On the Computation of Restricted Normal Cones

Mario Jelitte

Faculty of Mathematics, Technische Universität Dresden, 01062 Dresden, Germany
Mario.Jelitte@tu-dresden.de

Abstract. Restricted normal cones are of interest, for instance, in the theory of local error bounds, where they have recently been used to characterize the existence of a constrained Lipschitzian error bound. In this paper, we establish relations between two concepts for restricted normals. The first of these concepts was introduced in the late 1990s by Studniarski, the second more than ten years later by Bauschke et al. Under assumptions, suitable for the use in the theory of error bounds, we explain that the two concepts are the same. Furthermore, we develop several formulas that simplify the computation of restricted normals.

Keywords: Restricted Normal Cone · Mordukhovich Normal Cone · Switching Index Condition · Local Error Bound

1 Introduction

Mordukhovich's normal cone is one of the fundamental tools of variational analysis [15,16]. It is used to develop concepts of generalized differentiation of nonsmooth and even set-valued mappings [14], and as such, normal cones are used to characterize generalized concepts for the regularity of a mapping, like metric regularity [12,13] or metric subregularity [10,11]. In [17], restricted normal cones were introduced as a certain generalization of Mordukhovich's normal cone, and they are used in order to formulate sufficient conditions for a minimizer of an optimization problem to be *weak-sharp* [6]. The latter can be interpreted as a special kind of a (constrained) local error bound, which is the subject of interest in [8], where restricted normal cones play an important role. The usefulness of error bounds in mathematical programming is outlined, for example, in the latter three papers and references therein, and it is not the goal of this paper to deal with their applications. Instead, we focus on restricted normal cones and, in particular, aim at the development of formulas that simplify their computation. This proves to be a challenging task that has not been solved in [8,17]. A related concept for restricted normal cones is introduced and studied in [1,2,3], without reference to [17], however. We will see below in Sect. 2 that the restricted normal cone from [17] coincides with the one from [2], and this allows to obtain various new statements, related to [8,17]. Furthermore, we will develop some new results, complementing those in [2]. One such result (Proposition 1) allows for a simple computation of the restricted normal cone under an outer semicontinuity assumption. Subsequently, in Proposition 2, we will use the *switching index condition* [9] to guarantee the outer semicontinuity assumption. At this point, we want to mention the papers [4,7], where the switching index condition played a role, too. The rest of the paper is organized as follows. Section 2 contains a definition of the

restricted normal cone and the relations between existing concepts, mentioned in the paragraph above. A simplified representation of the cone is given in Lemma 1 under assumptions suitable for the use in [8]. The two propositions, mentioned above, are presented in Section 3.
Our notation follows standard textbooks [15,16], $\mathrm{cone}(A)$ refers to the conic hull of a set A, and $u \overset{A}{\to} y$ means that u converges to y with $u \in A$. Other notations are explained in the text as needed.

2 Restricted Normal Cones

For a set $\Omega \subset \mathbb{R}^n$, a closed subset $M \subset \Omega$, and a point $\bar{u} \in M$, let us introduce the *normal cone to M relative to Ω at $\bar{u}$* as

$$N_M(\bar{u};\Omega) := \operatorname*{Limsup}_{u \overset{\Omega}{\to} \bar{u}} \left(\mathrm{cone}\,(u - P_M(u))\right),$$

where Limsup denotes the outer limit in the sense of Painlevé-Kuratowski [16], and where $P_M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the (set-valued) projection onto the closed set M. We will often merely speak about the restricted normal cone, and by this we mean $N_M(\bar{u};\Omega)$.
The outer limit of a set-valued mapping is always closed [16, Proposition 4.4], and the outer limit of a cone-valued mapping is a cone [16, Exercise 4.14]. Thus, $N_M(\bar{u};\Omega)$ is a closed cone, and it can be easily checked that

$$N_M(\bar{u};\Omega) = \left\{ v \,\middle|\, \exists u^k \overset{\Omega}{\to} \bar{u}, \exists t_k \searrow 0, \exists \{y^k\} \subset M :\; y^k \in P_M(u^k) \forall k, \frac{u^k - y^k}{t_k} \to v \right\}$$

holds true. In other words, $N_M(\bar{u};\Omega)$ is the restricted normal cone from [8,17], and it agrees with Mordukhovich's normal cone $N_M(\bar{u})$, when $\Omega = \mathbb{R}^n$. By routine computations, one can show

$$N_M(\bar{u};\Omega) = \left\{ v \,\middle|\, \exists u^k \overset{M}{\to} \bar{u}, \exists t_k \searrow 0, \exists \{y^k\} \subset \Omega : y^k \in P_M^{-1}(u^k) \forall k, \frac{y^k - u^k}{t_k} \to v \right\},$$

i.e., we have

$$N_M(\bar{u};\Omega) = \operatorname*{Limsup}_{u \overset{M}{\to} \bar{u}} \left(\mathrm{cone}\left((\Omega - u) \bigcap \left(P_M^{-1}(u) - u\right)\right)\right). \tag{1}$$

But this means, in turn, that the restricted normal cone above also coincides with the one from [1,2,3], at least for our specific setting where $M \subset \Omega$. Elementary calculus rules for $N_M(\bar{u};\Omega)$ can thus be extracted from [2]. That paper also contains formulas for the computation of $N_M(\bar{u};\Omega)$ for some special cases, e.g., when Ω is a subspace or M is convex. In the next section, we want to find formulas, complementing the ones just mentioned, for the case where Ω is closed convex, and M is merely closed. The key to this is the subsequent lemma, which involves the mappings $R, N_M^{\mathrm{prox}}, \widehat{N}_M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, defined as

$$R(u) := \begin{cases} \varnothing & \text{if } u \notin M, \\ \mathrm{cone}\,(\Omega - u) & \text{if } u \in M, \end{cases} \tag{2}$$

$$N_M^{\mathrm{prox}}(u) := \mathrm{cone}\left(P_M^{-1}(u) - u\right), \tag{3}$$

$$\widehat{N}_M(u) := \left\{ v \,\middle|\, v^\top (y - u) \le o(\|y - u\|) \quad \text{for } y \in M \right\}. \tag{4}$$

The set $N_M^{\mathrm{prox}}(u)$ is the *proximal normal cone* to M at $u \in M$, and $\widehat{N}_M(u)$ is the *regular normal cone* to M at $u \in M$, cf. [5,16]. The two sets are empty, if $u \notin M$.

Lemma 1. *For a closed convex set $\Omega \subset \mathbb{R}^n$, a closed subset $M \subset \Omega$, and a point $\bar{u} \in M$, it holds that*

$$\begin{aligned} N_M(\bar{u};\Omega) &= \operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} \left(R(u) \bigcap N_M^{\mathrm{prox}}(u) \right) \\ &= \operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} \left(R(u) \bigcap \widehat{N}_M(u) \right) \\ &\subset \operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} \left(R(u) \right) \bigcap N_M(\bar{u}). \end{aligned}$$

Proof. Thanks to (1), we observe by [2, Lemma 1.5] that $N_M(\bar{u};\Omega)$ is contained in the outer limit of $R \cap N_M^{\mathrm{prox}}$. The converse inclusion follows by direct computations, relying on (1)–(3), and convexity of Ω.
Since $N_M^{\mathrm{prox}}(u) \subset \widehat{N}_M(u)$ is always true, the outer limit of $R \cap N_M^{\mathrm{prox}}$ is subset of the outer limit of $R \cap \widehat{N}_M$. The proof for the converse inclusion is more technical. Essentially, it relies on convexity of Ω, and an application of [15, Theorem 1.6, Step 2]. We omit further details for brevity. The remaining upper estimate for $N_M(\bar{u};\Omega)$ is a consequence of [16, formula 4(7)] and [2, Proposition 2.3]. □

In the language of [5], and with [5, Proposition 2.55] in mind, the value of R agrees for $u \in M$ with the *radial cone* to Ω at u, if Ω is closed convex, and then, its closure coincides with the usual *tangent cone* to Ω at u [16]. Thus, Lemma 1 says that the restricted normal cone can be identified with an outer limit of the intersection of the radial cone to Ω and the proximal normal cone to M. This observation complements [2, formulas (61a)–(61c) and Proposition 5.4]. In turn, the representation of $N_M(\bar{u};\Omega)$ as an outer limit of the intersection of the radial cone to Ω and the regular normal cone to M is new, and we will exploit this in the next section. Fig. 1 illustrates that the upper estimate for $N_M(\bar{u};\Omega)$ in the lemma can be sharp. There, M is the boundary of a convex polyhedral set Ω (colored in gray). The directions v_1, v_2 belong to the intersection of the outer limits, but not to $N_M(\bar{u};\Omega)$. The shaded area corresponds to the outer limit (relative to M) of R.

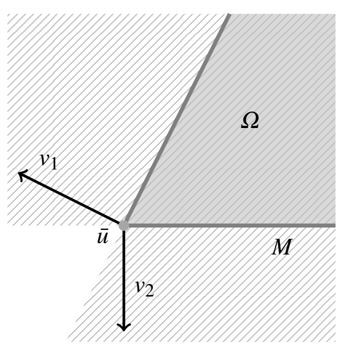

Fig. 1. Inclusion in Lemma 1 is sharp: $v_1, v_2 \notin N_M(\bar{u};\Omega)$.

3 Computation of the Restricted Normal Cone

Even under the assumptions of Lemma 1, the computation of the restricted normal cone can be a challenging task. This is due to the fact that $N_M(\bar{u};\Omega)$ is an outer limit of a nontrivial intersection of two sets. Therefore, a condition that allows to get rid of the outer limit can be beneficial, and this will be the *outer semicontinuity* of $R \cap \widehat{N}_M$ relative to the set M at $\bar{u}$. Recall [16] that the latter means validity of

$$\operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} \left(R(u) \bigcap \widehat{N}_M(u) \right) = R(\bar{u}) \bigcap \widehat{N}_M(\bar{u}). \tag{5}$$

As already mentioned in Sect. 2, the outer limit of a mapping is always closed. Hence, the latter semicontinuity condition implies that $R(\bar{u}) \cap \widehat{N}_M(\bar{u})$ is closed. It is known that $\widehat{N}_M(\bar{u})$ is always closed, but the set $R(\bar{u}) = \operatorname{cone}(\Omega - \bar{u})$ may not be closed for an arbitrary closed convex set Ω – just consider Ω as the unit ball, and a point $\bar{u}$ on its boundary.
The following is the main result of this section. It allows for direct computation of $N_M(\bar{u};\Omega)$ without having to explicitly evaluate an outer limit.

Proposition 1. *In the setting of Lemma 1, assume that $R \cap \widehat{N}_M$ is outer semicontinuous relative to M at $\bar{u}$, i.e.,* (5) *is satisfied. Then, it holds that*

$$N_M(\bar{u};\Omega) = \left\{ v \in \widehat{N}_M(\bar{u}) \,|\, \exists \tau > 0 : \quad \bar{u} + tv \in \Omega \; \forall t \in [0,\tau] \right\}. \tag{6}$$

In particular, if R and $\widehat{N}_M$ are both outer semicontinuous relative to M at $\bar{u}$, then so too is $R \cap \widehat{N}_M$, and it is further true that $N_M(\bar{u}) = \widehat{N}_M(\bar{u})$.

Proof. The equality in (6) is nothing else than $N_M(\bar{u};\Omega) = R(\bar{u}) \cap \widehat{N}_M(\bar{u})$. But then, thanks to Lemma 1 and (5), this equality is evidently fulfilled. If R and $\widehat{N}_M$ are both outer semicontinuous relative to M at $\bar{u}$, then we have

$$\begin{aligned} R(\bar{u}) \bigcap \widehat{N}_M(\bar{u}) \subset \operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} \left(R(u) \bigcap \widehat{N}_M(u) \right) &\subset \operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} (R(u)) \bigcap \operatorname*{Limsup}_{u \xrightarrow{M} \bar{u}} (\widehat{N}_M(u)) \\ &= R(\bar{u}) \bigcap \widehat{N}_M(\bar{u}), \end{aligned}$$

which implies (5), hence, outer semicontinuity of $R \cap \widehat{N}_M$. The remaining equality follows by [16, Corollary 6.29], recalling that M is closed. □

Since M is a closed set, the equality $N_M(\bar{u}) = \widehat{N}_M(\bar{u})$ corresponds [16] to the *Clarke regularity* of the set M at $\bar{u}$, and this is equivalent to outer semicontinuity of $\widehat{N}_M$. In presence of outer semicontinuity of R and $\widehat{N}_M$, the proposition implies that the inclusion in Lemma 1 holds as equation, and this complements [2, Proposition 5.5]. At this place, the reader should be aware that outer semicontinuity of R relative to M can only be guaranteed in quite special circumstances. For instance, it can never hold, when $\bar{u}$ is located at the (relative) boundary of Ω, and M contains a sequence $u^k \to \bar{u}$ that belongs to the (relative) interior of Ω. Another even more simple illustration for the absence of outer semicontinuity of R relative to M is given in Fig. 1. In special cases, however, it is possible to guarantee such outer semicontinuity, and a sufficient condition for this is the topic in the further course of this section.

As mentioned above, the set $R(\bar{u})$ need not be closed for arbitrary closed convex sets Ω, but this is essential for R to be outer semicontinuous at all. For this reason, we will be dealing with convex polyhedral sets in what follows, i.e., we suppose that

$$\Omega = \left\{ u \in \mathbb{R}^n \,\middle|\, a_i^\top u \le b^i \quad \forall i = 1, \ldots, m \right\} \tag{7}$$

holds for some $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b^1, \ldots, b^m \in \mathbb{R}$. In this case, [16, Exercise 6.47] implies that $R(\bar{u})$ coincides locally with $\Omega - \bar{u}$. Hence, $R(\bar{u})$ is itself convex polyhedral and as such, it is necessarily closed. In accordance with the wording in [9], we say that the *switching index condition* holds at $\bar{u}$ for M, if there is $\varepsilon > 0$, so that

$$a_i^\top \bar{u} = b^i \quad \Longrightarrow \quad a_i^\top u = b^i \tag{8}$$

for all $u \in M \cap (\bar{u} + \varepsilon\mathbb{B})$. We will now show that the combination of (7) and the switching index condition allows to guarantee outer semicontinuity of R.

Proposition 2. *In the setting of Lemma 1, suppose that Ω is given as in* (7) *for some $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b^1, \ldots, b^m \in \mathbb{R}$. If the switching index condition is satisfied at $\bar{u}$ for M, then R is outer semicontinuous relative to M at $\bar{u}$.*

Proof. Put $I_0 := \left\{ i \,\middle|\, a_i^\top \bar{u} = b^i \right\}$. Then, the switching index condition combined with [16, Theorem 6.46] yield $R(u) = \left\{ v \,\middle|\, a_i^\top v \le 0 \; \forall i \in I_0 \right\}$ for all $u \in M$ near $\bar{u}$, and this implies the desired outer semicontinuity of R. □

It may be possible to consider more general nonpolyhedral convex sets and establish outer semicontinuity with respect to M again under the switching index condition. This can be part of our future research.

Conclusion

In this paper, we established relations between existing concepts for restricted normal cones [2,17] for the first time to the best of our knowledge. Furthermore, we extended existing results on the computation of $N_M(\bar{u}; \Omega)$. In particular, our results can be used to compute restricted normals when Ω is a closed convex set, and M is merely some closed subset of Ω. Under an outer semicontinuity assumption, the computation of the restricted normal cone simplifies. For this reason, we adjusted the switching index condition from [9] in order to be able to guarantee the latter outer semicontinuity assumption, at least for the case where Ω is a convex polyhedral set. One application of our results is in the theory of local error bounds, for which the usefulness of restricted normal cones has been proved in [8,17].

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