A Polynomial Algorithm for the Lossless Battery Charging Problem

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Abstract. This study presents a polynomial time algorithm to solve the lossless battery charging problem. In this problem the optimal charging and discharging schedules are chosen to maximize total profit. Traditional solution approaches have relied on either approximations or exponential algorithms. By studying the optimality conditions of this problem, we are able to reduce it to a shortest path problem on acyclic graphs.

Keywords: Battery Energy Storage, Energy Arbitrage, Shortest Path.

1 Introduction

Energy storage is an increasingly attractive solution to reduce electricity costs and carbon footprints and to increase energy power systems' flexibility and reliability. In recent years the usage of energy storage has increased significantly as it has becoming more economically viable due to declining technological costs and surging renewable penetration into power grids. An energy storage system also provides energy arbitrage, which refers to purchasing, via charging, the energy when energy prices are low and selling, via discharging, energy when the prices are high. In order to maximize this revenue, a battery storage requires a proper management strategy able to make charge/discharge decisions according to price signals. Making these decisions efficiently in terms of time and computational efforts becomes more critical when energy storage is integrated within more complex optimization problems.
The standard mixed integer linear programming (MILP) model for the battery charging problem that participates in energy arbitrage can be formulated as

$$\max \sum_{t=1}^{T} (P_t^d d_t - P_t^c c_t) \tag{1a}$$

$$\text{s.t. } s_t = \eta s_{t-1} + c_t - d_t \qquad \forall t \in \mathcal{T}, \tag{1b}$$

$$s_0 = S_{\text{init}}, \tag{1c}$$

$$s_T = S_{\text{final}}, \tag{1d}$$

$$\underline{S} \leq s_t \leq \overline{S} \qquad \forall t \in \mathcal{T}, \tag{1e}$$

$$0 \leq c_t \leq \overline{C} u_t \qquad \forall t \in \mathcal{T}, \tag{1f}$$

$$0 \leq d_t \leq \overline{D} v_t \qquad \forall t \in \mathcal{T}, \tag{1g}$$

$$u_t + v_t = 1 \qquad \forall t \in \mathcal{T}, \tag{1h}$$

$$u_t, v_t \in \{0, 1\} \qquad \forall t \in \mathcal{T}. \tag{1i}$$

The above formulation is a state of charge (SOC) formulation with mutually exclusive charge and discharge modes over the time interval $[0, T]$. This interval is discretized into unit time periods resulting in the set $\mathcal{T} = \{1, 2, ..., T\}$. The storage parameters and variables are given in Table 1. The charging and discharging prices at each time t is denoted by P_t^c and P_t^d, respectively.

Table 1. Parameters and variables of formulation (1).

Parameters		**Variables**	
S_{init}	initial state of the charge	s_t	state of the charge at time period t
$\overline{S}$	maximum storage capacity	c_t	quantity of energy charged at time t
$\underline{S}$	minimum storage capacity	d_t	quantity of energy discharged at time t
$\overline{C}$	maximum charging capacity	u_t	charging status at time t
$\overline{D}$	maximum discharging capacity	v_t	discharging status at time t
η	storage efficiency		

The formulation 1 is at first glance very simple, and one might expect the solution would be easy to come by. If all of the prices are positive, there is no incentive for both c_t and d_t to be nonzero in any linear relaxation solution, meaning that optimal solution to the linear programming solution will have the same objective value as the MILP. However, the polyhedron defining the feasible region is quite complicated.

Observe that there are 2^T many vertices that represent the action of simply doing nothing throughout the entire interval, where $S_{\text{init}} = S_{\text{final}}$ and $c_t = d_t = 0$ for all t, and u_t (v_t) is either zero (one) or one (zero) arbitrarily. This complicated structure can impact solution times when prices become more general.
In our study we assume the storage efficiency η in the above formulation is one. This means that the we do not lose energy across time periods. We call this restriction the *lossless battery charging problem*. This assumption is necessary for the convex hull description and the polynomial time algorithm: we do not know if there are polynomial time algorithms when $\eta \neq 1$.
To avoid some complexity, some studies (1, 2) reformulate binary variables with weaker yet linear constraints to model an energy storage as a linear program. However, these studies are not viable when energy prices are negative. During negative price periods the storage operator is paid to consume electric energy, and will therefore attempt to charge the storage device as much as possible. If there is more than enough energy to fully charge the storage device, the LP solution is incentivized to charge and discharge simultaneously. Other approaches in literature to address this problem of fractional solutions are to ignore the non-simultaneous charge and discharge constraint during negative price periods (3) or exclude negative prices in the model in the first place (4). However, this is problematic since with the increasing penetration of renewable energy, the presence of negative prices are becoming more frequent (5).
While the above methods are interested in solving the battery charging problem directly, there is also a need to understand the convex hull of battery charging solutions. Often times a battery is a small component of a much larger system. In this context, branching on variables associated with the battery may have very little impact in changing the dual bound compared to variables associated with larger components; however, doing so may be required to arrive at an integer solution. By studying the optimality conditions of the battery charging problem, we are able to recast the battery charging problem as a shortest path problem, yielding both polynomial-time algorithms as well as convex hull descriptions.
Not much is known about the complexity of the energy storage problem. Adding non-zero and time-dependent minimum and maximum charge and discharge limits ($\underline{C}_t u_t \leq c_t \leq \overline{C}_t u_t$ and $\underline{D}_t v_t \leq d_t \leq \overline{D}_t v_t$) would make the above problem NP-hard (6). To the authors' knowledge, however, there have been no other results related to problem complexity.
While one often thinks of batteries when thinking of energy storage, we note that there are far more diverse types of energy storage. Large pumped storage hydro facilities operate in the same way, charging by pumping water into a reservoir and discharging by running the stored water through turbines. Fuel-constrained generators can be thought of as a (more complicated) storage device by considering delivery of the fuel as charging and the burning of fuel as discharging. At the micro-grid level, HVAC devices act as energy storage, where "charging" on a hot summer day is done by cooling the house to below the desired temperature in advance (7). Throughout this paper, however, we will use the terms energy storage device and battery interchangeably.
The rest of this paper is organized as follows. The optimality conditions and solution methods are given in section 2. Despite polynomial run times and convex hulls, the approach is likely not computationally tractable for real-world problems. We simplify the model by adding realistic assumptions on prices in Section 3. In Sections 4 and 5 we discuss our polyhedral and computational results. We conclude the paper in Section 6.

2 Optimality Conditions and Algorithms

In the following we study some properties of the convex hull of the formulation (1). We first restrict our study to a special type of vertices.

Definition 1. *We say a vertex $(s_t, c_t, d_t, u_t, v_t)_{t\in\mathcal{T}}$ in the convex hull of the formulation (1) is min or max on the boundary (MOB) if state of the charge meets its minimum or maximum bounds only on the boundary (and not in the interior). For the time interval $[0,T]$ this means*

$$s_0, s_T \in \{S_{init}, S_{final}\}, \tag{2}$$

$$\underline{S} < s_t < \overline{S} \qquad \forall t \in (0,T). \tag{3}$$

Theorem 1. *For a MOB vertex A in the convex hull of the formulation (1) there exists at most one time period t with $0 < t < T$ when the battery is either (strictly) partially charging or partially discharging; that is, there is at most one t such that either $0 < c_t < \overline{C}$ or $0 < d_t < \overline{D}$.*

Proof. Proof: Aiming at a contradiction, assume there are two time periods t and t' with $0 < t, t' < T$ and without loss of generality a partial charge at time t and a partial discharge at time t'. Since A is a MOB vertex, there is an $\epsilon > 0$ with $\underline{S} < s_t \pm \epsilon < \overline{S}$ for all $t \in (0,T)$. We also have $0 \leq c_t \pm \epsilon \leq \overline{C}$ and $0 \leq d_{t'} \pm \epsilon \leq \overline{D}$ (otherwise we can choose a smaller value for ϵ). Now, consider solutions B and C with almost identical charging and discharging actions as A, except that:

$$c_t^B = c_t^A + \epsilon\ , \quad d_{t'}^B = d_{t'}^A + \epsilon \tag{4}$$

$$c_t^C = c_t^A - \epsilon\ , \quad d_{t'}^C = d_{t'}^A - \epsilon. \tag{5}$$

As a result of our choice in ϵ, we have that B and C are feasible schedules and $A = \frac{1}{2}B + \frac{1}{2}C$. This contradicts with A being a vertex. □

2.1 Optimizing Over MOB Vertices

Consider a MOB vertex in interval $[0,T]$. As a result of the Theorem 1 possible optimal actions on the entire interval for this vertex are full charge or discharges, at most one partial charge or discharge and no charge and discharges (do nothings). Let n^c, n^d and n^n denote the number of time periods with full charge, full discharge, and do nothing, respectively. Then Theorem 1 implies that $n^c+n^d+n^n \in \{T-1,T\}$. Given these numbers, we compute the amount of the non-zero partial charge or discharge in the interval by the formula

$$p = S_{\text{final}} + \overline{D}n^d - \overline{C}n^c - S_{\text{init}}, \tag{6}$$

taking into account that $n^c+n^d+n^n = T-1$. Note that $p \in \left(-\overline{D},\overline{C}\right)\setminus\{0\}$, the negative values correspond to partial discharges while positive values correspond to partial charges. Also, the order that the charges and discharges are performed does not effect the calculation of the value p. This is due to the assumption $\eta = 1$. It is possible for p to take multiple distinct values for the given vertex, depending on the arrangement of the numbers n^c, n^d and n^n. In such cases, we consider each arrangement separately and calculate the value of p for each one. For example let $T = 4$, $S_{\text{init}} = \underline{S} = 0$, $S_{\text{final}} = \overline{S} = 10$, $\overline{C} = 6$, and $\overline{D} = 3$. If $n^c = 2$, $n^d = 1$ and $n^n = 0$, then according to the formula (6) $p = 1$ corresponding to partial charge. If $n^c = 2$, $n^d = 0$ and $n^n = 1$, then $p = -2$ which corresponds to partial discharge of value 2. We denote the set of all possible values of partial charge/discharge by $\mathcal{P}$. In the following Lemma we show that the size of the set $\mathcal{P}$ is linear in T.

Lemma 1. *The number of unique values of p amongst all MOB vertices is linear in terms of T.*

Proof. Proof: We show that for a fixed value of $n^n \in \{0,1,...,T\}$, there is at most one possible value for p. Towards a contradiction, let n_1^c and n_1^d be values such that

$$-\overline{D} < p_1 = S_{\text{final}} + \overline{D}n_1^d - \overline{C}n_1^c - S_{\text{init}} < \overline{C}, \tag{7}$$

and let $n_2^c \neq n_1^c$ and n_2^d be values such that

$$-\overline{D} < p_2 = S_{\text{final}} + \overline{D}n_2^d - \overline{C}n_2^c - S_{\text{init}} < \overline{C} \tag{8}$$

with $n_1^c + n_1^d = n_2^c + n_2^d = T - 1 - n^n$. Letting $n_2^c = n_1^c + j$ for $j \in \mathcal{T}$, we have $n_2^d = n_1^d - j$. Then

$$-\overline{D} < p_2 = S_{\text{final}} + \overline{D}(n_1^d - j) - \overline{C}(n_1^c + j) - S_{\text{init}} < \overline{C},$$

which leads to

$$(j-1)\overline{D} + j\overline{C} < p_1 < (j+1)\overline{C} + j\overline{D}, \quad j \in \mathcal{T}.$$

This is inconsistent with (7). □

Shortest path problem for MOB vertices We represent the space of all possible MOB solutions of the formulation (1) as a directed acyclic graph. Each node denotes the number of full charges, number of full discharges, number of do nothings, and partial charge/discharges that have occurred up to time t. we index the elements of the set $\mathcal{P}$ defined above by i, i.e., let p_i be the ith element in the set $\mathcal{P}$. We represent the nodes of the graph by

$$(n^c, n^d, n^n, \mathbf{0})_t \ \forall n^c, n^d, n^n \in \{0,\ \dots,\ T\},\ t = n^c + n^d + n^n \leq T - 1 \tag{9}$$

$$(n^c, n^d, n^n, e_i)_t \ \forall n^c, n^d, n^n \in \{0,\ \dots,\ T\},\ t - 1 = n^c + n^d + n^n \leq T - 1. \tag{10}$$

The zero vector, $\mathbf{0}$, of dimension $|\mathcal{P}|$ indicates that no partial charge/discharge has performed, whereas the vector e_i indicates that the ith partial charge in the set $\mathcal{P}$ has been performed. The arcs entering a node associated with time t, represents a possible action occurring at time t. To represent a full charge at time t, we add arcs leaving nodes of type $(n^c,n^d,n^n,\mathbf{0})_{t-1}$ and $(n^c,n^d,n^n,e_i)_{t-1}$ to nodes $(n^c+1,n^d,n^n,\mathbf{0})_t$ and $(n^c+1,n^d,n^n,e_i)_t$, respectively with the cost of $-\overline{C}P_t^c$. Construction of arcs reflecting full discharges, do nothings, and partial charge/discharges are analogous. We then add $\mathfrak{s} = (0,0,0,\mathbf{0})$ as the start node and $\mathfrak{t}$ as terminal node with arcs of respective cost leaving $\mathfrak{s}$ and entering $\mathfrak{t}$. Now the optimal MOB vertex can be found by solving the shortest $\mathfrak{s}-\mathfrak{t}$ path in the described graph, where the objective value is equal to the negative of the shortest path length. We enforce the constraints such as the initial and final state of the charge on the boundary of the time interval $[0,T]$ and the minimum and maximum state of the charges on interior time periods by removing the infeasible nodes from the graph through the following pruning rules. We remove nodes of types:

(a) $(n^c,n^d,n^n,\mathbf{0})^t$ if $S_{\text{init}} + \overline{C}n_t^c - \overline{D}n_t^d \notin (\underline{S},\overline{S})$,
(b) $(n^c,n^d,n^n,e_i)^t$ if $S_{\text{init}} + \overline{C}n_t^c - \overline{D}n_t^d + p_i \notin (\underline{S},\overline{S})$,
(c) $(n^c,n^d,n^n,e_i)^T$ if $S_{\text{init}} + \overline{C}n^c - \overline{D}n^d + p_i \neq S_{\text{final}}$,
(d) $(n^c,n^d,n^n,\mathbf{0})^t$ if $S_{\text{init}} + \overline{C}n_t^c - \overline{D}n_t^d \notin [S_{\text{final}} - \overline{C}(T-t), S_{\text{final}} + \overline{D}(T-t)]$,
(e) $(n^c,n^d,n^n,e_i)^t$ if $S_{\text{init}} + \overline{C}n_t^c - \overline{D}n_t^d + p_i \notin [S_{\text{final}} - \overline{C}(T-t), S_{\text{final}} + \overline{D}(T-t)]$.

Example 1. Let $T = 3$, $S_{\text{init}} = \underline{S} = 0$, $S_{\text{final}} = \overline{S} = 10$, $\overline{C} = 6$, and $\overline{D} = 4$. The charge and discharge prices are given in the following table.

time period	charging price	discharging price
1	3	2
2	5	4
3	7	5

We find all possible partial charge values, $|\mathcal{P}|$, by enumerating over the possible values of n^n taking into account the equation $n^n + n^c + n^d = T - 1$. For a fixed value of n^n, we find a unique value of partial charge/discharge that satisfies

$$p = 10 + 6n^d - 4n^c \ , \quad -4 < p < 6.$$

For $n^n = 0$ (and $n^c + n^d = 2$), we have $p = -2$ for $n^c = 2$ and $n^d = 0$. For $n^n = 1$ (and $n^c + n^d = 1$), we find a partial charge of value $p = 4$ for $n^c = 1$ and $n^d = 0$. The values $n^n = 2, 3$ are impossible since they lead to $n^c + n^d = 0$ and $n^c + n^d = -1$. Hence in this case $\mathcal{P} = \{-2, 4\}$. Figure 1 illustrates the associated directed graph. The $\mathfrak{s} - \mathfrak{t}$ shortest path,

$$\mathfrak{s} \to (1, 0, 0, \mathbf{0}) \to (1, 0, 1, \mathbf{0}) \to (1, 0, 1, e_2) \to \mathfrak{t},$$

represents the optimal schedule: a full charge at time 1, do nothing at time 2, and partial charge of value 4 at time 3. The optimal cost of this path is -46. We prove this optimality in the following theorem.

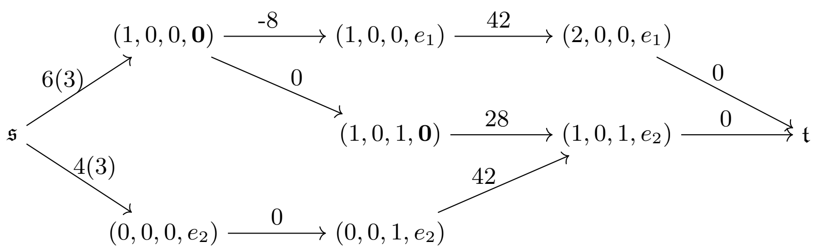

Fig. 1. Directed graph for optimizing over MOB vertices

Theorem 2. *Solving $\mathfrak{s} - \mathfrak{t}$ shortest path problem on the above graph provides a MOB vertex with the highest objective value relative to other MOBs.*

Proof. Proof: We show a one-to-one correspondence between $\mathfrak{s} - \mathfrak{t}$ paths in the directed graph and MOB vertices. First we prove that given a MOB vertex, there is an $\mathfrak{s} - \mathfrak{t}$ path with the opposite cost. Let $A = (s_t, c_t, d_t, u_t, v_t)_{t \in \mathcal{T}}$ be a MOB vertex with objective value of M_A. By Theorem 1, there is at most one time period $t_p \in \mathcal{T}$ such that either $0 < c_{t_p} < \overline{C}$ or $0 < d_{t_p} < \overline{D}$. Without loss of generality, we can assume that there is a time interval, t_p, with a partial charge, denoted by p_1. The implication of Theorem 1 is that for all $t \in \mathcal{T} \setminus \{t_p\}$ with $u_t = 1$ we have $c_t = \overline{C}$ and for all $t \in \mathcal{T}$ with $v_t = 1$ we have $d_t = \overline{D}$. Then the objective value of vertex A is as follows,

$$M_A = \sum_{t \in \mathcal{T}: d_t = \overline{D}} \overline{D} P_t^d \quad - \sum_{t \in \mathcal{T} \setminus \{t_p\}: c_t = \overline{C}} \overline{C} P_t^c \ - \ c_{t_p} P_{t_p}^c. \tag{11}$$

We now define the sequence of vertices in the respective $\mathfrak{s} - \mathfrak{t}$ path. Let n_t^c, n_t^d, n_t^n denote the number of full charges, full discharges, and no charges/discharges from time zero till time t. The corresponding $\mathfrak{s} - \mathfrak{t}$ path transverses vertices $(n_t^c, n_t^d, n_t^n, \mathbf{0})$ for all $t < t_p$ and vertices $(n_t^c, n_t^d, n_t^n, e_1)$ for all $t \geq t_p$. Because A is feasible, we have that $s_t = \overline{C} n_t^c + \overline{D} n_t^d \in (\underline{S}, \overline{S})$ for $t < t_p$ and $s_t = \overline{C} n_t^c + \overline{D} n_t^d + p_1 \in (\underline{S}, \overline{S})$ for $t >= t_p$, so none of these vertices were removed in the graph as a result of pruning rules (a) - (c). Similarly, we have that $(n_t^c, n_t^d, n_t^n, \mathbf{0}) - (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, \mathbf{0})$ for $t < t_p$, $(n_{t_p}^c, n_{t_p}^d, n_{t_p}^n, e_1) - (n_{t_p-1}^c, n_{t_p-1}^d, n_{t_p-1}^n, \mathbf{0})$, and $(n_t^c, n_t^d, n_t^n, e_1) - (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, e_1)$ for $t > t_p$ are all unit vectors, corresponding to edges in the graph. The costs of the edges are as follows:

$$p_t^c * \overline{C} \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_1 \to c_t = \overline{C} \tag{12}$$

$$-p_t^d * \overline{D} \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_2 \to d_t = \overline{D} \tag{13}$$

$$0 \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_3 \to c_t = d_t = 0 \tag{14}$$

$$c_{t_p} P_{t_p}^c \text{ if } t = t_p \tag{15}$$

Summing the costs of the edges used in the $\mathfrak{s}-\mathfrak{t}$ path gives the negative of (11). Now we prove that Given an $\mathfrak{s}-\mathfrak{t}$ path there is a MOB vertex with opposite cost. Let $\{(n_t^c, n_t^d, n_t^n, w_t)\}_{t\in T}$ be nodes that define an $\mathfrak{s}-\mathfrak{t}$ path where $w \in \mathbb{R}^{|\mathcal{P}|}$. We can construct a MOB vertex solution to (1) as follows.

$$c_t = \overline{C} \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_1 \tag{16}$$

$$d_t = \overline{D} \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_2 \tag{17}$$

$$c_t = 0 \text{ and } d_t = 0 \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_3 \tag{18}$$

$$c_t = p_j \text{ or } d_t = p_j \text{ if } (n_{t-1}^c, n_{t-1}^d, n_{t-1}^n, w_{t-1}) - (n_t^c, n_t^d, n_t^n, w_t) = e_{j+3}, \tag{19}$$

where the last value assignment is decided if p_j is a partial charge or discharge. We note that the s, u, and v variables can be uniquely determined by the charge and discharge values and that $s_0 = S_{\text{init}}$, $s_T = S_{\text{final}}$ and $s_t \in (\underline{S}, \overline{S})$ for all $0 < t < T$ as a result of the pruning rules for the nodes in the graph. As in the previous case, this vertex has the opposite cost of the $\mathfrak{s}-\mathfrak{t}$ path. □

Corollary 1. *The optimal MOB vertex can be found in polynomial time.*

Proof. Proof: There are no negative cost cycles in the graph. Considering the vertices of type (9) and (10), since $0 \le n^c, n^d, n^n, |\mathcal{P}| \le T$, then the number of nodes is of order $O(T^4)$. As a result, the number of edges is of order $O(T^8)$ in the worst case where we have a complete graph. Also, the cost of the edges are polynomial relative to the size of the inputs of the problem, since the computation requires summation and multiplication by constant numbers. □

Corollary 2. *There exists a polynomial-sized integer formulation for the minimum cost $\mathfrak{s}-\mathfrak{t}$ path for these graphs. This formulation corresponds to the convex hull of all MOB vertices in the given interval.*

2.2 Optimizing Over All Vertices

Here we show how to utilize the previous construction in order to optimize the base model over all vertices. The key idea is that any arbitrary vertex in the convex hull of the model can be decomposed into MOB vertices viewed as optimal solutions to sub-problems of the base model. Let $(s_t, c_t, d_t, u_t, v_t)_{t\in\mathcal{T}}$ be a generic vertex, and t_1 denote the first time period where s_{t_1} is either $\underline{S}$ or $\overline{S}$. We truncate all variables with indices greater than t_1 to obtain $(s_t, c_t, d_t, u_t, v_t)_{t\in\{1,\dots,t_1\}}$ which can be viewed as a MOB vertex and a solution to an instance of the base model defined over the time interval $[0, t_1]$ with $S_{\text{init}} = s_0$ and $S_{\text{final}} = s_{t_1}$. Similarly, if t_2 denotes the second time period where s_{t_2} is either $\underline{S}$ or $\overline{S}$, then projecting the vertex onto $(s_t, c_t, d_t, u_t, v_t)_{t\in\{t_1,\dots,t_2\}}$, gives us a MOB vertex as a solution to the subproblem defined over $[t_1, t_2]$ with $S_{\text{init}} = s_{t_1}$ and $S_{\text{final}} = s_{t_2}$. This process continues until we reach the last time period t_n with $s_{t_n} \in \{\underline{S}, \overline{S}\}$ and the MOB vertex $(s_t, c_t, d_t, u_t, v_t)_{t\in\{t_n,\dots,T\}}$. Taking advantage of this independent structure of the base model, we construct a graph for which solving the shortest path problem is equivalent to finding an optimal solution to the base model.

Shortest path problem for all vertices We define a directed graph where nodes are the time periods when the state of the charge of the battery meets it bounds. We denote the nodes by (t, b) where $t \in \mathcal{T}$ and $b \in \{\overline{S}, \underline{S}\}$. There is an arc from node (t, b) to node (t', b') if $t' > t$ and the MOB subproblem on interval $[t, t']$ with the initial state of charge b and final state of charge b' is feasible. The cost of this arc is the optimal objective value of the aforementioned subproblem. We add dummy nodes $\mathfrak{s} = (0, S_{\text{init}})$ and $\mathfrak{t}$ with arcs from $\mathfrak{s}$ to all feasible nodes (t, b) for $t \in \mathcal{T}$ and an arc from node (T, S_{final}) to $\mathfrak{t}$ (with the cost of zero).

Example 2. Consider the same parameters in example (1). Figure 2 illustrates the above graph where enforcing $S_{\text{final}} = 10$, the length of $\mathfrak{s}-\mathfrak{t}$ shortest path is 38. Then our optimal schedule with the objective value of -38 is: full charge at time 1, partial charge at time 2 and do nothing at time 3, which corresponds to the path

$$\mathfrak{s} \longrightarrow (2, 10) \longrightarrow (3, 10) \longrightarrow \mathfrak{t}.$$

The infeasible nodes are represented in gray and all arcs and paths traversing them are removed form the graph. Next theorem proves the optimality of this solution.

Theorem 3. *The $\mathfrak{s}-\mathfrak{t}$ shortest path on the above graph provides the optimal solution to the base model (1) with the cost of negative optimal objective value.*

Proof. Proof: We show for a given optimal solution of the base model with optimal objective value of z, there is an $\mathfrak{s}-\mathfrak{t}$ path in the graph with the cost of $-z$. This is trivial due to the construction of the graph and decomposition of the base model into independent subproblems using MOB vertices as described above. Now we show that the shortest $\mathfrak{s}-\mathfrak{t}$ path in the graph with the length of l provides an optimal solution to the base model with the optimal objective value of $-l$. Let $A = (s_t, c_t, d_t, u_t, v_t)_{t\in\mathcal{T}}$ be feasible solution to the base model corresponding to the shortest path. If this solution is not optimal, then there is a time period $t \in \mathcal{T}$ such that changing c_t or d_t provides an improvement to the objective value. Since t belongs to one of the subintervals of $[0, T]$ then it is possible to improve the optimal objective value of this subinterval. This leads to improvement of the length of the $\mathfrak{s}-\mathfrak{t}$ shortest path which is a contradiction. □

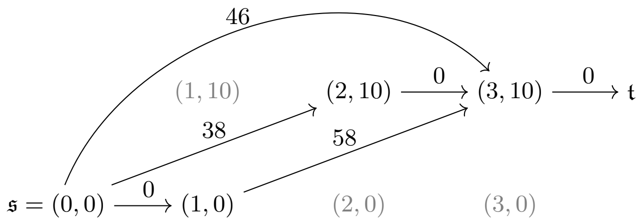

Fig. 2. Directed graph for optimizing over all vertices

Corollary 3. *The size of the above graph is polynomial.*

Proof. Proof: The number of vertices is of order $O(T)$ and the number of edges is $O(T^2)$. By theorem 1 processing each vertex is of polynomial time. Hence, the computation of the edges are polynomial relative to the size of the inputs. □

Corollary 4. *The base model (1) can be optimized over all vertices in polynomial time.*

Proof. Proof: The statement is true by Corollaries 1 and 3. □

3 Simplifications of problem (1) based on realistic pricing

Usually battery storage capacities are relatively small compared to conventional energy generators participating in the wholesale markets; thus, their impact on energy prices could be neglected and they are usually modelled as price takers (8). In this case it is common to consider equal and fixed prices for buying and selling at a given time period. However, this is the price of delivered energy. Charging and discharging inefficiencies affect how much energy is actually stored in the grid. Let P_m be the market price for buying and selling energy at a given time. Assume for the sake of argument that P_m is positive. From the perspective of electricity in the battery, and considering inefficiencies, discharging d units of energy *in the battery* delivers $< d$ units to the market, yielding a price of $< P_m$ per unit of energy in battery. Similarly, buying b from the market yields less than b units of electricity in the battery, making the price of charging, in terms of energy in the battery, strictly greater than P_m. With this in mind that we investigate how the model simplifies with this realistic pricing scenarios. For this section, we assume for all time periods $t \in \mathcal{T}$ we have

$$\frac{P_t^c}{P_t^d} > 1. \tag{20}$$

This implies that at each time t the charge and discharge prices have the same sign. Also, when prices are positive we have $P_t^c > P_t^d$, while when the prices are negative we have $P_t^c < P_t^d$.

Definition 2. *We say a feasible solution of the linear relaxation of formulation (1) is orthogonal at time period $t \in \mathcal{T}$ if $c_t d_t = 0$.*

Remark 1. An integer solution can be recovered from an orthogonal solution to the formulation (1).

Theorem 4. *If the charge and discharge prices at time period $t \in \mathcal{T}$ are strictly positive, then the LP relaxation of the formulation (1) returns an orthogonal solution at time t.*

Proof. Proof: Towards a contradiction assume that there is a time period t with $u_t v_t \neq 0$ in the solution of the LP relaxation. This implies that $c_t > 0$ and $d_t > 0$. For $\epsilon > 0$ we have that by reducing both c_t and d_t by ϵ, the problem remains feasible (as all other time periods are unaffected), but doing so increases the profit. □

Theorem 5. *If an optimal MOB solution to the battery scheduling problem has two or more time periods where the battery is neither fully charging or fully discharging (do nothing or partial charge/discharge), then in all time periods with negative prices, the battery is charging or discharging at full capacity.*

Proof. Proof: Aiming at a contradiction, assume t is a time period with negative prices; $P_t^c, P_t^d < 0$; when the system is not charging at full capacity, i.e. $c_t < \overline{C}$. Let t' be another time period without a full charge/discharge. Assume prices at time t' are negative, i.e. $P_{t'}^c, P_{t'}^d < 0$ (since otherwise increasing charge at time t, decreasing or remaining at zero charge at time t' is more profitable which contradictory to the optimality of the solution.) Now we consider two cases:
Case 1: in both two time periods t and t', charge and discharge variables are zero (do nothing in both times). Let $\epsilon > 0$ be given. In this case, increasing charge at time t and decreasing the charge at time t'

by ϵ will result in a feasible solution while changing the profit by $-\epsilon P^c_t + \epsilon P^d_{t'}$. Similarly, decreasing the charge at time t by ϵ and increasing the charge at time t' will result in a feasible solution while changing the profit by $\epsilon P^d_t - \epsilon P^c_{t'}$. As a result of the optimality, we must have

$$-\epsilon P^c_t + \epsilon P^d_{t'} \leq 0 \tag{21}$$

$$\epsilon P^d_t - \epsilon P^c_{t'} \leq 0. \tag{22}$$

From assumption (20), for negative prices we have $P^c_t < P^d_t$. However, the above equations lead to

$$P^d_{t'} \leq P^c_t < P^d_t \leq P^c_{t'}, \tag{23}$$

which contradicts (20).
Case 2: in time period t charge and discharge variables are zero (do nothing), and in time t' we have partial charge. (The case of partial charge in both times is impossible as a result of being a MOB vertex.) The same argument as above is valid in this case with taking into consideration that here we increase the discharge variable by ϵ at time t to have the inequality (22). □

The impact of the above theorem is the following. Suppose we decomposed the MOB shortest path problem into two different subproblems based on the following disjunction. We consider one subproblem where at most one partial/do-nothing is allowed and another where there are at least two (at most one of which is a partial charge). In the former subproblem, there is only one possible value for the partial charge. In addition, we can remove all arcs associated with do-nothing. In the latter we can remove all arcs associated with both partial and do-nothing from from nodes representing time periods with negative prices. Note that if all prices are negative, than the solution obtained by solving the later subproblem can not be optimal.

4 Polyhedral Results

We build off of techniques in (9), where polyhedral results for constrained sums of polyhedra are developed. The authors of this study prove the following theorem (Theorem 4) in https://optimization-online.org/?p=13608.

Theorem 6. *Consider m nonempty polyhedra $P^i := \{x \in \mathbb{R}^n \ : \ A^i x \leq b^i\}$, $i \in \{1, 2, ..., m\}$, and for each i, let Q^i be a (bounded) polyhedron in $\mathbb{R}^n$ and R^i be a (closed convex) cone in $\mathbb{R}^n$ such that $P^i = Q^i + R^i$. Let $\Lambda \in \mathbb{R}^m_+$ be a nonempty polyhedron. Consider the set $P := \bigcup_{\lambda \in \Lambda} \left(\sum_{i=1}^m \lambda^i Q^i + \sum_{i=1}^m R^i\right)$ and the polyhedron $Y \subset \mathbb{R}^n \times (\mathbb{R}^n)^m \times \mathbb{R}^m_+$ such that*

$$Y = \begin{cases} A^i x^i \leq b^i \lambda_i \quad 1 \leq i \leq m \\ \sum_{i=1}^m x^i = x \\ (\lambda_1, ..., \lambda_m) = \lambda \in \Lambda. \end{cases}$$

Then

$$P = \mathrm{Proj}_x(Y) = \{x \in \mathbb{R}^n \mid \exists (x^1, ..., x^n, \lambda) \in \mathbb{R}^{nm+m} \text{ such that } (x, x^1, ..., x^n, \lambda) \in Y\}.$$

In particular, P is a polyhedron.

Utilizing Theorem 6, we construct an extended formulation of the feasible region of problem (1). By construction, each vertex in the main graph corresponds to a MOB vertex to a given time interval, initial, and final criteria. We let polytope P^i represent the convex hull of MOB vertices given from Corollary 2 for $i \in I$, where

$$I = \left\{[t, t']^{b,b'} \mid t, t' \in \mathcal{T} \cup \{0\}, \ t < t', \ b, b' \in \{S_{\text{init}}, \underline{S}, \overline{S}, S_{\text{final}}\}\right\}$$

is the set of all feasible sub-intervals of $[0, T]$ (t and t' are integers). The polytope Λ can be though of as a flow formulation in the main graph. Let λ_i be a binary variable associated with the battery being in sub-interval i with the following constraints:

$$\sum_{t=1}^{\mathcal{T}} \lambda_{[0,t]^{S_{\text{init}},b}} = 1, \quad b \in \{\underline{S}, \ \overline{S}\} \tag{24a}$$

$$\lambda_{[0,t]^{S_{\text{init}},b}} + \sum_{b' \in \{\underline{S}, \overline{S}\}} \sum_{t'=1}^{t} \lambda_{[t',t]^{b',b}} = \sum_{b'' \in \{\underline{S}, \overline{S}\}} \sum_{t''=t+1}^{T} \lambda_{[t,t'']^{b,b''}}, \ \forall t \in \mathcal{T}, b \in \{\underline{S}, \overline{S}\}. \tag{24b}$$

These constraints model the flow through the main graph for optimizing over all vertices. The equation (24a) ensures that one unit of flow is leaving the starting node and the equation (24b) conserves the flow at each node. It is known that the constraints 24, by themselves, describe a totally unimodular matrix. As a result the polytope Λ is integral. Since each polytope P^i, for $i \in \{1, 2, ..., m\}$ is bounded and its recession cone is $\{0\}$, by theorem 6 we arrive at a polyhedral representation for the set $\bigcup_{\lambda \in \Lambda} \sum_{i=1}^m \lambda_i P^i$. Also, the polytope Y defined in the theorem provides a compact extended formulation for it.

Theorem 7. *The above extended formulation is a perfect formulation i.e. at every feasible vertex, the integer variables take integer values.*

Proof. Proof: By Theorem 7 of the aforementioned study, vertices of Y have λ_i components which are vertices of Λ. So if Λ is an integral polyhedron then the vertices of Y are also integer. □

5 Computational Results

In this section we represent the performance of the integer program (IP) given in (1) versus our shortest path (SP) approach. As mentioned at the beginning, in most cases the battery charging problem is very easy, even in the case of negative prices as long as there is a high variance in the prices (as one will always want to sell at the highest-priced times and sell at the lowest). The first two columns of Table 2 show results from negative priced instances where prices were chosen over a uniform distribution. None of these problems are difficult for Gurobi using the original formulation, and while both methods are fast, the integer programming approach is much faster than the polynomial algorithm. The hardest battery charging problems are those where the prices are negative and constant. This is shown in the last two columns, where Gurobi struggles to find optimal solutions to the longer-period problems, whereas the shortest-path approach optimizes them quickly.

Table 2. Performance of IP and SP for negative prices and parameters: $\underline{S} = 0, \overline{S} = 10, \overline{C} = 4, \overline{D} = 3, S_{\text{init}} = 2$.

	Random Prices		Constant Prices	
T	IP (s)	SP (s)	IP (s)	SP (s)
6	0.007	0.009	0.021	0.020
9	0.008	0.0356	0.011	0.035
12	0.0118	0.085	0.044	0.076
15	0.016	0.1296	0.094	0.141
18	0.018	0.285	0.103	0.268
21	0.0156	0.371	0.482	0.363
25	0.0132	0.612	1.16	0.578
28	0.0224	0.876	4.351	0.81
31	0.027	1.124	78.22	1.089
34	0.0314	1.490	89.809	1.511
37	0.039	1.978	> 180	1.941

6 Discussion and Conclusion

Motivated by increasing integration of energy storage into power systems and increasing occurrence of negative energy prices we investigate how to solve the lossless battery scheduling problem in polynomial time. We recast the MILP formulation (1) of the problem as a shortest path problem over directed acyclic graphs. Our approach provides convex hull description of the problem.

We view differently the existing possible actions of a battery storage, i.e. charging, discharging, and doing nothing at a given time period. In our approach, possible actions of a battery are: full charge/discharge, partial charge/discharge, and do nothing. This shift of viewpoint coupled with observations on the state of charge patterns allows us to recognize a special type of vertices; MOB vertices; in the MILP polytope. We find an optimal schedule over these special vertices through the use of the shortest path problem. Then we demonstrate how to decompose a generic vertex into MOB vertices and solve another shortest path in order to optimize over all vertices.

In regard of the MILP formulation (1) and its recast as a shortest path note some extensions that can still be solved in polynomial time:

- The minimum charge and discharge capacities in formulation (1) is assumed to be zero. In the case where these parameters are non zero, our method remains polynomial. A slightly modified Theorem 1 still hold for the nonzero parameters with the exception that there is only one time period where the charging and discharging rates are not at their bounds. In this case, the vertices for the MOB shortest path problem would include a reference to the number of minimum charges and discharges in addition to the number of full charges, full discharges, and partial charges.
- Minimum Up/Down time on the generators can be enforced in the shortest path polytope. This can be done by adding an additional index to each vertex in the MOB subproblem to reflect how many consecutive charges/discharges there have been.

However, some extensions may not necessarily be solvable in polynomial time.

- We assumed that there is no loss in the battery over time. The results still hold if the battery experiences a constant loss with respect to time. However, if the loss is proportional to the state of charge, the proposed algorithm would not work as the ordering of charge and discharge decisions the equation (6) are now important, making the resulting shortest path exponentially large.
- Similarly, we assume linear charge and discharge prices. Moving to piecewise linear makes the problem NP-Hard. This can be seen by (6), as they show that the fuel constrained unit commitment self scheduling problem with a piecewise-linear objective function is NP-Hard. In this context, the fuel constrained generator can be thought of as a battery with zero charge capacity. The main reason for the change in complexity is that Theorem 1 is not consistent with piecewise linear objective functions.

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