2,3} & \cdots & \gamma_{2,I-1} & \gamma_{2,I} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \gamma_{I,1} & \gamma_{I,2} & \gamma_{I,3} & \cdots & \gamma_{I,I-1} & -\beta_I \end{bmatrix}.$$

This implies that

$$\nabla^2 R(\mathbf{p}) = -2\mathbf{M}_{\boldsymbol{\beta},\boldsymbol{\gamma}}.$$

The function R is therefore concave if the matrix $\mathbf{M}_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ is positive semidefinite. Therefore, if $\mathcal{U}$ is such that $\inf\{\lambda_{\min}(\mathbf{M}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) \mid (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathcal{U}\} \geq 0$, where $\lambda_{\min}(\mathbf{A})$ denotes the minimum eigenvalue of a symmetric matrix $\mathbf{A}$, then the robust price optimization problem is randomization proof. □

Example 5. (Single-product pricing under semi-log demand). For the single-product pricing problem under semi-log demand, $d(p) = \exp(\alpha - \beta p)$ is the demand function given the parameter vector $\mathbf{u} = (\alpha, \beta)$. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an uncertainty set of possible values of (α, β), and assume that β is bounded away from zero, that is, $\inf\{\beta \mid (\alpha,\beta) \in \mathcal{U}\} \geq 0$. $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the revenue function uncertainty set. For a given $R \in \mathcal{R}$, its second derivative is $R''(p) = R''(p) = \beta(\beta p - 2)e^{\alpha - \beta p}$. Thus, for $R''(p)$ to be nonpositive, we need $\beta p - 2 \leq 0$ or equivalently $\beta p \leq 2$ (since β is assumed to be nonnegative) for all $p \in \mathcal{P}$ in order for $R(p)$ to be concave. Thus, if $\sup_{p \in \mathcal{P}} \sup_{(\alpha,\beta)\in\mathcal{U}} \{\beta p\} \leq 2$ and $\inf\{\beta \mid (\alpha,\beta) \in \mathcal{U}\} \geq 0$, then the RPO problem is randomization-proof. □

Example 6. (Single-product pricing under log-log demand). For the single-product pricing problem under log-log demand, $d(p) = \exp(\alpha - \beta \log p) = e^{\alpha} \cdot p^{-\beta}$ is the demand function and $\mathbf{u} = (\alpha, \beta) \in \mathbb{R}^2$ is the vector of uncertain demand model parameters. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an uncertainty set of possible values of (α, β), and assume that β is bounded away from zero from below, that is, $\inf\{\beta \mid (\alpha,\beta) \in \mathcal{U}\} \geq 0$. Let $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the revenue function uncertainty set. For a given $R \in \mathcal{R}$, its second derivative is $R''(p) = e^{\alpha} \cdot (\beta - 1)(\beta) \cdot p^{-\beta-1}$. Thus, for $R''(p)$ to be nonpositive, we need $\beta - 1 \leq 0$, or equivalently $\beta \leq 1$. Thus, if $\sup_{(\alpha,\beta)\in\mathcal{U}} \beta \leq 1$ and $\inf_{(\alpha,\beta)\in\mathcal{U}} \beta \geq 0$, then the RPO problem is randomization-proof. □

4.2. Quasiconcavity in p and quasiconvexity in u

The second result we establish concerns the RRPO problem when there is a demand parameter uncertainty set $\mathcal{U}$. In this case, the RRPO and DRPO problems are

$$\begin{aligned} \text{RRPO}: &\max_{F \in \mathcal{F}} \min_{\mathbf{u} \in \mathcal{U}} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) \; dF(\mathbf{p}), \\ \text{DRPO}: &\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}). \end{aligned}$$

We make the following assumption about R.

Assumption 1. *R is a continuous function of* $(\mathbf{p}, \mathbf{u})$.

Under these assumptions, we obtain the following result.

Theorem 2. *Suppose that Assumptions 1 holds. Suppose that $\mathcal{P} \subseteq \mathbb{R}^I$ and $\mathcal{U} \subseteq \mathbb{R}^d$ are compact convex sets. Suppose that $\int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u})\, dF(\mathbf{p})$ is a quasiconvex function of $\mathbf{u}$ on $\mathcal{U}$ for any $F \in \mathcal{F}$. Suppose that $R(\mathbf{p}, \mathbf{u})$ is quasiconcave in $\mathbf{p}$ on $\mathcal{P}$ for any $\mathbf{u} \in \mathcal{U}$ and quasiconvex in $\mathbf{u}$ on $\mathcal{U}$ for any $\mathbf{p} \in \mathcal{P}$. Then, the robust price optimization problem is randomization-proof, that is, $Z^*_{\mathrm{DR}} = Z^*_{\mathrm{RR}}$.*

The proof of Theorem 2 (see Section EC.1.2 of the ecompanion) follows from applying Sion's minimax theorem twice. This result allows us to show that a larger number of RPO problems are randomization-proof. We provide a few examples below.

Example 7. Consider a single-product price optimization problem where the demand follows a semi-log model. The uncertain parameter is therefore $\mathbf{u} = (\alpha, \beta)$.

Observe that $R(p, \mathbf{u}) = pe^{\alpha - \beta p}$ is convex in $\mathbf{u}$. Thus, it is also quasiconvex in $\mathbf{u}$ for a fixed p. Additionally, for any distribution F over $\mathcal{P}$, we have that the function $\int_{\mathcal{P}} R(p, \mathbf{u})\, dF(p)$ is convex in $\mathbf{u}$ (it is a nonnegative weighted combination of the functions $\mathbf{u} = (\alpha, \beta) \mapsto pe^{\alpha - \beta p}$, each of which is convex), and is thus also quasiconvex in $\mathbf{u}$.

Note also that the function R is quasi-concave in p. To see this, observe that $\log R(p, \mathbf{u}) = \log p + \alpha - \beta p$, which is concave in p; this means that R is log-concave in p. Since any log-concave function is quasiconcave, it follows that R is quasiconcave in p.

Thus, if $\mathcal{P} \subseteq \mathbb{R}$ and $\mathcal{U} \subseteq \mathbb{R}^2$ are compact and convex, then Theorem 2 asserts that the RPO problem is randomization-proof. $\square$

Example 8. Consider a single-product price optimization problem where the demand follows a log-log model. The uncertain parameter is $\mathbf{u} = (\alpha, \beta)$, and $R(p, \mathbf{u}) = pe^{\alpha - \beta \log p}$. Assume that $\mathcal{P} \subseteq \mathbb{R}$ is a compact convex set, and that $\min\{p \mid p \in \mathcal{P}\} > 0$.

Observe that $R(p, \mathbf{u}) = p \cdot e^{\alpha - \beta \log p}$ is convex in $\mathbf{u}$, and therefore quasiconvex in $\mathbf{u}$ for a fixed p. Additionally, for any distribution F over $\mathcal{P}$, we have that the function $\int_{\mathcal{P}} R(p, \mathbf{u})\, dF(p)$ is convex in $\mathbf{u}$ and therefore also quasiconvex in $\mathbf{u}$ for a fixed F.

Lastly, with regard to quasiconcavity in p, observe that $\log R(p, \mathbf{u}) = \log p + \alpha - \beta \log p = (1 - \beta) \log p + \alpha$, which means that R is log-concave in p whenever $1 - \beta > 0$ or equivalently $\beta < 1$. Therefore, R will also be quasiconcave whenever $\beta < 1$.

Thus, if $\mathcal{P} \subseteq \mathbb{R}$ and $\mathcal{U} \subseteq \mathbb{R}^2$ are compact and convex, and $\max\{\beta \mid (\alpha, \beta) \in \mathcal{U}\} < 1$, then Theorem 2 guarantees that the RPO problem is randomization-proof. $\square$

With regard to the above two examples, we note that in general, the revenue function for a semi-log or a log-log demand model is not concave in p. Thus, Theorem 1 cannot be used in these cases, and we must use Theorem 2. Note, however, that the two examples above critically rely on the revenue function being log-concave and therefore quasiconcave, which is only the case for single product price optimization problems. Log-concavity and quasiconcavity are in general not preserved under addition (i.e., the sum of quasiconcave functions is not always quasiconcave, and the sum of log-concave functions is not always log-concave), and so Theorem 2 will in general not be applicable for multiproduct pricing problems involving the semi-log or log-log demand model.

4.3. Finite price set $\mathcal{P}$

In this section, we analyze randomization-receptiveness when $\mathcal{P}$ is a finite set. To study this setting, let us define the set $\mathcal{Q}$ as the set of all probability distributions supported on $\mathcal{U}$. We note that these results are adaptations of several results from Wang et al. (2020) to the pricing setting that we study, which develop analogous conditions for randomization-proofness for the robust assortment optimization problem.

Our first result establishes that randomization-proofness is equivalent to the existence of a distribution Q over $\mathcal{U}$ under which any price vector's expected performance is no better than the deterministic robust optimal value.

THEOREM 3. *Suppose that $R(\mathbf{p},\mathbf{u})$ is continuous in $\mathbf{u}$ for any fixed $\mathbf{p}\in\mathcal{P}$. A robust price optimization problem with finite $\mathcal{P}$ is randomization-proof if and only if there exists a distribution $Q\in\mathcal{Q}$ such that for all $\mathbf{p}\in\mathcal{P}$,*

$$\int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}) \le Z^*_{\mathrm{DR}}.$$

To prove this result, we use Sion's minimax theorem to establish that

$$Z^*_{\mathrm{RR}} = \inf_{Q\in\mathcal{Q}} \max_{\mathbf{p}\in\mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}); \tag{16}$$

with that result in hand, the condition in Theorem 3 is equivalent to establishing that

$$Z^*_{\mathrm{DR}} \ge \inf_{Q\in\mathcal{Q}} \max_{\mathbf{p}\in\mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}) = Z^*_{\mathrm{RR}},$$

which, together with the inequality $Z^*_{\mathrm{DR}} \le Z^*_{\mathrm{RR}}$ immediately yields randomization-proofness. We note that this result is analogous to Theorem 1 in Wang et al. (2020), which provides a similar necessary and sufficient condition for randomization-proofness in the context of robust assortment optimization. Our proof, which relies on Sion's minimax theorem, is perhaps slightly more direct than the proof of Theorem 1 in Wang et al. (2020), although this is a matter of taste.

Our next two results are consequences of this theorem. The first essentially states that a price optimization problem proof will be randomization-proof if the robust price optimization problem obeys strong duality. The second states that, under some conditions, a robust price optimization problem is randomization-receptive if and only if the deterministic robust price vector $\mathbf{p}^*_{\mathrm{DR}}$ is not an optimal solution of the nominal price optimization problem under the worst-case $\mathbf{u}^*$ that attains the worst-case objective under $\mathbf{p}^*_{\mathrm{DR}}$. We note that these results are both analogous to Corollaries 1 and 2 in Wang et al. (2020).

COROLLARY 1. *Suppose that $R(\mathbf{p},\mathbf{u})$ is a continuous function of $\mathbf{u}$ for every $\mathbf{p}\in\mathcal{P}$. A robust price optimization problem with finite $\mathcal{P}$ is randomization-proof if and only if it satisfies strong duality:*

$$\max_{\mathbf{p}\in\mathcal{P}}\min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p},\mathbf{u}) = \min_{\mathbf{u}\in\mathcal{U}}\max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}). \tag{17}$$

COROLLARY 2. *Suppose that $\mathcal{U}$ is a compact subset of $\mathbb{R}^d$, and that $R(\mathbf{p},\mathbf{u})$ is a continuous function of $\mathbf{u}$ for every $\mathbf{p}\in\mathcal{P}$. Suppose that $\mathbf{p}^*_{\mathrm{DR}}\in\arg\max_{\mathbf{p}\in\mathcal{P}}\min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p},\mathbf{u})$ is an optimal solution of the deterministic robust price optimization problem, and suppose that $\min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p}^*_{\mathrm{DR}};\mathbf{u})$ has a unique solution $\mathbf{u}^*$. Then the robust price optimization is randomization-receptive if and only if $\mathbf{p}^*_{\mathrm{DR}}\notin\arg\max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}^*)$.*

With regard to Corollary 2, we note that the uniqueness requirement for $\mathbf{u}^*$ cannot in general be relaxed. In Section EC.1.6 of the ecompanion, we show an instance where $\min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p}^*_{\mathrm{DR}},\mathbf{u})$ has multiple optimal solutions, $\mathbf{p}^*_{\mathrm{DR}}\notin\arg\max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}')$ for every $\mathbf{u}'$ that solves $\min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p}^*_{\mathrm{DR}},\mathbf{u})$, and yet the problem is randomization-proof, i.e., $Z^*_{\mathrm{DR}} = Z^*_{\mathrm{RR}}$.

We remark that the necessary and sufficient conditions for randomization-proofness in Corollaries 1 and 2 are rather stringent and demanding. With regard to Corollary 1, strong duality is in general unlikely to hold given that $\mathcal{P}$ is a finite set. With regard to Corollary 2, we note that in general, the solution of the deterministic robust price optimization problem is unlikely to also be an optimal solution of an appropriately defined nominal price optimization problem; this is frequently not the case in many applications of robust optimization outside of pricing. Given this, these conditions are suggestive of the fact that most robust price optimization problems will be randomization-receptive. This motivates our study of solution algorithms for numerically solving the RRPO problem in the next two sections.

5. Solution algorithm for finite price set $\mathcal{P}$, convex uncertainty set $\mathcal{U}$

In this section, we describe a general solution algorithm for solving the RRPO problem when the price set $\mathcal{P}$ is a finite set, and the uncertainty set $\mathcal{U}$ is a general convex uncertainty set. Section 5.1 describes the general solution algorithm, which is a constraint generation algorithm that involves solving a nominal pricing problem over $\mathcal{P}$ as a subroutine. Sections 5.2, 5.3 and 5.4 describe how the solution algorithm specializes to the cases of the linear, semi-log and log-log demand models, respectively, and in particular, how the nominal pricing problem can be solved for each of these three cases; the formulations we present for the semi-log and log-log models here may be of independent interest as they are, to the best of our knowledge, the first exact mixed-integer convex formulations for the multi-product pricing problem under a finite price set for these demand models.

5.1. General solution approach

The first general solution scheme that we consider is when $\mathcal{P}$ is a discrete set and the uncertainty set $\mathcal{U}$ is a convex uncertainty set. In this case, if the revenue function $R(\mathbf{p},\mathbf{u})$ is quasiconvex and continuous in $\mathbf{u}\in\mathcal{U}$, then the RRPO problem can be reformulated as follows:

$$\begin{aligned}
&\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}}\min_{\mathbf{u}\in\mathcal{U}}\sum_{\mathbf{p}\in\mathcal{P}}\pi_{\mathbf{p}}R(\mathbf{p},\mathbf{u})\\
&=\min_{\mathbf{u}\in\mathcal{U}}\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}}\sum_{\mathbf{p}\in\mathcal{P}}\pi_{\mathbf{p}}R(\mathbf{p},\mathbf{u})\\
&=\min_{\mathbf{u}\in\mathcal{U}}\max_{\mathbf{p}\in\mathcal{P}}R(\mathbf{p},\mathbf{u}),
\end{aligned}$$

where the first equality follows by Sion's minimax theorem, and the second equality follows by the fact that the inner maximum is attained by setting $\pi_{\mathbf{p}}=1$ for some $\mathbf{p}$ and setting $\pi_{\mathbf{p}'}=0$ for all $\mathbf{p}'\neq\mathbf{p}$. This last problem can be written in epigraph form as

$$\begin{aligned}
\underset{\mathbf{u},t}{\text{minimize}}\quad & t && \text{(18a)}\\
\text{subject to}\quad & t\geq R(\mathbf{p},\mathbf{u}),\quad\forall\ \mathbf{p}\in\mathcal{P}, && \text{(18b)}\\
& \mathbf{u}\in\mathcal{U}. && \text{(18c)}
\end{aligned}$$

Problem (18) can be solved using constraint generation. In such a scheme, we start with constraint (18b) enforced only at a subset $\hat{\mathcal{P}}\subset\mathcal{P}$, and solve problem (18) to obtain a solution $(\mathbf{u},t)$. At this solution, we solve the problem $\max_{\mathbf{p}\in\mathcal{P}}R(\mathbf{p},\mathbf{u})$, and compare this objective value to the current value of t. If it is less than or equal to t, we conclude that $(\mathbf{u},t)$ satisfies constraint (18b) and terminate with $(\mathbf{u},t)$ as the optimal solution. Otherwise, if it is greater than t, we have identified a $\mathbf{p}$ for which constraint (18b) and we add the new constraint to $\hat{\mathcal{P}}$. We then re-solve the problem to obtain a new solution $(\mathbf{u},t)$ and repeat the process until we can no longer identify any violated constraints. To recover the optimal randomization scheme from the solution of this problem (i.e., the distribution $\boldsymbol{\pi}$), we simply consider the optimal dual variable of each constraint $t\leq R(\mathbf{p},\mathbf{u})$.

The viability of this solution approach critically depends on our ability to solve the separation problem $\max_{\mathbf{p}\in\mathcal{P}}R(\mathbf{p},\mathbf{u})$ efficiently, and to solve the problem (18) efficiently for a fixed subset $\hat{\mathcal{P}}\subset\mathcal{P}$. In what follows, we shall demonstrate that this problem can actually be solved practically for the linear, semi-log and log-log problems.

To develop our approaches for the linear, semi-log and log-log models, we will make the following assumption about the price set $\mathcal{P}$, which simply states that $\mathcal{P}$ is a Cartesian product of finite sets of prices for each of the products.

Assumption 2. *$\mathcal{P}=\mathcal{P}_1\times\cdots\times\mathcal{P}_I$, where $\mathcal{P}_i$ is a finite subset of $\mathbb{R}_+$ for each i.*

5.2. Linear demand model

We begin by showing how our solution approach for convex $\mathcal{U}$ applies to the linear demand model case. Recall that the linear model revenue function is

$$R(\mathbf{p},\mathbf{u}) = \sum_{i=1}^{I} p_i(\alpha_i - \beta_i p_i + \sum_{j\neq i} \gamma_{i,j} p_j). \tag{19}$$

For a fixed $\mathbf{p}$, the function $R(\mathbf{p},\mathbf{u})$ is linear and therefore convex and quasiconvex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Thus, given a subset $\hat{\mathcal{P}} \subset \mathcal{P}$, the problem (18) should be easy to solve, assuming that $\mathcal{U}$ is also a sufficiently tractable convex set. For example, if $\mathcal{U}$ is a polyhedron, then since each constraint (18b) is linear in $\mathbf{u}$, problem (18) would be a linear program.

The separation problem for the linear demand model case is

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \sum_{i=1}^{I} p_i(\alpha_i - \beta_i p_i + \sum_{j\neq i} \gamma_{i,j} p_j) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \sum_{i=1}^{I} p_i \alpha_i - \sum_{i=1}^{I} \beta_i p_i^2 + \sum_{i=1}^{I} \sum_{j\neq i} \gamma_{i,j} p_i p_j
\end{aligned}$$

Since $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_I$, we can formulate this as a mixed-integer program. Let $x_{i,t}$ be a binary variable that is 1 if product i has price $t \in \mathcal{P}_i$, and 0 otherwise. Similarly, let y_{i,j,t_1,t_2} be a binary decision variable that is 1 if product i is given price t_1 and product j is given price t_2 for $i \neq j$, and 0 otherwise. Then the separation problem can be straightforwardly written as

$$\begin{aligned}
\underset{\mathbf{x},\mathbf{y}}{\text{maximize}} \quad & \sum_{i=1}^{I} \sum_{t\in\mathcal{P}_i} \alpha_i \cdot t \cdot x_{i,t} + \sum_{i=1}^{I} \sum_{t\in\mathcal{P}_i} t^2 \cdot \beta_i \cdot x_{i,t} + \sum_{i=1}^{I} \sum_{j\neq i} \sum_{t_1\in\mathcal{P}_i} \sum_{t_2\in\mathcal{P}_j} \gamma_{i,j} \cdot t_1 \cdot t_2 \cdot y_{i,j,t_1,t_2} && \text{(20a)} \\
\text{subject to} \quad & \sum_{t\in\mathcal{P}_i} x_{i,t} = 1, \quad \forall\, i \in [I], && \text{(20b)} \\
& \sum_{t_2\in\mathcal{P}_j} y_{i,j,t_1,t_2} = x_{i,t_1}, \quad \forall\, i,j \in [I], j \neq i, t_1 \in \mathcal{P}_i, && \text{(20c)} \\
& \sum_{t_1\in\mathcal{P}_i} y_{i,j,t_1,t_2} = x_{i,t_2}, \quad \forall\, i,j \in [I], j \neq i, t_2 \in \mathcal{P}_j, && \text{(20d)} \\
& x_{i,t} \in \{0,1\}, \quad \forall\, i \in [I],\ t \in \mathcal{P}_i, && \text{(20e)} \\
& y_{i,j,t_1,t_2} \in \{0,1\}, \quad \forall\, i,j \in [I], i \neq j, t_1 \in \mathcal{P}_i, t_2 \in \mathcal{P}_j, && \text{(20f)}
\end{aligned}$$

where the first constraint simply enforces that exactly one price is chosen for each product, while the second and third constraints require that the y_{i,j,t_1,t_2} variables are essentially equal to $x_{i,t_1} \cdot x_{j,t_2}$.

5.3. Semi-log demand model

We will now show how the solution approach we have defined earlier applies to the semi-log demand model. Recall that the semi-log revenue function is

$$R(\mathbf{p},\mathbf{u}) = \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j}p_j}. \tag{21}$$

Observe that for a fixed $\mathbf{p}$, the function $R(\mathbf{p},\mathbf{u})$ is convex (and therefore quasiconvex) in $\mathbf{u}$, since it is the nonnegative weighted combination of exponentials of linear functions of $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Thus, given a subset $\hat{\mathcal{P}} \subset \mathcal{P}$, solving problem (18) should again be "easy", assuming also that $\mathcal{U}$ is a sufficiently tractable convex set. (In particular, the function $R(\mathbf{p},\mathbf{u})$ can be represented using I exponential cones; assuming that $\mathcal{U}$ is also representable using conic constraints, problem (18) will thus be some type of continuous conic program.)

We now turn our attention to the separation problem, $\max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u})$. Specifically, this problem is

$$\max_{\mathbf{p}\in\mathcal{P}} \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j}p_j}.$$

Observe that since the function $f(t) = \log(t)$ is monotonic, the set of optimal solutions remains unchanged if we consider the same problem with a log-transformed objective

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \log\left(\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j}p_j}\right) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \log\left(\sum_{i=1}^{I} e^{\alpha_i + \log p_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j}p_j}\right).
\end{aligned} \tag{22}$$

To now further re-formulate this problem, we observe that the objective function can be re-written using the function $g(\mathbf{y}) = \log(\sum_{i=1}^{I} e^{y_i})$. The function g is what is known as the *log-sum-exp* function, which is a convex function (Boyd and Vandenberghe 2004). More importantly, a standard result in convex analysis is that any proper, lower semi-continuous, convex function is equivalent to its *biconjugate* function, which is the convex conjugate of its convex conjugate (Rockafellar 1970). For the log-sum-exp function, this in particular means that $g(\mathbf{y})$ can be written as

$$g(\mathbf{y}) = \max_{\boldsymbol{\mu}\in\Delta_{[I]}} \{\boldsymbol{\mu}^T\mathbf{y} - \sum_{i=1}^{I} \mu_i \log \mu_i\}.$$

The function $h(x) = x\log x$ is the negative entropy function (Boyd and Vandenberghe 2004), and is a convex function; thus, the function inside the $\max\{\cdot\}$ is a linear function minus a sum of convex functions, and is a concave function.

For our problem, this means that (22) can be re-written as

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \log R(\mathbf{p},\mathbf{u}) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \log\left(\sum_{i=1}^{I} e^{\alpha_i + \log p_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j} p_j}\right) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \max_{\boldsymbol{\mu}\in\Delta_{[I]}} \left\{ \sum_{i=1}^{I} \mu_i(\alpha_i + \log p_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j} p_j) - \sum_{i=1}^{I}\mu_i \log \mu_i \right\} \\
&= \max_{\mathbf{p}\in\mathcal{P},\ \boldsymbol{\mu}\in\Delta_{[I]}} \left\{ \sum_{i=1}^{I} \mu_i(\alpha_i + \log p_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j} p_j) - \sum_{i=1}^{I}\mu_i \log \mu_i \right\}. \qquad (23)
\end{aligned}$$

To further reformulate this problem, we now make use of Assumption 2, which states that $\mathcal{P}$ is the Cartesian product of finite sets. Let us introduce a new binary decision variable $x_{i,t}$ which is 1 if product i's price is set to price $t \in \mathcal{P}_i$, and 0 otherwise. Using this new decision variable, observe that we can replace p_i wherever it occurs with $\sum_{t\in\mathcal{P}_i} t \cdot x_{i,t}$. We can also similarly replace $\log p_i$ with $\sum_{t\in\mathcal{P}_i} \log t \cdot x_{i,t}$. Therefore, problem (23) can be further reformulated as

$$\begin{aligned}
\underset{\mathbf{x},\boldsymbol{\mu}}{\text{maximize}} \quad & \sum_{i=1}^{I} \mu_i \left(\alpha_i + \sum_{t\in\mathcal{P}_i} \log t \cdot x_{i,t} - \beta_i \cdot \sum_{t\in\mathcal{P}_i} t\cdot x_{i,t} + \sum_{j\neq i} \gamma_{i,j} \sum_{t\in\mathcal{P}_j} t \cdot x_{j,t} \right) - \sum_{i=1}^{I} \mu_i \log \mu_i && (24a) \\
\text{subject to} \quad & \sum_{i=1}^{I} \mu_i = 1, && (24b) \\
& \sum_{t\in\mathcal{P}_i} x_{i,t} = 1, \quad \forall\, i \in [I], && (24c) \\
& x_{i,t} \in \{0,1\}, \quad \forall\, i\in[I],\ t\in\mathcal{P}_i, && (24d) \\
& \mu_i \geq 0, \quad \forall\, i \in [I]. && (24e)
\end{aligned}$$

This last problem is *almost* a mixed-integer convex program: as noted earlier, the expression $-\sum_{i=1}^{I} \mu_i \log \mu_i$ is concave in μ. The main wrinkle is the presence of the bilinear terms in the objective function, specifically terms of the form $\mu_i \cdot x_{j,t}$. Fortunately, we can circumvent this difficulty by introducing a new decision variable, $w_{i,j,t}$, which is the linearization of $\mu_i \cdot x_{j,t}$, for each $i, j \in [I]$, $t \in \mathcal{P}_j$. By adding this new decision variable and additional constraints, we arrive at our final formulation, which is a mixed-integer convex program.

$$\begin{aligned}
\underset{\boldsymbol{\mu},\mathbf{w},\mathbf{x}}{\text{maximize}} \quad & \sum_{i=1}^{I} \mu_i\alpha_i + \sum_{i=1}^{I}\sum_{t\in\mathcal{P}_i} \log t \cdot w_{i,i,t} - \sum_{i=1}^{I} \beta_i \cdot \sum_{t\in\mathcal{P}_i} t\cdot w_{i,i,t} + \sum_{i=1}^{I}\sum_{j\neq i} \gamma_{i,j} \cdot (\sum_{t\in\mathcal{P}_j} t\cdot w_{i,j,t}) - \sum_{i=1}^{I} \mu_i \log \mu_i && (25a) \\
\text{subject to} \quad & \sum_{t\in\mathcal{P}_j} w_{i,j,t} = \mu_i, \quad \forall\, i\in[I],\ j\in[I], && (25b) \\
& \sum_{i=1}^{I} w_{i,j,t} = x_{j,t}, \quad \forall\, j\in[I],\ t\in\mathcal{P}_j, && (25c)
\end{aligned}$$

$$\sum_{i=1}^{I} \mu_i = 1, \tag{25d}$$

$$\sum_{t \in \mathcal{P}_i} x_{i,t} = 1, \quad \forall\, i \in [I], \tag{25e}$$

$$w_{i,j,t} \geq 0, \quad \forall\, i \in [I],\ j \in [I], t \in \mathcal{P}_j, \tag{25f}$$

$$x_{i,t} \in \{0,1\}, \quad \forall\, i \in [I],\ t \in \mathcal{P}_i, \tag{25g}$$

$$\mu_i \geq 0, \quad \forall\, i \in [I]. \tag{25h}$$

There are a few important points to observe about this formulation. First, note that because the μ_i's sum to 1 over i, and the $x_{j,t}$'s are binary and sum to 1 over $t \in \mathcal{P}_j$ for any j, then ensuring that $w_{i,j,t} = \mu_i \cdot x_{j,t}$ can be done simply through constraints (25b) and (25c). This is different from the usual McCormick envelope-style linearization technique, which in this case would involve the four inequalities:

$$w_{i,j,t} \leq x_{j,t}, \tag{26}$$

$$w_{i,j,t} \leq \mu_i, \tag{27}$$

$$w_{i,j,t} \geq x_{j,t} + \mu_i - 1, \tag{28}$$

$$w_{i,j,t} \geq 0, \tag{29}$$

for every $i \in [I]$, $j \in [I]$, $t \in \mathcal{P}_j$. It is not difficult to show that these constraints are implied by constraints (25b), (25c) and (25f).

Second, at the risk of belaboring the obvious, the optimal objective value of problem (25) is the value of $\max_{\mathbf{p} \in \mathcal{P}} \log R(\mathbf{p}, \mathbf{u})$, where R is the semi-log revenue function. Upon solving problem (25) to obtain the objective value Z', we can obtain the optimal objective value of the untransformed problem $\max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$ as $e^{Z'}$.

Third, this formulation is notable because, to our knowledge, this is the first exact mixed-integer convex formulation of the nominal multi-product pricing problem under semi-log demand and a price set defined as the Cartesian product of finite sets. To date, virtually all research that has considered solving this type of problem in the marketing and operations management literatures has involved heuristics (see, for example, Section EC.3 of Mišić 2020, which solves log-log and semi-log multi-product pricing problems for a collection of stores using local search). From this perspective, although we developed this formulation as part of the overall solution approach for the RRPO problem, we believe it is of more general interest.

Building on the previous point, problem (25) can be formulated as a mixed-integer exponential cone program. Such problems are garnering increasing attention from the academic and industry sides. In particular, since 2019, the MOSEK solver (ApS 2022) supports the exponential cone and can solve mixed-integer conic programs that involve the exponential cone to global optimality. Although the solution technology for mixed-integer conic programs is not as developed as that of mixed-integer linear programs (as exemplified by state-of-the-art solvers such as Gurobi and CPLEX), it is reasonable to expect that these solvers will continue to improve and allow larger and larger problem instances to be solved to optimality in the future.

Lastly, we comment that the same reformulation technique used above – taking the logarithm, replacing the log-sum-exp function with its biconjugate, and then linearizing the products of the binary decision variables and the probability mass function values (the μ_i variables) that arise from the biconjugate – can also be used to derive an exact formulation of the deterministic robust price optimization problem. By taking the same approach, one obtains a max-min-max problem, and one can use Sion's minimax theorem again to swap the inner maximization over μ with the minimization over $\mathbf{u}$ to obtain a robust counterpart that can then be further reformulated using duality or otherwise solved using delayed constraint generation. We provide the details of this derivation in Section EC.2.1 of the ecompanion.

5.4. Log-log demand model

To now show how the solution scheme in Section 5.1 applies to the log-log approach, we again recall the form of the log-log revenue function:

$$R(\mathbf{p},\mathbf{u}) = \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j} \tag{30}$$

$$= \sum_{i=1}^{I} e^{\alpha_i + \log p_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j} \tag{31}$$

Using the same biconjugate trick as with the semi-log approach, we can show that

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \log R(\mathbf{p},\mathbf{u}) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \log\left(\sum_{i=1}^{I} e^{\alpha_i + \log p_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j}\right) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \max_{\boldsymbol{\mu}\in\Delta_{[I]}} \left\{ \sum_{i=1}^{I} \mu_i(\alpha_i + \log p_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j) - \sum_{i=1}^{I} \mu_i \log \mu_i \right\}
\end{aligned} \tag{32}$$

If we now invoke Assumption 2, then we can introduce the same decision variables $x_{i,t}$ and $w_{i,j,t}$ as in problem (25) to obtain a mixed-integer convex formulation of the log-log price optimization problem, which has the same feasible region as the semi-log formulation (25):

$$\underset{\mathbf{w},\mathbf{x},\boldsymbol{\mu}}{\text{maximize}} \quad \sum_{i=1}^{I} \mu_i \alpha_i + \sum_{i=1}^{I} \sum_{t\in\mathcal{P}_i} \log t \cdot w_{i,i,t} - \sum_{i=1}^{I} \beta_i \cdot \sum_{t\in\mathcal{P}_i} \log t \cdot w_{i,i,t} + \sum_{i=1}^{I} \sum_{j\neq i} \gamma_{i,j} \sum_{t\in\mathcal{P}_j} \log t \cdot w_{i,j,t} - \sum_{i=1}^{I} \mu_i \log \mu_i \tag{33a}$$

$$\text{subject to} \quad \text{constraints (25b) – (25h).} \tag{33b}$$

While the feasible region of problem (33) is the same as that of (25), the objective function of (33) is different. Just like problem (25), problem (33) can be written as a mixed-integer exponential cone program, and similarly, to the best of our knowledge, this is the first exact mixed-integer convex formulation of the log-log multi-product price optimization problem under a Cartesian product price set. Lastly, just like the semi-log problem (25), one can easily modify the formulation to obtain an exact formulation of the deterministic robust price optimization problem under log-log demand (see Section EC.2.2 of the ecompanion).

We note that the log-log separation problem has an interesting property, which is that there exist optimal solutions that are extreme, in the sense that each product's price is set to either its lowest or highest allowable price. This property is formalized in the following proposition (see Section EC.1.7 for the proof).

PROPOSITION 1. *Suppose that Assumption 2 holds. Let* $(\boldsymbol{\mu}, \mathbf{p})$ *be an optimal solution of problem* (32)*. Then there exists an optimal solution* $(\boldsymbol{\mu}, \mathbf{p}')$*, such that for each* $i \in [I]$*, either* $p'_i = \min \mathcal{P}_i$ *or* $p'_i = \max \mathcal{P}_i$*.*

6. Solution method for finite $\mathcal{P}$, finite $\mathcal{U}$

In addition to the case where $\mathcal{U}$ is convex, we also consider the case where $\mathcal{U}$ is a finite discrete set. Due to page limitations, our presentation of our solution method for this case is relegated to Section EC.3 of the ecompanion. At a high level, the foundation of our approach is double column generation, which alternates between solving the primal version of the RRPO problem, which is $\max_{\boldsymbol{\pi} \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u})$, and the dual version of the RRPO problem, which is $\min_{\boldsymbol{\lambda} \in \Delta_{\mathcal{U}}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \mathcal{U}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u})$. In each iteration, we solve the primal problem with $\mathcal{P}$ replaced by a subset $\hat{\mathcal{P}} \subseteq \mathcal{P}$, where we use constraint generation to handle the inner minimization over $\mathbf{u} \in \mathcal{U}$; this gives rise to a finite set of uncertainty realizations $\hat{\mathcal{U}} \subseteq \mathcal{U}$. We then solve the dual problem with $\mathcal{U}$ replaced by $\hat{\mathcal{U}}$, where we use constraint generation to handle the inner maximization over $\mathbf{p} \in \mathcal{P}$, which gives rise to a finite set of price vectors $\hat{\mathcal{P}} \subseteq \mathcal{P}$. At each step of the algorithm, the objective value of the primal problem restricted to $\hat{\mathcal{P}}$ is a lower bound on the true optimal objective, while the objective value of the dual problem restricted to $\hat{\mathcal{U}}$ is an upper bound on the optimal objective; the algorithm terminates when these two bounds are equal or are otherwise within a pre-specified tolerance.

To implement this approach for the demand models that we consider, one needs to be able to solve the primal separation problem (solve $\min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u})$) and the dual separation problem (solve $\max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot R(\mathbf{p}, \mathbf{u})$). We show how both of these problems can be reformulated as mixed-integer exponential cone programs for the semi-log and log-log demand models, and as mixed-integer linear programs for the linear demand model.

7. Numerical experiments

In this section, we conduct several sets of experiments involving synthetic problem instances to understand the tractability of the RRPO approach and the improvement in worst-case revenue of the randomized robust pricing strategy over the deterministic robust pricing strategy. In Section 7.1, we consider instances involving the linear, semi-log and log-log models where the uncertainty set $\mathcal{U}$ is a convex set. In Section 7.2, we consider instances involving the linear, semi-log and log-log models where the uncertainty set $\mathcal{U}$ is a discrete set. Finally, in Section 7.3, we consider log-log and semi-log robust price optimization instances derived from a real data set on sales of orange juice products from a grocery store chain.

All of our code is implemented in the Julia programming language (Bezanson et al. 2017). All optimization models are implemented using the JuMP package (Lubin and Dunning 2015). All linear and mixed-integer linear programs are solved using Gurobi Gurobi Optimization, LLC (2023) and all mixed-integer exponential cone programs are solved using Mosek (ApS 2022), with a maximum of 8 threads per program. All of our experiments are conducted on Amazon Elastic Compute Cloud (EC2), on a single instance of type `m6a.48xlarge` (AMD EPYC 7R13 processor, with 192 virtual CPUs and 768 GB of memory).

7.1. Experiments with convex $\mathcal{U}$ and linear, log-log and semi-log demand models

In our first set of experiments, we consider the log-log and semi-log demand models, and specifically consider a $L1$-norm uncertainty set $\mathcal{U}$:

$$\mathcal{U} = \{\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \mid \|\tilde{\mathbf{u}}\|_1 \leq \theta, [\tilde{\mathbf{u}}_k = \frac{\mathbf{u}_k - \mathbf{u}_{0k}}{\mathbf{u}_{0k}} \ \forall k \in \{1, .., I + I^2\}]\}, \tag{34}$$

where $\mathbf{u}_0$ is the vector of nominal values of the uncertain parameters $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, and θ is the budget of the uncertainty set.

For each of the three demand models (linear, semi-log and log-log), we vary the number of products I varies in $\{5, 10, 15, 20\}$. For each value of I, we generate 24 random instances, where the values of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are independently randomly generated as follows:

1. *Linear demand.* Each $\alpha_i \sim \text{Uniform}(200, 300)$, $\beta_i \sim \text{Uniform}(5, 15)$, $\gamma_{i,j} \sim \text{Uniform}(-0.1, +0.1)$.
2. *Semi-log demand.* Each $\alpha_i \sim \text{Uniform}(4, 7)$, $\beta_i \sim \text{Uniform}(1, 1.5)$, $\gamma_{i,j} \sim \text{Uniform}(-0.4, +0.4)$.
3. *Log-log demand.* Each $\alpha_i \sim \text{Uniform}(10, 14)$, $\beta_i \sim \text{Uniform}(1, 2)$, $\gamma_{i,j} \sim \text{Uniform}(-0.6, +0.6)$.

For each product $i \in [I]$, we set $\mathcal{P}_i = \{1, 2, 3, 4, 5\}$.

For the uncertainty set $\mathcal{U}$, the budget parameter θ varies in $\{0.1, 0, 5, 1, 1.5, 2\}$ for each instance.

For each instance, we solve the nominal problem, the DRPO problem and the RRPO problem. For DRPO and RRPO, we vary the budget parameter θ that defines the uncertainty within the set $\{0.1, 0.5, 1, 1.5, 2\}$. To solve the RRPO problem for each instance, we execute the constraint generation solution algorithm described in Section 5. For instances with log-log demand, we take advantage of Proposition 1 and thus simplify the price set $\mathcal{P}$ to contain the highest and lowest price levels for each product only. To solve the DRPO problem, we formulate it as either a mixed-integer linear program (for linear demand) or a mixed-integer exponential cone program (for semi-log and log-log demand) via the log-sum-exp biconjugate-based technique described in Section EC.2 of the ecompanion, and use standard LP duality techniques to reformulate the objective function of the resulting problem (formulation (EC.7) and formulation (EC.8) in Section EC.2). Due to the prohibitive computation times that we encountered for the DRPO problem with log-log and semi-log demand, we impose a computation time limit of 20 minutes. From our experimentation with the DRPO problem for log-log and semi-log, it is often the case that an optimal or nearly optimal solution is found early on, and the bulk of the remaining computation time, which can be in the hours, is required by Mosek to prove optimality and close the gap. Finally, to solve the nominal problem for each instance, we also use the same biconjugate-based technique to formulate the nominal price optimization problem as a mixed-integer exponential cone program.

We present the objective value as well as the computation time of each RRPO, DRPO and nominal problem. We additionally compute several other metrics. We compute $\mathbb{E}[R(\mathbf{p}^*_{\mathrm{RR}}, \mathbf{u}_0)]$, which is the expected revenue of the randomized RPO solution assuming that the nominal parameter values are realized. We also compute $R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}_0)$, the nominal revenue of DRPO solution, and $Z_{\mathrm{N,WC}} = \min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p}^*_{\mathrm{N}}, \mathbf{u})$, the worst-case revenue of the nominal solution. We use the following metric to show the benefit of randomized strategy in robust price optimization:

$$\mathrm{RI} = (Z^*_{\mathrm{RR}} - Z^*_{\mathrm{DR}})/Z^*_{\mathrm{DR}} \times 100\% \tag{35}$$

For each metric, we compute its average over the 24 instances for each value of I and θ.

Tables 1, 2 and 3 shows the results for the linear, semi-log and log-log demand models, respectively. For linear demand, we find that the improvement by randomized robust pricing over deterministic robust pricing is modest; the largest average improvement is 4.63% for $I = 5$, $\theta = 2$. We note that we experimented with other forms of uncertainty sets and choices of the nominal parameter values, but we generally did not encounter large improvements of the same size as we did for the other two demand models.

Besides linear demand, these results also show that for semi-log and log-log demand, there can be a very large difference between the randomized and deterministic robust pricing schemes. The benefit of randomization, quantified by the metric RI, ranges from about 3% to as much as 1320% for semi-log instances, and from about 7% to 243% for log-log instances. Note that the magnitude of RI for the semi-log instances is larger than that for log-log, because the logarithm of demand in

I	θ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\mathrm{RR}}, \mathbf{u}_0)]$	t_{DR}	Z^*_{DR}	RI(%)	$R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\mathrm{N,WC}}$
5	0.1	7.77	4825.94	4967.01	0.40	4825.94	0.00	4967.01	0.34	4967.01	4825.94
5	0.5	0.06	4261.65	4967.01	0.03	4261.65	0.00	4967.01	–	–	4261.65
5	1.0	0.14	3574.50	4929.31	0.03	3557.12	0.52	4960.78	–	–	3556.29
5	1.5	0.22	2912.85	4816.66	0.03	2859.49	1.98	4899.71	–	–	2878.88
5	2.0	0.32	2293.09	4686.41	0.03	2195.89	4.63	4717.99	–	–	2201.48
10	0.1	0.17	9851.95	9998.29	0.10	9851.95	0.00	9998.29	0.08	9998.29	9851.95
10	0.5	0.18	9266.59	9998.29	0.10	9266.59	0.00	9998.29	–	–	9266.59
10	1.0	0.43	8552.00	9958.68	0.11	8534.89	0.20	9998.29	–	–	8534.89
10	1.5	0.62	7853.61	9911.18	0.11	7803.34	0.65	9990.94	–	–	7831.55
10	2.0	0.81	7173.52	9856.00	0.12	7077.66	1.37	9941.59	–	–	7128.21
15	0.1	0.34	14855.50	15002.93	0.21	14855.50	0.00	15002.93	0.16	15002.93	14855.50
15	0.5	0.34	14265.78	15002.93	0.21	14265.78	0.00	15002.93	–	–	14265.78
15	1.0	0.85	13539.69	14975.91	0.23	13528.63	0.08	15002.93	–	–	13528.63
15	1.5	1.20	12825.97	14931.12	0.24	12791.49	0.27	15002.93	–	–	12810.52
15	2.0	1.60	12126.11	14916.28	0.24	12054.34	0.61	15002.93	–	–	12092.41
20	0.1	0.57	19776.82	19923.37	0.36	19776.82	0.00	19923.37	0.28	19923.37	19776.82
20	0.5	0.55	19190.63	19923.37	0.37	19190.63	0.00	19923.37	–	–	19190.63
20	1.0	1.41	18464.77	19910.10	0.39	18457.88	0.04	19923.37	–	–	18457.88
20	1.5	2.08	17744.39	19895.41	0.42	17725.13	0.11	19923.37	–	–	17735.21
20	2.0	2.89	17030.17	19859.43	0.43	16992.38	0.22	19923.37	–	–	17012.55

Table 1 Results for linear instances with convex $\mathcal{U}$.

I	θ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\mathrm{RR}}, \mathbf{u}_0)]$	t_{DR}	Z^*_{DR}	RI(%)	$R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\mathrm{N,WC}}$
5	0.1	12.04	2.60×10^3	4.53×10^3	0.45	2.55×10^3	2.86	4.56×10^3	0.37	4.56×10^3	2.54×10^3
5	0.5	0.28	5.40×10^2	2.92×10^3	0.18	4.26×10^2	30.89	3.67×10^3	–	–	3.25×10^2
5	1.0	0.34	2.14×10^2	2.33×10^3	0.19	1.52×10^2	42.14	1.81×10^3	–	–	66.06
5	1.5	0.38	1.09×10^2	1.77×10^3	0.21	72.97	49.46	9.42×10^2	–	–	27.85
5	2.0	0.39	60.06	1.67×10^3	0.22	38.76	53.11	9.17×10^2	–	–	14.54
10	0.1	0.80	2.29×10^5	4.05×10^5	0.45	2.27×10^5	5.14	4.08×10^5	0.20	4.08×10^5	2.27×10^5
10	0.5	1.55	5.44×10^4	2.44×10^5	1.77	2.83×10^4	86.68	2.84×10^5	–	–	2.44×10^4
10	1.0	2.35	1.93×10^4	2.00×10^5	4.78	6.22×10^3	178.14	1.41×10^5	–	–	3.24×10^3
10	1.5	3.36	8.37×10^3	1.59×10^5	12.76	2.19×10^3	242.84	8.94×10^4	–	–	1.20×10^3
10	2.0	5.03	4.62×10^3	1.29×10^5	21.73	1.07×10^3	273.72	1.42×10^5	–	–	6.58×10^2
15	0.1	1.83	7.26×10^6	1.33×10^7	1.93	7.24×10^6	2.79	1.34×10^7	0.75	1.34×10^7	7.24×10^6
15	0.5	4.50	1.18×10^6	9.18×10^6	11.57	7.12×10^5	91.22	1.29×10^7	–	–	6.63×10^5
15	1.0	10.18	3.41×10^5	6.11×10^6	123.07	9.89×10^4	302.39	6.11×10^6	–	–	5.26×10^4
15	1.5	20.74	1.47×10^5	5.46×10^6	499.02	3.64×10^4	406.11	6.59×10^6	–	–	2.40×10^4
15	2.0	32.79	7.67×10^4	4.79×10^6	799.69	1.66×10^4	502.85	7.32×10^6	–	–	1.21×10^4
20	0.1	6.65	2.17×10^8	4.13×10^8	5.74	2.16×10^8	3.87	4.13×10^8	1.66	4.13×10^8	2.16×10^8
20	0.5	25.98	2.33×10^7	1.96×10^8	204.50	1.65×10^7	178.27	4.11×10^8	–	–	1.63×10^7
20	1.0	36.70	6.93×10^6	1.83×10^8	795.72	1.11×10^6	715.40	7.17×10^7	–	–	7.42×10^5
20	1.5	59.76	3.01×10^6	1.30×10^8	1036.22	3.64×10^5	1000.01	3.66×10^8	–	–	3.1×10^5
20	2.0	75.66	1.56×10^6	2.05×10^8	1129.75	1.72×10^5	1320.44	2.52×10^8	–	–	1.43×10^5

Table 2 Results for semi-log instances with convex $\mathcal{U}$.

the semi-log model has a linear dependence on price which results in an exponential dependence of demand on price, but in log-log, the logarithm of demand is linear in the logarithm of price, resulting

I	θ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\text{RR}}, \mathbf{u}_0)]$	t_{DR}	Z^*_{DR}	RI(%)	$R(\mathbf{p}^*_{\text{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\text{N,WC}}$
5	0.5	11.93	3.47×10^5	2.01×10^6	0.39	3.30×10^5	6.90	1.95×10^6	0.55	2.85×10^6	2.34×10^5
5	1.0	0.16	9.74×10^4	1.94×10^6	0.14	9.23×10^4	7.24	1.88×10^6	–	–	6.03×10^4
5	1.5	0.17	2.87×10^4	1.94×10^6	0.14	2.70×10^4	7.93	1.83×10^6	–	–	1.76×10^4
5	2.0	0.17	8.52×10^3	1.94×10^6	0.14	7.99×10^3	7.93	1.83×10^6	–	–	5.20×10^3
10	0.5	1.53	2.78×10^6	1.20×10^7	25.53	2.07×10^6	34.28	1.07×10^7	0.55	2.24×10^7	1.37×10^6
10	1.0	2.21	1.15×10^6	9.94×10^6	44.96	8.27×10^5	38.84	7.59×10^6	–	–	5.04×10^5
10	1.5	2.94	5.67×10^5	9.20×10^6	54.57	4.03×10^5	41.13	8.15×10^6	–	–	2.41×10^5
10	2.0	2.93	3.00×10^5	8.74×10^6	61.97	2.11×10^5	43.68	7.21×10^6	–	–	1.24×10^5
15	0.5	11.42	1.18×10^7	5.42×10^7	934.31	7.45×10^6	70.56	5.77×10^7	4.63	1.28×10^8	4.45×10^6
15	1.0	22.91	5.32×10^6	3.77×10^7	1193.31	2.98×10^6	85.48	3.43×10^7	–	–	1.68×10^6
15	1.5	32.63	3.01×10^6	3.28×10^7	1200.57	1.58×10^6	93.71	2.51×10^7	–	–	8.68×10^5
15	2.0	38.84	1.83×10^6	3.06×10^7	1200.61	9.36×10^5	99.16	2.05×10^7	–	–	5.00×10^5
20	0.5	42.59	5.16×10^7	2.40×10^8	1200.50	1.89×10^7	178.79	1.41×10^8	21.44	6.77×10^8	9.07×10^6
20	1.0	115.78	2.37×10^7	1.62×10^8	1200.88	8.04×10^6	209.87	1.04×10^8	–	–	3.65×10^6
20	1.5	197.12	1.40×10^7	1.33×10^8	1200.97	4.40×10^6	229.16	8.81×10^7	–	–	2.04×10^6
20	2.0	257.13	8.95×10^6	1.21×10^8	1201.03	2.66×10^6	243.71	8.34×10^7	–	–	1.27×10^6

Table 3 Results for log-log instances with convex $\mathcal{U}$.

in a milder polynomial dependence of demand on price. For semi-log and log-log demand, both the worst-case revenue of RRPO solution and the worst-case revenue of DRPO solution decrease as the uncertainty set becomes larger, and the rate of reduction becomes less as the uncertainty budget θ is larger. In addition, for linear, semi-log and log-log demand, the RI generally increases as the uncertainty budget θ increases. Also, as we expect, $Z^*_{\text{RR}} \geq Z^*_{\text{DR}} \geq Z_{\text{N,WC}}$. Interestingly, the randomized robust pricing scheme can achieve better performance than the deterministic robust scheme under the nominal demand model (for example, compare $\mathbb{E}[R(\mathbf{p}^*_{\text{RR}}, \mathbf{u}_0)]$ and $R(\mathbf{p}^*_{\text{DR}}, \mathbf{u}_0)$ for log-log demand with $I = 10$); this appears to be the case for almost all (I, θ) combinations for log-log, and for a smaller set of (I, θ) combinations for semi-log.

With regard to the computation time, we observe that the computation time generally grows with the number of products for both RRPO and DRPO. For linear demand, both RRPO and DRPO can be solved extremely quickly (no more than 3 seconds on average, even with $I = 20$ products). For log-log and semi-log, when the number of products is held constant, the amount of time required to solve either RRPO generally becomes larger as the uncertainty set becomes larger. However, what we find is that for both log-log and semi-log demand, RRPO generally requires much less time to solve to complete optimality than DRPO; this is likely because the nominal problem (which is a key piece of the constraint generation method for RRPO when $\mathcal{U}$ is convex) can be solved rapidly, whereas the robust version of this mixed-integer exponential cone program is more challenging for Mosek.

7.2. Experiments with discrete $\mathcal{U}$ and log-log and semi-log demand models

In our second set of experiments, we consider linear, log-log and semi-log demand models, where uncertainty is modeled through a discrete uncertainty set. We specifically consider a discrete budget uncertainty set $\mathcal{U}$ here:

$$\mathcal{U} = \{\mathbf{u} = \mathbf{u}_0 - (\mathbf{u}_0 - \bar{\mathbf{u}}) \circ \xi - (\mathbf{u}_0 - \underline{\mathbf{u}}) \circ \eta \mid \mathbf{e}^\top \xi + \mathbf{e}^\top \eta \leq \Gamma, \xi + \eta \leq \mathbf{1}, \xi, \eta \in \{0,1\}^{I+I^2}\}, \tag{36}$$

where $\underline{\mathbf{u}}$ and $\bar{\mathbf{u}}$ are respectively the component-wise lower and upper bounds of $\mathbf{u}$, $\mathbf{u}_0$ is the nominal value of the uncertain parameter vector $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, Γ is the budget of uncertainty and $I + I + I(I-1) = I + I^2$ is the total number of demand model parameters. Under the budget uncertainty set $\mathcal{U}$, up to Γ parameters can attain their lower bounds or upper bounds, whereas the remaining parameters can only attain their nominal values. We shall assume that the lower bound vector $\underline{\mathbf{u}}$ and upper bound vector $\bar{\mathbf{u}}$ are defined as $\underline{\mathbf{u}} = 0.7\mathbf{u}_0$ and $\bar{\mathbf{u}} = 1.3\mathbf{u}_0$, where $\mathbf{u}_0$ is the vector of nominal parameters.

For each of the three demand models (linear, semi-log and log-log), we vary the number of products I in $\{5, 10, 15\}$. For each value of I, we generate 24 random instances, where the values of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are independently randomly generated as follows:

1. *Linear demand*. Each $\alpha_i \sim \text{Uniform}(100, 200)$, $\beta_i \sim \text{Uniform}(5, 15)$, $\gamma_{i,j} \sim \text{Uniform}(-0.1, +0.1)$.
2. *Semi-log demand*. Each $\alpha_i \sim \text{Uniform}(8, 10)$, $\beta_i \sim \text{Uniform}(1.5, 2)$, $\gamma_{i,j} \sim \text{Uniform}(-0.5, +0.5)$.
3. *Log-log demand*. Each $\alpha_i \sim \text{Uniform}(10, 14)$, $\beta_i \sim \text{Uniform}(1.5, 2)$, $\gamma_{i,j} \sim \text{Uniform}(-0.8, +0.8)$.

We set the price set of each $i \in [I]$ as $\mathcal{P}_i = \{1, 2, 3, 4, 5\}$.

For each instance, we solve the nominal problem, the DRPO problem and the RRPO problem. For both RRPO and DRPO, we test a different collection of Γ values for the uncertainty set depending on the value of I.

To solve the RRPO problem for each instance, we execute the double column generation algorithm described in Section EC.3. In our preliminary experimentation with the restricted dual problem, we observed that exactly solving the dual separation problem (EC.32) (for semi-log demand) or (EC.38) (for log-log demand) via Mosek takes quite a long time. Therefore, to reduce the computation time of RRPO with discrete $\mathcal{U}$, we instead use a random improvement heuristic to obtain the solution of dual separation problem. Specifically, we randomly select a price vector $\mathbf{p}^0$ as a starting point. We start with changing the price of product $i = 1$ and keeping the prices of all other products unchanged, to search for a price vector $\mathbf{p}^1$ that makes the objective value of the dual separation problem the largest. Then based on the current price vector $\mathbf{p}^1$, we change the price of product $i = 2$ and keep the prices of all other products unchanged, to search for a better price vector $\mathbf{p}^2$. We repeat this for all of the products, yielding the price vector $\mathbf{p}^I$. We repeat this procedure with 100 random starting points, and retain the best solution over these 100 repetitions. Although this approximate method cannot guarantee that the overall double column generation procedure converges to a provably optimal solution, our preliminary experimentation with small instances suggests that it obtains the exact solution of RRPO that one would obtain if the dual separation problem were solved to provable optimality. For the linear demand model, we solve both primal and dual separation problems as mixed-integer programs in Gurobi.

With regard to the DRPO problem for each log-log and semi-log instance, we note that we do not have a solution algorithm or formulation to solve it exactly. Therefore, we again use the same random improvement heuristic to obtain an approximate solution of DRPO with these demand models. We randomly pick a starting price vector, and change the price of one product at a time to improve the worst case objective value until we no longer get an improvement. We repeat this procedure 50 times and select the best resulting price vector from these 50 repetitions as the approximate solution of DRPO. We note that we use a smaller number of repetitions because each repetition involves solving worst-case problem over $\mathbf{u} \in \mathcal{U}$ repeatedly in order to evaluate the robust objective of each candidate price vector; this contributes to a large overall computation time for this approach. With regard to the DRPO problem for linear demand, we observe that the objective function of DRPO is linear in the uncertain parameter vector $\mathbf{u}$, and that the description of the set polyhedron (36) is integral (i.e., extreme points of this polyhedron naturally correspond to $\boldsymbol{\xi}, \boldsymbol{\eta} \in \{0,1\}^{2I+I^2}$). Therefore, DRPO can be solved exactly by relaxing the requirement $\boldsymbol{\xi}, \boldsymbol{\eta} \in \{0,1\}^{2I+I^2}$ in the uncertainty set (36), and reformulating the worst-case objective using LP duality, leading to a mixed-integer linear program.

Lastly, for the nominal problem for each instance, we use the biconjugate technique to formulate it as a mixed-integer exponential cone program.

We report the same metrics as in Section 7.1, with two minor modifications. We use $\hat{Z}_{\mathrm{DR}}$ and $\hat{\mathbf{p}}_{\mathrm{DR}}$ to denote the approximate objective value and solution of DRPO given by the random improvement heuristic. The approximate improvement percentage is then $\hat{\mathrm{RI}} = (Z^*_{\mathrm{RR}} - \hat{Z}_{\mathrm{DR}})/\hat{Z}_{\mathrm{DR}} \times 100\%$.

Table 4 shows the results for the linear demand model. Here, we interestingly find that the vast majority of instances are randomization-proof, i.e., the average RI is below 1%, if not exactly 0%. We note here that we tested other families of instances where $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and $\mathcal{P}_i$ are generated differently, but in virtually every case we found that the relative improvement of randomized over deterministic robust pricing was very small. These results, together with those for the convex $\mathcal{U}$ case, suggest that randomized pricing is of limited benefit compared to deterministic pricing for the uncertain linear demand model case.

Tables 5 and 6 show how the results vary for different values of discrete uncertainty budget Γ for semi-log and log-log demand.[1] We can see that, in most of the cases we test, the randomized robust pricing strategy provides a substantial benefit over the deterministic robust price solution. The percentage improvement given by randomization ranges from 0% to as much as 488.59% for semi-log instances, and from 0% to 175.18% for log-log instances. Similar to the cases with convex $\mathcal{U}$, both Z^*_{RR} and $\hat{Z}_{\mathrm{DR}}$ decrease as the uncertainty set becomes larger. While the RI metric generally decreases as Γ increases, in some instances it can be increasing in Γ at small values of Γ (this is visible in the average results metrics for $I = 10$ with semi-log demand). When Γ is large enough, the RI metric often becomes very small or even zero. This makes sense when interpreted through Corollary 2. Specifically, when nature is able to make a large number of demand model parameters take their worst values, it is likely that at the $\mathbf{u}^*$ at which the optimal objective of DRPO is attained is such that the price vector for the nominal problem with $\mathbf{u}^*$ coincides with the optimal price vector for DRPO. Thus, by Corollary 2, the problem will be randomization-proof.

With regard to the computation time, the computation time for both RRPO and DRPO increases with I. Interestingly, the computation time required by RRPO does not necessarily increase as the discrete uncertainty budget Γ increases; in some cases, when Γ is large, the RRPO solution degenerates to the DRPO solution, allowing the double column generation algorithm to terminate quickly. By comparing t_{RR} and t_{DR}, we can see that RRPO in general takes less time than DRPO. The computation time of the RRPO problem in semi-log instances is no more than approximately two minutes on average ($I = 15$, $\Gamma = 60$), while in log-log instances, solving RRPO requires no more than 1.5 minutes on average ($I = 15$, $\Gamma = 18$). Lastly, for linear demand, the computation time for RRPO is extremely small, requiring no more than a few seconds on average.

7.3. Results using real data instances

In our last set of experiments, we evaluate the effectiveness of solution algorithms on problem instances calibrated with real data. For these experiments, we consider the `orangeJuice` data set from Montgomery (1997), which was accessed via the `bayesm` package in R (Rossi 2022). This data set contains price and sales data for $I = 11$ different orange juice brands at the Dominick's Finer Foods chain of grocery stores in the Chicago area. Each observation in the data set consists of: the store s; the week t; the log of the number of units sold $\log(q_{t,s,i})$ for brand i; the prices $p_{t,s,1} \ldots p_{t,s,11}$ of the eleven orange juice brands; the dummy variable $d_{t,s,i}$ indicating whether brand

[1] We note here that for the log-log model, we encountered one instance ($I = 10$, $\Gamma = 44$) where $\hat{Z}_{\mathrm{DR}}$ was higher than Z^*_{RR}; in general, Z^*_{RR} should be higher than Z^*_{DR}. We have verified that the reason for this anomaly was a numerical error in the solution of the worst-case subproblem in Mosek within the DRPO random improvement heuristic. This instance is omitted in our calculation of $\hat{Z}_{\mathrm{DR}}$, RI and $\mathbb{E}[R(\mathbf{p}_{\mathrm{DR}}, \mathbf{u}_0)]$, and the affected entries are indicated by * in Table 6.

I	Γ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\text{RR}}, \mathbf{u}_0)]$	t_{DR}	Z^*_{DR}	RI(%)	$R(\mathbf{p}^*_{\text{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\text{N,WC}}$
5	3	8.76	1724.38	2458.17	0.43	1723.46	0.06	2462.66	0.32	2473.30	1719.29
5	6	0.27	1316.39	2382.10	0.04	1313.68	0.23	2383.35	–	–	1261.84
5	9	0.27	1159.33	2294.44	0.03	1157.98	0.12	2293.84	–	–	1039.24
5	12	0.19	1126.85	2259.05	0.03	1126.85	0.00	2259.05	–	–	987.29
5	18	0.20	1124.87	2256.23	0.03	1124.87	0.00	2256.23	–	–	984.10
5	24	0.17	1123.78	2256.23	0.03	1123.78	0.00	2256.23	–	–	982.18
10	5	0.53	3717.70	4990.89	0.11	3714.08	0.10	4997.68	0.09	5009.03	3711.50
10	7	0.59	3315.59	4972.42	0.11	3312.91	0.08	4973.49	–	–	3297.07
10	9	0.70	2982.02	4941.26	0.11	2979.68	0.08	4944.40	–	–	2942.80
10	14	1.35	2589.70	4782.85	0.12	2583.93	0.23	4791.83	–	–	2437.09
10	19	0.84	2372.37	4662.16	0.11	2371.21	0.05	4657.17	–	–	2133.38
10	26	0.56	2342.57	4636.61	0.10	2342.57	0.00	4636.61	–	–	2086.03
10	33	0.57	2339.18	4633.87	0.11	2339.18	0.00	4636.61	–	–	2081.46
10	44	0.57	2334.90	4627.78	0.10	2334.90	0.00	4627.78	–	–	2075.15
15	6	0.92	5920.62	7495.77	0.22	5918.51	0.04	7506.26	0.17	7513.95	5917.60
15	12	1.79	4706.04	7435.56	0.25	4701.16	0.11	7442.32	–	–	4668.21
15	18	3.16	4051.33	7250.57	0.27	4045.35	0.15	7248.00	–	–	3889.25
15	24	4.50	3722.40	7071.15	0.29	3713.86	0.23	7068.49	–	–	3428.59
15	36	1.12	3500.73	6889.94	0.21	3500.73	0.00	6889.47	–	–	3112.83

Table 4 **Results for linear instances with discrete $\mathcal{U}$.**

I	Γ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\text{RR}}, \mathbf{u}_0)]$	t_{DR}	$\hat{Z}_{\text{DR}}$	RI(%)	$R(\hat{\mathbf{p}}_{\text{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\text{N,WC}}$
5	3	13.23	8.65×10^3	1.57×10^5	20.35	5.56×10^3	63.01	2.17×10^5	0.93	2.27×10^5	5.04×10^3
5	6	0.43	3.06×10^3	1.81×10^5	20.52	1.92×10^3	66.89	2.20×10^5	–	–	1.78×10^3
5	9	0.55	1.84×10^3	2.09×10^5	22.96	1.64×10^3	18.42	2.22×10^5	–	–	1.63×10^3
5	12	0.63	1.65×10^3	2.22×10^5	25.53	1.60×10^3	4.39	2.25×10^5	–	–	1.60×10^3
5	18	0.27	1.59×10^3	2.27×10^5	23.87	1.59×10^3	0.11	2.27×10^5	–	–	1.59×10^3
5	24	0.08	1.59×10^3	2.27×10^5	13.53	1.59×10^3	0.00	2.27×10^5	–	–	1.59×10^3
10	5	1.59	1.21×10^6	5.30×10^7	117.69	4.95×10^5	173.47	6.73×10^7	0.15	7.16×10^7	4.48×10^5
10	7	2.59	6.50×10^5	4.10×10^7	119.28	1.87×10^5	233.11	5.69×10^7	–	–	1.88×10^5
10	9	3.49	4.32×10^5	3.64×10^7	121.43	1.09×10^5	250.12	5.41×10^7	–	–	1.05×10^5
10	14	6.44	1.80×10^5	3.67×10^7	180.63	6.69×10^4	161.08	6.96×10^7	–	–	6.78×10^4
10	19	10.23	9.63×10^4	5.29×10^7	362.29	6.36×10^4	67.80	7.15×10^7	–	–	6.36×10^4
10	26	12.14	6.69×10^4	6.40×10^7	559.63	6.32×10^4	19.70	7.15×10^7	–	–	6.33×10^4
10	33	11.37	6.37×10^4	7.11×10^7	699.06	6.33×10^4	6.47	7.15×10^7	–	–	6.33×10^4
10	44	7.84	6.33×10^4	7.15×10^7	799.81	6.33×10^4	0.09	7.16×10^7	–	–	6.33×10^4
15	6	37.63	3.55×10^8	3.22×10^9	331.13	1.59×10^7	488.59	4.52×10^9	0.44	5.05×10^9	1.42×10^7
15	12	18.86	8.69×10^6	3.24×10^9	344.94	2.15×10^6	471.14	3.51×10^9	–	–	1.77×10^6
15	18	33.08	3.11×10^6	3.08×10^9	653.64	1.18×10^6	323.61	4.91×10^9	–	–	1.16×10^6
15	24	48.52	1.75×10^6	3.67×10^9	1525.41	1.12×10^6	165.75	5.03×10^9	–	–	1.11×10^6
15	36	101.38	1.21×10^6	4.82×10^9	3400.17	1.1×10^6	38.58	5.02×10^9	–	–	1.10×10^6

Table 5 **Results for semi-log instances with discrete $\mathcal{U}$.**

i had any in-store displays at store s in week t; and the variable $f_{t,s,i}$ indicating if brand i was featured/advertised at store s in week t. We fit log-log and semi-log regression models for each

I	Γ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\mathrm{RR}}, \mathbf{u}_0)]$	t_{DR}	$\hat{Z}_{\mathrm{DR}}$	RI(%)	$R(\hat{\mathbf{p}}_{\mathrm{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\mathrm{N,WC}}$
5	3	13.33	3.26×10^5	1.86×10^6	31.23	2.80×10^5	22.03	1.89×10^6	0.96	4.31×10^6	1.49×10^5
5	6	0.20	6.37×10^4	4.03×10^6	20.39	6.15×10^4	3.16	4.29×10^6	–	–	6.1×10^4
5	9	0.27	4.91×10^4	4.19×10^6	18.39	4.87×10^4	0.65	4.24×10^6	–	–	4.77×10^4
5	12	0.35	4.71×10^4	4.20×10^6	16.82	4.70×10^4	0.20	4.18×10^6	–	–	4.57×10^4
5	18	0.17	4.66×10^4	4.18×10^6	11.67	4.66×10^4	0.01	4.18×10^6	–	–	4.52×10^4
5	24	0.14	4.66×10^4	4.18×10^6	8.73	4.66×10^4	0.05	4.16×10^6	–	–	4.52×10^4
10	5	4.64	2.33×10^6	3.89×10^7	161.57	1.38×10^6	83.54	6.45×10^7	0.31	7.66×10^7	1.23×10^6
10	7	5.39	1.31×10^6	5.22×10^7	165.21	8.59×10^5	63.34	7.12×10^7	–	–	8.39×10^5
10	9	5.30	8.74×10^5	5.15×10^7	149.63	6.23×10^5	38.99	7.28×10^7	–	–	6.12×10^5
10	14	5.10	5.07×10^5	5.67×10^7	138.67	3.82×10^5	29.31	7.30×10^7	–	–	3.80×10^5
10	19	3.52	3.74×10^5	6.35×10^7	136.70	3.31×10^5	12.90	7.51×10^7	–	–	3.33×10^5
10	26	3.56	3.27×10^5	7.38×10^7	138.77	3.18×10^5	4.28	7.59×10^7	–	–	3.21×10^5
10	33	4.79	3.18×10^5	7.60×10^7	137.07	3.15×10^5	2.34	7.59×10^7	–	–	3.17×10^5
10	44	3.19	3.15×10^5	7.62×10^7	131.47	3.19×10^5*	1.19*	7.53×10^7*	–	–	3.14×10^5
15	6	35.68	1.71×10^7	4.33×10^8	552.64	7.75×10^6	175.18	7.83×10^8	1.20	8.38×10^8	7.20×10^6
15	12	62.70	5.55×10^6	4.93×10^8	582.20	2.94×10^6	102.06	7.78×10^8	–	–	2.76×10^6
15	18	76.60	2.92×10^6	4.55×10^8	561.96	1.92×10^6	65.73	8.24×10^8	–	–	1.88×10^6
15	24	54.30	2.03×10^6	6.79×10^8	572.08	1.71×10^6	32.76	8.19×10^8	–	–	1.71×10^6
15	36	48.34	1.69×10^6	8.11×10^8	600.30	1.63×10^6	8.07	8.32×10^8	–	–	1.63×10^6

Table 6 **Results for log-log instances with discrete $\mathcal{U}$.**

brand i according to the following specifications:

$$\text{(semi-log)} \quad \log(q_{t,s,i}) = \alpha_i - \beta_i p_{t,s,i} + \sum_{j \neq i} \gamma_{ij} p_{t,s,j} + \theta_i d_{t,s,i} + \mu_i f_{t,s,i} + \epsilon_{t,s,i}, \tag{37}$$

$$\text{(log-log)} \quad \log(q_{t,s,i}) = \alpha_i - \beta_i \log(p_{t,s,i}) + \sum_{j \neq i} \gamma_{ij} \log(p_{t,s,j}) + \theta_i d_{t,s,i} + \mu_i f_{t,s,i} + \epsilon_{t,s,i}, \tag{38}$$

where $\{\epsilon_{t,s,i}\}_{t,s,i}$ is a collection of IID normally distributed error terms. The point estimates of the model parameters are provided in Section EC.4.1 of the ecompanion. We note that prior work has considered the estimation of both of these types of models (see the examples in Rossi 2022; see also Montgomery 1997 and Mišić 2020).

We consider the problem of obtaining a price vector $\mathbf{p} = (p_1, p_2, \ldots, p_{11})$ for this collection of 11 products. To formulate the price vector set $\mathcal{P}$, we assume that each product i has five allowable prices, which are shown in Table 7. These prices correspond to the 0th (i.e., minimum), 25th, 50th, 75th and 100th (i.e., maximum) percentiles of the observed prices in the dataset.

Product	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
	1.29	2.86	1.25	0.99	0.88	2.76	0.91	0.91	0.69	0.52	1.99
	2.49	4.19	2.69	1.99	1.99	3.67	1.99	1.99	1.79	1.58	2.99
	2.99	4.75	2.89	2.35	2.17	3.96	2.39	2.19	1.99	1.59	3.59
	3.19	4.99	3.12	2.49	2.49	4.49	2.56	2.39	2.36	1.99	3.99
	3.87	5.82	3.35	3.06	3.17	5.09	3.07	2.69	3.08	2.69	4.99

Table 7 **Possible price levels for products in `orangeJuice` experiment instances.**

For each type of demand model, we consider two forms of uncertainty set: a convex $L1$-norm uncertainty set (as in equation (34)) and a discrete budget uncertainty set (as in equation (36)). We vary the budget θ of the $L1$-norm uncertainty set and present the results in Tables 8 and 9. We also vary the budget Γ of the discrete budget uncertainty set and present the results in tables EC.3 and EC.4. Specifically, for discrete budget uncertainty set, we assume that $\bar{\boldsymbol{\alpha}} = 1.2\boldsymbol{\alpha}$, $\underline{\boldsymbol{\alpha}} = 0.8\boldsymbol{\alpha}$, $\bar{\boldsymbol{\beta}} = 1.3\boldsymbol{\beta}$, $\underline{\boldsymbol{\beta}} = 0.7\boldsymbol{\beta}$, $\bar{\boldsymbol{\gamma}} = 1.4\boldsymbol{\gamma}$, and $\underline{\boldsymbol{\gamma}} = 0.6\boldsymbol{\gamma}$. Tables 8 and 9 below present the results under the convex $L1$-norm uncertainty set for the semi-log and log-log demand models, respectively. Due to page considerations, the results for the discrete $\mathcal{U}$ case are provided in Section EC.4.2 of the ecompanion.

We can see from Tables 8 and 9 that the randomized pricing strategy performs significantly better than the deterministic pricing solution under the worst-case demand model, with the RI ranging from 17.86% to 47.81% for semi-log demand and from 27.71% to 92.31% for log-log demand. In addition, for the same demand type and uncertainty set, the computation time of RRPO is comparable to that of DRPO. With regard to the discrete uncertainty set case, the results shown in Section EC.4.2 are qualitatively similar, with the randomized robust pricing strategy similarly outperforming the deterministic robust solution. We do also observe that under bothdemand models, solving RRPO with the discrete uncertainty set requires more time than solving it with convex uncertainty set, although the overall time is still reasonable (in the most extreme case, RRPO for the discrete uncertainty set can take up to approximately 300 seconds, and DRPO requires up to 600 second, compared to 60 seconds for both RRPO and DRPO for the $L1$-norm uncertainty set).

θ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\mathrm{RR}}, \mathbf{u}_0)]$	t_{DR}	Z^*_{DR}	RI(%)	$R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\mathrm{N,WC}}$
0.10	16.86	342357.06	481125.25	15.98	290474.67	17.86	590546.51	0.83	590547.01	290474.76
0.50	41.42	197517.06	373973.40	47.65	147748.35	33.68	294483.28	–	–	96016.90
0.80	42.61	149709.04	352742.22	67.83	105734.14	41.59	294483.28	–	–	67924.78
1.00	67.47	125987.02	349644.33	74.03	86977.24	44.85	265173.68	–	–	55394.70
1.50	61.70	82880.96	348467.74	44.65	56474.64	46.76	265173.68	–	–	34864.43
2.00	65.21	54665.15	348466.26	52.10	37164.75	47.09	265173.68	–	–	22615.70

Table 8 Results for `orangeJuice` pricing problem with semi-log demand and convex $\mathcal{U}$.

θ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\mathrm{RR}}, \mathbf{u}_0)]$	t_{DR}	Z^*_{DR}	RI(%)	$R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\mathrm{N,WC}}$
0.10	0.71	722647.22	1051269.80	1.97	565866.71	27.71	922172.97	0.88	1112050.59	560812.30
0.50	1.42	342614.34	687632.84	4.99	233387.10	46.80	782893.68	–	–	152881.89
0.80	1.91	260049.66	672481.74	6.77	162276.97	60.25	782893.68	–	–	102893.20
1.00	2.08	217580.86	683576.99	9.41	128220.45	69.69	782893.68	–	–	81427.57
1.50	2.26	142307.66	670759.12	13.02	75897.66	87.50	599315.12	–	–	48983.56
2.00	2.39	94847.37	670758.38	13.49	49319.21	92.31	377932.71	–	–	31055.19

Table 9 Results for `orangeJuice` pricing problem with log-log demand and convex $\mathcal{U}$.

Lastly, it is also interesting to compare the randomized robust pricing strategy to the deterministic robust price vector. Taking the log-log demand model and the convex $L1$ uncertainty set with $\theta = 0.8$ as an example, the solution of the RRPO problem is the following randomized pricing strategy:

$$\mathbf{p} = \begin{cases} (3.87, 5.82, 1.25, 0.99, 3.17, 5.09, 3.07, 0.91, 0.69, 2.69, 1.99) & \text{w.p. } 0.1628, \\ (1.29, 5.82, 3.35, 3.06, 0.88, 2.76, 3.07, 2.69, 3.08, 2.69, 4.99) & \text{w.p. } 0.1752, \\ (3.87, 2.86, 1.25, 3.06, 3.17, 5.09, 0.91, 2.69, 3.08, 0.52, 4.99) & \text{w.p. } 0.2658, \\ (3.87, 5.82, 3.35, 3.06, 0.88, 2.76, 3.07, 0.91, 3.08, 2.69, 4.99) & \text{w.p. } 0.0381, \\ (3.87, 2.86, 1.25, 3.06, 3.17, 2.76, 3.07, 2.69, 0.69, 2.69, 4.99) & \text{w.p. } 0.3258, \\ (3.87, 2.86, 1.25, 3.06, 3.17, 5.09, 0.91, 0.91, 3.08, 0.52, 4.99) & \text{w.p. } 0.0323. \end{cases} \tag{39}$$

Observe that in this randomized pricing strategy, each price vector is such that the product is set to either its lowest or highest allowable price. This is congruent with Proposition 1, which suggests that the nominal problem under the log-log demand model will always have a solution that involves setting each product to its highest or lowest price; since our solution algorithm is based on constraint generation using this nominal problem as a separation procedure, it makes sense that the randomized price vector will be supported on such extremal price vectors. On the other hand, the solution of the DRPO problem is the price vector $\mathbf{p}_{\text{DR}} = (3.87, 2.86, 1.25, 3.06, 3.17, 2.76, 0.91, 2.69, 0.69, 0.52, 4.99)$, for which we observe that the chosen prices are also either the lowest or highest for each product.

8. Conclusions

In this paper, we considered the problem of designing randomized robust pricing strategies to maximize worst-case revenue. We presented idealized conditions under which such randomized pricing strategies fare no better than the deterministic robust pricing approach, and subsequently we developed solution methods for obtaining the randomized pricing strategies in different settings (when the price set is finite, and when the uncertainty set is either convex or discrete). We showed using both synthetic instances and real data instances that such randomized pricing strategies can lead to large improvements in worst-case revenue over deterministic robust price prescriptions.

With regard to future research, an interesting direction is to consider a version of the robust price optimization problem that incorporates contextual information. For example, in the ecommerce setting, different customers who log onto a retailer's website will differ in characteristics (age, web browser, operating system, etc.). This information could be used to craft a richer uncertainty set, and to motivate randomization strategies that randomize differently based on user characteristics. More generally, we hope that this work, which was inspired by the paper of Wang et al. (2020), motivates further study in how randomization can be used to mitigate risk in revenue management applications.

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EC.1. Omitted proofs

EC.1.1. Proof of Theorem 1

To prove this, we prove that $Z^*_{\mathrm{DR}} \geq Z^*_{\mathrm{RR}}$. For any $R \in \mathcal{R}$, and any distribution F supported on $\mathcal{P}$, we have

$$\int_{\mathcal{P}} R(\mathbf{p})\, dF(\mathbf{p}) \leq R\left(\int_{\mathcal{P}} \mathbf{p}\, dF(\mathbf{p})\right), \tag{EC.1}$$

which follows by Jensen's inequality and the concavity of R. This implies that for any $F \in \mathcal{F}$,

$$\inf_{R\in\mathcal{R}} \int_{\mathcal{P}} R(\mathbf{p})\, dF(\mathbf{p}) \leq \inf_{R\in\mathcal{R}} R\left(\int_{\mathcal{P}} \mathbf{p}\, dF(\mathbf{p})\right). \tag{EC.2}$$

Therefore, we have that

$$\begin{aligned}
Z^*_{\mathrm{RR}} &= \max_{F\in\mathcal{F}} \left\{ \inf_{R\in\mathcal{R}} \int_{\mathcal{P}} R(\mathbf{p})\, dF(\mathbf{p}) \right\} \\
&\leq \max_{F\in\mathcal{F}} \left\{ \inf_{R\in\mathcal{R}} R\left(\int_{\mathcal{P}} \mathbf{p}\, dF(\mathbf{p})\right) \right\} \\
&\leq \max_{\mathbf{p}\in\mathcal{P}} \left\{ \inf_{R\in\mathcal{R}} R\left(\mathbf{p}\right)) \right\} \\
&= Z^*_{\mathrm{DR}},
\end{aligned}$$

where the second inequality follows because $\mathcal{P}$ is assumed to be convex, and thus for any $F \in \mathcal{F}$, $\int_{\mathcal{P}} \mathbf{p}\, dF(\mathbf{p})$ is contained in $\mathcal{P}$. □

EC.1.2. Proof of Theorem 2

Before we begin, we recall Sion's minimax theorem:

THEOREM EC.1 **(Sion's minimax theorem (Corollary 3.3 of Sion 1958))**. *Let M and N be convex spaces, with at least one of the two spaces being compact Let $f: M \times N \to \mathbb{R}$ be a function such that $f(\mu,\nu)$ is quasiconcave and upper semi-continuous in μ for any fixed ν, and quasiconvex and lower semi-continuous in ν for any fixed μ. Then $\sup_{\mu\in M} \inf_{\nu\in N} f(\mu,\nu) = \inf_{\nu\in N} \sup_{\mu\in M} f(\mu,\nu)$.*

We have:

$$\begin{aligned}
Z^*_{\mathrm{RR}} &= \max_{F\in\mathcal{F}} \min_{\mathbf{u}\in\mathcal{U}} \int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\, dF(\mathbf{p}) \\
&= \min_{\mathbf{u}\in\mathcal{U}} \max_{F\in\mathcal{F}} \int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\, dF(\mathbf{p}) \\
&= \min_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p},\mathbf{u}) \\
&= Z^*_{\mathrm{DR}}.
\end{aligned}$$

In the above, the steps are justified as follows. The first step follows by the definition of Z^*_{RR}.

The second step follows by applying Sion's minimax theorem to interchange the order of minimization over $\mathcal{U}$ and maximization over $\mathcal{F}$. The justification for applying Sion's minimax theorem here is that (1) the set $\mathcal{F}$ of distributions supported on $\mathcal{P}$ is a convex set; (2) $\int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\,dF(\mathbf{p})$ is linear in F when $\mathbf{u}$ is fixed; and (3) $\int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})dF(\mathbf{p})$ is quasiconvex in $\mathbf{u}$ when F is fixed by the hypotheses of the theorem. Note that $\int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\,dF(\mathbf{p})$ is continuous in F if the set of measures $\mathcal{F}$ is endowed with the topology of weak convergence. Additionally, note that $\int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\,dF(\mathbf{p})$ is continuous in $\mathbf{u}$. This is guaranteed because, by compactness of $\mathcal{P}$ and $\mathcal{U}$ and continuity of R in $(\mathbf{p},\mathbf{u})$ from Assumption 1, there exists a constant C such that $|R(\mathbf{p},\mathbf{u})| < C$ for all $\mathbf{p}\in\mathcal{P}$ and $\mathbf{u}\in\mathcal{U}$; thus, by the bounded convergence theorem, for any sequence $(\mathbf{u}_k)_{k=1}^{\infty}$ such that $\mathbf{u}_k \to \mathbf{u}$, we shall also have $\int_{\mathcal{P}} R(\mathbf{p},\mathbf{u}_k)\,dF(\mathbf{p}) \to \int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\,dF(\mathbf{p})$.

The third step follows by the fact that $\max_{F\in\mathcal{F}} \int_{\mathcal{P}} R(\mathbf{p},\mathbf{u})\,dF(\mathbf{p}) = \max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u})$, since $\mathcal{F}$ includes the distribution the Dirac delta distribution $\delta_{\mathbf{p}'}$ that places unit probability mass on $\mathbf{p}'$, for every $\mathbf{p}'\in\mathcal{P}$.

The fourth step follows by applying Sion's minimax theorem again, using the hypotheses that $R(\mathbf{p},\mathbf{u})$ is quasiconvex in $\mathbf{u}$ and quasiconcave in $\mathbf{p}$, and additionally that R is continuous in both $\mathbf{u}$ and $\mathbf{p}$ (Assumption 1). $\square$

EC.1.3. Proof of Theorem 3

To establish this result, observe that

$$
\begin{aligned}
&\inf_{Q\in\mathcal{Q}} \max_{\mathbf{p}\in\mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}) \\
&= \inf_{Q\in\mathcal{Q}} \max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}} \sum_{\mathbf{p}\in\mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}) \\
&= \max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}} \inf_{Q\in\mathcal{Q}} \sum_{\mathbf{p}\in\mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}) \\
&= \max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}} \inf_{Q\in\mathcal{Q}} \int_{\mathcal{U}} \sum_{\mathbf{p}\in\mathcal{P}} \pi(\mathbf{p}) \cdot R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u}) \\
&= \max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}} \min_{\mathbf{u}\in\mathcal{U}} \sum_{\mathbf{p}\in\mathcal{P}} \pi(\mathbf{p}) \cdot R(\mathbf{p},\mathbf{u}) \\
&= Z^*_{\mathrm{RR}}.
\end{aligned}
$$

In the above, the steps are justified as follows. The first step follows because maximizing a function of $\mathbf{p}$ over the finite set $\mathcal{P}$ is the same as maximizing the expected value of that same function with respect to all probability mass functions $\boldsymbol{\pi}$ supported on $\mathcal{P}$.

The second step follows by Sion's minimax theorem, because the quantity $\sum_{\mathbf{p}\in\mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\,dQ(\mathbf{u})$ is linear and therefore quasiconcave in $\boldsymbol{\pi}$ for a fixed Q, and is linear and therefore quasiconvex in Q for a fixed $\boldsymbol{\pi}$; additionally, the set $\Delta_{\mathcal{P}} = \{\boldsymbol{\pi} \in \mathbb{R}^{|\mathcal{P}|} \mid \mathbf{1}^T\boldsymbol{\pi} = 1, \boldsymbol{\pi} \geq \mathbf{0}\}$ is a compact convex set, and $\mathcal{Q}$ is a convex set. Additionally, note that $\sum_{\mathbf{p}\in\mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\, dQ(\mathbf{u})$ is clearly continuous in $\boldsymbol{\pi}$. It is also continuous in Q, because each term $\int_{\mathcal{U}} R(\mathbf{p},\mathbf{u})\, dQ(\mathbf{u})$ is continuous in Q when $\mathcal{Q}$ is endowed with the topology of weak convergence, and there are finitely many such terms.

The third step follows by the linearity of integration. The fourth step follows by the fact that $\mathcal{Q}$ contains the Dirac delta distribution that places unit mass on $\mathbf{u}$, for every $\mathbf{u} \in \mathcal{U}$. The final step just follows from the definition of Z^*_{RR}.

With this result in hand, observe that the existence of a $Q \in \mathcal{Q}$ such that for all $\mathbf{p} \in \mathcal{P}$, $\int_{\mathcal{U}} R(\mathbf{p},\mathbf{u}) dQ(\mathbf{u}) \leq Z^*_{\mathrm{DR}}$ is equivalent to the existence of $Q \in \mathcal{Q}$ such that

$$\max_{\mathbf{p}\in\mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u}) dQ(\mathbf{u}) \leq Z^*_{\mathrm{DR}},$$

which is equivalent to

$$\inf_{Q\in\mathcal{Q}} \max_{\mathbf{p}\in\mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u}) dQ(\mathbf{u}) \leq Z^*_{\mathrm{DR}}.$$

Since the left-hand side of this inequality is equal to Z^*_{RR}, the existence of the distribution $Q \in \mathcal{Q}$ as in the theorem statement is equivalent to $Z^*_{\mathrm{RR}} \leq Z^*_{\mathrm{DR}}$; since it is always the case that $Z^*_{\mathrm{RR}} \geq Z^*_{\mathrm{DR}}$, this is equivalent to the problem being randomization-proof. □

EC.1.4. Proof of Corollary 1

Observe that since $\max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p},\mathbf{u}) \geq \min_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u})$ always holds, equation (17) is equivalent to

$$\max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p},\mathbf{u}) \leq \min_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}),$$

or equivalently,

$$Z^*_{\mathrm{DR}} \geq \min_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}).$$

Observe that the condition $\min_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{p}\in\mathcal{P}} R(\mathbf{p},\mathbf{u}) \leq Z^*_{\mathrm{DR}}$ is exactly equivalent to the condition that there exists a $\mathbf{u} \in \mathcal{U}$ such that for all $\mathbf{p} \in \mathcal{P}$, $R(\mathbf{p},\mathbf{u}) \leq Z^*_{\mathrm{DR}}$.

To connect this to Theorem 3, consider $Q = \delta_{\mathbf{u}}$, where $\delta_{\mathbf{u}}$ is the Dirac delta distribution centered at $\mathbf{u}$. For any $\mathbf{p} \in \mathcal{P}$, $R(\mathbf{p},\mathbf{u}) = \int_{\mathcal{U}} R(\mathbf{p},\mathbf{u}')\, dQ(\mathbf{u}')$. Thus, for this choice of Q, it is the case that for all $\mathbf{p} \in \mathcal{P}$, $\int_{\mathcal{U}} R(\mathbf{p},\mathbf{u}')\, dQ(\mathbf{u}') \leq Z^*_{\mathrm{DR}}$. By Theorem 3, this is equivalent to randomization-proofness. Thus, it follows that the strong duality condition (17) is equivalent to the RPO problem being randomization-proof. □

EC.1.5. Proof of Corollary 2

To prove the $\Rightarrow$ direction of the equivalence, suppose that the robust price optimization problem is randomization-receptive. From Theorem 3, a robust price optimization problem is randomization-proof if and only if there exists a distribution $Q \in \mathcal{Q}$ such that for all $\mathbf{p} \in \mathcal{P}$, $\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u})\, dQ(\mathbf{u}) \leq Z^*_{\mathrm{DR}}$. The negation of this latter statement is the following statement:

$$\forall Q \in \mathcal{Q},\ \exists \mathbf{p} \in \mathcal{P} \text{ such that } \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u})\, dQ(\mathbf{u}) > Z^*_{\mathrm{DR}}. \tag{EC.3}$$

We need to show that $\mathbf{p}^*_{\mathrm{DR}} \notin \arg\max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}^*)$. To establish this, it is sufficient to show that there exists a $\tilde{\mathbf{p}} \in \mathcal{P}$ such that $R(\tilde{\mathbf{p}}, \mathbf{u}^*) > R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*)$. Let $\tilde{Q} = \delta_{\mathbf{u}^*}$, where $\delta_{\mathbf{u}^*}$ denotes the Dirac delta distribution centered at $\mathbf{u}^*$. By invoking (EC.3), we are assured of the existence of a price vector $\tilde{\mathbf{p}}$ such that $\int_{\mathcal{U}} R(\tilde{\mathbf{p}}, \mathbf{u})\, d\tilde{Q}(\mathbf{u}) > Z^*_{\mathrm{DR}}$. Since $\tilde{Q} = \delta_{\mathbf{u}^*}$, we have that $\int_{\mathcal{U}} R(\tilde{\mathbf{p}}, \mathbf{u})\, d\tilde{Q}(\mathbf{u}) = R(\tilde{\mathbf{p}}, \mathbf{u}^*)$, and thus we have that

$$R(\tilde{\mathbf{p}}, \mathbf{u}^*) > R(\tilde{\mathbf{p}}_{\mathrm{DR}}, \mathbf{u}^*),$$

exactly as needed. Thus, it follows that $\mathbf{p}^*_{\mathrm{DR}} \notin \arg\max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}^*)$.

To prove the $\Leftarrow$ direction of the equivalence, let $\tilde{\mathbf{p}} \in \mathcal{P}$ be a price vector for which $R(\tilde{\mathbf{p}}, \mathbf{u}^*) > R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*)$. To establish that the problem is randomization-receptive, we shall again use the condition (EC.3).

Let $Q \in \mathcal{Q}$ be an arbitrary distribution. We need to show that there exists a $\mathbf{p} \in \mathcal{P}$ that satisfies $\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u})\, dQ(\mathbf{u}) > Z^*_{\mathrm{DR}}$. There are two mutually exclusive and collectively exhaustive cases to consider:

Case 1: There exists a closed set $B \subseteq \mathcal{U}$ such that $\mathbf{u}^* \notin B$ and $Q(B) > 0$. The candidate price vector we will consider in this case is $\mathbf{p}^*_{\mathrm{DR}}$. In this case, observe that since $\mathcal{U}$ is compact, then B is also compact, and together with the extreme value theorem we can assert that $\min_{\mathbf{u} \in B} R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}) = R(\mathbf{p}^*_{\mathrm{DR}}, \tilde{\mathbf{u}})$ for some $\tilde{\mathbf{u}} \in B$. Additionally, since $\min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u})$ has a unique solution $\mathbf{u}^*$ and $\mathbf{u}^* \notin B$, we are assured that $R(\mathbf{p}^*_{\mathrm{DR}}, \tilde{\mathbf{u}}) > R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*)$. Armed with these facts, we have that

$$\begin{aligned}
&\int_{\mathcal{U}} R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u})\, dQ(\mathbf{u}) \\
&= \int_{B} R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u})\, dQ(\mathbf{u}) + \int_{\mathcal{U} \setminus B} R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u})\, dQ(\mathbf{u}) \\
&\geq R(\mathbf{p}^*_{\mathrm{DR}}, \tilde{\mathbf{u}}) \cdot Q(B) + R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) \cdot (1 - Q(B)) \\
&> R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) \cdot (Q(B) + 1 - Q(B)) \\
&= R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) \\
&= Z^*_{\mathrm{DR}},
\end{aligned}$$

which establishes that condition (EC.3) holds in Case 1.

Case 2: For every closed set $B \subseteq \mathcal{U}$, either $\mathbf{u}^* \in B$ or $Q(B) = 0$. The candidate price vector in this case will be $\tilde{\mathbf{p}}$.

To establish condition (EC.3) in this case, let ϵ be any number such that $0 < \epsilon < R(\tilde{\mathbf{p}}, \mathbf{u}^*) - R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*)$. The assumption about the continuity of $R(\mathbf{p}, \mathbf{u})$ in $\mathbf{u}$ implies that there must exist a $\delta > 0$ such that for any $\mathbf{u} \in \mathcal{U}$ with $\|\mathbf{u} - \mathbf{u}^*\| < \delta$, $R(\tilde{\mathbf{p}}, \mathbf{u}) > R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) + \epsilon$.

Let $C = \{\mathbf{u} \in \mathcal{U} \mid \|\mathbf{u} - \mathbf{u}^*\| < \delta\}$, which is an open set. Additionally, let $B = \mathcal{U} \setminus C = \{\mathbf{u} \in \mathcal{U} \mid \|\mathbf{u} - \mathbf{u}^*\| \geq \delta\}$ be the complement of C, which must be a closed set. By the assumption of Case 2, any closed subset of $\mathcal{U}$ must be such that either $\mathbf{u}^*$ is inside that set, or the measure of that set under Q is zero. Here, by construction, B cannot contain $\mathbf{u}^*$; therefore, we must have that $Q(B) = 0$. Since C and B are complements, it must also be the case that $Q(C) = 1$.

Armed with these facts, we now have that

$$
\begin{aligned}
&\int_{\mathcal{U}} R(\tilde{\mathbf{p}}, \mathbf{u})\, dQ(\mathbf{u}) \\
&= \int_C R(\tilde{\mathbf{p}}, \mathbf{u})\, dQ(\mathbf{u}) + \int_B R(\tilde{\mathbf{p}}, \mathbf{u})\, dQ(\mathbf{u}) \\
&\geq [R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) + \epsilon] \cdot Q(C) + 0 \\
&= R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) + \epsilon \\
&> R(\mathbf{p}^*_{\mathrm{DR}}, \mathbf{u}^*) \\
&= Z^*_{\mathrm{DR}},
\end{aligned}
$$

which again establishes that condition (EC.3) holds.

Since we have shown that condition (EC.3) holds in these two mutually exclusive and collectively exhaustive cases, it follows that the problem is randomization-receptive, as required. □

EC.1.6. Example of necessity of uniqueness assumption in Corollary 2

Consider a single product pricing instance, i.e., $I = 1$, which we define as follows. Let $\mathcal{P} = \{p_1, p_2, p_3\}$ where $p_1 = 5$, $p_2 = 8$, $p_3 = 9$. Let the demand model d be a linear demand model, so that the uncertain parameter $\mathbf{u} = (\alpha, \beta)$ and $d(p, \mathbf{u}) = \alpha - \beta p$. Finally, let $\mathcal{U} = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$, where $(\alpha_1, \beta_1) = (10, 1)$, $(\alpha_2, \beta_2) = (3, 0.1)$, $(\alpha_3, \beta_3) = (3.6, 0.2)$.

We first calculate $\min_{\mathbf{u} \in \mathcal{U}} R(p, \mathbf{u})$ for each $p \in \mathcal{P}$. We have:

- For $p_1 = 5$: $p_1(\alpha_2 - \beta_2 p_1) = 12.5 < p_1(\alpha_3 - \beta_3 p_1) = 13 < p_1(\alpha_1, \beta_1 p_1) = 25$. Hence, $\min_{\mathbf{u} \in \mathcal{U}} R(p_1, \mathbf{u}) = \min\{12.5, 13, 25\} = 12.5$.
- For $p_2 = 8$: $p_2(\alpha_1 - \beta_1 p_2) = p_2(\alpha_3 - \beta_3 p_2) = 16 < p_2(\alpha_2 - \beta_2 p_2) = 17.6$. Hence, $\min_{\mathbf{u} \in \mathcal{U}} R(p_2, \mathbf{u}) = \min\{16, 16, 17.6\} = 16$, and note also that the minimizing $\mathbf{u}$ is not unique (the minimum is attained at both (α_1, β_1) and (α_3, β_3)).

- For $p_3 = 9$: $p_3(\alpha_1 - \beta_1 p_3) = 9 < p_3(\alpha_3 - \beta_3 p_3) = 16.2 < p_3(\alpha_2 - \beta_2 p_3) = 18.9$. Hence, $\min_{\mathbf{u}\in\mathcal{U}} R(p_3,\mathbf{u}) = \min\{9, 16.2, 18.9\} = 9$.

From this, we can see that the optimal deterministic robust price is $p^*_{\mathrm{DR}} = p_2 = 8$ and the optimal deterministic robust objective value is $Z^*_{\mathrm{DR}} = 16$. At $p = 8$, we can see that $\arg\min_{\mathbf{u}\in\mathcal{U}} R(p_2,\mathbf{u}) = \{(\alpha_1,\beta_1),(\alpha_3,\beta_3)\}$.

Let us now consider the RRPO problem. When we write the problem $\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}} \min_{\mathbf{u}\in\mathcal{U}} \sum_{\mathbf{p}\in\mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p},\mathbf{u})$ as a linear program, we get the following problem:

$$\begin{aligned}
\underset{\boldsymbol{\pi},t}{\text{maximize}} \quad & t && \text{(EC.4a)}\\
\text{subject to} \quad & t \le \pi_{p_1}\cdot p_1(\alpha_1-\beta_1 p_1) + \pi_{p_2}\cdot p_2(\alpha_1-\beta_1 p_2) + \pi_{p_3}\cdot p_3(\alpha_1-\beta_1 p_3), && \text{(EC.4b)}\\
& t \le \pi_{p_1}\cdot p_1(\alpha_2-\beta_2 p_1) + \pi_{p_2}\cdot p_2(\alpha_2-\beta_2 p_2) + \pi_{p_3}\cdot p_3(\alpha_2-\beta_2 p_3), && \text{(EC.4c)}\\
& t \le \pi_{p_1}\cdot p_1(\alpha_3-\beta_3 p_1) + \pi_{p_2}\cdot p_2(\alpha_3-\beta_3 p_2) + \pi_{p_3}\cdot p_3(\alpha_3-\beta_3 p_3), && \text{(EC.4d)}\\
& \pi_{p_1} + \pi_{p_2} + \pi_{p_3} = 1, && \text{(EC.4e)}\\
& \pi_{p_1}, \pi_{p_2}, \pi_{p_3} \ge 0, && \text{(EC.4f)}
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\underset{\boldsymbol{\pi},t}{\text{maximize}} \quad & t && \text{(EC.5a)}\\
\text{subject to} \quad & t \le 25\pi_{p_1} + 16\pi_{p_2} + 9\pi_{p_3} && \text{(EC.5b)}\\
& t \le 12.5\pi_{p_1} + 17.6\pi_{p_2} + 18.9\pi_{p_3}, && \text{(EC.5c)}\\
& t \le 13\pi_{p_1} + 16\pi_{p_2} + 16.2\pi_{p_3}, && \text{(EC.5d)}\\
& \pi_{p_1} + \pi_{p_2} + \pi_{p_3} = 1, && \text{(EC.5e)}\\
& \pi_{p_1}, \pi_{p_2}, \pi_{p_3} \ge 0, && \text{(EC.5f)}
\end{aligned}$$

for which the optimal objective value is $Z^*_{\mathrm{RR}} = 16$, which is the same as Z^*_{DR}. Thus, if the uniqueness condition on $\min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p}^*_{\mathrm{DR}},\mathbf{u})$ is relaxed, then it is possible for the problem to be randomization proof.

EC.1.7. Proof of Proposition 1

Observe that the objective function in (32) can be re-arranged as

$$\begin{aligned}
&\max_{\boldsymbol{\mu}\in\Delta_{[I]},\mathbf{p}\in\mathcal{P}} \left\{ \sum_{i=1}^{I} \mu_i(\alpha_i + \log p_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j}\log p_j) - \sum_{i=1}^{I} \mu_i \log \mu_i \right\}\\
&= \max_{\boldsymbol{\mu}\in\Delta_{[I]}} \max_{\mathbf{p}\in\mathcal{P}} \left\{ \sum_{i=1}^{I} \mu_i\cdot\alpha_i + \sum_{i=1}^{I} \mu_i \cdot \left[1 - \beta_i + \sum_{j\neq i}\gamma_{j,i}\right]\cdot \log p_i - \sum_{i=1}^{I}\mu_i\log\mu_i \right\}\\
&= \max_{\boldsymbol{\mu}\in\Delta_{[I]}} \left[\sum_{i=1}^{I}\mu_i\cdot\alpha_i + \sum_{i=1}^{I} \max_{p_i\in\mathcal{P}_i} \left\{ \mu_i\cdot\left[1-\beta_i+\sum_{j\neq i}\gamma_{j,i}\right]\cdot\log p_i \right\} - \sum_{i=1}^{I}\mu_i\log\mu_i \right]
\end{aligned}$$

where the first step follows by algebra, and the second by the separability of the objective in $p_1, \ldots, p_I$ and Assumption 2 (since the price set is a Cartesian product and the objective is separable, each product's price can be optimized independently). Thus, when $\boldsymbol{\mu}$ is fixed, the optimal value of p_i for the above objective depends on the sign of $(1 - \beta_i + \sum_{j \neq i} \gamma_{j,i})$. If this coefficient is positive, then since $\log p_i$ is increasing in p_i, it is optimal to set $p'_i = \max \mathcal{P}_i$. If this coefficient is negative, then it is optimal to set $p'_i = \min \mathcal{P}_i$. It thus follows that for any $\boldsymbol{\mu}$ for which we can find a price vector $\mathbf{p}$ such that $(\boldsymbol{\mu}, \mathbf{p})$ is optimal, it will be the case that $(\boldsymbol{\mu}, \mathbf{p}')$ will also be optimal. $\square$

EC.2. Deterministic robust price optimization for finite $\mathcal{P}$, convex $\mathcal{U}$ under the semi-log and log-log demand models

In this section, we describe how to formulate the DRPO problem as a mixed-integer exponential cone program for the semi-log and log-log demand models. In both cases, we assume that $\mathcal{U}$ is a convex uncertainty set, and that Assumption 2 on the structure of $\mathcal{P}$ holds.

EC.2.1. Semi-log model

For the semi-log demand model, we can write the DRPO problem as

$$\begin{aligned} &\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) \\ &= \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j}. \end{aligned} \tag{EC.6}$$

To accomplish our reformulation, we will make use of the fact that the optimal solution set of the DRPO problem is unchanged upon log-transformation, that is,

$$\arg \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) = \arg \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log R(\mathbf{p}, \mathbf{u}).$$

Thus, instead of problem (EC.6), we can focus on the following problem:

$$\begin{aligned} &\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log \left(\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right) \\ &= \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log \left(\sum_{i=1}^{I} e^{\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right) \end{aligned}$$

Here, we can again use the log-sum-exp biconjugate technique to reformulate the objective function in the following way:

$$\begin{aligned} &\log \left(\sum_{i=1}^{I} e^{\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right) \\ &= \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \left\{ \sum_{i=1}^{I} \mu_i \cdot (\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) - \sum_{i=1}^{I} \mu_i \log \mu_i \right\}. \end{aligned}$$

Thus, the overall problem becomes the following max-min-max problem:

$$\max_{\mathbf{p}\in\mathcal{P}}\min_{\mathbf{u}\in\mathcal{U}}\max_{\boldsymbol{\mu}\in\Delta_{[I]}}\left\{\sum_{i=1}^{I}\mu_i\cdot(\log p_i+\alpha_i-\beta_i p_i+\sum_{j\neq i}\gamma_{i,j}p_j)-\sum_{i=1}^{I}\mu_i\log\mu_i\right\}.$$

Here, we observe that the objective function is linear in $\mathbf{u}=(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$, and is concave in $\boldsymbol{\mu}$; additionally, the feasible region of $\mathbf{u}$ is assumed to be convex, and the feasible region of $\boldsymbol{\mu}$ is convex and compact (being just the $(|I|-1)$-dimensional unit simplex). Therefore, we can use Sion's minimax theorem to interchange the minimization over $\mathbf{u}$ and the maximization over $\boldsymbol{\mu}$, which gives us

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}}\min_{\mathbf{u}\in\mathcal{U}}\max_{\boldsymbol{\mu}\in\Delta_{[I]}}\left\{\sum_{i=1}^{I}\mu_i\cdot(\log p_i+\alpha_i-\beta_i p_i+\sum_{j\neq i}\gamma_{i,j}p_j)-\sum_{i=1}^{I}\mu_i\log\mu_i\right\}\\
&=\max_{\mathbf{p}\in\mathcal{P}}\max_{\boldsymbol{\mu}\in\Delta_{[I]}}\min_{\mathbf{u}\in\mathcal{U}}\left\{\sum_{i=1}^{I}\mu_i\cdot(\log p_i+\alpha_i-\beta_i p_i+\sum_{j\neq i}\gamma_{i,j}p_j)-\sum_{i=1}^{I}\mu_i\log\mu_i\right\}\\
&=\max_{\mathbf{p}\in\mathcal{P},\boldsymbol{\mu}\in\Delta_{[I]}}\min_{\mathbf{u}\in\mathcal{U}}\left\{\sum_{i=1}^{I}\mu_i\cdot(\log p_i+\alpha_i-\beta_i p_i+\sum_{j\neq i}\gamma_{i,j}p_j)-\sum_{i=1}^{I}\mu_i\log\mu_i\right\}
\end{aligned}$$

Under Assumption 2, this final problem can then be reformulated as robust mixed-integer exponential cone program, just as in Section 5.3. We introduce the same binary decision variable $x_{i,t}$ which is 1 if product i is offered at price $t\in\mathcal{P}_i$, and 0 otherwise, and we use $w_{i,j,t}$ to denote the linearization of $\mu_i\cdot x_{j,t}$ for $i,j\in[I]$, $t\in\mathcal{P}_j$. This gives rise to the following program:

$$\underset{\boldsymbol{\mu},\mathbf{w},\mathbf{x}}{\text{maximize}}\quad \min_{\mathbf{u}\in\mathcal{U}}\left\{\sum_{i=1}^{I}\mu_i\alpha_i+\sum_{i=1}^{I}\left(\sum_{t\in\mathcal{P}_i}\log t\cdot w_{i,i,t}-\beta_i\cdot\sum_{t\in\mathcal{P}_i}t\cdot w_{i,i,t}+\sum_{j\neq i}\gamma_{i,j}\sum_{t\in\mathcal{P}_j}t\cdot w_{i,j,t}\right)-\sum_{i=1}^{I}\mu_i\log\mu_i\right\} \tag{EC.7a}$$

$$\text{subject to}\quad \sum_{t\in\mathcal{P}_j}w_{i,j,t}=\mu_i,\quad\forall\, i\in[I],\ j\in[I], \tag{EC.7b}$$

$$\sum_{i=1}^{I}w_{i,j,t}=x_{j,t},\quad\forall\, j\in[I],\ t\in\mathcal{P}_j, \tag{EC.7c}$$

$$\sum_{i=1}^{I}\mu_i=1, \tag{EC.7d}$$

$$\sum_{t\in\mathcal{P}_i}x_{i,t}=1,\quad\forall\, i\in[I], \tag{EC.7e}$$

$$w_{i,j,t}\geq 0,\quad\forall\, i\in[I],\ j\in[I],t\in\mathcal{P}_j, \tag{EC.7f}$$

$$x_{i,t}\in\{0,1\},\quad\forall\, i\in[I],\ t\in\mathcal{P}_i, \tag{EC.7g}$$

$$\mu_i\geq 0,\quad\forall\, i\in[I]. \tag{EC.7h}$$

Note that the feasible region of this problem is identical to that of problem (25), which appeared in our discussion of the separation problem for the RRPO problem when $\mathcal{U}$ is convex and $\mathcal{P}$ is finite. The difference here is that the objective is now a robust objective; it is the worst-case value of the objective of problem (25), taken over the convex uncertainty set $\mathcal{U}$. Depending on the structure of $\mathcal{U}$, the overall problem can remain in the mixed-integer convex program problem class. For example, if $\mathcal{U}$ is a polyhedron, then one can use LP duality to reformulate the robust problem exactly by introducing additional variables and constraints, as is normally done in robust optimization (Bertsimas and Sim 2004, Ben-Tal and Nemirovski 2000, Bertsimas et al. 2011). Similarly, if $\mathcal{U}$ is a second-order cone representable set, then one can again use conic duality to reformulate the problem. Alternatively, one can also consider solving the problem using a cutting plane method/delayed constraint generation approach, whereby one reformulates the program in epigraph form and then solves the inner minimization over $\mathbf{u}$ to identify new cuts to add (Bertsimas et al. 2016a).

EC.2.2. Log-log model

For the log-log demand model, we can write the DRPO problem as

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} R(\mathbf{p},\mathbf{u}) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i}\gamma_{i,j}\log p_j} \\
&= \max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} \sum_{i=1}^{I} e^{\alpha_i + (1-\beta_i)\log p_i + \sum_{j\neq i}\gamma_{i,j}\log p_j}
\end{aligned}$$

Again, as with the semi-log model, solving the above problem is equivalent to solving the same problem with a log-transformed objective. Taking this log-transformed problem as our starting point, replacing the log-sum-exp function with its biconjugate and applying Sion's minimax theorem gives us:

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} \log\left(\sum_{i=1}^{I} e^{\alpha_i + (1-\beta_i)\log p_i + \sum_{j\neq i}\gamma_{i,j}\log p_j}\right) \\
&= \max_{\mathbf{p}\in\mathcal{P}} \min_{\mathbf{u}\in\mathcal{U}} \max_{\boldsymbol{\mu}\in\Delta_{[I]}} \left\{\sum_{i=1}^{I}\left[\alpha_i\mu_i + \sum_{i=1}^{I}(1-\beta_i)\mu_i\cdot\log p_i + \sum_{j\neq i}\gamma_{i,j}\mu_i\cdot\log p_j\right] - \sum_{i=1}^{I}\mu_i\log\mu_i\right\} \\
&= \max_{\mathbf{p}\in\mathcal{P},\boldsymbol{\mu}\in\Delta_{[I]}} \min_{\mathbf{u}\in\mathcal{U}} \left\{\sum_{i=1}^{I}\left[\alpha_i\mu_i + \sum_{i=1}^{I}(1-\beta_i)\mu_i\cdot\log p_i + \sum_{j\neq i}\gamma_{i,j}\mu_i\cdot\log p_j\right] - \sum_{i=1}^{I}\mu_i\log\mu_i\right\}.
\end{aligned}$$

Under Assumption 2, this last problem can be re-written as the following robust version of problem (33), with the decision variables defined identically:

$$\underset{\boldsymbol{\mu},\mathbf{w},\mathbf{x}}{\text{maximize}} \quad \min_{\mathbf{u}\in\mathcal{U}} \left\{\sum_{i=1}^{I}\mu_i\alpha_i + \sum_{i=1}^{I}\left(\sum_{t\in\mathcal{P}_i}\log t\cdot w_{i,i,t} - \beta_i\cdot\sum_{t\in\mathcal{P}_i}\log t\cdot w_{i,i,t} + \sum_{j\neq i}\gamma_{i,j}\sum_{t\in\mathcal{P}_j}\log t\cdot w_{i,j,t}\right) - \sum_{i=1}^{I}\mu_i\log\mu_i\right\} \tag{EC.8a}$$

$$\text{subject to} \quad \text{constraints (25b) – (25h).} \tag{EC.8b}$$

Again, this problem has exactly the same feasible region as the log-log separation problem (33) and the semi-log separation problem (25). Additionally, just as with the deterministic robust problem (EC.7) for the semi-log model, this problem can be further reformulated by exploiting the structure of $\mathcal{U}$, or otherwise one can design a cutting plane method that generates violated uncertain parameter vectors $\mathbf{u}\in\mathcal{U}$ on the fly.

EC.3. Solution method for finite $\mathcal{P}$, finite $\mathcal{U}$

The second solution approach we consider is for the case where both $\mathcal{P}$ and $\mathcal{U}$ are finite sets. In particular, we assume that the uncertainty set $\mathcal{U}$ is a binary representable set. For fixed positive integers m and n, we let $\mathcal{U}$ be defined as

$$\mathcal{U}=\{\mathbf{u}=\mathbf{F}\mathbf{z}\mid \mathbf{A}\mathbf{z}\leq \mathbf{b},\mathbf{z}\in\{0,1\}^n\}, \tag{EC.9}$$

where $\mathbf{b}$ is a m dimensional real vector, $\mathbf{A}$ is a m-by-n real matrix and $\mathbf{F}$ is a d-by-n real matrix, where d is the dimension of the uncertain parameter vector $\mathbf{u}$.

Recall that when $\mathcal{P}$ is finite, then the RRPO problem is

$$Z^*_{\mathrm{RR}}=\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}}\min_{\mathbf{u}\in\mathcal{U}}\sum_{\mathbf{p}\in\mathcal{P}}\pi_{\mathbf{p}}R(\mathbf{p},\mathbf{u}). \tag{EC.10}$$

We can transform this problem into a dual problem where the outer problem is to optimize a distribution over uncertain parameter vectors, and the inner problem is to optimize over the price vector, as follows:

$$\begin{aligned}
Z^*_{\mathrm{RR}}&=\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}}\min_{\mathbf{u}\in\mathcal{U}}\sum_{\mathbf{p}\in\mathcal{P}}\pi_{\mathbf{p}}R(\mathbf{p},\mathbf{u}) && \text{(EC.11)}\\
&=\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}}\min_{\boldsymbol{\lambda}\in\Delta_{\mathcal{U}}}\sum_{\mathbf{p}\in\mathcal{P}}\sum_{\mathbf{u}\in\mathcal{U}}\pi_{\mathbf{p}}\lambda_{\mathbf{u}}R(\mathbf{p},\mathbf{u}) && \text{(EC.12)}\\
&=\min_{\boldsymbol{\lambda}\in\Delta_{\mathcal{U}}}\max_{\boldsymbol{\pi}\in\Delta_{\mathcal{P}}}\sum_{\mathbf{p}\in\mathcal{P}}\sum_{\mathbf{u}\in\mathcal{U}}\pi_{\mathbf{p}}\lambda_{\mathbf{u}}R(\mathbf{p},\mathbf{u}) && \text{(EC.13)}\\
&=\min_{\boldsymbol{\lambda}\in\Delta_{\mathcal{U}}}\max_{\mathbf{p}\in\mathcal{P}}\sum_{\mathbf{u}\in\mathcal{U}}\lambda_{\mathbf{u}}R(\mathbf{p},\mathbf{u}), && \text{(EC.14)}
\end{aligned}$$

where the first equality follows because minimization of a function of $\mathbf{u}$ over the finite set $\mathcal{U}$ is the same as minimizing the expected value of that function over all probability mass functions supported on $\mathcal{U}$; the second equality follows by linear programming duality; and the final equality follows because maximization of a function of $\mathbf{p}$ over $\mathcal{P}$ is the same as maximizing the expected value of that function over all probability mass functions supported on $\mathcal{P}$. We refer to problem (EC.10) as the *primal* problem and (EC.14) as the *dual* problem.

Consider now the *restricted primal problem*, where we replace $\mathcal{P}$ with a subset $\hat{\mathcal{P}} \subseteq \mathcal{P}$ in problem (EC.10), and the *restricted dual problem*, where we replace $\mathcal{U}$ with a subset $\hat{\mathcal{U}} \subseteq \mathcal{U}$ in problem (EC.14). Let us denote the objective values of these two problems with $Z_{P,\hat{\mathcal{P}}}$ and $Z_{D,\hat{\mathcal{U}}}$, respectively. These two problems are:

$$Z_{P,\hat{\mathcal{P}}} = \max_{\boldsymbol{\pi} \in \Delta_{\hat{\mathcal{P}}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \tag{EC.15}$$

$$Z_{D,\hat{\mathcal{U}}} = \min_{\boldsymbol{\lambda} \in \Delta_{\hat{\mathcal{U}}}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}). \tag{EC.16}$$

Observe that $Z_{P,\hat{\mathcal{P}}}$ and $Z_{D,\hat{\mathcal{U}}}$ bound Z^*_{RR} from below and above, that is,

$$Z_{P,\hat{\mathcal{P}}} \le Z^*_{\mathrm{RR}} \le Z_{D,\hat{\mathcal{U}}}.$$

In the above, the justification for the first inequality is because maximizing over distributions supported on the smaller set of price vectors $\hat{\mathcal{P}}$ cannot result in a higher worst-case objective than solving the full primal problem with $\mathcal{P}$, which gives the value Z^*_{RR}. The second inequality similarly follows because minimizing over distributions supported on the smaller set of uncertainty realizations $\hat{\mathcal{U}}$ cannot result in a lower worst-case objective than solving the full dual problem with $\mathcal{U}$, which gives Z^*_{RR}.

The idea of double column generation is as follows. Let us pick some subset of price vectors $\hat{\mathcal{P}} \subseteq \mathcal{P}$ and some subset of uncertainty realizations $\hat{\mathcal{U}} \subseteq \mathcal{U}$. Observe that the restricted primal problem (EC.15) for $\hat{\mathcal{P}}$ can be written in epigraph form as

$$\begin{aligned}
\underset{\boldsymbol{\pi}, t}{\text{maximize}} \quad & t && \text{(EC.17a)}\\
\text{subject to} \quad & t \le \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad \forall\, \mathbf{u} \in \mathcal{U}, && \text{(EC.17b)}\\
& \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} = 1, && \text{(EC.17c)}\\
& \pi_{\mathbf{p}} \ge 0, \quad \forall\, \mathbf{p} \in \hat{\mathcal{P}}. && \text{(EC.17d)}
\end{aligned}$$

This problem has a huge number of constraints (one for each $\mathbf{u} \in \mathcal{U}$). However, we can solve it using delayed constraint generation, starting from the set $\hat{\mathcal{U}}$. Upon solving it in this way, at termination we will have a subset $\mathcal{U}'$ of uncertainty realizations from $\mathcal{U}$ that were found during the constraint generation process. We update $\hat{\mathcal{U}}$ to be equal to $\mathcal{U}'$.

With this (updated) subset $\hat{\mathcal{U}}$ in hand, we now solve the restricted dual problem (EC.16) for $\hat{\mathcal{U}}$. This problem can be written in epigraph form as

$$\underset{\boldsymbol{\lambda}, \rho}{\text{minimize}} \quad \rho \tag{EC.18a}$$

$$\text{subject to} \quad \rho \geq \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} R(\mathbf{p},\mathbf{u}), \quad \forall\ \mathbf{p}\in\mathcal{P}, \tag{EC.18b}$$

$$\sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} = 1, \tag{EC.18c}$$

$$\lambda_{\mathbf{u}} \geq 0, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}}. \tag{EC.18d}$$

This problem also has a huge number of constraints, but again we can solve it using delayed constraint generation, with the initial subset of price vectors set to $\hat{\mathcal{P}}$. At termination, we will have a new subset $\mathcal{P}'$ of price vectors, which will contain the original set of price vectors in $\hat{\mathcal{P}}$. We then update $\hat{\mathcal{P}}$ to $\mathcal{P}'$, and go back to solving the restricted primal problem. The process then repeats: after solving the restricted primal, we will have a new (bigger) $\hat{\mathcal{U}}$; we then solve the restricted dual, after which we have a new (bigger) $\hat{\mathcal{P}}$; we then go back to the restricted primal, and so on. After each iteration of solving the restricted primal and restricted dual, the set $\hat{\mathcal{P}}$ expands and the set $\hat{\mathcal{U}}$ expands. Thus, the bounds $Z_{P,\hat{\mathcal{P}}}$ and $Z_{D,\hat{\mathcal{U}}}$ get closer and closer to Z^*_{RR}. The algorithm can then be terminated either when $Z_{P,\hat{\mathcal{P}}} = Z_{D,\hat{\mathcal{U}}}$, which would imply that both restricted primal and restricted dual objective values exactly coincide with Z^*_{RR}; or otherwise, one can terminate when $Z_{D,\hat{\mathcal{U}}} - Z_{P,\hat{\mathcal{P}}} < \epsilon$, where $\epsilon > 0$ is a user specified tolerance.

The overall algorithmic approach is formalized as Algorithm 1. This algorithm invokes two procedures, PrimalCG (Algorithm 2) and DualCG (Algorithm 3), which are delayed constraint generation algorithms for solving the restricted primal and dual problems respectively. We note that Algorithm 1 is an adaptation of the double column generation algorithm of Wang et al. (2020) for the randomized robust assortment optimization problem, which is itself adapted from the double column generation algorithm of Delage and Saif (2022) for solving mixed-integer distributionally robust optimization problems. The proof of correctness of this procedure follows similarly to Delage and Saif (2022), and is omitted for brevity. The novelty in our approach lies in how we handle the separation problems which are at the heart of PrimalCG and DualCG, which we discuss next.

Note that the doubly restricted primal and dual problems (EC.19) and (EC.21) solved in PrimalCG and DualCG are both linear programs, and can be thus be solved easily. The principal difficulty in these procedures comes from the primal and dual separation problems (EC.20) and (EC.22), which require optimizing over a price vector $\mathbf{p}\in\mathcal{P}$ and an uncertain parameter vector $\mathbf{u}\in\mathcal{U}$ respectively. In the following sections, we discuss how these two separation problems can be tackled for the linear, semi-log and log-log demand models. Note that in all three sections, we continue to make Assumption 2, which states that $\mathcal{P}$ can be written as the Cartesian product of finite sets of prices for each product, i.e., $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_I$, where $\mathcal{P}_1, \ldots, \mathcal{P}_I$ are finite sets.

Algorithm 1 Double column generation method for solving the finite $\mathcal{P}$, finite $\mathcal{U}$ RRPO problem.

Initialize $\hat{\mathcal{P}}$ to be a non-empty subset of $\mathcal{P}$, and $\hat{\mathcal{U}}$ to be a non-empty subset of $\mathcal{U}$.
Set LB $\leftarrow -\infty$, UB $\leftarrow +\infty$
repeat
  Run PrimalCG($\hat{\mathcal{P}}$,$\hat{\mathcal{U}}$) to solve the restricted primal problem with $\hat{\mathcal{P}}$ and with $\hat{\mathcal{U}}$ as the initial uncertainty set. Let the objective value be $Z_{P,\hat{\mathcal{P}}}$ and the new uncertainty set be $\mathcal{U}'$.
  Set $\hat{\mathcal{U}} \leftarrow \mathcal{U}'$.
  Set LB $\leftarrow Z_{P,\hat{\mathcal{P}}}$.
  Run DualCG($\hat{\mathcal{P}}$,$\hat{\mathcal{U}}$) to solve the restricted dual problem with $\hat{\mathcal{U}}$ and with $\hat{\mathcal{P}}$ as the initial price vector set. Let the objective value be $Z_{D,\hat{\mathcal{P}}}$ and the new price vector set be $\mathcal{P}'$.
  Set $\hat{\mathcal{P}} \leftarrow \mathcal{P}'$.
  Set UB $\leftarrow Z_{D,\hat{\mathcal{U}}}$.
until UB $-$ LB $< \epsilon$

EC.3.1. Primal and dual subproblems for linear demand model

For the linear demand model, the primal separation problem is

$$\min_{\mathbf{u}\in\mathcal{U}} \sum_{\mathbf{p}\in\hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^{I} p_i \cdot (\alpha_i - \beta_i p_i + \sum_{j\neq i} \gamma_{i,j} p_j) \right]. \tag{EC.23}$$

Note that this objective function is linear in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Therefore, the whole problem can be expressed as

$$\begin{aligned}
\text{minimize} \quad & \sum_{\mathbf{p}\in\hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^{I} p_i \cdot (\alpha_i - \beta_i p_i + \sum_{j\neq i} \gamma_{i,j} p_j) \right] && \text{(EC.24a)}\\
\text{subject to} \quad & \mathbf{u} = \mathbf{F}\mathbf{z}, && \text{(EC.24b)}\\
& \mathbf{A}\mathbf{z} \leq \mathbf{z}, && \text{(EC.24c)}\\
& \mathbf{z} \in \{0,1\}^n, && \text{(EC.24d)}
\end{aligned}$$

which is a mixed-integer linear program.

The dual separation problem is

$$\max_{\mathbf{p}\in\mathcal{P}} \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^{I} p_i(\alpha_i - \beta_i p_i + \sum_{j\neq i} \gamma_{i,j} p_j) \right]. \tag{EC.25}$$

Algorithm 2 PrimalCG procedure.

Initialize $\mathcal{U}' \leftarrow \hat{\mathcal{U}}$

repeat

  Solve the doubly restricted primal problem:

$$\begin{aligned}
\underset{\boldsymbol{\pi}, t}{\text{maximize}} \quad & t && \text{(EC.19a)}\\
\text{subject to} \quad & t \leq \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{U}', && \text{(EC.19b)}\\
& \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} = 1, && \text{(EC.19c)}\\
& \pi_{\mathbf{p}} \geq 0, \quad \forall\ \mathbf{p} \in \hat{\mathcal{P}}. && \text{(EC.19d)}
\end{aligned}$$

  Let $(\boldsymbol{\pi}, t^*)$ be the optimal solution of the doubly restricted problem.

  Solve the primal separation problem:

$$\min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot R(\mathbf{p}, \mathbf{u}). \tag{EC.20}$$

  Let t' and $\mathbf{u}^*$ be the optimal objective value and solution of this separation problem.

  if $t^* > t'$ **then**

    Set $\mathcal{U}' \leftarrow \mathcal{U}' \cup \{\mathbf{u}^*\}$

  end if

until $t^* \leq t'$

Set $Z_{P,\hat{\mathcal{P}}} \leftarrow t^*$

return $(Z_{P,\hat{\mathcal{P}}}, \mathcal{U}')$.

By introducing the same binary variables as in the separation problem (20) (the linear demand separation problem for the convex $\mathcal{U}$ setting), we obtain the following mixed-integer linear program:

$$\begin{aligned}
\underset{\mathbf{x}, \mathbf{y}}{\text{maximize}} \quad & \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^{I} \sum_{t \in \mathcal{P}_i} \alpha_i \cdot t \cdot x_{i,t} + \sum_{i=1}^{I} \sum_{t \in \mathcal{P}_i} t \cdot \beta_i \cdot x_{i,t} + \sum_{i=1}^{I} \sum_{j \neq i} \sum_{t_1 \in \mathcal{P}_i} \sum_{t_2 \in \mathcal{P}_j} \gamma_{i,j} \cdot t_1 \cdot t_2 \cdot y_{i,j,t_1,t_2} \right] && \text{(EC.26a)}\\
\text{subject to} \quad & \sum_{t \in \mathcal{P}_i} x_{i,t} = 1, \quad \forall\ i \in [I], && \text{(EC.26b)}\\
& \sum_{t_2 \in \mathcal{P}_j} y_{i,j,t_1,t_2} = x_{i,t_1}, \quad \forall\ i, j \in [I], j \neq i, t_2 \in \mathcal{P}_j, && \text{(EC.26c)}\\
& \sum_{t_1 \in \mathcal{P}_i} y_{i,j,t_1,t_2} = x_{i,t_1}, \quad \forall\ i, j \in [I], j \neq i, t_1 \in \mathcal{P}_i, && \text{(EC.26d)}\\
& x_{i,t} \in \{0, 1\}, \quad \forall\ i \in [I],\ t \in \mathcal{P}_i, && \text{(EC.26e)}
\end{aligned}$$

Algorithm 3 DUALCG procedure.

Initialize $\mathcal{P}' \leftarrow \hat{\mathcal{P}}$

repeat

  Solve the doubly restricted dual problem:

$$\begin{aligned} \underset{\boldsymbol{\lambda},\rho}{\text{minimize}} \quad & \rho & \text{(EC.21a)} \\ \text{subject to} \quad & \rho \geq \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} R(\mathbf{p},\mathbf{u}), \quad \forall \mathbf{p}\in\mathcal{P}', & \text{(EC.21b)} \\ & \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} = 1, & \text{(EC.21c)} \\ & \lambda_{\mathbf{u}} \geq 0, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}}. & \text{(EC.21d)} \end{aligned}$$

  Let $(\boldsymbol{\lambda}, \rho^*)$ be the optimal solution of the doubly restricted problem.

  Solve the dual separation problem:

$$\max_{\mathbf{p}\in\mathcal{P}} \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot R(\mathbf{p},\mathbf{u}) \tag{EC.22}$$

  Let ρ' and $\mathbf{p}^*$ be the optimal objective value and solution of this separation problem.

  if $\rho^* < \rho'$ **then**

    Set $\mathcal{P}' \leftarrow \mathcal{P}' \cup \{\mathbf{p}^*\}$

  end if

until $\rho^* \geq \rho'$

Set $Z_{D,\hat{\mathcal{U}}} \leftarrow \rho^*$

return $(Z_{D,\hat{\mathcal{U}}}, \mathcal{P}')$.

$$x_{i,j,t_1,t_2} \in \{0,1\}, \quad \forall\ i,j\in[I], i\neq j, t_1\in\mathcal{P}_i, t_2\in\mathcal{P}_j. \tag{EC.26f}$$

Importantly, note that the size of this problem does not scale with the number of uncertainty realizations inside $\hat{\mathcal{U}}$; the form of this problem is equivalent to problem (20) where $\mathbf{u}$ is replaced with $\sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \mathbf{u}$ (the "average" uncertain demand parameter). As we will see in the next couple of sections, the same will not be true for the semi-log and log-log demand models.

EC.3.2. Primal and dual subproblems for semi-log demand model

For the semi-log demand model, the primal separation problem is

$$\min_{\mathbf{u}\in\mathcal{U}} \sum_{\mathbf{p}\in\hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j\neq i}\gamma_{i,j}p_j} \right]. \tag{EC.27}$$

Note that this objective function is convex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, because the weights $\pi_{\mathbf{p}}$ and p_i for a given $\mathbf{p} \in \mathcal{P}$ and $i \in [I]$ are nonnegative, and because the function $e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j}$ is convex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Thus, the whole problem can be expressed as

$$\begin{aligned}
\underset{\mathbf{u},\mathbf{z}}{\text{minimize}} \quad & \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right] && \text{(EC.28a)} \\
\text{subject to} \quad & \mathbf{u} = \mathbf{F}\mathbf{z}, && \text{(EC.28b)} \\
& \mathbf{A}\mathbf{z} \leq \mathbf{b}, && \text{(EC.28c)} \\
& \mathbf{z} \in \{0,1\}^n, && \text{(EC.28d)}
\end{aligned}$$

which can be re-written as a mixed-integer exponential cone program.

The dual separation problem is

$$\max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right]. \tag{EC.29}$$

The objective function of this problem is in general not concave in $\mathbf{p}$. However, just as in Section 5.3, the related problem of optimizing the logarithm of this objective, which is

$$\begin{aligned}
& \max_{\mathbf{p} \in \mathcal{P}} \log \left[\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right] \right] && \text{(EC.30)} \\
& = \max_{\mathbf{p} \in \mathcal{P}} \log \left[\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^{I} e^{\log \lambda_{\mathbf{u}} + \log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right] && \text{(EC.31)}
\end{aligned}$$

can be reformulated as a mixed-integer exponential cone program using the same biconjugate-based technique in Section 5.3. In particular, when Assumption 2 holds, then problem (EC.31) is equivalent to

$$
\begin{aligned}
\underset{\mathbf{w},\mathbf{x},\boldsymbol{\mu}}{\text{maximize}} \quad & \sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\mu_{\mathbf{u},i}\cdot(\log\lambda_{\mathbf{u}}+\alpha_i) \\
& +\sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\sum_{t\in\mathcal{P}_i}\log t w_{\mathbf{u},i,i,t} \\
& +\sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\sum_{t\in\mathcal{P}_i}(-\beta_i)\cdot t\cdot w_{\mathbf{u},i,i,t} \\
& +\sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\sum_{j\neq i}\gamma_{i,j}\cdot\sum_{t\in\mathcal{P}_j}t\cdot w_{\mathbf{u},i,j,t} \\
& -\sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\mu_{\mathbf{u},i}\log\mu_{\mathbf{u},i} && \text{(EC.32a)} \\
\text{subject to} \quad & \sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\mu_{\mathbf{u},i}=1, && \text{(EC.32b)} \\
& \sum_{t\in\mathcal{P}_j}w_{\mathbf{u},i,j,t}=\mu_{\mathbf{u},i}, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}},\ i,j\in[I], && \text{(EC.32c)} \\
& \sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}w_{\mathbf{u},i,j,t}=x_{j,t}, \quad \forall\ j\in[I],\ t\in\mathcal{P}_j, && \text{(EC.32d)} \\
& \sum_{t\in\mathcal{P}_j}x_{j,t}=1, \quad \forall j\in[I], && \text{(EC.32e)} \\
& w_{\mathbf{u},i,j,t}\geq 0, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}},\ i,j\in[I],t\in\mathcal{P}_j, && \text{(EC.32f)} \\
& \mu_{\mathbf{u},i}\geq 0, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}},\ i\in[I], && \text{(EC.32g)} \\
& x_{j,t}\in\{0,1\}, \quad \forall\ j\in[I],\ t\in\mathcal{P}_j, && \text{(EC.32h)}
\end{aligned}
$$

where $x_{j,t}$ is a binary decision variable that is 1 if product j is offered at price $t\in\mathcal{P}_j$, and 0 otherwise; $\mu_{\mathbf{u},i}$ is a nonnegative decision variable introduced as part of the biconjugate-based reformulation; and $w_{\mathbf{u},i,j,t}$ is a decision variable that represents the linearization of $\mu_{\mathbf{u},i}\cdot x_{j,t}$ for all $\mathbf{u}\in\hat{\mathcal{U}}$, $i,j\in[I]$, and $t\in\mathcal{P}_j$.

As with problem (25), this problem can be expressed as a mixed-integer exponential cone program. One notable difference between formulation (EC.32) and formulation (25) from earlier is that the number of decision variables and constraints is larger because the decision variable $\mu_{\mathbf{u},i}$ is introduced for every combination of an uncertainty realization in $\hat{\mathcal{U}}$ and each product i; thus, $\boldsymbol{\mu}$ represents a probability mass function over the set $\hat{\mathcal{U}}\times[I]$.

EC.3.3. Primal and dual subproblems for log-log demand model

For the log-log demand model, the primal separation problem is

$$\min_{\mathbf{u}\in\mathcal{U}} \sum_{\mathbf{p}\in\hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j}. \tag{EC.33}$$

Note that the objective function is convex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$; it is the nonnegative weighted combination of terms of the form $e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j}$, each of which are convex in $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Thus, the overall problem, which can be stated as

$$\begin{aligned}
\underset{\mathbf{z}}{\text{minimize}} \quad & \sum_{\mathbf{p}\in\hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j} && \text{(EC.34a)}\\
\text{subject to} \quad & \mathbf{u} = \mathbf{F}\mathbf{z}, && \text{(EC.34b)}\\
& \mathbf{A}\mathbf{z} \leq \mathbf{b}, && \text{(EC.34c)}\\
& \mathbf{z} \in \{0,1\}^n, && \text{(EC.34d)}
\end{aligned}$$

is a mixed-integer convex program, and can be expressed as a mixed-integer exponential cone program.

The dual separation problem is

$$\max_{\mathbf{p}\in\mathcal{P}} \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j} \right]. \tag{EC.35}$$

The objective function of this problem is in general not concave in $\mathbf{p}$. However, following the same method as in Section 5.4, we can reformulate the related problem of maximizing the logarithm, which is

$$\begin{aligned}
&\max_{\mathbf{p}\in\mathcal{P}} \log \left(\sum_{\mathbf{u}\in\hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^{I} p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j} \right] \right) && \text{(EC.36)}\\
&= \max_{\mathbf{p}\in\mathcal{P}} \log \left(\sum_{\mathbf{u}\in\hat{\mathcal{U}}} \sum_{i=1}^{I} e^{\log \lambda_{\mathbf{u}} + \log p_i + \alpha_i - \beta_i \log p_i + \sum_{j\neq i} \gamma_{i,j} \log p_j} \right) && \text{(EC.37)}
\end{aligned}$$

as a mixed-integer exponential cone program. Under Assumption 2, the resulting formulation is

$$\begin{aligned}
\underset{\mathbf{x},\mathbf{w},\boldsymbol{\mu}}{\text{maximize}} \quad & \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \sum_{i=1}^{I} \mu_{\mathbf{u},i} \cdot (\log \lambda_{\mathbf{u}} + \alpha_i)\\
& + \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \sum_{i=1}^{I} \sum_{t\in\mathcal{P}_i} (1-\beta_i) \log t \cdot w_{\mathbf{u},i,i,t}\\
& + \sum_{\mathbf{u}\in\hat{\mathcal{U}}} \sum_{i=1}^{I} \sum_{j\neq i} \gamma_{i,j} \cdot \sum_{t\in\mathcal{P}_j} \log t \cdot w_{\mathbf{u},i,j,t}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\mu_{\mathbf{u},i}\log\mu_{\mathbf{u},i} && \text{(EC.38a)}\\
\text{subject to}\quad & \sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I}\mu_{\mathbf{u},i}=1, && \text{(EC.38b)}\\
& \sum_{t\in\mathcal{P}_j} w_{\mathbf{u},i,j,t}=\mu_{\mathbf{u},i}, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}},\ i,j\in[I], && \text{(EC.38c)}\\
& \sum_{\mathbf{u}\in\hat{\mathcal{U}}}\sum_{i=1}^{I} w_{\mathbf{u},i,j,t}=x_{j,t}, \quad \forall\ j\in[I],\ t\in\mathcal{P}_j, && \text{(EC.38d)}\\
& \sum_{t\in\mathcal{P}_j} x_{j,t}=1, \quad \forall j\in[I], && \text{(EC.38e)}\\
& w_{\mathbf{u},i,j,t}\geq 0, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}},\ i,j\in[I], t\in\mathcal{P}_j, && \text{(EC.38f)}\\
& \mu_{\mathbf{u},i}\geq 0, \quad \forall\ \mathbf{u}\in\hat{\mathcal{U}},\ i\in[I], && \text{(EC.38g)}\\
& x_{j,t}\in\{0,1\}, \quad \forall\ j\in[I],\ t\in\mathcal{P}_j, && \text{(EC.38h)}
\end{aligned}
$$

where the decision variables have the same meaning as those in formulation (EC.32).

EC.4. Additional numerical results

EC.4.1. Estimation results for `orangeJuice` data set

Tables EC.1 and EC.2 display the point estimates of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ for the semi-log and log-log demand models for the `orangeJuice` data set.

	Product										
Parameters	1	2	3	4	5	6	7	8	9	10	11
$\boldsymbol{\alpha}_{\text{semi-log}}$	9.873	9.829	8.598	9.504	9.024	9.828	8.582	7.901	7.152	11.161	10.896
$\boldsymbol{\beta}_{\text{semi-log}}$	1.0222	0.4581	1.2735	1.7888	1.3354	0.6507	1.6491	1.3945	2.0809	1.6290	0.0383
$\boldsymbol{\alpha}_{\text{log-log}}$	10.140	10.956	8.266	8.421	9.045	10.613	7.832	7.127	6.563	11.326	11.198
$\boldsymbol{\beta}_{\text{log-log}}$	2.7195	2.0410	3.3037	3.8855	2.9357	2.6101	3.6063	2.8209	3.9717	2.7942	0.1542

Table EC.1 Estimation results for α and β.

EC.4.2. Performance results for `orangeJuice` data set

Tables EC.3 and EC.4 below compare the performance of the nominal, deterministic robust and randomized robust pricing solutions under a discrete budget uncertainty set for the `orangeJuice` data set.

$\gamma_{\text{semi-log}}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$
$i=1$	–	0.0571	0.0813	0.0966	0.0193	-0.0232	0.1305	0.1904	0.1490	0.0582	0.0815
$i=2$	0.1384	–	0.0041	0.0009	0.0204	0.0153	0.0090	0.1040	-0.0023	0.0491	0.0394
$i=3$	0.3386	0.0916	–	0.1943	0.0702	-0.0062	0.0051	0.0950	-0.0310	0.0690	0.0950
$i=4$	0.4313	0.0976	-0.1112	–	0.4089	0.3518	0.2085	-0.0777	-0.0352	0.0383	-0.2290
$i=5$	0.1916	0.0490	0.3026	0.2966	–	-0.1538	0.1547	-0.0314	0.1034	0.3338	0.0370
$i=6$	0.0211	0.0493	-0.0194	-0.0018	0.0888	–	0.0340	0.0472	-0.0167	0.0297	0.1119
$i=7$	0.2007	0.0388	0.0706	0.0672	0.3233	0.0837	–	0.0377	0.2216	-0.0504	0.1405
$i=8$	0.0117	0.0119	0.0932	0.0757	0.1023	-0.0160	0.1345	–	0.1372	0.2143	0.2699
$i=9$	0.0955	0.0373	-0.0211	0.3651	0.4176	0.0358	0.2127	0.1462	–	0.2337	0.1627
$i=10$	0.0412	-0.3941	0.0764	0.4867	0.4810	0.0109	-0.0814	-0.1047	0.0878	–	0.0274
$i=11$	-0.0893	-0.1587	-0.1358	-0.0252	-0.0690	0.0079	-0.0574	-0.1117	-0.1271	0.0809	–
$\gamma_{\text{log-log}}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$
$i=1$	–	0.2196	0.1631	0.2129	0.0646	-0.0577	0.2576	0.3338	0.2494	0.0621	0.2939
$i=2$	0.3474	–	0.0403	0.0004	0.0338	0.0492	0.0193	0.1879	-0.0042	0.0739	0.1257
$i=3$	0.8673	0.5123	–	0.4400	0.1482	-0.0338	0.0527	0.1480	-0.1001	0.1267	0.3683
$i=4$	1.1581	0.3822	-0.2283	–	0.8367	1.2659	0.4495	-0.1569	-0.0115	0.1003	-0.7321
$i=5$	0.4624	0.2241	0.8344	0.6406	–	-0.6800	0.3223	0.0646	0.1426	0.5815	0.1782
$i=6$	0.0462	0.2424	-0.0343	-0.0173	0.2086	–	0.0975	0.1187	-0.0364	0.0561	0.4159
$i=7$	0.4644	0.2531	0.0971	0.1204	0.6997	0.2946	–	0.1728	0.4912	-0.0564	0.4497
$i=8$	0.0652	0.1430	0.1980	0.1587	0.2705	-0.0988	0.3198	–	0.3034	0.2992	0.9436
$i=9$	0.2971	-0.1190	-0.0216	0.7986	0.8825	0.3045	0.5869	0.1706	0.0	0.3171	0.4223
$i=10$	0.1406	-1.7987	0.1061	1.0453	1.0852	0.0811	-0.1213	-0.1760	0.0454	–	-0.0371
$i=11$	-0.2246	-0.7519	-0.3177	-0.0483	-0.1635	-0.0570	-0.1100	-0.1780	-0.2646	0.1341	–

Table EC.2 Estimation results for γ for `orangeJuice` data set.

Γ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\text{RR}}, \mathbf{u}_0)]$	t_{DR}	$\hat{Z}_{\text{DR}}$	RI(%)	$R(\hat{\mathbf{p}}_{\text{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\text{N,WC}}$
5	14.64	162753.97	290939.28	225.36	102626.41	58.59	260321.17	0.81	590547.01	85304.36
10	6.29	70401.48	404458.22	208.42	47969.46	46.76	350396.25	–	–	38815.61
15	3.93	39567.50	349936.30	209.14	32757.43	20.79	334211.84	–	–	22798.44
20	16.70	31438.76	328664.19	197.37	25348.77	24.02	299970.20	–	–	15940.16

Table EC.3 Results for `orangeJuice` pricing problem with semi-log demand and discrete $\mathcal{U}$.

Γ	t_{RR}	Z^*_{RR}	$\mathbb{E}[R(\mathbf{p}^*_{\text{RR}}, \mathbf{u}_0)]$	t_{DR}	$\hat{Z}_{\text{DR}}$	RI(%)	$R(\hat{\mathbf{p}}_{\text{DR}}, \mathbf{u}_0)$	t_{N}	Z^*_{N}	$Z_{\text{N,WC}}$
5	12.85	272399.89	605265.00	306.63	174478.12	56.12	811254.69	0.87	1110000.00	117186.58
10	10.41	135761.31	750084.92	260.96	77297.42	75.63	896972.12	–	–	50458.15
15	15.59	72930.45	761785.89	193.13	44914.20	62.38	896972.12	–	–	27389.15
20	8.56	45153.74	770505.32	190.76	27675.07	63.16	409330.70	–	–	17502.32

Table EC.4 Results for `orangeJuice` pricing problem with log-log demand and discrete $\mathcal{U}$