Stability of Markovian Stochastic Programming

David Wozabal
Vrije Universiteit Amsterdam, Amsterdam, Netherlands, d.wozabal@vu.nl

Multi-stage stochastic programming is notoriously hard, since solution methods suffer from the curse of dimensionality. Recently, stochastic dual dynamic programming has shown promising results for Markovian problems with many stages and a moderately large state space. In order to numerically solve these problems simple discrete representations of Markov processes are required but a convincing theoretical foundation for the generation of these approximations is still lacking. This paper aims to fill this gap and proposes a framework to analyze quantitative stability for multi-stage stochastic optimization problems with a Markovian structure. The results show how the objective values change if the underlying stochastic process is approximated by a simpler one. The resulting bound is formulated using the Fortet-Mourier distance and works for problems whose value functions are locally Lipschitz continuous in the random data. The framework is applicable for important classes of stochastic optimization problems and the results motivate approximations of general Markovian processes by discrete scenario lattices that can be used to obtain numerical solutions. We propose a computationally cheap stochastic gradient descent algorithm for building lattices and show that out-of-sample objectives as well as decisions converge to the respective quantities of the original problem as the approximation gets finer. A numerical study of a multi-period newsvendor problem provides a practical proof of concept of the proposed ideas.

1. Introduction

Multi-stage stochastic programming deals with optimization problems that encompass several sequential decisions in an uncertain future. In every stage of the problem, the decisions depend on updated information about a stochastic process $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_T)$ that models the random data of the problem. Since most real-life optimization problems cannot be solved analytically, numerical solutions are usually obtained by building discrete approximations $\tilde{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$. While for two-stage stochastic programs discretizations merely consist of a set of scenarios for the realization of the second stage randomness, the situation in problems with multiple stages is complicated by the requirement to model the distributions of $\boldsymbol{\xi}_t$ conditional on the history of the process $(\xi_1, \dots, \xi_{t-1})$.

The most general approach to build discretizations for stochastic processes are scenario trees.

There is an extensive literature on the topic of tree generation and on stability results that provide bounds for errors induced by approximating a process by a tree in a stochastic programming problem. In particular, Shapiro (2003, 2008) investigates trees that are built using Monte-Carlo sampling, Høyland and Wallace (2001), Høyland et al. (2003), Kaut and Wallace (2003) match the first moments of the tree with the moments of $\boldsymbol{\xi}$, and Pennanen (2005, 2009) use integration quadratures as guiding principle for tree generation. Another approach is based on the principle of bound-based constructions, see Edirisinghe (1996), Frauendorfer (1996), Kuhn (2005), Casey and Sen (2005). The most common approach in the contemporary literature employs probability metrics to measure the distance between the original and the approximated processes (e.g. Pflug 2001, Dupacová et al. 2003, Heitsch and Römisch 2003, Pflug 2009, Pflug and Pichler 2012).

In a scenario tree there is a unique path from the root node to any other node, which implies that the tree contains information about the whole history of the process. While this property has theoretical merits and enables the most general formulation of stochastic programming problems, it results in a trade-off between accurate modeling of conditional distributions and the size of the tree which determines the computational complexity of the approximate problem. In particular, scenario trees that have no deterministic transitions between nodes necessarily grow exponentially in the number of stages. Consequently, trees for problems with more than a handful of stages always have to have a large number of deterministic state transitions in order to be computationally tractable.

One way to avoid these difficulties is to restrict the attention to more specific problems, where conditional distributions for $\boldsymbol{\xi}_t$ do not depend on the whole history $(\xi_1, \ldots, \xi_{t-1})$, which in turn allows for more compact representations $\tilde{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$. The simplest case in this regard are problems with stage-wise independent randomness, which only require one set of scenarios for every stage, implying a linear growth in the number of required discrete states as the number of stages increases. The next more complicated problem class that still permits a lean discretization are Markov processes, where the conditional distributions of $\boldsymbol{\xi}_{t+1}$ only depend on the current state ξ_t and discretizations take the form of *scenario lattices*. The resulting class of problems presents an interesting compromise between the expressiveness of scenario trees and the simplicitiy of stage-wise independence. Both of these cases have been successfully used in conjunction with *stochastic dual dynamic programming* (SDDP) for problems with a large number of stages (e.g., Pereira and Pinto 1991, Löhndorf et al. 2013, Dowson et al. 2019, Löhndorf and Shapiro 2019, Löhndorf and Wozabal 2021, Bakker et al. 2021, Shinde et al. 2022).

In this paper, we are interested in finite time Markovian stochastic programming based on scenario lattices as the discretization of general multivariate Markovian randomness. In contrast to the sizable literature on scenario trees and despite the increasing popularity of the model class, there is very little work on how to build scenario lattices for stochastic programming problems.

Scenario lattices have long been used in options pricing where binary lattices are routinely used as discretizations of pricing measures. This stream of literature started with the seminal contribution of Cox et al. (1979), who approximate one dimensional continuous-time diffusion processes by discrete-time binomial lattices that weakly converge to the true process as the temporal resolution gets finer. Consequently, evaluating the expectation of option values on lattice process asymptotically yields the true option value and avoids the need for closed form solutions. In the literature on real options pricing, these initial ideas have been substantially refined to incorporate information from observed market prices (Rubinstein 1994) and cover mean reverting processes (e.g. Hahn and Dyer 2008) as well as processes with complex volatilities and jumps (e.g. Harikae et al. 2021, Wang and Dyer 2010). In Wang and Dyer (2010), Chourdakis (2004) multinomial lattice approximations are explored. The general idea of these papers is construct lattices in such a way that the resulting process matches certain characteristics of the original process, mostly volatility and drift.

Broadie and Glasserman (2004) consider discrete time problems based on fairly general stochastic processes that are approximated by multinomial lattices called *stochastic meshes*. Lattices are build using a sampling scheme that is closely motivated by the intended options pricing application. The papers in the real options literature that come closest to our approach are Felix and Weber (2012) and Bardou et al. (2009). Both papers consider pricing problems with complex decisions and use the Wasserstein metric to measure the difference between the original process and the lattice. Felix and Weber (2012) employs k-means clustering on simulated realizations of general processes to minimize the distance between the lattice and the process, while Bardou et al. (2009) restrict their attention to Gaussion randomness and use known optimal quantizers to build their lattices.

Bally and Pagès (2003) refer to lattices as *vector quantization trees* and propose a learning algorithm that minimizes the Wasserstein distance between the unconditional distribution of a Markov chain and the nodes of a lattice. Löhndorf and Wozabal (2021) use a similar approach in a problem of gas storage optimization. However, the focus on unconditional distributions neglects important aspects of the problem and the papers do not contain quantitative stability results. In a recent paper, Kiszka and Wozabal (2022) propose a *lattice distance* and show that the objective values of certain linear stochastic programming problems are Lipschitz continuous with respect to the proposed distance. However, their approach is rather complicated and does not directly lend itself to computationally efficient algorithmic implementation.

We mention that there is a large and well developed theory on the approximations of *Markov decision processes* (MDPs) that is concerned with similar questions as this article. Typical formulations of MDP problems feature finite state and action spaces as well as homogeneous Markov processes describing the randomness, which is potentially influenced by the actions. Furthermore, much of the MDP literature treats infinite horizon problems.

The papers in the MDP literature that come closest to our approach are Müller (1997), Dufour and Prieto-Rumeau (2012, 2013), Saldi et al. (2017). These papers use the Wasserstein metric to impose continuity conditions similar to the ones used in this paper on the Markov kernels and require Lipschitz continuous value functions. In the tradition of the MDP literature, the stability results are with respect to the discretization of the whole state and action space and are thus fully susceptible to the curse of dimensionality. In order to show that value functions are Lipschitz, typically stronger regularity assumptions are required than in this work.

We contribute to the literature by deriving stability results for Markovian stochastic programming problems that can be used as a guiding principle to construct scenario lattices and motivate a fast and straightforward stochastic gradient (SGD) method to generate discretizations based on samples from the original stochastic process. In particular, our contributions are the following:

1. We use the Fortet-Mourier metric to prove a quantitative stability result for multi-stage Markovian stochastic programming problems with locally Lipschitz continuous value functions. We argue that the Fortet-Mourier metric is more flexible and covers a significantly larger set of problems than the Wasserstein metric that is frequently used in the extant literature on tree and lattice generation. In particular, the treatment of problems with randomness in the constraints is possible in the proposed framework. While the Fortet-Mourier metric has been used for two-stage problems (Rachev and Römisch 2002, Han and Chen 2015, Chen and Jiang 2020), to the best of our knowledge, this is the first attempt to employ it to the stability analysis of multi-stage stochastic programs.

2. We demonstrate that the condition of local Lipschitz continuity is fulfilled for a large number of practically relevant problems. In particular, we show that problems with randomness only in the objective, problems with compactly supported randomness, and linear problems with right hand side and objective function randomness all have locally Lipschitz value functions, provided that a natural continuity condition on the conditional distributions of the underlying stochastic process is fulfilled. Our analysis furthermore provides guidance on the optimal topology of scenario lattices, i.e., how many nodes the lattice should have in which stage.

3. We show that as the discretizations get finer, the optimal policies found for the approximated problems can be transferred to the actual problem and the resulting out-of-sample objective values as well as the optimal solutions converge to the respective quantities of the true problem.

4. We propose a computationally inexpensive SGD algorithm that uses simulations from the original process to build approximating scenario lattices.

This paper is organized as follows: In Section 2, we define the problem class of Markovian stochastic optimization problems, review basic facts about probability metrics, and introduce a smoothness assumption for stochastic processes, which is central to our approach. In Section

3, we derive a stability result for problems with locally Lipschitz value functions and show that this property holds for a large set of practically relevant optimization problems. Section 4 is devoted to the generation of scenario lattices using SGD for non-smooth, nonconvex problems, while Section 5 contains results about the convergence of the approximated problems as the discretization gets finer. In Section 6, we use a multi-stage newsvendor example to numerically evaluate in-sample and out-of-sample performance of the proposed approach and compare it to an established method based on the Wasserstein distance. Section 7 concludes the paper.

Notation: We use a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that permits a uniform random variable on $[0,1]$ and denote random vectors $\boldsymbol{\xi}_t : \Omega \to \mathbb{R}^n$ by bold letters while printing their realizations ξ_t in normal font. For a random vector $\boldsymbol{\xi}_t$, we write $\mathrm{supp}(\boldsymbol{\xi}_t)$ for its support. Furthermore, for a random process $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_T)$, we denote the value of $\boldsymbol{\xi}_t$ conditional on the event $\{\omega \in \Omega : \boldsymbol{\xi}_{t-1}(\omega) = \xi_{t-1}\}$ as $\boldsymbol{\xi}_t | \xi_{t-1}$ and for a function $f : \mathbb{R}^n \to \mathbb{R}$, the expectation of $f(\boldsymbol{\xi}_t)$ given $\{\omega \in \Omega : \boldsymbol{\xi}_{t-1}(\omega) = \xi_{t-1}\}$ as $\mathbb{E}[f(\boldsymbol{\xi}_t)|\xi_{t-1}]$. In particular, this implies that for two random processes $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\xi}}$, $\mathbb{E}[f(\boldsymbol{\xi}_t)|\tilde{\xi}_{t-1}]$ is the expectation conditional on the event $\{\omega \in \Omega : \boldsymbol{\xi}_{t-1}(\omega) = \tilde{\xi}_{t-1}\}$ and not on the event $\{\omega \in \Omega : \tilde{\boldsymbol{\xi}}_{t-1}(\omega) = \tilde{\xi}_{t-1}\}$. We use a similar convention for $P(\boldsymbol{\xi}_t | \tilde{\xi}_{t-1})$ and $\boldsymbol{\xi}_t | \tilde{\xi}_{t-1}$. Finally, we denote the set $\{1, \dots, n\}$ by $[n]$ and write *a.s.* for almost surely.

2. Problem Definition and Basic Facts about Probability Metrics

We define the class of Markovian stochastic optimization problems and introduce the required notation in Section 2.1. In Section 2.2, we introduce the Fortet-Mourier distance along with some of its most important properties, while Section 2.3 is devoted to an assumption about the continuity of conditional distributions with respect to the Fortet-Mourier distance, which will be essential in establishing Lipschitz continuity of value functions in Section 3.

2.1. Markovian Stochastic Programming Problems

Let $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_T)$ be a discrete time Markov process where $\boldsymbol{\xi}_t : \Omega \to \mathbb{R}^{N_t}$ and the starting state ξ_1 is deterministic. Consider the following finite horizon Markovian multi-stage stochastic programming problem with T stages

$$V_1(x_0, \xi_1) = \begin{cases} \min\limits_{x_1, x_2, \dots, x_T} & \mathbb{E}\left[\sum_{t=1}^{T} \Pi_t(x_t, \boldsymbol{\xi}_t)\right] \\ \text{s.t.} & x_t \in \mathcal{X}_t(x_{t-1}, \boldsymbol{\xi}_t), \quad t \in [T], \text{ a.s.}, \end{cases} \tag{1}$$

where $\Pi_t(x_t, \boldsymbol{\xi}_t)$ is the immediate profit in stage t, which depends on the decision x_t in stage t as well as the realization of the randomness $\boldsymbol{\xi}_t$. Decisions x_t are required to be in the feasible sets $\mathcal{X}_t(x_{t-1}, \boldsymbol{\xi}_t) \subseteq \mathbb{R}^{m_t}$, which depend on the random data $\boldsymbol{\xi}_t$ as well as the last stage decision x_{t-1}. Note that due to the stochasticity of the problem the decisions $x_t : \Omega \to \mathbb{R}^{M_t}$ are random variables that depend on the realization of the stochastic process. The parameter x_0 together with ξ_1 constitutes the known initial state. We call the problem Markovian, since $\boldsymbol{\xi}$ is Markov and the decisions only depend on the last state of the problem and not explicitly on its entire history.

We will analyze the problem via its dynamic programming equations

$$V_t(x_{t-1}, \xi_t) = \begin{cases} \max\limits_{x_t \in \mathbb{R}^{M_t}} & \Pi_t(\xi_t, x_t) + \mathbb{E}\left[V_{t+1}(x_t, \boldsymbol{\xi}_{t+1}) | \xi_t\right] \\ \text{s.t.} & x_t \in \mathcal{X}(\xi_t, x_{t-1}) \end{cases}, \quad \forall t \in [T] \tag{2}$$

with terminal condition $V_{T+1} \equiv 0$.[1] For the following, it will be convenient to define so-called *post-decision* value functions (see Powell 2011) as

$$\mathcal{V}_t(x_t, \xi_t) = \mathbb{E}\left[V_{t+1}(x_t, \boldsymbol{\xi}_{t+1}) | \xi_t\right], \quad \forall t \in [T].$$

Note that we divide the state space into an *environmental state* ξ_t and a *resource state* x_{t-1} (e.g. Löhndorf et al. 2013, Löhndorf and Wozabal 2021). The former is assumed to be independent of the decisions and typically models external factors such as prices, demand for a product, equipment failure, or environmental variables. The *resource state* x_t, on the other hand, describes the part of the state space that is influenced by the decision maker. Examples include inventory levels, states of machinery, and contractual obligations. In most real-life problems the resource state x_{t-1} is of substantially lower dimension than the decisions taken in stage $t-1$. In the dynamic programming literature the *actions* in stage $t-1$ and the initial state in t are therefore routinely treated as separate. Since for our considerations we do not need to distinguish the state from the actions, we write our problems just in terms of x_{t-1} in order not to over-complicate the notation.

[1] All results in the paper hold, if V_{T+1} is replaced by an arbitrary given concave salvage value $V_{T+1} : \mathbb{R}^{M_T} \to \mathbb{R}$ of x_T. However, for the sake of simplicity, we assume $V_{T+1} \equiv 0$ in what follows.

In what follows, we will approximate problem (1) by replacing $\boldsymbol{\xi}$ by a simpler Markov process $\tilde{\boldsymbol{\xi}} = (\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_T)$ with $\tilde{\boldsymbol{\xi}}_t : \Omega \to \mathbb{R}^{N_t}$ and $\tilde{\xi}_1 = \xi_1$. To that end, we will denote the value functions of the approximated problem where $\boldsymbol{\xi}$ is replaced by $\tilde{\boldsymbol{\xi}}$ as $\tilde{V}_t(x_{t-1}, \tilde{\xi}_t)$ and $\tilde{\mathcal{V}}_t(x_t, \tilde{\xi}_t)$. If the problem is to be solved numerically, a discretely supported process $\tilde{\boldsymbol{\xi}}$ has to be used. However, note that while the results on quantitative stability in Section 3 do not require $\tilde{\boldsymbol{\xi}}$ to be finitely supported, we will need the following assumption, which we will assume to hold throughout.

Assumption 1. *The support of all conditional distributions of $\tilde{\boldsymbol{\xi}}$ is contained in the support of the conditional distributions of $\boldsymbol{\xi}$. More specifically,*

$$\operatorname{supp}(\tilde{\boldsymbol{\xi}}_t | \tilde{\xi}_{t-1}) \subseteq \operatorname{supp}(\boldsymbol{\xi}_t | \tilde{\xi}_{t-1}), \quad \forall \tilde{\xi}_{t-1} \in \operatorname{supp}(\tilde{\boldsymbol{\xi}}_{t-1}), \quad \forall t : 2 \leq t \leq T.$$

The above assumption basically states that the original *complicated* process $\boldsymbol{\xi}$ is *finer* than the *simpler* approximating process $\tilde{\boldsymbol{\xi}}$ in that it can take more values, which introduces an asymmetry between $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\xi}}$. In particular, this implies that we can evaluate $V_t(x_{t-1}, \tilde{\xi}_t)$ for all $\tilde{\xi}_t \in \operatorname{supp}(\tilde{\boldsymbol{\xi}}_t)$, i.e., it makes sense to plug realizations of $\tilde{\boldsymbol{\xi}}$ into the value functions for the process $\boldsymbol{\xi}$. Note that the reverse is not necessarily always possible, i.e., $\tilde{V}_t(x_{t-1}, \xi_t)$ need not be defined for all $\xi_t \in \operatorname{supp}(\boldsymbol{\xi}_t)$.

2.2. The Fortet-Mourier Metric

We analyze the stability of (1) using the Fortet-Mourier metric, which has been used previously in analysis of the stability of two-stage stochastic programming problems (e.g., Rachev and Römisch 2002, Han and Chen 2015, Chen and Jiang 2020).

For two probability measures P_1 and P_2 on $\mathbb{R}^N$, the Fortet-Mourier metric is defined as

$$d^p_{FM}(P_1, P_2) = \sup \left\{ \int f(x)\, d(P_1 - P_2) : f \in \operatorname{Lip}^p(\mathbb{R}^N) \right\}$$

where

$$\operatorname{Lip}^p(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \to \mathbb{R} \text{ borel} : |f(x) - f(y)| \leq \max(1, ||x||, ||y||)^{p-1}\, ||x - y|| \right\}$$

are the functions that are locally Lipschitz with moduli that fulfill a polynomial growth condition of order p. Note that by using $\operatorname{Lip}^p(\mathbb{R}^N)$ as *test functions*, the Fortet-Mourier metric falls into the class of metrics with a ζ-structure that metricizes the weak topology on the set of probability measures with finite p-th moment (Rachev and Römisch 2002). In the following, we will frequently write $d^p_{FM}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ to mean $d^p_{FM}(P_{\boldsymbol{\xi}_1}, P_{\boldsymbol{\xi}_2})$ with $P_{\boldsymbol{\xi}_1}$ and $P_{\boldsymbol{\xi}_2}$ the image measures of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, respectively.

Given two measures P_1 and P_2 on $\mathbb{R}^N$, the Fortet-Mourier metric has a dual representation as the following *transshipment problem* (Rachev and Römisch 2002)

$$d^p_{FM}(P_1, P_2) = \begin{cases} \inf_z \int \max(1, ||x||, ||y||)^{p-1} ||x-y||\, z(dx, dy) \\ \text{s.t.} \;\; z(B \times \mathbb{R}^N) - P_1(B) = z(\mathbb{R}^N \times B) - P_2(B), \quad \forall B \;\text{ Borel}, \end{cases} \tag{3}$$

where z is a Borel measure on $\mathbb{R}^N \times \mathbb{R}^N$. In the special case of $p = 1$, d^p_{FM} reduces to the well-known Wasserstein metric, for which (3) is a transport problem, i.e., the constraints on the joint measure can be written as $z(B \times \mathbb{R}^N) = P_1(B)$ and $z(\mathbb{R}^N \times B) = P_2(B)$. The reason for this is that the first term in the objective vanishes, making it suboptimal to transport mass indirectly via intermediate transshipment nodes instead of directly between the source and the target.

2.3. An Assumption on Conditional Distributions

To facilitate our analysis in the next section and prove that value functions of Markovian problems are indeed locally Lipschitz and therefore amendable to analysis with the Fortet-Mourier metric, we require the following continuity assumption on the stochastic process $\boldsymbol{\xi}$. Similar assumptions using the Wasserstein distance can be found in the MDP literature (e.g., Hinderer 2005, Dufour and Prieto-Rumeau 2012, 2013, Saldi et al. 2017).

Assumption 2. *There are constants $L^C_t > 0$ and $p^C_t \in \mathbb{N}$ such that*

$$d^{p^C_t}_{FM}(\boldsymbol{\xi}_{t+1}|\xi_t, \boldsymbol{\xi}_{t+1}|\xi'_t) \le L^C_t \, \max(1, ||\xi_t||, ||\xi'_t||)^{p^C_t - 1} \, ||\xi_t - \xi'_t||, \quad \forall \xi_t, \xi'_t \in \operatorname{supp}(\boldsymbol{\xi}_t). \tag{4}$$

This essentially means that conditional distributions for $\boldsymbol{\xi}_{t+1}$ are required to be locally Lipschitz continuous in ξ_t with respect to the Fortet-Mourier distance. This property enables to bound the error that is made when representing conditional distributions of $\boldsymbol{\xi}_{t+1}$ for different ξ_t by a single distribution $\tilde{\boldsymbol{\xi}}_{t+1}|\tilde{\xi}_t$.

Assumption 2 is natural because a violation would imply that varying ξ_t only slightly may lead to vastly different conditional distributions of $\boldsymbol{\xi}_{t+1}$ which would make a discrete approximation practically impossible. Luckily this condition is fulfilled for all Markov processes commonly used in stochastic programming. This is illustrated by the sufficient condition in the next result.

THEOREM 1. *Let $\boldsymbol{\xi}$ be Markov with transition functions $\boldsymbol{\xi}_{t+1} = f_{t+1}(\xi_t, \boldsymbol{\varepsilon}_{t+1})$ depending affinely on ξ_t and a random innovation $\boldsymbol{\varepsilon}_{t+1} : \Omega \to \mathbb{R}^k$ with image measure P_{t+1}, i.e.,*

$$f_{t+1}(\xi_t, \boldsymbol{\varepsilon}_{t+1}) = a_{t+1}(\boldsymbol{\varepsilon}_{t+1}) + A_{t+1}(\boldsymbol{\varepsilon}_{t+1})\xi_t \tag{5}$$

where $a_{t+1}(\varepsilon_{t+1}) : \mathbb{R}^k \to \mathbb{R}^{N_t}$ and $A_{t+1}(\varepsilon_{t+1}) : \mathbb{R}^k \to \mathbb{R}^{N_{t+1} \times N_t}$. If additionally

$$\int \left(||a_{t+1}(\varepsilon_{t+1})||^{p_t^C - 1} + \max(1, ||A_{t+1}(\varepsilon_{t+1})||)^{p_t^C - 1} \right) ||A_{t+1}(\varepsilon_{t+1})|| \, P_{t+1}(d\varepsilon_{t+1}) < \infty \tag{6}$$

for $||A_{t+1}||$ the operator-norm of A_{t+1}, then property (4) is satisfied.

Proof. Define z by the transport map $T(x) = x$ between $\boldsymbol{\xi}_{t+1}|\xi_t$ and $\boldsymbol{\xi}_{t+1}|\xi_t'$, i.e., $z(A \times B) = P_{t+1}(A \cap B)$ and further denote

$$\Lambda(\varepsilon_{t+1}) = \max(1, ||a_{t+1}(\varepsilon_{t+1}) + A_{t+1}(\varepsilon_{t+1})\xi_t||, ||a_{t+1}(\varepsilon_{t+1}) + A_{t+1}(\varepsilon_{t+1})\xi_t'||)^{p_t^C - 1}.$$

We then can bound the Fortet-Mourier distance by

$$\begin{aligned}
d_{FM}^{p_t^C}(\boldsymbol{\xi}_{t+1}|\xi_t, \boldsymbol{\xi}_{t+1}|\xi_t') &\leq \int \max(1, ||\xi_{t+1}||, ||\xi_{t+1}'||)^{p_t^C - 1} \, ||\xi_{t+1} - \xi_{t+1}'|| \, z(d\xi_{t+1}, d\xi_{t+1}') \\
&= \int \Lambda(\varepsilon_{t+1}) \, ||A_{t+1}(\varepsilon_{t+1})(\xi_t - \xi_t')|| \, P_{t+1}(d\varepsilon_{t+1}) \\
&\leq \int \Lambda(\varepsilon_{t+1}) ||A_{t+1}(\varepsilon_{t+1})|| \, P_{t+1}(d\varepsilon_{t+1}) \, ||\xi_t - \xi_t'||.
\end{aligned}$$

We continue by noting that

$$\begin{aligned}
||a_{t+1}(\varepsilon_{t+1}) + A_{t+1}(\varepsilon_{t+1})\xi_t||^{p_t^C - 1} &= 2^{p_t^C - 1} \left\| \frac{a_{t+1}(\varepsilon_{t+1})}{2} + \frac{A_{t+1}(\varepsilon_{t+1})\xi_t}{2} \right\|^{p_t^C - 1} \\
&\leq 2^{p_t^C - 2} (||a_{t+1}(\varepsilon_{t+1})||^{p_t^C - 1} + ||A_{t+1}(\varepsilon_{t+1})\xi_t||^{p_t^C - 1}) \\
&\leq 2^{p_t^C - 2} (||a_{t+1}(\varepsilon_{t+1})||^{p_t^C - 1} + ||A_{t+1}(\varepsilon_{t+1})||^{p_t^C - 1} ||\xi_t||^{p_t^C - 1}),
\end{aligned}$$

where the second line follows from Jensen's inequality. It therefore holds that

$$\Lambda(\varepsilon_{t+1}) \leq 2^{p_t^C - 2} ||a_{t+1}(\varepsilon_{t+1})||^{p_t^C - 1} + 2^{p_t^C - 2} \max(1, ||A_{t+1}(\varepsilon_{t+1})||)^{p_t^C - 1} \max(1, ||\xi_t||, ||\xi_t'||)^{p_t^C - 1}$$

$$\leq 2^{p_t^C-2}\left(||a_{t+1}(\varepsilon_{t+1})||^{p_t^C-1}+\max(1,||A_{t+1}(\varepsilon_t)||)^{p_t^C-1}\max(1,||\xi_t||,||\xi_t'||)^{p_t^C-1}\right)$$

and

$$d_{FM}^{p_t^C}(\boldsymbol{\xi}_{t+1}|\xi_t,\boldsymbol{\xi}_{t+1}|\xi_t')\leq L_t^C\ \max(1,||\xi_t||,||\xi_t'||)^{p_t^C-1}||\xi_t-\xi_t'||$$

with

$$L_t^C=2^{p_t^C-2}\int(||a_{t+1}(\varepsilon_{t+1})||^{p_t^C-1}+\max(1,||A_{t+1}(\varepsilon_{t+1})||)^{p_t^C-1})||A_{t+1}(\varepsilon_{t+1})||\,P_{t+1}(d\varepsilon_{t+1})<\infty,$$

which finishes the proof. □

The above result covers most of the classic models of arithmetic and geometric randomness that are routinely used in stochastic programming as shown in the examples below.

Example 1 (Arithmetic Randomness). Consider a process of the form (5) with

$$\boldsymbol{\xi}_{t+1}=\phi(t+1)+A_{t+1}\boldsymbol{\xi}_t+\boldsymbol{\varepsilon}_{t+1}$$

with ϕ a deterministic seasonality, $A_{t+1}\in\mathbb{R}^{N_{t+1}\times N_t}$, and $\boldsymbol{\varepsilon}_{t+1}$ independent innovations. Clearly, the above definition covers the cases where the deseasonalized process $\xi_t-\phi(t)$ follows a Markovian time series model like VARMA or GARCH and, in particular, time discretized versions of arithmetic Itô processes as long as the integrability condition (6) for $\boldsymbol{\varepsilon}_{t+1}$ is fulfilled.

Example 2 (Geometric Randomness). Processes $\boldsymbol{\xi}$ of the form

$$\boldsymbol{\xi}_{t+1}=\boldsymbol{\xi}_tA_{t+1}(\boldsymbol{\varepsilon}_{t+1})$$

fulfill (5) and cover processes of the *geometric* type like the time discretized geometric Brownian motion and similar processes. Again (4) holds because of Theorem 1 if the moment condition (6) is fulfilled for $\boldsymbol{\varepsilon}_{t+1}$.

3. An Error Bound for Markovian Stochastic Programs

In Section 3.1, we derive a quantitative stability result for Markovian stochastic problems using the Fortet-Mourier distance based on the assumption that the value functions $V_t(x_{t-1},\xi_t)$ of the problem are locally Lipschitz continuous in ξ_t uniformly in x_{t-1}. In Section 3.2, we show the uniform Lipschitz property for three classes of problems based on the continuity assumption for the conditional distributions discussed in Section 2.3.

3.1. A Stability Result Based on Fortet-Mourier Distances

In the following, we will prove a quantitative stability result for problems of the form (1) given that the value functions $V_t(x_{t-1},\xi_t)$ are uniformly locally Lipschitz in x_{t-1}, i.e.,

$$|V_t(x_{t-1},\xi_t) - V_t(x_{t-1},\xi_t')| \le L_t \, \max(1,||\xi_t||,||\xi_t'||)^{p_t-1} \, ||\xi_t - \xi_t'||, \quad \forall x_{t-1}. \tag{7}$$

This means that $\xi_t \mapsto L_t^{-1} \, V_t(x_{t-1},\xi_t) \in \mathrm{Lip}^{p_t}(\mathbb{R}^{N_t})$ for all x_{t-1}, which allows us to use $d_{FM}^{p_t}$ to bound differences in objective values of stochastic programs. The recursive application of this principle backwards in time yields the following result.

THEOREM 2. *If Assumption 1 and 2 hold for problem* (1)*, then*

$$|V_t(x_{t-1},\xi_t) - \tilde{V}_t(x_{t-1},\tilde{\xi}_t)| \le \sum_{s=t+1}^{T} L_s \, \mathbb{E}[d_{FM}^{p_s}(\boldsymbol{\xi}_s|\tilde{\boldsymbol{\xi}}_{s-1},\tilde{\boldsymbol{\xi}}_s|\tilde{\boldsymbol{\xi}}_{s-1})|\tilde{\xi}_t], \quad \forall t=1,\dots,T-1. \tag{8}$$

Proof. At stage T, $V_T(x_{T-1},\tilde{\xi}_T) = \tilde{V}_T(x_{T-1},\tilde{\xi}_T)$ and therefore we get for stage $T-1$

$$\begin{aligned} V_{T-1}(x_{T-2},\tilde{\xi}_{T-1}) - \tilde{V}_{T-1}(x_{T-2},\tilde{\xi}_{T-1}) &\le \mathbb{E}[V_T(x_{T-1}^*,\boldsymbol{\xi}_T)|\tilde{\xi}_{T-1}] - \mathbb{E}[\tilde{V}_T(x_{T-1}^*,\tilde{\boldsymbol{\xi}}_T)|\tilde{\xi}_{T-1}] \\ &= \mathbb{E}[V_T(x_{T-1}^*,\boldsymbol{\xi}_T)|\tilde{\xi}_{T-1}] - \mathbb{E}[V_T(x_{T-1}^*,\tilde{\boldsymbol{\xi}}_T)|\tilde{\xi}_{T-1}] \\ &\le L_T \, d_{FM}^{p_T}(\boldsymbol{\xi}_T|\tilde{\xi}_{T-1},\tilde{\boldsymbol{\xi}}_T|\tilde{\xi}_{T-1}), \end{aligned}$$

where x_{T-1}^* is an optimal decision for $V_{T-1}(x_{T-2},\tilde{\xi}_{T-1})$ and the second inequality follows from the definition of the Fortet-Mourier metric and (7). Repeating the argument with reversed roles of V_{T-1} and $\tilde{V}_{T-1}$ yields

$$|V_{T-1}(x_{T-2},\tilde{\xi}_{T-1}) - \tilde{V}_{T-1}(x_{T-2},\tilde{\xi}_{T-1})| \le L_T \, d_{FM}^{p_T}(\boldsymbol{\xi}_T|\tilde{\xi}_{T-1},\tilde{\boldsymbol{\xi}}_T|\tilde{\xi}_{T-1}).$$

We now assume that the result was already proven for stage t and write

$$\begin{aligned} &V_{t-1}(x_{t-2},\tilde{\xi}_{t-1}) - \tilde{V}_{t-1}(x_{t-2},\tilde{\xi}_{t-1}) \\ &\le \mathbb{E}[V_t(x_{t-1}^*,\boldsymbol{\xi}_t)|\tilde{\xi}_{t-1}] - \mathbb{E}[\tilde{V}_t(x_{t-1}^*,\tilde{\boldsymbol{\xi}}_t)|\tilde{\xi}_{t-1}] \\ &= \mathbb{E}[V_t(x_{t-1}^*,\boldsymbol{\xi}_t)|\tilde{\xi}_{t-1}] - \mathbb{E}[V_t(x_{t-1}^*,\tilde{\boldsymbol{\xi}}_t)|\tilde{\xi}_{t-1}] + \mathbb{E}[V_t(x_{t-1}^*,\tilde{\boldsymbol{\xi}}_t) - \tilde{V}_t(x_{t-1}^*,\tilde{\boldsymbol{\xi}}_t)|\tilde{\xi}_{t-1}] \\ &\le L_t \, d_{FM}^{p_t}(\boldsymbol{\xi}_t|\tilde{\xi}_{t-1},\tilde{\boldsymbol{\xi}}_t|\tilde{\xi}_{t-1}) + \sum_{s=t+1}^{T} L_s \, \mathbb{E}[d_{FM}^{p_s}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s-1},\tilde{\boldsymbol{\xi}}_s|\tilde{\boldsymbol{\xi}}_{s-1})|\tilde{\xi}_{t-1}] \end{aligned}$$

$$= \sum_{s=t}^{T} L_s \, \mathbb{E}[d_{FM}^{p_s}(\boldsymbol{\xi}_s|\tilde{\boldsymbol{\xi}}_{s-1}, \tilde{\boldsymbol{\xi}}_s|\tilde{\boldsymbol{\xi}}_{s-1})|\tilde{\xi}_{t-1}],$$

where x^*_{t-1} is optimal for $V_{t-1}(x_{t-2}, \tilde{\xi}_{t-1})$ and the last inequality is due to the Lipschitz continuity of V_t, Assumption 2, and the induction hypothesis. Reversing the roles of V_{t-1} and $\tilde{V}_{t-1}$ as above yields the desired result for $t-1$ and concludes the proof. □

Plugin in $t = 1$ in (8) yields a bound between the objective values of the original problem and its approximated counterpart. Note that due to Assumption 1, the roles of $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\xi}}$ are not symmetric in the above result. Note further that, in contrast to many other stability results in stochastic programming, the bound does not depend on any convexity assumptions. However, as we will see below, convexity is often convenient in establishing (7) for concrete problems.

3.2. Problems with Locally Lipschitz Value Functions

In the following, we will show (7) for several important cases that cover most of the problems encountered in real-life applications: In Theorem 3 the objective is random while the feasible set is deterministic, Theorem 4 treats the case where the randomness is compactly supported, and lastly Theorem 5 discusses the case of linear problems. Furthermore, note that local Lipschitz continuity is a natural property of value functions (e.g. Berkovitz 1989, Clarke 1990, Veliov 1997), hence, it is plausible that are a variety of other cases in which (7) is fulfilled as well.

We start discussing problems where randomness only enters in the objective function. These problems received a fair share of attention, since many stability results in the literature using scenario trees are restricted to this setting (see e.g., Pflug and Pichler 2014). Note that we do not make any assumptions on the convexity of the problem.

THEOREM 3. *If the feasible sets $\mathcal{X}_t(x_{t-1}, \xi_t) = \mathcal{X}_t(x_{t-1})$ are deterministic, and the profit functions are uniformly locally Lipschitz, i.e.,*

$$|\Pi_t(x_t, \xi_t) - \Pi_t(x_t, \xi_t')| \le L_t^{\Pi} \, \max(1, ||\xi_t||, ||\xi_t'||)^{p_t^{\Pi}-1} \, ||\xi_t - \xi_t'||, \quad \forall x_t, \tag{9}$$

and Assumption 2 holds with $p_t^C \ge \max(\max_{s>t} p_s^C, \max_{s>t} p_s^{\Pi})$, then (7) holds with $p_t = \max(p_t^C, p_t^{\Pi})$.

Proof. Clearly, for $t=T$, we get by (9)

$$
\begin{aligned}
V_T(x_{T-1},\xi_T) - V_T(x_{T-1},\xi'_T) &\leq \Pi_T(x^*_T,\xi_T) - \Pi_t(x^*_T,\xi'_T) \\
&\leq L^{\Pi}_T \max(1,||\xi_T||,||\xi'_T||)^{p^{\Pi}_t-1}\,||\xi_T-\xi'_T||,
\end{aligned}
$$

where x^*_T is the optimal solution for the problem $V_T(x_{T-1},\xi_T)$. Reversing the roles of $V_T(x_{T-1},\xi_T)$ and $V_T(x_{T-1},\xi'_T)$ proves the result for the last stage.

Proceeding by backward induction and assuming that (7) was already shown for V_{t+1} with Lipschitz constant L_{t+1}, we get, again using (9) and Assumption 2,

$$
\begin{aligned}
&V_t(x_{t-1},\xi_t) - V_t(x_{t-1},\xi'_t) \\
&\leq \Pi_t(x^*_t,\xi_t) + \mathbb{E}[V_{t+1}(x^*_t,\boldsymbol{\xi}_{t+1})|\xi_t] - (\Pi_t(x^*_t,\xi'_t) + \mathbb{E}[V_{t+1}(x^*_t,\boldsymbol{\xi}_{t+1})|\xi'_t]) \\
&\leq L^{\Pi}_t \max(1||\xi_t||,||\xi'_t||)^{p^{\Pi}_t-1}\,||\xi_t-\xi'_t|| + L_{t+1} d^{p_{t+1}}_{FM}(\boldsymbol{\xi}_{t+1}|\xi_t,\boldsymbol{\xi}_{t+1}|\xi'_t) \\
&\leq L^{\Pi}_t \max(1||\xi_t||,||\xi'_t||)^{p^{\Pi}_t-1}\,||\xi_t-\xi'_t|| + L_{t+1}L^C_t \max(1||\xi_t||,||\xi'_t||)^{p^C_t-1}\,||\xi_t-\xi'_t|| \\
&= \left(L^{\Pi}_t + L_{t+1}L^C_t\right)\max(1||\xi_t||,||\xi'_t||)^{p_t-1}\,||\xi_t-\xi'_t||.
\end{aligned}
$$

with $p_t = \max(p^C_t, p^{\Pi}_t)$. Reversing the roles of ξ_t and ξ'_t concludes the proof. □

Remark 1. The coefficient p_t of the local Lipschitz condition (7) is weakly increasing backwards in time, i.e., earlier stages cannot have lower moduli than later stages. However, if all profit functions are globally Lipschitz and condition (4) holds with $p^C_t = 1$ for all $t \in [T]$, then the Wasserstein distance can be used instead of the Fortet-Mourier distance in Theorem 2.

Remark 2. The constants L_t are increasing towards the first stage, which together with the previous remark implies that the earlier stages require a *finer* representation in order to guarantee good results when approximating the process $\boldsymbol{\xi}$. This gives a hint at how to optimally design scenario lattices and, in particular, how many nodes in which stage to use. This modeling issue regularly arises in practical applications, but there seem to exist no theoretical results in the literature that could guide the optimal choice of a topology for scenario trees or lattices.

For the next two models classes, we require the following compactness assumption for the feasible sets, which is usually innocuous in real-world applications.

Assumption 3. *For every stage $t \in [T]$ the feasible sets are contained in a $||.||_\infty$-ball, i.e., for every $t \in [T]$ there are $D_t > 0$ such that*

$$\sup_{x_t \in \mathcal{X}_t(x_{t-1},\xi_t)} ||x_t||_\infty < D_t, \quad \forall \xi_t \in \operatorname{supp}(\boldsymbol{\xi}_t), \quad \forall x_{t-1}.$$

Using this assumption, we continue with the case where additionally $\boldsymbol{\xi}_t$ is compactly supported in every stage, i.e.,

$$\forall t \in [T]\, \exists C_t \subseteq \mathbb{R}^{N_t} \text{ compact} : \mathbb{P}(\boldsymbol{\xi}_t > 0 \cap C_t) = 1. \tag{10}$$

In what follows, we will vary ξ_t in V_t in the objective and the constraints of the respective problems separately. To that end, we define three parameter versions of the value functions as

$$V_t(x_{t-1},\xi_t^1,\xi_t^2) = \begin{cases} \max_{x_t} & \Pi_t(x_t,\xi_t^2) + \mathcal{V}_t(x_t,\xi_t^2) \\ \text{s.t} & x_t \in \mathcal{X}_t(x_{t-1},\xi_t^1), \end{cases}$$

where ξ_t^1 models the impact of the randomness on the constraints, while ξ_t^2 determines the value of ξ_t in the objective. Naturally, $V_t(x_{t-1},\xi_t) = V_t(x_{t-1},\xi_t,\xi_t)$.

Theorem 4. *If Assumption 2 holds with $p_t^C \geq p_{t+1}^C$, Assumption 3 is fulfilled,* (10) *holds, $(x_t,\xi_t) \mapsto \Pi_t(x_t,\xi_t)$ is continuous, and the feasible sets are of the form*

$$\mathcal{X}_t(x_{t-1},\xi_t) = \{x_t : f_t(x_t,x_{t-1},\xi_t) \leq 0\},$$

with $(x_t,\xi_t) \mapsto f_t(x_t,x_{t-1},\xi_t)$ convex for all x_{t-1}, then (7) *follows with $p_t = p_t^C$*

Proof. Note that by the compactness of $C_t \times \mathcal{X}_t(x_{t-1},\xi_t)$

$$\xi_t \mapsto \Pi_t(x_t,\xi_t)$$

is Lipschitz with a constant L_t^Π uniformly in x_t on the support of $\boldsymbol{\xi}_t$.

For stage T and realizations ξ_T and ξ_T', we get

$$\begin{aligned} V_T(x_{T-1},\xi_T) = \Pi_T(x_T^*,\xi_T) &\leq \Pi_T(x_T^*,\xi_T') + L_T^\Pi\, ||\xi_T - \xi_T'|| \\ &\leq V_T(x_{T-1},\xi_T,\xi_T') + L_T^\Pi\, ||\xi_T - \xi_T'|| \end{aligned}$$

and therefore

$$|V_T(x_{T-1},\xi_T) - V_T(x_{T-1},\xi_T,\xi_T')| \leq L_T^\Pi\, ||\xi_T - \xi_T'||. \tag{11}$$

From our assumptions on $\mathcal{X}_t(x_{t-1},\xi_t)$ and Theorem 5.2 in Still (2018) it follows that the mapping $(x_{T-1},\xi_T)\mapsto V_T(x_{T-1},\xi_T,\xi_T')$ is continuous and therefore we get by (11) that for every $\varepsilon>0$ there is a $\delta>0$ such that for $||x_{T-1}-x'_{T-1}||+||\xi_T-\xi'_T||<\delta$, we have

$$\begin{aligned}
|V_T(x_{T-1},\xi_T)-V_T(x'_{T-1},\xi'_T)| &= |V_T(x_{T-1},\xi_T)-V_T(x'_{T-1},\xi'_T,\xi'_T)|\\
&\le |V_T(x_{T-1},\xi_T,\xi'_T)-V_T(x'_{T-1},\xi'_T,\xi'_T)|+L_T^{\Pi}\,||\xi_T-\xi'_T||\\
&\le \varepsilon+L_T^{\Pi}\,||\xi_T-\xi'_T||\\
&\le 2\varepsilon
\end{aligned}$$

for $||\xi_T-\xi'_T||$ sufficiently small. Hence, continuity of V_T follows and from that Lipschitz continuity can be deduced from the compactness of $C_T\times\mathcal{X}_T(x_{T-1},\xi_T)$.

For $t<T$, we assume that the result was already shown for V_{t+1} with Lipschitz constant L_{t+1}. We therefore get by Assumption 2

$$\begin{aligned}
V_t(x_{t-1},\xi_t) &= \Pi_t(x_t^*,\xi_t)+\mathbb{E}[V_{t+1}(x_t^*,\boldsymbol{\xi}_{t+1})|\xi_t]\\
&\le \Pi_t(x_t^*,\xi'_t)+\mathbb{E}[V_{t+1}(x_t,\boldsymbol{\xi}_{t+1})|\xi'_t]+L_t\,\max(1,||\xi_t||,||\xi'_t||)^{p_t-1}\,||\xi_t-\xi'_t||\\
&\le V_t(x_{t-1},\xi_t,\xi'_t)+L_t\;\max(1,||\xi_t||,||\xi'_t)^{p_t-1}\,||\xi_t-\xi'_t||,
\end{aligned}$$

with $L_t=(L_{t+1}L_t^C+L_t^{\Pi})$. We again deduce from Theorem 5.2 in Still (2018) that $(x_{t-1},\xi_t)\mapsto V_t(x_{t-1},\xi_t,\xi'_t)$ is continuous so that the rest of the argument follows like the case for T and backward induction finishes the proof. □

Remark 3. Note that the convexity assumption on the set $\mathcal{X}_t(x_{t-1},\xi_t)$ holds in particular for convex problems if the randomness is *on the right hand side* of the constraint, i.e., if $f_t(x_{t-1},x_t,\xi_t)=f_t^1(x_{t-1},x_t)+f_t^2(\xi_t)$ is separable in ξ_t and f_t^2 is convex.

Remark 4. The modulus of continuity is only determined by the modulus of continuity in Assumption 2 and remains constant over the stages, if p_t^C are the same for all t. This, in particular, implies that, like in Theorem 3, the Wasserstein distance could be used instead of the Fortet-Mourier distance if $p_t^C=1$ for all $t\in[T]$. However, as in the case of Theorem 3, the Lipschitz constant grows backward in time, indicating that, everything else being equal, a finer discretization is required for earlier stages.

The last case we are discussing is linear multi-stage stochastic programming with fixed recourse, i.e., we assume the value functions are of the form

$$V_t(x_{t-1},\xi_t) = \begin{cases} \max_{x_t} & \langle \pi_t(\xi_t), x_t\rangle + \mathcal{V}_t(x_t,\xi_t) \\ \text{s.t} & A_t x_t \le T_t x_{t-1} + R_t \xi_t, \end{cases} \tag{12}$$

where $\pi_t(\xi_t) \in \mathbb{R}^{M_t}$ and $A_t \in \mathbb{R}^{n\times M_t}$, $T_t \in \mathbb{R}^{n\times M_{t-1}}$ and $R_t \in \mathbb{R}^{n\times N_t}$ describe the feasible sets $\mathcal{X}(x_{t-1},\xi_t)$ by n linear constraints.

We start by showing some results about the local Lipschitz continuity of V_t for the case that for every ξ_t $\mathcal{V}_t(x_t,\xi_t)$ can be exactly represented as a piecewise linear function of x_t, i.e., is of the form

$$\mathcal{V}_t(x_t,\xi_t) = \min_{1\le i\le I} a_i(\xi_t) + \langle b_i(\xi_t), x_t\rangle \tag{13}$$

In this case, the problems in the stages become linear programs, which enables the use of classic results about the Lipschitz continuity of linear systems as in Mangasarian and Shiau (1987) or Theorem 7.13 in Shapiro et al. (2009).

Lemma 1. *Let $\mathcal{V}_t$ be of the form (13) resulting in*

$$V_t(x_{t-1},\xi_t) = \begin{cases} \max\limits_{x_t,\theta} & \langle \pi_t(\xi_t), x_t\rangle + \theta \\ s.t & A_t x_t \le c \\ & a_i(\xi_t) + \langle b_i(\xi_t), x_t\rangle \ge \theta, \quad \forall i \in [I] \end{cases}$$

with $c = T_t x_{t-1} + R_t\xi_t$. Denoting $\mathcal{S}(c)$ as the set of optimal solutions as a function of the right hand side, we have

$$d_H(\mathcal{S}(c),\mathcal{S}(c')) \le K\,||c-c'||_1,$$

where d_H is the Hausdorff distance and K only depends on the matrix A_t and the Lipschitz constant of $x_t \mapsto \mathcal{V}_t(x_t,\xi_t)$.

Proof. Defining Φ to be the matrix with $b_i(\xi_t)$ as rows and

$$B = \begin{pmatrix} A_t & 0 \\ -\Phi & 1 \end{pmatrix}$$

we get by Mangasarian and Shiau (1987) that

$$d_H(\mathcal{S}(c),\mathcal{S}(c')) \le \mu(A_t,\Phi)||c-c'||_1$$

where

$$\mu(A_t,\Phi) = \sup \left\{ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\infty \;\middle|\; \begin{array}{l} \|(u,v)B\|_1 \le 1 \\ u \ge 0,\ v \ge 0 \\ \text{the rows of } B \text{ corresponding} \\ \text{to non-zero elements of } (u,v) \\ \text{are linearly independent} \end{array} \right\}.$$

Since the last element of $(u,v)B$ equals $\sum v_i$, we get that $\sum_i v_i \le 1$ and consequently

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\infty \le ||u||_\infty + ||v||_\infty \le ||u||_\infty + 1.$$

Furthermore, we get that $||uA_t - v\Phi||_1 \le 1$ implies that

$$||uA_t||_1 \le ||uA_t - v\Phi||_1 + ||v\Phi||_1 \le 1 + L$$

where L is the Lipschitz constant of the value function.

We therefore have

$$\mu(A_t,\Phi) \le \mu(A_t) = \sup \left\{ \|u\|_\infty \;\middle|\; \begin{array}{l} \|uA_t\|_1 \le 1 + L \\ u \ge 0 \\ \text{the rows of } A \text{ corresponding} \\ \text{to non-zero elements of } u \\ \text{are linearly independent} \end{array} \right\}.$$

Hence, the result follows with $K = \mu(A_t)$. □

We use the above lemma in the following two results, once for changes in x_{t-1} and once for changes in ξ_t in the constraints.

LEMMA 2. *If $V_t(x_{t-1},\xi_t)$ and $\mathcal{V}_t(x_t,\xi_t)$ are of the form* (12) *and* (13)*, respectively, there is an $L_t^x < \infty$ and p_t^x such that*

$$|\mathcal{V}_t(x_t,\xi_t) - \mathcal{V}_t(x_t',\xi_t)| \le L_t^x \max(1, ||\xi_t||_1)^{p_t^x - 1}\, ||x_t - x_t'||_1 \tag{14}$$

and

$$||\pi_t(\xi_t)||_2 \le C_t^\pi ||\xi_t||_1^{p_t^\pi - 1}, \tag{15}$$

then there is a $L_t^\xi < \infty$ such that for $p_t^\xi = \max(p_t^x, p_t^\pi)$ and arbitrary ξ_t and ξ_t'

$$|V_t(x_{t-1},\xi_t,\xi_t) - V_t(x_{t-1},\xi_t',\xi_t)| \le L_t^\xi \max(1, ||\xi_t||)^{p_t^\xi - 1}\, ||\xi_t - \xi_t'||_1, \quad \forall x_{t-1}. \tag{16}$$

Proof. We get by Lemma 1 that for an optimal solution x_t^* of $V_t(x_{t-1},\xi_t',\xi_t)$

$$\mathrm{dist}(x_t^*, \mathcal{S}(\xi_t')) \le K_2\, ||\xi_t - \xi_t'||_1. \tag{17}$$

where $\mathcal{S}(\xi_t')$ is the set of optimal solutions to $V_t(x_{t-1}, \xi_t', \xi_t)$ and the constant K_2 only depends on the matrix A_t and in particular is independent of ξ_t and ξ_t'.

To translate the above bound to the distance between optimal values for different ξ_t in the constraints, holding ξ_t in the objective fixed, we have

$$\begin{aligned}
&|V_t(x_{t-1}, \xi_t, \xi_t) - V_t(x_{t-1}, \xi_t', \xi_t)| \\
&= |\langle \pi_t(\xi_t), x_t^* - x_t^{*\prime} \rangle + (\mathcal{V}_t(x_t^*, \xi_t) - \mathcal{V}_t(x_t^{*\prime}, \xi_t))| \\
&\leq ||\pi_t(\xi_t)||_2 \, ||x_t^* - x_t^{*\prime}||_2 + L_t^x \, \max(1, ||\xi_t||_1)^{p_t^x - 1} \, ||x_t^* - x_t^{*\prime}||_1 \\
&\leq C_t^\pi ||\xi_t||_1^{p_t^\pi - 1} \, ||x_t^* - x_t^{*\prime}||_2 + L_t^x \, \max(1, ||\xi_t||_1)^{p_t^x - 1} \, ||x_t^* - x_t^{*\prime}||_1 \\
&\leq (C_t^{\pi\prime} + L_t^x) \, \max(1, ||\xi_t||_1)^{p_t^\xi - 1} \, ||x_t^* - x_t^{*\prime}||_1 \\
&\leq (C_t^{\pi\prime} + L_t^x) \, \max(1, ||\xi_t||_1)^{p_t^\xi - 1} \, K_2 \, ||\xi_t - \xi_t'||_1.
\end{aligned}$$

Note that the first inequality follows from (14), the second follows from (15), while the last one is due to (17). The result thus follows with $L_t^\xi = (C_t^\pi + L_t^x) K_2$ and $p_t^\xi = \max(p_t^x, p_t^\pi)$. ☐

In the lemma above, we required the post-decision value functions $\mathcal{V}_t$ to fulfill (14). The next lemma provides sufficient conditions for this to be true.

LEMMA 3. *If the problem is of the form* (12)*, Assumption 3 holds, there are constants* $C_t^\pi > 0$ *and* p_t^π *with*

$$||\pi_t(\xi_t)||_2 \leq C_t^\pi \, \max(1, ||\xi_t||_1)^{p_t^\pi - 1}, \quad \forall \xi_t \in \operatorname{supp}(\boldsymbol{\xi}_t), \, \forall t \in [T], \tag{18}$$

and the stochastic process is such that

$$\mathbb{E}[\max(1, ||\boldsymbol{\xi}_{t+1}||_1)^{p_{t+1}^\pi - 1} | \xi_t] \leq C_t^\xi \, \max(1, ||\xi_t||_1)^{p_t^\pi - 1}, \quad \forall \xi_t \in \operatorname{supp}(\boldsymbol{\xi}_t), \, \forall t \in [T] \tag{19}$$

holds for some $C_t^\xi > 0$*, then there is a* p_t^x *with*

$$|\mathcal{V}_t(x_t, \xi_t) - \mathcal{V}_t(x_t', \xi_t)| \leq L_t^x \, \max(1, ||\xi_t||_1)^{p_t^x - 1} \, ||x_t - x_t'||_1, \quad \forall \xi_t, \forall t = 1, \ldots, T-1.$$

Proof. We start at stage T and consider the function $x_{T-1} \mapsto V_T(x_{T-1}, \xi_T)$. For two values x_{T-1} and x_{T-1}' we get by Lemma 1 that there is a K only dependent on A_T such that

$$d_H(\mathcal{S}(x_{T-1}), \mathcal{S}(x_{T-1}')) \leq K \, ||x_{T-1} - x_{T-1}'||_1.$$

It follows that there is an $L<\infty$ such that

$$
\begin{aligned}
|V_T(x_{T-1},\xi_T)-V_T(x'_{T-1},\xi_T)| = |\langle \pi(\xi_T), x^*_T - x'^*_T\rangle| &\le ||\pi_T(\xi_T)||_2\,||x^*_T - x'^*_T||_2 \\
&\le L\,\max(1,||\xi_T||_1)^{p^\pi_T-1}\,||x_{T-1}-x'_{T-1}||_1,
\end{aligned}
$$

where x^*_T and x'^*_T are optimal for $V_{T-1}(x_{T-1},\xi_T)$ and $V_{T-1}(x'_{T-1},\xi_T)$.

Using this, we can bound the difference in $\mathcal{V}_{T-1}$ as follows

$$
\begin{aligned}
|\mathcal{V}_{T-1}(x_{T-1},\xi_{T-1})-\mathcal{V}_{T-1}(x'_{T-1},\xi_{T-1})| &\le \mathbb{E}\left[|V_T(x_{T-1},\boldsymbol{\xi}_t)-V_T(x'_{T-1},\boldsymbol{\xi}_T)|\middle|\xi_{T-1}\right] \\
&\le L\,\mathbb{E}[\max(1,||\boldsymbol{\xi}_T||_1)^{p^\pi_T-1}\,|\xi_{T-1}]\,||x_{T-1}-x'_{T-1}||_1 \\
&\le LC^\xi_{T-1}\,\max(1,||\xi_{T-1}||_1)^{p^\pi_{T-1}-1}\,||x_{T-1}-x'_{T-1}||_1.
\end{aligned}
$$

Hence, the result follows for $T-1$ with $L^x_{T-1}=LC^\xi_{T-1}$ and $p^x_{T-1}=p^\pi_{T-1}$.

Moving on to $T-1$, in order to be able to apply Lemma 1, we approximate $x_{T-1}\mapsto \mathcal{V}_{T-1}(x_{T-1},\xi_{T-1})$ by a piece-wise linear upper approximation $\tilde{\mathcal{V}}_{T-1}$ by supporting hyperplanes such that

$$
|\mathcal{V}_{T-1}(x_{T-1},\xi_{T-1})-\tilde{\mathcal{V}}_{T-1}(x_{T-1},\xi_{T-1})|\le\varepsilon,\quad \forall x_{T-1}.
$$

Note that such an approximation exists for every $\varepsilon>0$ due to the compactness of the feasible sets. Further note that since $\mathcal{V}_{T-1}$ is concave in x_{T-1}, $\tilde{\mathcal{V}}_{T-1}$ can be chosen such that the Lipschitz constant of the approximation is always lower than that of $\mathcal{V}_{T-1}$. Using $\tilde{\mathcal{V}}_{T-1}$, we define $\tilde{V}_{T-1}(x_{T-2},\xi_{T-1})$ as $V_{T-1}(x_{T-2},\xi_{T-1})$ with $\mathcal{V}_{T-1}$ replaced by $\tilde{\mathcal{V}}_{T-1}$.

We therefore get

$$
\begin{aligned}
&|V_{T-1}(x_{T-2},\xi_{T-1})-V_{T-1}(x'_{T-2},\xi_{T-1})| \\
&\le |\tilde{V}_{T-1}(x_{T-2},\xi_{T-1})-\tilde{V}_{T-1}(x'_{T-2},\xi_{T-1})|+2\varepsilon \\
&\le |\langle\pi_{T-1}(\xi_{T-1}),x^*_{T-1}-x'^*_{T-1}\rangle|+|\tilde{\mathcal{V}}_{T-1}(x^*_{T-1},\xi_{T-1})-\tilde{\mathcal{V}}_{T-1}(x'^*_{T-1},\xi_{T-1})|+2\varepsilon \\
&\le ||\pi_{T-1}(\xi_{T-1})||_2\,||x^*_{T-1}-x'^*_{T-1}||_2+L^x_{T-1}\,\max(1,||\xi_{T-1}||_1)^{p^x_{T-1}-1}\,||x^*_{T-1}-x'^*_{T-1}||_1+2\varepsilon \\
&\le (C^{\pi\prime}_{T-1}+L^x_{T-1})K\,\max(1,||\xi_{T-1}||_1)^{p'_{T-1}-1}\,||x_{T-2}-x'_{T-2}||_1+2\epsilon,
\end{aligned}
$$

where $p'_{T-1}=\max(p^\pi_{T-1},p^x_{T-1})$, the second last inequality follows from the result for $T-1$ and the last inequality follows because, again, the optimal values x^*_{T-1} and $x^{*\prime}_{T-1}$ of $\tilde{V}_{T-1}(x_{T-2},\xi_{T-1})$

and $\tilde{V}_{T-1}(x'_{T-2}, \xi_{T-1})$ can be chosen such that $K||x_{T-2} - x'_{T-2}||_1$ with K only depending on the matrix A_{T-1} and (in particular) not on $\epsilon > 0$ due to Lemma 1. Note that the change from C^{π}_{T-1} to $C^{\pi\prime}_{T-1}$ accounts for the change from $||\cdot||_2$ to $||\cdot||_1$ in the last step.

Since $\varepsilon > 0$ was arbitrary, the same argument as above with $L_{T-2} = (C^{\pi\prime}_{T-1} + L^{x}_{T-1})K$ leads to the result for $T-2$. The result follows by backward induction. □

We are now in the position to prove the following result, establishing the local Lipschitz continuity for linear problems of the form (12).

Theorem 5. *If the problems are linear as in* (12)*, we have* (15)*,* (18)*,* (19)*, and*

$$||\pi_t(\xi_t) - \pi_t(\xi'_t)||_1 \le C^{\pi}_t \, \max(1, ||\xi_t||_1, |\xi'_t||_1)^{p^{\pi}_t - 1} \, ||\xi_t - \xi'_t||_1, \quad \forall t \in [T], \tag{20}$$

Assumption 2 holds with $p^C_t \ge \max(\max_{s>t} p^C_s, \max_{s\ge t} p^{\pi}_s)$ *and Assumption 3 is fulfilled, then the value functions are uniformly Lipschitz as in* (7) *with* $p_t = p^C_t$*.*

Proof. We start in the last stage and conclude by (20) and Assumption 3

$$\begin{aligned} V_T(x_{T-1}, \xi_T) &= \langle \pi_T(\xi_T), x^*_T \rangle \\ &= \langle \pi_T(\xi'_T), x^*_T \rangle + \langle \pi_T(\xi_T) - \pi_T(\xi'_T), x^*_T \rangle \\ &\le \langle \pi_T(\xi'_T), x^*_T \rangle + D_T \, ||\pi_T(\xi_T) - \pi_T(\xi'_T)||_1 \\ &\le \langle \pi_T(\xi'_T), x^*_T \rangle + D_T C^{\pi}_T \, \max(1, ||\xi_T||_1, ||\xi'_T||_1)^{p^{\pi}_T - 1} \, ||\xi_T - \xi'_T||_1 \\ &\le V_T(x_{T-1}, \xi'_T) + (L^{\xi}_T + D_T C^{\pi}_T) \, \max(1, ||\xi_T||_1, ||\xi'_T||_1)^{p_T - 1} \, ||\xi_T - \xi'_T||_1, \end{aligned}$$

where x^*_T is the optimal solution to $V_T(x_{T-1}, \xi_T)$ and the last inequality follows because of Lemma 2. Reversing the roles of ξ_T and ξ'_T, yields the results for stage T with $L_T = L^{\xi}_T + D_T C^{\pi}_T$ and $p_T = p^{\pi}_T$).

We proceed by induction assuming that the result was already shown for $t+1$ and by fixing $\varepsilon > 0$ as well as arbitrary ξ_t, ξ'_t, x_{t-1}, and a piecewise linear approximation $\tilde{\mathcal{V}}_t$ such that

$$|\mathcal{V}_t(x_t, \xi'_t) - \tilde{\mathcal{V}}_t(x_t, \xi'_t)| \le \varepsilon, \quad \forall x_t.$$

Denoting by x^*_t the solution to $V_t(x_{t-1}, \xi_t)$, $p_t = \max(p^C_t, p^{\pi}_t, p^{\xi}_t)$, we write

$$V_t(x_{t-1}, \xi_t) = \langle \pi_t(\xi_t), x^*_t \rangle + \mathbb{E}[V_{t+1}(x^*_t, \boldsymbol{\xi}_{t+1}) | \xi_t]$$

$$
\begin{aligned}
&\leq \langle \pi_t(\xi_t'), x_t^* \rangle + \mathbb{E}[V_{t+1}(x_t^*, \boldsymbol{\xi}_{t+1})|\xi_t'] + (L_{t+1}L_t^C + C_t^\pi D_t) \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1} \, ||\xi_t - \xi_t'||_1 \\
&\leq \langle \pi_t(\xi_t'), x_t^* \rangle + \tilde{\mathcal{V}}_t(x_t^*, \xi_t') + (L_{t+1}L_t^C + C_t^\pi D_t) \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1} \, ||\xi_t - \xi_t'||_1 + \varepsilon \\
&\leq \tilde{V}_t(x_{t-1}, \xi_t, \xi_t') + (L_{t+1}L_t^C + C_t^\pi D_t) \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1} \, ||\xi_t - \xi_t'||_1 + \varepsilon \\
&\leq \tilde{V}_t(x_{t-1}, \xi_t', \xi_t') + (L_{t+1}L_t^C + C_t^\pi D_t + L_t^\xi) \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1} \, ||\xi_t - \xi_t'||_1 + \varepsilon \\
&= \tilde{V}_t(x_{t-1}, \xi_t') + (L_{t+1}L_t^C + C_t^\pi D_t + L_t^\xi) \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1} \, ||\xi_t - \xi_t'||_1 + \varepsilon \\
&\leq V_t(x_{t-1}, \xi_t') + (L_{t+1}L_t^C + C_t^\pi D_t + L_t^\xi) \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1} \, ||\xi_t - \xi_t'||_1 + 2\varepsilon
\end{aligned}
$$

where, as before, $\tilde{V}_t$ is the function V_t where $\mathcal{V}_t$ in the objective is replaced with $\tilde{\mathcal{V}}_t$. Note that the first inequality follows by (20) and the induction hypothesis, the second by the definition of $\tilde{\mathcal{V}}_t$, the fourth by Lemma 2, which is applicable because of Lemma 3

To see that the constant L_t^ξ above does not depend on ε, note that the Lipschitz constant of $\tilde{\mathcal{V}}_t$ is bounded above by the Lipschitz constant of $\mathcal{V}_t$. Reversing the roles of ξ_t and ξ_t' and noting that both the Lipschitz constant and $x_t \mapsto \mathcal{V}_t(x_t, \xi_t)$ and $x_t \mapsto \mathcal{V}_t(x_t, \xi_t')$ can be bounded by $L_t^x \max(1, ||\xi_t||_1, ||\xi_t'||_1)^{p_t-1}$ by Lemma 3 finishes the proof, since $\varepsilon > 0$ was arbitrary. □

Remark 5. Note that opposed to the situation in Theorem 3 and Theorem 4, the modulus of continuity does not only depend on p_t^C but also on the properties of the function $\pi_t(\xi_t)$. In particular, in case of the two earlier results global Lipschitz continuity of the value functions is possible as long as (4) holds with $p_t = 1$. This, in case of Theorem 5, only works if (15) holds with $p_t^\pi = 1$, i.e., for the case of a deterministic π_t or if $\pi_t(\xi_t)$ is bounded as a function of ξ_t. This shows, that Lipschitz continuous value functions are possible if either only the objective is random as in Theorem 2 or only the right hand side of the constraints is random, i.e., $p_t^\pi = 1$. The combination of both, seems to necessitate a higher order modulus of continuity.

Remark 6. The above result cannot be extended to problems where there is also randomness in the left-hand side of the constraints, since in this case the value function might be discontinuous (e.g. Terça and Wozabal 2021).

Remark 7. Since all norms are equivalent in $\mathbb{R}^n$, $||\cdot||_1$ in Theorem 5 can be replaced by arbitrary norms by an appropriate change in constants.

4. Building Scenario Lattices

In this section, we describe how to build scenario lattices as natural discrete approximations of discrete time Markov processes. First, we review the concept of lattices in Section 4.1 and then we proceed to the description and theoretical analysis of an SGD algorithm in Section 4.2 that generates lattices from simulations of Markov processes while minimizing (8) and thereby guaranteeing the stability of the stochastic program when the original process is replaced by the lattice approximation.

4.1. Scenario Lattices

Formally, a scenario lattice is a graph organized in a finite number of layers. Each layer is associated with a discrete point in time t, models the process $\tilde{\boldsymbol{\xi}}_t$ at stage t, and contains a finite number of nodes $\tilde{\xi}_{tn}$. Successive layers are connected by arcs. A node $\tilde{\xi}_{tn}$ represents a possible state of the stochastic process, and an arc indicates the possibility of a state transition between the two connected nodes. Each arc connecting two nodes $\tilde{\xi}_{tn}$ and $\tilde{\xi}_{t+1,m}$ is associated with a probability weight p_{tnm} and the weights of outgoing arcs of a node add up to one. Note that this definition in particular covers inhomogeneous Markov processes where conditional distributions change over time. A scenario tree differs from a scenario lattice by the additional requirement that every node in stage $t+1$ has only one predecessor in stage t. See Figure 1 for a graphical comparison of a scenario lattice with a scenario tree.[2]

Denote $\mathcal{N}_t$ as the set of nodes in stage t, $p_t,\ t=1,\dots,T-1$ as the $|\mathcal{N}_t| \times |\mathcal{N}_{t+1}|$ transition matrix with elements p_{tnm}, and

$$q_{tm} = \sum_{k \in \mathcal{N}_{T-1}} q_{t-1,k} p_{t-1,km},\ m \in \mathcal{N}_t,\ t = 2,\dots,T.$$

as the unconditional probability of node n in stage t. We assume that the initial state of the lattice is deterministic, i.e., $q_{1,1} = 1$.

A scenario tree is a general representation of a discrete stochastic process and a lattice is a general representation of a discrete Markovian process. Hence, while scenario trees could

[2] Note that the topology of both the lattice and the tree is chosen for illustrative purposes. In particular, the number of nodes per stage in the lattices need not grow linearly in time and the branching factor of the tree need not be as regular as depicted, let alone binary.

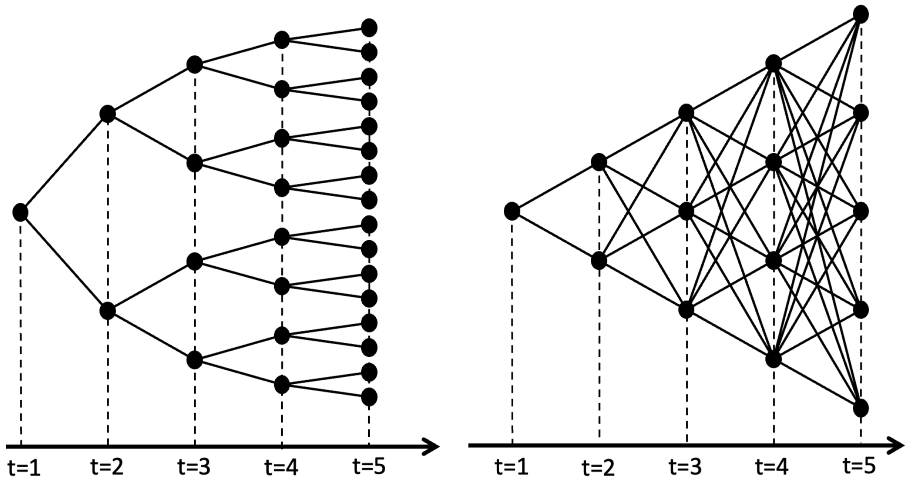

Figure 1 A tree with 31 nodes representing 16 scenarios on the left and a lattice with 15 nodes representing 120 scenarios on the right.

be used as representations of Markov processes, the advantage of scenario lattices is that the number of nodes grows slower in the number of stages as is visually demonstrated in Figure 1. In particular, in contrast to scenario lattices, for trees with a branching factor of at least two, the number of nodes necessarily grows exponentially in the number of stages.

The reason for this difference is that every node in a tree contains information on the entire history of the process, while this is not the case for a scenario lattice. This implies that a scenario lattice can only be used in Markovian stochastic programming, where neither the history of the decisions (beyond the information encapsulated in the resource state) nor the history of the randomness is relevant for making decisions. The advantage of this property is that it allows to mitigate the curse of dimensionality caused by an increasing number of stages in a multi-stage problem. The intuition suggested by Figure 1 is made precise for the special case of stagewise independent randomness in Lan (2020).

An algorithmic disadvantage of scenario lattices is rooted in the fact that a node does not represent the whole history of the decision process as well. In particular, a lattice only uniquely models the environmental part of the state space. Consequently, there might be multiple paths of $\tilde{\boldsymbol{\xi}}$ traversing the same node $\tilde{\xi}_{tn}$ that have different optimal resource states x_t^* associated with them. It is therefore in general not possible to assign a single optimal decision to nodes of a scenario lattice. This makes it impossible to solve the overall problem directly by its deterministic equivalent as can be done with scenario trees. Instead one needs the value functions V_t

that implicitly define a policy for every lattice node given the resource state, which necessitates a solution strategy like SDDP.

4.2. A Stochastic Gradient Descent Algorithm

We propose an algorithm that constructs a scenario lattice minimizing the bound in Theorem 2 in order to assure that the error in the first stage objective function is small when replacing $\boldsymbol{\xi}$ with the easier to handle $\tilde{\boldsymbol{\xi}}$. To this end, we note that

$$\mathbb{E}[d_{FM}^{p_t}(\boldsymbol{\xi}_t|\tilde{\boldsymbol{\xi}}_{t-1}, \tilde{\boldsymbol{\xi}}_t|\tilde{\boldsymbol{\xi}}_{t-1})] = \sum_{n\in\mathcal{N}_{t-1}} q_{t-1,n} d_{FM}^{p_t}(\boldsymbol{\xi}_t|\tilde{\xi}_{t-1,n}, \tilde{\boldsymbol{\xi}}_t|\tilde{\xi}_{t-1,n}),$$

which allows us to rewrite the left hand side of (8) for $t=1$ as

$$\sum_{t=2}^{T} L_t \sum_{n\in\mathcal{N}_{t-1}} q_{t-1,n} d_{FM}^{p_t}(\boldsymbol{\xi}_t|\tilde{\xi}_{t-1,n}, \tilde{\boldsymbol{\xi}}_t|\tilde{\xi}_{t-1,n}). \tag{21}$$

In the algorithm proposed below, we minimize $d_{FM}^{p_t}(\boldsymbol{\xi}_t|\tilde{\xi}_{t-1,n}, \tilde{\boldsymbol{\xi}}_t|\tilde{\xi}_{t-1,n})$ building the lattice stage by stage, i.e., solve the *quantization problem* of finding a good discrete approximating distribution $\tilde{\xi}_t$ for ξ_t, given that both the processes take the value $\tilde{\xi}_{t-1,n}$ stored on the lattice node $n\in\mathcal{N}_{t-1}$.

Clearly, to make the problem well defined we have to fix the number of atoms of $\tilde{\xi}_t$. Note that we do not explicitly use the constants L_t in the lattice generation algorithm, since, given a fixed number of nodes for each stage, the L_t do not have an influence on the result by the logic of the stagewise construction. However, as pointed out in Section 3, estimates of the constants L_t can be used to determine how many nodes the lattice should have in which stage.

SGD-type algorithms proved to be successful when building trees and lattices based on the Wasserstein distance (e.g. Pflug and Pichler 2015, Löhndorf and Wozabal 2021). However, there are two problems with directly transferring this approach to the more general case of the Fortet-Mourier metric: Firstly, for $p_t>1$ the distance $d_{FM}^{p_t}$ does not have a dual representation as a mass transport problem, which is the basis for all the aforementioned algorithms. Secondly, the maximum in the definition of the Fortet-Mourier distance makes the problem non-smooth and therefore incompatible with standard SDG that relies on gradients to compute updates.

We start by tackling the first problem. Luckily, although the dual representation of the Fortet-Mourier distance is a transshipment problem, it can be modeled as transportation problem when constructing optimal discretizations as shown in the following result.

Theorem 6. *Denote by $\mathcal{P}_N$ the set of discrete random vectors $\tilde{\xi}:\Omega\to\mathbb{R}^n$ with N atoms. Given a random vector $\xi:\Omega\to\mathbb{R}^n$ the problem*

$$\min_{\tilde{\xi}\in\mathcal{P}_N} d^p_{FM}(\xi,\tilde{\xi}) \tag{22}$$

is equivalent to

$$\min_{\tilde{\xi}_1,\ldots,\tilde{\xi}_N} \int \min_i \max(1,||\xi||,||\tilde{\xi}_i||)^{p-1}\,||\xi-\tilde{\xi}_i||\,P(d\xi). \tag{23}$$

Furthermore, the optimization problem that defines $d^p_{FM}(\xi,\tilde{\xi})$ in (22) can, without loss of generality, be viewed as a transportation problem.

Proof. Suppose the optimal measure is supported on points $\tilde{\xi}_1,\ldots,\tilde{\xi}_N$ and π^* is the optimal transshipment measure for (22).

Define the sets

$$A_i=\left\{\xi\in\mathbb{R}^n : i\in\arg\min_i \max(1,||\xi||,||\tilde{\xi}_i||)\,||\xi-\tilde{\xi}_i||\right\}\setminus\bigcup_{j<i}A_j$$

which form a partition of $\mathbb{R}^n$.

We then have

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} \max(1,||\xi||,||\tilde{\xi}||)^{p-1}\,||\xi-\tilde{\xi}||\,\pi^*(d\xi,d\tilde{\xi})\\
&=\sum_i\sum_j\int_{A_i\times A_j} \max(1,||\xi||,||\tilde{\xi}||)^{p-1}\,||\xi-\tilde{\xi}||\,\pi^*(d\xi,d\tilde{\xi})\\
&\geq\sum_i\int_{A_i} \max(1,||\xi||,||\tilde{\xi}_i||)^{p-1}\,||\xi-\tilde{\xi}_i||\,P(d\xi)\\
&=\int_{\mathbb{R}^n\times\mathbb{R}^n} \max(1,||\xi||,||\tilde{\xi}||)^{p-1}\,||\xi-\tilde{\xi}||\,\bar{\pi}(d\xi,d\tilde{\xi}),
\end{aligned} \tag{24}$$

where $\bar{\pi}(A,B)=P(A\cap T^{-1}(B))$ is the transport measure induced by the transport map $T(\xi)=\tilde{\xi}(\xi)=\xi_{tn}$ with $n=\min\arg\min_i\,\max(1,||\xi||,||\tilde{\xi}_i||)^{p-1}\,||\xi-\tilde{\xi}_i||$. Hence, the transport measure $\bar{\pi}$ is also optimal for (22). Lastly, (24) implies that maximizing over transport plans to measures in $\mathcal{P}_N$ is equivalent to solving (23), which concludes the proof. ☐

In order to solve problem (23) using SGD, we have to also deal with the second problem mentioned above, namely the non-smoothness of the objective. To this end, we use a method pioneered in Norkin (1980) that is based on generalized gradients instead of gradients.

DEFINITION 1 (GENERALIZED GRADIENTS). A function $f : \mathbb{R}^n \to \mathbb{R}$ is *generalized differentiable* at a point $x \in \mathbb{R}^n$ if in some neighborhood U of x there exists a multi-valued mapping ∂f upper semicontinuous at x with $\partial f(y) \subseteq \mathbb{R}^n$ closed and convex $\forall y \in U$ for which the following approximation holds

$$f(y) = f(x) + \langle \lambda, y - x \rangle + o(x, y, \lambda),$$

where

$$\frac{o(x, y_k, \lambda_k)}{||y_k - x||} \xrightarrow{k\to\infty} 0 \quad \text{for all } (y_k) \subseteq U \text{ with } y_k \to x \text{ and } \lambda_k \in \partial f(y_k).$$

$\partial f(x)$ is the *generalized gradient* of f at x.

Generalized gradients can be computed for a wide range nonconvex and nonsmooth functions. We summarize some facts in the following result.

THEOREM 7. 1. *All convex functions $f : \mathbb{R}^n \to \mathbb{R}$ are generalized differentiable with ∂f equal to the set of subgradients.*

2. *Differentiable functions are generalized differentiable and $\partial f(x) = \{\nabla f(x)\}$.*

3. *If $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are generalized differentiable then $f = \max(f_1, \dots, f_m)$ is generalized differentiable with*

$$\partial f(x) = \operatorname{conv} \{\lambda \in \mathbb{R}^n : \lambda \in \partial f_i(x),\, f_i(x) = f(x),\, 1 \le i \le m\}.$$

4. *If $f_0 : \mathbb{R}^m \to \mathbb{R}$ as well as $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are generalized differentiable, then the composition $f = f_0(f_1(x), \dots, f_m(x))$ is a generalized differentiable function with*

$$\partial f(x) = \operatorname{conv} \{\lambda \in \mathbb{R}^n : \lambda = [\lambda_1, \dots, \lambda_m]\lambda_0,\, \lambda_0 \in \partial f_0(z),\, \lambda_i \in \partial f_i(x),\, 1 \le i \le m\},$$

where $z(x) = (f_1(x), \dots, f_m(x))^\top \in \mathbb{R}^m$ and $[\lambda_1, \dots, \lambda_m] \in \mathbb{R}^{n\times m}$ is the matrix with λ_i as column vectors.

5. *If x^* is a local minimum of a generalized differrentiable function f, then $0 \in \partial f(x^*)$.*

6. *Let $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$, (Ω, σ, P) be a probability space with finite measure, $f(x, \cdot)$ integrable for every $x \in \mathbb{R}^n$, and $f(\cdot, \omega)$ generalized differentiable with measurable gradient maps. If further for every compact set $K \subseteq \mathbb{R}^n$ there exist an integrable function $L_K(\omega)$ such that*

$$\sup\{||\lambda|| : \lambda \in \partial f(\cdot, \omega)(x),\ x \in K\} \le L_K(\omega)$$

then $f(x) := \int f(x, \omega)\, P(d\omega)$ is generalized differentiable and

$$\partial f(x) = \int \partial f(\cdot, \omega)(x)\, P(d\omega), \tag{25}$$

i.e., integration and taking generalized gradients can be interchanged.

Proof. For (1)-(5), see Norkin (1980), (6) is proven in Norkin (1986). Note that in (25) the integral is the Aumann integral for set valued functions (see, e.g., Aumann 1965). □

Based on the above, we can prove the following for the quantization problem (23).

THEOREM 8. 1. *For $p \ge 1$, and $y \in \mathbb{R}^{n \times k}$ with $y = (y_1, \dots, y_k)$, $y_i \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$, the function*

$$y \mapsto \min_i \max(1, ||x||, ||y_i||)^{p-1} ||x - y_i|| \tag{26}$$

is generalized differentiable. Furthermore, if $i = \min \arg\min_i \max(1, ||x||, ||y_i||)^{p-1}||x - y_i||$, the sets

$$\Lambda(y) = \begin{cases} -\partial || \cdot ||(x - y_i) \max(1, ||x||)^{p-1}, & ||y_i|| \le 1 \\ -\partial || \cdot ||(x - y_i)||y_i||^{p-1} + (p-1)||y_i||^{p-2} \partial || \cdot ||(y_i)||x - y_i||, & ||y_i|| > 1 \end{cases} \tag{27}$$

are subsets of the generalized gradient of the function in (26) at y.

2. *If the random variable $||\xi||^p$ is integrable, then the generalized gradient set for the function*

$$(\tilde{\xi}_1, \dots, \tilde{\xi}_N) \mapsto \int \min_i \max(1, ||\xi||, ||\tilde{\xi}_i||)^{p-1}\, ||\xi - \tilde{\xi}_i||\, P(d\xi)$$

equals

$$\int \partial \min_i \max(1, ||\xi||, ||\tilde{\xi}_i||)^{p-1}\, ||\xi - \tilde{\xi}_i||\, P(d\xi). \tag{28}$$

Proof. To proof 1. note that the norm is convex and therefore generalized differentiable because of Theorem 7.1, the maximum raised to the power $p-1$ is generalized differentiable because of Theorem 7.2-4 and finally the whole term is generalized differentiable because of Theorem 7.2 and 7.4 and the fact that $(t,s) \mapsto ts$ is continuously differentiable for $s,\ t \in \mathbb{R}$. Formula (27) follows from straightforward calculation.

To show the second point, we verify the conditions of Theorem 7.6. For fixed $\tilde{\xi}$ we have

$$
\begin{aligned}
&\int \min_i \max(1, ||\xi||, ||\tilde{\xi}_i||)^{p-1}\, ||\xi - \tilde{\xi}_i||\, P(d\xi) \\
&\le \max(1, ||\tilde{\xi}_1||)^{p-1} \left(\int_{||\xi|| \le 1} ||\xi - \tilde{\xi}_1||\, P(d\xi) + \int_{||\xi||>1} ||\xi||^{p-1}\, ||\xi - \tilde{\xi}_1||\, P(d\xi) \right) \\
&\le \max(1, ||\tilde{\xi}_1||)^{p-1} \left(||\tilde{\xi}_1|| + \int ||\xi||\, P(d\xi) + ||\tilde{\xi}_1|| \int ||\xi||^{p-1}\, P(d\xi) + \int ||\xi||^{p}\, P(d\xi) \right) \\
&< \infty
\end{aligned}
$$

by the assumption on the integrability of $||\xi||^p$. The boundedness of the subgradient set can be shown in the same way. Lastly, the measurability of the gradient maps follows from Remark 1 in Norkin (1986). □

Remark 8. If the probability distribution P is absolutely continuous with respect to the Lebesgue measure, the generalized gradient sets in (28) are almost everywhere singletons and the points where this is not the case do not affect the value of the integral.

After these preparations, we have everything in place to apply the SDG algorithm in Ermol'ev and Norkin (1998) to solve (22). In particular, to find an optimal solution, we start at an arbitrary point $\tilde{\xi}_0 \in \mathbb{R}^{n\times N}$, sample points $\bar{\xi}^k \in \mathbb{R}^n$, and use the following updates to generate candidate solutions $\tilde{\xi}_{k+1}$ for $k \ge 0$

$$
g_k \in \Lambda(\bar{\xi}^k) \tag{29}
$$

$$
\tilde{\xi}_i^{k+1} = \begin{cases} \tilde{\xi}_i^k - \rho_k g_k, & \tilde{\xi}(\bar{\xi}^k) = \tilde{\xi}_i \\ \tilde{\xi}_i^k, & \text{otherwise,} \end{cases} \tag{30}
$$

where the learning rates ρ_k fulfills the Robbins-Monroe conditions

$$
\sum_{k=0}^{\infty} \rho_k = \infty, \quad \sum_{k=0}^{\infty} \rho_k^2 = 0.
$$

The above procedure converges to a critical point as shown in the following result.

THEOREM 9 **(Ermol'ev and Norkin (1998))**. *All limit points of the sequence $\tilde{\xi}_k$ generated in* (30)*, are almost surely contained in the set*

$$\{\tilde{\xi} \in \mathbb{R}^{N\times n} : 0 \in \partial d^p_{FM}(\xi, \tilde{\xi})\},$$

i.e., fulfill the necessary conditions for local minima.

In order to employ the above result to generate a scenario lattice, we have to re-formulate the problem of minimizing (21) to fit the framework in Theorem 9. In order, to do so we split the problem into stage-wise subproblems. Note that this strategy could potentially be improved by considering the whole problem as a single optimization problem over all stages.

We start by defining the root node $\tilde{\xi}_1$ as the deterministic starting state ξ_1 of the process. Then we proceed in a stage-wise manner assuming that the nodes $\tilde{\xi}_{tn}$, $n \in \mathcal{N}_t$ at stage t are already found. We find $|\mathcal{N}_{t+1}|$ nodes $\tilde{\xi}_{t+1,m}$ such that the probability weighted sum of the transportation problems is minimized, i.e.,

$$\begin{array}{lll} \min_{z_n,\tilde{\xi}_{t+1}} & \sum_{n\in\mathcal{N}_t} q_{tn} \int \max(1, ||\xi_{t+1}||, ||\tilde{\xi}_{t+1}||)^{p_{t+1}-1}\, ||\xi_{t+1} - \tilde{\xi}_{t+1}||\, z_n(d\tilde{\xi}_{t+1}, d\xi_{t+1}) & \\ \text{s.t.} & z_n(\mathbb{R}^{N_{t+1}} \times B) = P_{t+1}(B|\xi_t = \tilde{\xi}_{tn}), \qquad \forall B \text{ borel}, \forall n \in \mathcal{N}_t & \\ & z_n(\{\tilde{\xi}_{t+1,m}\} \times \mathbb{R}^{N_{t+1}}) = P_{t+1}(A_{t+1,i}|\xi_t = \tilde{\xi}_{tn}),\ \forall m \in \mathcal{N}_{t+1},\ \forall n \in \mathcal{N}_t, & \end{array} \tag{31}$$

where z_n is the transportation measure between the image measure of $\boldsymbol{\xi}_{t+1}$ given $\xi_t = \tilde{\xi}_{tn}$ and the image measure of $\tilde{\boldsymbol{\xi}}_{t+1}$ given $\tilde{\xi}_t = \tilde{\xi}_{tn}$ on the lattice and $A_{t+1,i}$ are the sets defined in the proof of Theorem 6 for the random variable $\tilde{\boldsymbol{\xi}}_{t+1}$. The constraints of the above problem ensure that the conditional distributions of $\boldsymbol{\xi}_{t+1}$ given the discretization in stage t are captured as accurately as possible, whereby the weighting with the unconditional probabilities q_{tn} of the nodes in stage t captures the trade-off between the $|\mathcal{N}_t|$ different transportation problems.

To rewrite this weighted sum of optimization problems to a single problem of the form (22), denote by $P_{t+1}(\cdot|\tilde{\xi}_{tn})$ the image measure of ξ_{t+1} in $\mathbb{R}^{N_{t+1}}$ given $\xi_t = \tilde{\xi}_{tn}$ and define a measure $\hat{P}_{t+1}$ for $\boldsymbol{\xi}_{t+1}$ conditional on the nodes of the lattice at stage t with unconditional probabilities q_{tn}, i.e.,

$$\hat{P}_{t+1}(B) = \sum_{n\in\mathcal{N}_t} q_{tn} P(B|\tilde{\xi}_{tn}), \quad \forall B \subseteq \mathbb{R}^{N_{t+1}} \text{ borel}. \tag{32}$$

Hence, $\hat{P}_{t+1}$ is the image measure of $\boldsymbol{\xi}_{t+1}$, given a random starting state $\tilde{\xi}_{tn}$ with probability q_{tn} in period t. Note that by this conditioning $\hat{P}_{t+1}$ is different from the original unconditional measure P_{t+1}.

Using the measure $\hat{P}_{t+1}$, we reformulate (31) using the following proposition.

THEOREM 10. *Problem* (31) *is equivalent to* (22) *for the measure* $\hat{P}_{t+1}$*, i.e., the problem*

$$\begin{array}{lll}\min_{z,\tilde{\xi}_{t+1}} & \int \max(1,||\xi_{t+1}||,||\tilde{\xi}_{t+1}||)^{p_{t+1}-1}\,||\xi_{t+1}-\tilde{\xi}_{t+1}||\,z(d\tilde{\xi}_{t+1},d\xi_{t+1}) & \\ s.t. & z(\mathbb{R}^{N_{t+1}}\times B)=\hat{P}_{t+1}(B), & \forall B,\ \forall n\in\mathcal{N}_t \\ & z(\{\tilde{\xi}_{t+1,m}\}\times\mathbb{R}^{N_{t+1}})=\hat{P}_{t+1}(A_{t+1,m}),\ \forall m\in\mathcal{N}_{t+1}. & \end{array} \tag{33}$$

Proof. We start by showing that every solution of (33) can be transformed to a solution of (31) with the same objective. To this end, we disintegrate z with respect to the projection on the second coordinate into a measure λ on the second coordinate and conditional distributions $z_{\xi_{t+1}}$ for the first coordinate. By the first constraint in (33) it follows that $\lambda=\hat{P}_{t+1}$ and we can rewrite the objective as

$$\begin{aligned}&\int \max(1,||\xi_{t+1}||,||\tilde{\xi}_{t+1}||)^{p_{t+1}-1}\,||\xi_{t+1}-\tilde{\xi}_{t+1}||\,z(d\tilde{\xi}_{t+1},d\xi_{t+1})\\ &\quad=\iint \max(1,||\xi||_{t+1},||\tilde{\xi}||_{t+1})^{p_{t+1}-1}\,||\xi_{t+1}-\tilde{\xi}_{t+1}||\,z_{\xi_{t+1}}(d\tilde{\xi}_{t+1})\,\hat{P}_{t+1}(d\xi_{t+1})\\ &\quad=\sum_{n\in\mathcal{N}_t} q_{tn}\iint \max(1,||\xi||_{t+1},||\tilde{\xi}||_{t+1})^{p_{t+1}-1}\,||\xi_{t+1}-\tilde{\xi}_{t+1}||\,z_{\xi_{t+1}}(d\tilde{\xi}_{t+1})\,P_{t+1}(d\xi_{t+1}|\tilde{\xi}_{tn}) \qquad (34)\\ &\quad=\sum_{n\in\mathcal{N}_t} q_{tn}\int \max(1,||\xi||_{t+1},||\tilde{\xi}||_{t+1})^{p_{t+1}-1}\,||\xi_{t+1}-\tilde{\xi}_{t+1}||\,z_n(d\tilde{\xi}_{t+1},d\xi_{t+1})\end{aligned}$$

with

$$z_n(\tilde{B}\times B)=\iint \mathbb{1}_{\tilde{B}\times B}(\tilde{\xi}_{t+1},\xi_{t+1})\,z_{\xi_{t+1}}(d\tilde{\xi}_{t+1})P_{t+1}(d\xi_{t+1}|\tilde{\xi}_{tn}),$$

for measurable sets B and $\tilde{B}$ where (34) follows from (32).

Since

$$\begin{aligned}z_n(\mathbb{R}^{N_{t+1}}\times B)&=\iint \mathbb{1}_{\mathbb{R}^{N_{t+1}}\times B}(\tilde{\xi}_{t+1},\xi_{t+1})\;z_{\xi_{t+1}}(d\tilde{\xi}_{t+1})\,P_{t+1}(d\xi_{t+1}|\tilde{\xi}_{tn})\\ &=\int \mathbb{1}_B(\xi_{t+1})\,P_{t+1}(d\xi_{t+1}|\tilde{\xi}_{tn})\\ &=P_{t+1}(B|\tilde{\xi}_{tn})\\ z_n(\{\hat{\xi}_{t+1,m}\}\times\mathbb{R}^{N_{t+1}})&=\iint \mathbb{1}_{\{\hat{\xi}_{t+1,m}\}\times\mathbb{R}^{N_{t+1}}}(\tilde{\xi}_{t+1},\xi_{t+1})\,z_{\xi_{t+1}}(d\tilde{\xi}_{t+1})\,P_{t+1}(d\xi_{t+1}|\tilde{\xi}_{tn})\\ &=\int z_{\xi_{t+1}}(\{\tilde{\xi}_{t+1,m}\})\,P_{t+1}(d\xi_{t+1}|\tilde{\xi}_{tn})\\ &=P_{t+1}(A_{t+1,m}|\tilde{\xi}_{tn})\end{aligned}$$

the z_n are feasible for (31), where the last equality follows from Theorem 6, because it is optimal to transport $\xi_{t+1}\in A_{t+1,n}$ to $\tilde{\xi}_{t+1,n}$.

To show the other direction, let $(z_n)_{n\in N_t}$ be a solution of (33) and define

$$z(\tilde{B}\times B) = \sum_{n\in\mathcal{N}_t} q_{tn} z_n(\tilde{B}\times B).$$

Note that the objective function (33) for z equals

$$\int \max(1, ||\xi_{t+1}||, ||\tilde{\xi}_{t+1}||)^{p_{t+1}-1}\, ||\tilde{\xi}_{t+1} - \xi_{t+1}||\ z(d\tilde{\xi}_{t+1}, d\xi_{t+1}).$$

and thus is the same as the objective for (31) with $(z_n)_{n\in\mathcal{N}_t}$.

To show that z is a feasible solution to (33), note that

$$z(\mathbb{R}^{N_{t+1}}\times B) = \sum_{n\in\mathcal{N}_t} q_{tn} z_n(\mathbb{R}^{N_{t+1}}\times B) = \sum_{n\in\mathcal{N}_t} q_{tn} P_{t+1}(B|\tilde{\xi}_{tn}) = \hat{P}_{t+1}(B),$$

where the second equality follows from the first constraint in (31).

Similarly, we have

$$\begin{aligned} z(\{\tilde{\xi}_{t+1,m}\}\times\mathbb{R}^{N_{t+1}}) &= \sum_{n\in\mathcal{N}_t} q_{tn} z_n(\{\tilde{\xi}_{t+1,m}\}\times\mathbb{R}^{N_{t+1}}) = \sum_{n\in\mathcal{N}_t} q_{tn} P_{t+1}(A_{t+1,m}|\tilde{\xi}_{tn}) \\ &= \hat{P}_{t+1}(A_{t+1,m}), \end{aligned}$$

which finishes the proof. □

We therefore can use the SGD algorithm in (29) for the distribution $\hat{P}_t$ stage by stage to generate a scenario lattice whose nodes on each stage solve the problem (31). We summarize the overall algorithm for the generation of a scenario lattice over T stages in Algorithm 1. Note that after we find the nodes of the lattice in a stage, we estimate the conditional probabilities to reach these nodes from nodes in the previous stage by generating K_{t2} samples from the distributions of ξ_t given the values $\tilde{\xi}_{t-1,i}$ on the nodes of the previous stage in line 10 to 17.

5. Out-of-sample Evaluation of Policies

In this section, we will show that the scenario lattices generated in the previous section lead to approximate decision problems that in some sense converge to the true problems. In particular, we apply the optimal policies for the approximate problem to realizations of the real processes by rounding the environmental state $\boldsymbol{\xi}_t$ to the next lattice node and use the value function associated to that node to make decisions. In this way the policy, which is implicitly encoded in

Data: $\mathcal{N}_t$, ξ, K_{t1}, K_{t2}, S

$\tilde{\xi}_1 \leftarrow \xi_1$

`/* Loop over stages */`

for $t \leftarrow 2$ **to** T **do**

  Set $\tilde{\xi}_{tn}$ randomly for $n \in \mathcal{N}_t$ by sampling from $\hat{P}_t$ defined in (32)

  `/* Learn nodes of next stage using the SGD algorithm */`

  for $k \leftarrow 1$ **to** K_{t1} **do**

    Sample $\bar{\xi}_t^k$ from $\hat{P}_t$ defined in (32)

    Obtain $g_t^k \in \Lambda(\bar{\xi}_t^k)$ as defined in (29)

    $i^* \leftarrow \min \arg\min_i \max(1, ||\tilde{\xi}_{ti}||, ||\bar{\xi}_t^k||)^{p_t - 1} ||\tilde{\xi}_{ti} - \bar{\xi}_t^k||$

    $\tilde{\xi}_{ti^*} \leftarrow \tilde{\xi}_{ti^*} - \rho_t^k g_t^k$

  end

  `/* Calculate conditional probabilities */`

  for $n \in \mathcal{N}_{t-1}$ **do**

    $p_{tni} = 0$ for all $i \in \mathcal{N}_t$

    for $k \leftarrow 1$ **to** K_{t2} **do**

      Sample $\bar{\xi}_t^k$ from $\xi_t | \tilde{\xi}_{t-1,n}$

      $i = \max \arg\min_j \max(1, ||\bar{\xi}_t^k||, ||\tilde{\xi}_{tj}||)^{p_t - 1} ||\bar{\xi}_t^k - \tilde{\xi}_{tj}||$

      $p_{t-1,ni} \leftarrow p_{t-1,ni} + {}^1/_{K_{t2}}$

    end

  end

  `/* Calculate unconditional probabilities */`

  for $n_2 \in \mathcal{N}_t$ **do**

    $q_{tn_2} \leftarrow \sum_{n_1 \in \mathcal{N}_{t-1}} q_{t-1,n_1} p_{t-1,n_1 n_2}$

  end

end

Algorithm 1: Stage-wise scenario lattice generation algorithm.

the value function, can be applied to the true problem. The analysis extends similar arguments presented in Terça and Wozabal (2021) for the more restrictive case of compactly supported randomness.

Note that the ensuing convergence analysis is different from the results in Section 3, since it does not merely approximate the optimal value of the original problem, but additionally demonstrates that the approximate policy is actually useful in the original problem, i.e., is close to the true optimal policy and, what is even more important, yields an objective that is close to the true optimal objective. Although, strictly speaking, these results are the pre-requisite for employing the decisions of the approximate problem in the actual problem, this perspective is rarely taken in the literature on stability of stochastic programs.

To transfer the solution from the scenario lattice back to the original problem and obtain a solution for an observed trajectory $\bar{\xi}_1, \dots, \bar{\xi}_T$ of the original process, we sequentially compute decisions as follows

$$x_t^* \in \arg\max_{x_t} \left\{ \Pi_t(\bar{\xi}_t, x_t) + \tilde{\mathcal{V}}_{t+1}(x_t, \tilde{\xi}_t(\bar{\xi}_t)) : x_t \in \mathcal{X}_t(x_{t-1}^*, \bar{\xi}_t) \right\}. \tag{35}$$

This implies that the problem is solved with the data determined by the sample $\bar{\xi}_t$ using the post-decision value function $x_t \mapsto \tilde{\mathcal{V}}_{t+1}(x_t, \tilde{\xi}_t(\bar{\xi}_t))$ of the approximated problem associated with the lattice node $\tilde{\xi}_t(\bar{\xi}_t)$ that is closest to $\bar{\xi}_t$ in the respective notion of *distance* implied by the Fortet-Mourier metric. Given that $\tilde{\boldsymbol{\xi}}_t$ has S atoms, we denote by $\bar{V}_t^S(x_{t-1}, \bar{\xi}_t)$ the objective value of the problem in (35) and refer to this procedure as *rounding to a lattice node*. Rounding to a lattice node is made possible by the fact that once the problem is solved for an approximating scenario lattice, the value functions can, in principle, be used to make decisions for all possible resource states.

For the purpose of the next results, we denote by $\tilde{\xi}_t^{S_t}$, the optimal approximation of ξ_t with $S_t = |\mathcal{N}_t|$ centers minimizing (21). Note that we use the superscript in $\tilde{\xi}_t^{S_t}$ to indicate the number of nodes and not, as in Section 4, the number of iterations of the SGD algorithm.

In the following, we prove Lemma 5, which is adapted from Terça and Wozabal (2021) where it is proven for the Wasserstein distance. We adapt the proof for the Fortet-Mourier distance and the lattices used in this paper. As a preparation, we show the following auxiliary result, which establishes that optimal discretizations with respect to the Fortet-Mourier metric are asymptotically correct and lead to almost sure convergence of realizations to the next center of the discrete distribution.

Lemma 4. *Given a random vector* $\boldsymbol{\xi}:\Omega\to\mathbb{R}^n$ *and discretizations* $\tilde{\boldsymbol{\xi}}^S:\Omega\to\mathbb{R}^n$ *that live on* S *atoms such that*

$$\tilde{\boldsymbol{\xi}}^S \in \underset{\tilde{\boldsymbol{\xi}}'\in\mathcal{P}_S}{\arg\min}\, d_{FM}^p(\tilde{\boldsymbol{\xi}}',\boldsymbol{\xi}).$$

It follows that

$$d_{FM}^p(\tilde{\boldsymbol{\xi}}^S,\boldsymbol{\xi}) \xrightarrow{S\to\infty} 0 \tag{36}$$

and $\tilde{\boldsymbol{\xi}}^S(\xi)\xrightarrow{a.s.}\boldsymbol{\xi}$.

Proof. The result in (36) follows in the same way as the corresponding result for the Wasserstein distance in Graf and Luschgy (2000), Lemma 6.1.

Now suppose that almost sure convergence would not be true, i.e., that there is a $\xi'\in\operatorname{supp}(\boldsymbol{\xi})$, a sequence $S_k\xrightarrow{k\to\infty}\infty$, and an $\varepsilon>0$ for which

$$||\xi'-\tilde{\xi}^{S_k}(\xi')||>\varepsilon,\quad \forall k.$$

Define $A=\{z\in\mathbb{R}^n:||z-\xi'||<\epsilon/2\}$ and $\delta=P(A)$. If $\delta>0$, then

$$d_{FM}^p(\tilde{\boldsymbol{\xi}}^{S_k},\boldsymbol{\xi})\geq\int_A \max(1,||\xi||,||\xi^{S_k}(\xi)||)^{p-1}||\xi-\tilde{\xi}^{S_k}(\xi)||\,P(d\xi)>\delta\frac{\varepsilon}{2},\quad\forall k\in\mathbb{N},$$

which is in contradiction to (36) and therefore proves almost sure convergence. □

We now establish that, with an increasing number of nodes of the lattice, the optimal discretization $\tilde{\xi}_t^{S_t}$ converges almost surely to ξ_t and that this property carries over to the conditional distributions at stage t.

Lemma 5. *Let Assumption 2 hold and the problem be convex. Let* $\tilde{\boldsymbol{\xi}}_t^{S_t}$ *be the optimal discretization of* $\boldsymbol{\xi}_t$ *with* S_t *centers according to Algorithm 1. Then it holds that:*

1. $d_{FM}^{p_t}(\boldsymbol{\xi}_t,\tilde{\boldsymbol{\xi}}_t^{S_t})\xrightarrow{S_t\to\infty}0$ *and the discretization is almost surely asymptotically correct, i.e.,*

$$\tilde{\boldsymbol{\xi}}_t^{S_t}(\xi_t)\xrightarrow{S_t\to\infty}\boldsymbol{\xi}_t,\quad a.s. \tag{37}$$

2. *The distances between the conditional distributions* $\tilde{\boldsymbol{\xi}}_t^{S_t}|\tilde{\xi}_{t-1}^{S_{t-1}}(\xi_t)$ *and the true conditional distributions vanishes as* S_t *grows, i.e.,*

$$d_{FM}^{p_t}\left(\tilde{\boldsymbol{\xi}}_t^{S_t}|\bar{\xi}_{t-1}^{S_{t-1}}(\xi_{t-1}),\boldsymbol{\xi}_t|\xi_{t-1}\right)\xrightarrow{S_t\to\infty}0.$$

Proof. We proof 1. starting from the first stage. Clearly, $\tilde{\boldsymbol{\xi}}_1^{S_1} = \boldsymbol{\xi}_1$ and $d_{FM}^{p_2}(\boldsymbol{\xi}_2, \tilde{\boldsymbol{\xi}}_2^{S_2}) \to 0$ as $S_2 \to \infty$ and $\tilde{\xi}_2^{S_2}(\xi_t) \to \xi_2$ almost surely by Lemma 4.

For the induction step from t to $t+1$, we choose $\varepsilon > 0$ and write

$$\begin{aligned} d_{FM}^{p_t}(\boldsymbol{\xi}_{t+1}, \tilde{\boldsymbol{\xi}}_{t+1}^{S_t}) &= \sup_f \int f\, P_{t+1}(d\xi_{t+1}) - \int f\, \tilde{P}_{t+1}^{S_{t+1}}(d\tilde{\xi}_{t+1}^{S_{t+1}}) \\ &= \sup_f \iint f\, P_{t+1,\xi_t}(d\xi_{t+1}) P_t(d\xi_t) - \iint f\, \tilde{P}^{S_{t+1}}_{t+1,\tilde{\xi}_t^{S_t}}(d\tilde{\xi}_{t+1}^{S_{t+1}}) \tilde{P}_t^{S_t}(d\tilde{\xi}_t^{S_t}) \\ &\le \sup_f \int \left(\int f\, P_{t+1,\tilde{\xi}_t^{S_t}(\xi_t)}(d\xi_{t+1}) - \int f\, \tilde{P}^{S_{t+1}}_{t+1,\tilde{\xi}_t^{S_t}}(d\tilde{\xi}_{t+1}^{S_{t+1}}) \right) \tilde{P}_t^{S_t}(d\tilde{\xi}_t^{S_t}) + \varepsilon \\ &\le \int \sup_f \left(\int f\, P_{t+1,\tilde{\xi}_t^{S_t}(\xi_t)}(d\xi_{t+1}) - \int f\, \tilde{P}^{S_{t+1}}_{t+1,\tilde{\xi}_t^{S_t}}(d\tilde{\xi}_{t+1}^{S_{t+1}}) \right) \tilde{P}_t^{S_t}(d\tilde{\xi}_t^{S_t}) + \varepsilon \qquad (38) \end{aligned}$$

where the first inequality follows the fact that the inner integral is in $Lip^{p_t^C}$, since

$$\begin{aligned} \left| \int f\, P_{t+1,\xi_t}(d\xi_{t+1}) - \int f\, P_{t+1,\xi_t'}(d\xi_{t+1}) \right| &\le d_{FM}^{p_t}(\boldsymbol{\xi}_{t+1}|\xi_t, \boldsymbol{\xi}_{t+1}|\xi_t') \\ &\le L_t^C \min(1, ||\xi_t||, ||\xi_t'||)^{p_t^C - 1} ||\xi_t - \xi_t'||, \end{aligned}$$

which together with the induction hypothesis and Assumption 2 yields the inequality for large enough S_t.

Note that (38) is minimized in (21) and that this problem can be written as a quantization problem of a single distribution due to Theorem 10. Since $\varepsilon > 0$ was arbitrary and due to Lemma 4 the right hand side of (38) thus converges to zero and it follows that $\tilde{\xi}_{t+1}^{S_{t+1}}(\xi_{t+1}) \to \xi_{t+1}$ almost surely as in Lemma 4, which proves 1.

To prove the second point, we use Assumption 2 to write

$$\begin{aligned} & d_{FM}^{p_{t-1}^C}(\boldsymbol{\xi}_t|\xi_{t-1}, \tilde{\boldsymbol{\xi}}_t^{S_t}|\tilde{\xi}_{t-1}^{S_{t-1}}(\xi_{t-1})) \le d_{FM}^{p_{t-1}^C}(\boldsymbol{\xi}_t|\tilde{\xi}_{t-1}^{S_{t-1}}(\xi_{t-1}), \tilde{\boldsymbol{\xi}}_t^{S_t}|\tilde{\xi}_t^{S_{t-1}}(\xi_{t-1})) \\ & + L_t^C \max(1, ||\xi_{t-1}||, ||\tilde{\xi}_{t-1}^{S_{t-1}}(\xi_{t-1})||)^{p_{t-1}^C - 1} ||\xi_{t-1} - \tilde{\xi}_{t-1}^{S_{t-1}}(\xi_{t-1})||, \end{aligned}$$

where the first term on the right hand side vanishes because (21) goes to zero due to Lemma 4 as S_t increases and the second term vanishes, since $\tilde{\xi}_{t-1}^{S_{t-1}}(\xi_{t-1}) \to \xi_{t-1}$ almost surely because of 1. $\square$

We are now in a position to prove the following theorem, which shows that the approximation of the value functions progressively gets better as the scenario lattice gets finer, i.e., as the number of nodes per stage increases. In particular, we establish that the out-of-sample value generated by rounding to the next lattice node converges to the true optimal value of the problem in the number of nodes.

THEOREM 11. *Consider a sequence of approximating random variables* $(\tilde{\boldsymbol{\xi}}_1^S, \ldots, \tilde{\boldsymbol{\xi}}_T^S)_{S \in \mathbb{N}}$ *such that for each* $\tilde{\boldsymbol{\xi}}_t^S$ *the number of atoms* $S_t(S)$ *of* $\tilde{\boldsymbol{\xi}}_t^S$ *goes to infinity as* $S \to \infty$ *and the atoms of* $\tilde{\boldsymbol{\xi}}_t^S$ *are chosen to minimize* (21). *Let the problem be convex and the value functions be locally Lipschitz as in* (7), *Assumption 2 be true, and assume that there are integrable bounding functions* Ψ_t *such that*

$$V_t(x_{t-1}, \xi_t) \le \Psi_t(x_{t-1}, \xi_t), \quad a.s., \forall t \in [T] \tag{39}$$

$$\tilde{V}_t^S(x_{t-1}, \tilde{\xi}_t^S) \le \Psi_t(x_{t-1}, \tilde{\xi}_t^S), \quad a.s., \forall t \in [T] \tag{40}$$

Then it holds that

$$\bar{V}_t^S(x_{t-1}, \xi_t) \xrightarrow{S \to \infty} V_t(x_{t-1}, \xi_t), \quad a.s., \quad \forall x_{t-1}, \forall t \in [T] \tag{41}$$

and

$$\tilde{\mathcal{V}}_t^S(x_t, \tilde{\xi}_t^S(\xi_t)) \xrightarrow{S \to \infty} \mathcal{V}_t(x_t, \xi_t), \quad a.s., \forall x_t, \forall t \in [T]. \tag{42}$$

Proof. For the final stage T, we have

$$\bar{V}_T^S(x_{T-1}, \xi_T) = V_T(x_{T-1}, \xi_T), \quad \forall x_{T-1} \in \mathbb{R}^{M_t}.$$

For all other time periods, it follows because of Theorem 2 and Lemma 5 that

$$\tilde{V}_t^S(x_{t-1}, \tilde{\xi}_t^S) \xrightarrow{S \to \infty} V_t(x_{t-1}, \tilde{\xi}_t^S), \quad \forall \tilde{\xi}_t^S \in \operatorname{supp}(\tilde{\boldsymbol{\xi}}_t^S), \forall x_t. \tag{43}$$

We then can write

$$\begin{aligned}
&|\tilde{\mathcal{V}}_t^S(x_t, \tilde{\xi}_t^S(\xi_t)) - \mathcal{V}_t(x_t, \xi_t)| \\
&= |\mathbb{E}[\tilde{V}_{t+1}^S(x_t, \tilde{\xi}_{t+1}^S)|\tilde{\xi}_t^S(\xi_t)] - \mathbb{E}[V_{t+1}(x_t, \xi_{t+1})|\xi_t]| \\
&\le |\mathbb{E}[\tilde{V}_{t+1}^S(x_t, \tilde{\xi}_{t+1}^S)|\tilde{\xi}_t^S(\xi_t)] - \mathbb{E}[V_{t+1}(x_t, \xi_{t+1})|\tilde{\xi}_t^S(\xi_t)]| \\
&+ L_t^C L_t \max(1, ||\xi_t||, ||\tilde{\xi}_t^S(\xi_t)||)^{p_t^C - 1} ||\xi_t - \tilde{\xi}_t^S(\xi_t)|| \\
&\le |\mathbb{E}[\tilde{V}_{t+1}^S(x_t, \tilde{\xi}_{t+1}^S) - V_{t+1}(x_t, \tilde{\xi}_{t+1}^S)|\tilde{\xi}_t^S(\xi_t)]|
\end{aligned}$$

$$+ L_t^C L_t \max(1, ||\xi_t||, ||\tilde{\xi}_t^S(\xi_t)||)^{p_t^C - 1} \, ||\xi_t - \tilde{\xi}_t^S(\xi_t)|| + L_t \, d_{FM}^{p_t}(\boldsymbol{\xi}_{t+1}|\tilde{\xi}_t^S(\xi_t), \tilde{\boldsymbol{\xi}}_{t+1}^S|\tilde{\xi}_t^S(\xi_t))$$

$$\xrightarrow{C\to\infty} 0$$

where the first inequality follows, since V_{t+1} is locally Lipschitz and Assumption 2 and the second one follows again because of the local Lipschitz continuity of V_{t+1}. Finally, the last line goes to zero because the Fortet-Mourier term goes to zero and $\tilde{\xi}_t^S(\xi_t) \to \xi_t$ because of Lemma 5 and Theorem 2 while the second last line goes to zero by (43) and the dominated convergence theorem which is applicable because of (39) and (40). This proves (42).

Finally, because of (42)

$$\bar{V}_t^S(x_{t-1}, \xi_t) - V_t(x_{t-1}, \xi_t) \le \tilde{\mathcal{V}}_t^S(\bar{x}_t^*, \tilde{\xi}_t^S(\xi_t)) - \mathcal{V}_t(\bar{x}_t^*, \xi_t) \xrightarrow{S\to\infty} 0,$$

where $\bar{x}_t^*$ is optimal for $\bar{V}_t^S(x_{t-1}, \xi_t)$. Reversing the roles of $\bar{V}_t^S$ and V_t shows (41) and concludes the proof. □

REMARK 9. Conditions (39) and (40) on the existence of a bounding function Ψ_t are usually unproblematic. In particular, if $||\xi_t||^{p_t}$ is integrable, the existence of Ψ_t can be easily shown in the settings of Theorem 3, Theorem 4, and Theorem 5.

THEOREM 12. *If the assumptions of Theorem 11 hold and the Hausdorff distance of the feasible sets converge, i.e.,*

$$d_H(\mathcal{X}_t(x_{t-1}, \xi_t), \mathcal{X}_t(x'_{t-1}, \xi_t)) \xrightarrow{x'_t \to x_t} 0, \quad \forall x_{t-1}, \, \forall \xi_t \in \operatorname{supp}(\boldsymbol{\xi}_t), \tag{44}$$

then every limit point of the optimal policy $\bar{x}_t^{S*}$ of the approximated problems converge almost surely to a policy x_t^* that is optimal for the original problem as $S \to \infty$.

Proof. For a given sample path $\bar{\xi}_1, \ldots, \bar{\xi}_T$ of the stochastic process $\boldsymbol{\xi}$, we compare the solutions $\bar{x}_t^{S*}$ to (35) to the solutions x_t^* of the original problem for the sample path.

From Theorem 11, we know that

$$\tilde{\mathcal{V}}_{t+1}^S(x_t, \tilde{\xi}_t(\xi_t)) \xrightarrow{S\to\infty} \mathcal{V}_{t+1}(x_t, \xi_t), \quad \forall x_t, \ a.s.$$

For the first stage, the constraints of problem $\tilde{V}_1^S$ and V_1 are the same and therefore the objective functions epi-converge because of Theorem 7.31 in Shapiro et al. (2009). It thus follows from

Proposition 7.30 in Shapiro et al. (2009) that every limit point of the first stage solutions $\bar{x}_1^{S*}$ converge to a first stage solution x_1^*.

Proceeding by induction over t, we notice that for a given sample $\bar{\xi}_1, \ldots, \bar{\xi}_T$ the feasible sets of the problems $\bar{V}_t^S$ and V_t differ due to the difference between x_{t-1}^* and $\bar{x}_{t-1}^{S*}$. Consider an arbitrary point x with distance $\epsilon > 0$ to the boundary of $\mathcal{X}_t(x_{t-1}^*, \bar{\xi}_t)$. Because of (44), we get that, eventually, $x \in \mathcal{X}_t(x_{t-1}^*, \bar{\xi}_t) \cap \mathcal{X}_t(\bar{x}_{t-1}^{S*}, \bar{\xi}_t)$ or $x \notin \mathcal{X}_t(x_{t-1}^*, \bar{\xi}_t) \cap \mathcal{X}_t(\bar{x}_{t-1}^{S*}, \bar{\xi}_t)$, due to the fact that by the induction hypothesis every limit point of $\bar{x}_{t-1}^{S*}$ converge to an optimal solution x_{t-1}^*.

Since the complement of the boundary of $\mathcal{X}_t$ is dense in $\mathbb{R}^{M_t}$, it follows by Proposition 7.31 in Shapiro et al. (2009) that for every convergent subsequence $\bar{x}_{t-1}^{S_k*}$

$$\Pi_t(x_t, \bar{\xi}_t) + \bar{\mathcal{V}}_t^S(x_t, \tilde{\xi}_t^S(\xi_t)) + \mathbb{1}_{\mathcal{X}_t(\bar{x}_{t-1}^{S_k*}, \bar{\xi}_t)}$$

epi-converges to

$$\Pi(x_t, \bar{\xi}_t) + \mathcal{V}_t(x_t, \bar{\xi}_t) + \mathbb{1}_{\mathcal{X}_t(x_{t-1}^*, \bar{\xi}_t)}$$

as $S \to \infty$ for x_{t-1}^* an optimal solution for the problem in $t-1$. Therefore, every limit point of $\bar{x}_t^{S*}$ converges to an optimal solution x_t^* due to Proposition 7.30 in Shapiro et al. (2009), which finishes the proof. $\square$

Remark 10. The condition on the Hausdorff distance of the feasible sets holds for all problem classes discussed in Section 3: The case with deterministic feasible sets treated in Theorem 3 trivially fulfills the condition. Problems of the type treated in Theorem 4 fulfill the condition because of the convexity of the function f. Finally, the linear problems in Theorem 5 fulfill the property because of the Lipschitz continuity of solutions of linear inequalities (Mangasarian and Shiau 1987).

6. A Numerical Example

In this section, we put the proposed methods to the test using a multistage extension of the classic newsvendor problem with uncertain demands and prices and benchmark the results with a common lattice generation method which approximates unconditional distributions and is, for example, used in Bally and Pagès (2003), Löhndorf and Shapiro (2019), Löhndorf and Wozabal (2021).

We consider a supplier of a single perishable good who faces a random demand D_t. The supplier buys the goods at a random price P_t and sells it to its customers with a markup $m > 0$ for $(1+m)P_t$. Goods ordered in stage t arrive at stage $t+1$. The supplier owns a storage of size $\bar{L}$ with the storage balance connecting the decision stages. Due to the perishable nature of the goods, a fraction $\eta \in (0,1)$ of the stock in period t perishes until period $t+1$.

Denoting the order decision in period t by o_t, the sell decision by s_t, and the storage level by l_t, the problem can be written as

$$\begin{array}{lll} \max\limits_{o,s,l} & \mathbb{E}\left[\sum_{t=1}^{T}(P_t+m)s_t - P_t o_t\right] & \\ \text{s.t.} & l_t \leq (1-\eta)l_{t-1} + o_{t-1} - s_t, & a.s., \forall t \in [T] \\ & l_t \leq \bar{L}, & a.s., \forall t \in [T] \\ & l_t, s_t, o_t \geq 0, & a.s., \forall t \in [T], \end{array} \tag{45}$$

with initial values l_0 and o_0. For the sake of simplicity, we set the salvage value at the end of the planning horizon to 0, i.e., remaining goods are discarded. Note that (45) falls in the class of problems treated in Theorem 5 with $p_t = 2$, since the objective is linear in the randomness.

We choose the following stochastic processes for prices and demands

$$D_t = \max(0, D_{t-1} + \sigma_D \varepsilon_{Dt})$$

$$P_t = P_{t-1}\exp(\sigma_P \varepsilon_{Pt})$$

with $(\varepsilon_{Dt}, \varepsilon_{Pt})$ bivariate normal with mean 0 and correlation α.

For our numerical experiments, we fix $T = 20$, $\sigma_D = 10$, $\sigma_P = 0.1$, $\alpha = 0.5$, $P_0 = D_0 = 100$, $o_0 = 0$, $l_0 = 5$, $m = 0.1$, $\bar{L} = 10$, and $\eta = 0.1$. We choose $\rho_k = {}^{3}/_{30+k}$ as learning rate for the SGD, which ensures that early updates do not push atoms $\tilde{\xi}_{ti}$ to far out locations where they will never again be reached by subsequent samples. The latter happens for the naive choice $\rho_k = k^{-1}$.

As mentioned in Section 3, the Lipschitz constants L_t of the problem keep growing, suggesting a finer approximation, i.e., more lattice nodes, in earlier stages as compared to the later stages. On the other hand, both the stochastic process for demand and price are nonstationary with increasing variance. Since these two factors have an opposing effect and to keep the study simple, we keep the number of nodes of the scenario lattices constant for all non-root stages and postpone a more detailed study of the optimal number of nodes to future work.

	Fortet-Mourier			**Wasserstein**			**Difference**	
	UB	**OOS**	**Gap**	**UB**	**OOS**	**Gap**	**Δ**	**Std**
5 nodes	15593	11443	36.27	14853	6032	146.23	47.28	33.02
10 nodes	16604	13793	20.38	17274	10864	59.00	21.23	34.24
20 nodes	17455	16105	8.38	18704	13329	40.33	17.24	39.60
50 nodes	18311	17999	1.73	19290	17121	12.67	4.88	43.64
100 nodes	18786	18795	-0.05	19549	18739	4.32	0.30	47.14

Table 1 Comparison of results for problem (45) using lattices generated from Algorithm 1 and lattices generated minimizing the Wasserstein distance between the unconditional distributions of $\boldsymbol{\xi}_t$. **UB**: SDPP upper bound. **OOS**: average profit on samples of the continuous stochastic process. **Gap**: gap between UB and OOS measured in percentages of UB. **Δ**: gap between the OOS results of the two methods measured in percentages of the Fourtet-Mourier OOS values. **Std**: Standard deviation of the difference between the OOS values.

For our numerical experiments, we vary the number of nodes in order to study how the approximations improve with progressively finer scenario lattices. In particular, we use lattices with 5, 10, 20, 50, and 100 nodes per stage and $K_{t1} = 10^5$ and $K_{t2} = 10^4$ samples for lattice generation for all $t = 2, \dots, T$. For the lattices that are based on the minimization of the Wasserstein distance between the unconditional distributions ξ_t and $\tilde{\xi}_t$, we use the same learning rate and 10^5 samples as well. For the analysis of the out-of-sample performance of the trained policies, we generate 10^5 samples from the original process and solve the problems $\bar{V}_t$, rounding to the next lattice node, as described in Section 5. Note that even for lattices with only 5 nodes per stage the equivalent scenario trees would have $\sum_{t=1}^{20} 5^{t-1} = 2.38 \times 10^{13}$ nodes and therefore would lead to computationally intractable problems.

We solve problem (45) using the *Quasar*[3] implementation of SDDP and run the solver for a fixed number of 500 iterations, which is easily enough to ensure convergence in the above example. The code that generated the results in this section is available at *Google Colab*.[4]

Table 1 summarizes the results of the numerical study. As expected the quality of the approximation gets better with the number of lattice nodes. This can be seen for both methods in the increasing out-of-sample performance of the respective policies and the decreasing gaps between the SDDP upper bounds and the out of sample values. In particular, for the method based on the Fortet-Mourier metric the gap closes for 100 nodes per stage implying that the

[3] See `https://www.quantego.com/quasar`.

[4] See `https://colab.research.google.com/drive/1FhpII8aRPdUzX55e_o6Z3ZUXYz5Nfaer`.

approximate policy delivers an out-of-sample value close to the objective of the approximated stochastic program.

The comparison of the two methods reveals that the out of sample results as well as the gap of the out-of-sample profit to the SDDP upper bound are significantly better for the method proposed in this paper – particularly for more coarse discretization by smaller lattices, where the unconditional Wasserstein lattices perform rather poorly. This can be seen as a promising result for problems with higher dimensional stochastic processes $\boldsymbol{\xi}$, where even discretizations with thousands of nodes are necessarily *coarse* due to the curse of dimensionality.

7. Conclusion

This paper proposes a bound for the objective function of a Markovian stochastic optimization problem when the distribution of the stochastic process varies. The result uses Fortet-Mourier distances between conditional distributions and holds whenever the value functions of the problems are locally Lipschitz in the resource state, which is the case for a range of important problems provided a rather natural continuity assumption on the stochastic process holds.

Based on these results, we propose a nonconvex and nonsmooth SGD method using generalized gradients that generates scenario lattices from samples of the stochastic process. We detail how the solution obtained for the scenario lattice can be transferred back to the original process and show that under mild regularity conditions the discretization error in the objective and the solutions of the problem vanishes as the approximation gets finer.

Finally, we show in a numerical study for a multi-stage newsvendor problem that the proposed method outperforms a state-of-the-art lattice generation method that has been used in the extant literature. In particular, the obtained solutions exhibit better out-of-sample solutions and a smaller gap between the estimated insample objective and the realized out of sample profits, especially for smaller lattices.

Interesting topics for future research are more extensive numerical tests with a wide range of examples and the exploration of different choices for lattice topologies guided by the relative size of the constants L_t in different stages. Furthermore, it would interesting to explore which other problem classes lead to locally Lipsschitz continuous value functions and to possibly obtain more general results on Lipschitz moduli.

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