An Extension of the Bertsimas & Sim Result for Discrete, Linear, and $\Gamma$-Robust Min-Max Problems

Yasmine Beck, Ivana Ljubić, and Martin Schmidt

Abstract. Due to their nested structure, bilevel problems are intrinsically hard to solve—even if all variables are continuous and all parameters of the problem are exactly known. In this paper, we study mixed-integer linear bilevel problems with lower-level objective uncertainty, which we address using the notion of $\Gamma$-robustness. We provide an extension of the famous result by Bertsimas and Sim for $\Gamma$-robust single-level problems stating that the $\Gamma$-robust counterpart of a min-max bilevel problem can be solved by solving a polynomial number of min-max problems of the nominal type. Moreover, we discuss the situation in which the problems of the nominal type are not solved exactly but in which either an $\alpha$-approximation algorithm or a primal heuristic together with a lower bounding scheme is used. For general $\Gamma$-robust bilevel problems, however, we illustrate that the previous ideas cannot be carried over completely. Nevertheless, we present a primal heuristic that is based on the solution of a polynomial number of problems of the nominal type. To assess the performance of the presented methods, we perform a computational study on 560 instances for both settings. For the primal heuristic, we observe that the optimality gap is closed for a significant portion of the considered instances. If our result for the min-max setting is exploited, we report speed-up factors exceeding 400 and 32 in the mean and the median, respectively, when compared to recently published problem-tailored solution approaches.

1. Introduction

Bilevel optimization is a rather young but very active field of research, having its game-theoretic roots dating back to the seminal publications of von Stackelberg (1932, 1954). Over the last years and decades, bilevel problems have gained increasing attention due to their ability to model hierarchical decision making processes. For an overview of the many applications of bilevel optimization, we refer to the annotated bibliography by Dempe (2020) as well as to the recent surveys by Kleinert et al. (2021) and by Beck et al. (2023a). The latter focuses on bilevel problems under uncertainty, which is also at the core of this paper.

Due to their hierarchical structure, bilevel problems are intrinsically hard to solve—even if all objective functions and constraints are linear, all variables are continuous, and all parameters of the problem are exactly known (Hansen et al. 1992). However, the situation becomes more challenging if, e.g., (i) discrete variables are introduced and (ii) problems under uncertainty are considered. In mathematical optimization, there are two main approaches to deal with uncertainties: stochastic optimization (Birge and Louveaux 2011; Kall and Wallace 1994) and robust optimization (Ben-Tal and Nemirovski 1998; Ben-Tal, Ghaoui, et al. 2009; Bertsimas, Brown, et al. 2011; Soyster 1973). While, in the context of bilevel optimization, stochastic approaches to deal with uncertainties are more thoroughly studied, robust bilevel optimization is still in its infancy; see, e.g., Beck et al. (2022, 2023a) for more detailed discussions.

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The contributions of this paper are the following. We consider mixed-integer linear bilevel problems with a binary lower-level problem that is affected by objective uncertainty. To deal with this kind of uncertainty, we pursue a $\Gamma$-robust approach (Bertsimas and Sim 2003; Sim 2004) in which the follower only hedges against a subset of the uncertain parameters that adversely influence the solution to the problem. In particular, we exploit the main result presented by Bertsimas and Sim (2003) and Sim (2004) for $\Gamma$-robust single-level problems, namely, that the $\Gamma$-robust counterpart of a binary problem can be solved by solving a number of binary problems of the nominal type that is polynomial in the problem data. A similar setting has been considered in our previous work (Beck et al. 2023b), on which we will build on in this paper. We show that optimal $\Gamma$-robust solutions to min-max problems with a binary lower level can be obtained by solving a polynomial number of min-max problems of the nominal type. The latter naturally extends the main result by Bertsimas and Sim (2003) and Sim (2004) to the min-max setting, which is an important special case of general bilevel optimization. Our result is of particular relevance for both theory and practice since (i) it allows to use state-of-the-art as well as off-the-shelf solvers for the solution of the problems of the nominal type such that (ii) no problem-tailored approaches (such as branch-and-cut) are necessary to solve the $\Gamma$-robust counterpart of the bilevel problem, (iii) the solution of the problems of the nominal type can be parallelized, and (iv) one can further use the result to obtain approximation results. In particular, (i) and (ii) make a huge difference when considering $\Gamma$-robust min-max problems computationally. In our numerical study, we solve 560 $\Gamma$-robust knapsack interdiction problems for which we find speed-up factors exceeding 400 and 32 in the mean and the median, respectively, if our result is exploited. With respect to (iv), we discuss the settings in which the problems of the nominal type are not solved exactly but in which either an $\alpha$-approximation algorithm or a primal heuristic together with a lower bounding scheme is used. For general $\Gamma$-robust mixed-integer linear bilevel problems, however, the situation becomes much more challenging such that the extensions for the min-max case cannot be carried over completely. We illustrate the latter using a counterexample. Moreover, we show that, in the general setting, even obtaining feasibility is (computationally) more expensive compared to the min-max case. Nevertheless, using the same ideas as before, we present a primal heuristic for general $\Gamma$-robust mixed-integer linear bilevel problems. In particular and to the best of our knowledge, we are the first to consider this type of problem—theoretically as well as computationally—in both our previous (Beck et al. 2023b) as well as the current work.

The remainder of this paper is organized as follows. In Section 2, we describe the overall problem statement and present the main result by Bertsimas and Sim (2003) and Sim (2004), which we apply to the $\Gamma$-robust lower-level problem. In Section 3, we focus on mixed-integer linear $\Gamma$-robust min-max problems for which we show an extension of the result by Bertsimas and Sim. A primal heuristic for the general $\Gamma$-robust bilevel setting that exploits the solution of a polynomial number of mixed-integer linear bilevel problems of the nominal type is presented in Section 4. In Section 5, we perform a computational study to assess the performance of the methods presented in this paper. Finally, we conclude in Section 6.
2. Problem Statement

In this paper, we consider mixed-integer linear bilevel problems of the form

\[ \min_{x,y} \ c^\top x + d^\top y \]
\[ \text{s.t.} \quad x \in X, \quad y \in \arg\max_{y'} \{ f^\top y': y' \in Y(x) \}, \]

where \( Y(x) \subseteq \{0,1\}^{n_y} \) and \( X := \{ x \in \mathbb{R}^{n_C} \times \mathbb{Z}^{n_D} : Ax \geq a \} \) with \( n_x = n_C + n_D \), \( c \in \mathbb{R}^{n_x} \), \( d, f \in \mathbb{R}^{n_y} \), \( A \in \mathbb{R}^{m \times n_x} \), and \( a \in \mathbb{R}^m \). We refer to the first two lines of (BMIP) as the upper-level (or the leader’s) problem. The last constraint in (BMIP) is the so-called lower-level (or follower’s) problem. The variables \( x \) and \( y \) are the leader’s and the follower’s variables, respectively. Here, we consider the optimistic approach to bilevel optimization; see, e.g., Dempe (2002). This means that, whenever the set of optimal solutions to the lower-level problem is not a singleton, the follower decides such as to favor the leader w.r.t. her \(^1\) objective function value. This is expressed in (BMIP) by optimizing not only over the leader’s variables \( x \) but also over the follower’s variables \( y \). Throughout this paper, we impose the following.

**Assumption 1.**

(i) For every \( x \in X \), the lower-level feasible set \( Y(x) \) is non-empty.

(ii) The shared constraint set \( \{(x,y) : x \in X, y \in Y(x)\} \) is non-empty and compact.

(iii) All linking variables, i.e., all variables of the leader that appear in the lower-level constraints, are bounded integers.

Assumption 1 is necessary to ensure that (BMIP) has a solution; see, e.g., Section 5.1 in Kleinert et al. (2021) and the references therein for a detailed discussion. For \( x \in X \), we further define the lower-level optimal-value function

\[ \Phi(x) = \max_y \{ f^\top y : y \in Y(x) \} \]

(1)

to re-write (BMIP) as the single-level problem

\[ \min_{x,y} \ c^\top x + d^\top y \]
\[ \text{s.t.} \quad x \in X, \ y \in Y(x), \]
\[ f^\top y \geq \Phi(x). \]

In this paper, we are interested in bilevel problems of the above form, which are, however, affected by lower-level data uncertainty. We focus on uncertainties in the lower-level objective function coefficients, i.e., for all \( i \in [n_y] := \{1, \ldots, n_y\} \), we consider the coefficients \( f_i \) with \( f_i \in [f_i - \Delta f_i, f_i] \) instead of \( f_i \). Here, we denote \( f_i \) as the nominal value of the \( i \)th lower-level objective function coefficient and \( \Delta f_i \) as its maximum deviation from the nominal value. For a discussion of the case with uncertainties in a single packing-type constraint in the lower level, we refer to Beck et al. (2023b).

To deal with lower-level data uncertainty, we pursue a \( \Gamma \)-robust approach (Bertsimas and Sim 2003, 2004) in which the follower hedges against at most \( \Gamma \in [n_y] \) deviations in his objective function coefficients. This leads us to considering the bilevel problem

\[ \min_{x,y} \ c^\top x + d^\top y \]
\[ \text{s.t.} \quad x \in X, \ y \in S_\Gamma(x), \]

\[ (\Gamma \text{-BMIP}) \]

\(^1\)Throughout this paper, we use “her” for the leader and “his” for the follower.
where \( S_{\Gamma}(x) \) is the set of optimal solutions to the \( x \)-parameterized \( \Gamma \)-robust lower-level problem

\[
\max_y f^T y - \max_{\{S \subseteq [n_y] : |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y_i \quad \text{s.t.} \quad y \in Y(x).
\]

For a feasible upper-level decision \( x \in X \), we define the optimal-value function of the \( \Gamma \)-robust lower level as

\[
\Phi_{\Gamma}(x) = \max_{y \in Y(x)} \left\{ f^T y - \max_{\{S \subseteq [n_y] : |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y_i \right\}
\]

such that the \( \Gamma \)-robust counterpart (\( \Gamma \)-BMIP) of the bilevel problem can be written as

\[
\min_{x,y} c^T x + d^T y \\
\text{s.t.} \quad x \in X, \ y \in Y(x), \\
f^T y - \max_{\{S \subseteq [n_y] : |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y_i \geq \Phi_{\Gamma}(x).
\]

For the validity of the techniques we present in this paper, we further impose the following.

**Assumption 2.**

(i) The deviations are non-negative, i.e., \( \Delta f_i \geq 0 \) for all \( i \in [n_y] \).

(ii) The indices are ordered such that the deviations are given in non-increasing order, i.e., \( \Delta f_i \geq \Delta f_{i+1} \) for all \( i \in [n_y] \) with \( \Delta f_{n_y+1} = 0 \).

Assumption 2 is w.l.o.g. but necessary to exploit Theorem 3 in Bertsimas and Sim (2003), which is what we do in the next lemma.

**Lemma 1.** Let \( x \in X \) be a feasible upper-level decision. Under Assumption 2, solving the \( \Gamma \)-robust counterpart (2) of the lower-level problem is equivalent to solving \( n_y + 1 \) problems of the nominal type, i.e.,

\[
\Phi_{\Gamma}(x) = \max_{\ell \in [n_y+1]} \left\{ \Phi_\ell(x) \right\}
\]

holds, where for all \( \ell \in [n_y+1] \), we have

\[
\Phi_\ell(x) = -\Gamma \Delta f_\ell + \max_{y \in Y(x)} \left\{ \tilde{f}(\ell)^T y \right\}
\]

with

\[
\tilde{f}(\ell)_i = \begin{cases} 
  f_i - (\Delta f_i - \Delta f_\ell), & 1 \leq i \leq \ell, \\
  f_{\ell+1}, & \ell + 1 \leq i \leq n_y.
\end{cases}
\]

In Álvarez-Miranda et al. (2013), the authors present an improvement of the previous result by reducing the number of problems of the nominal type to be solved to \( n_y - \Gamma + 2 \). Further reductions have been established by Lee and Kwon (2014) by showing that it suffices to solve

\[
\Phi_{\Gamma}(x) = \max_{\ell \in \mathcal{L}} \left\{ \Phi_\ell(x) \right\},
\]

with

\[
\mathcal{L} = \{\Gamma + 1, \Gamma + 3, \Gamma + 5, \ldots, \Gamma + \gamma, n_y + 1\}
\]

and \( \gamma \) is the largest odd integer such that \( \Gamma + \gamma < n_y + 1 \) holds. Hence, only \( \lceil (n_y - \Gamma)/2 \rceil + 1 \) problems of the nominal type need to be considered. We will hold on to the result by Lee and Kwon (2014) throughout this paper.
Table 1. Central Notation.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi : X \rightarrow \mathbb{R}$</td>
<td>Optimal-value function of the nominal lower level; see (1)</td>
</tr>
<tr>
<td>$\Phi_\Gamma : X \rightarrow \mathbb{R}$</td>
<td>Optimal-value function of the $\Gamma$-robust lower level; see (2)</td>
</tr>
<tr>
<td>$\Phi_{\ell} : X \rightarrow \mathbb{R}$</td>
<td>Optimal-value function of the $\ell$th lower-level sub-problem; see (3)</td>
</tr>
<tr>
<td>$v_\Gamma \in \mathbb{R}$</td>
<td>Optimal objective value of the $\Gamma$-robust min-max problem</td>
</tr>
<tr>
<td>$v_\ell \in \mathbb{R}$</td>
<td>Optimal objective value of the $\ell$th min-max problem of the nominal type; see ($\ell$-Min-Max)</td>
</tr>
<tr>
<td>$v_\alpha \in \mathbb{R}$</td>
<td>Objective value of an $\alpha$-approximate solution to the $\Gamma$-robust min-max problem</td>
</tr>
<tr>
<td>$v_{\alpha,\ell} \in \mathbb{R}$</td>
<td>Objective value of an $\alpha$-approximate solution to the $\ell$th min-max problem of the nominal type</td>
</tr>
<tr>
<td>$v_{h,\ell} \in \mathbb{R}$</td>
<td>Objective value of a heuristic solution to the $\ell$th min-max problem of the nominal type</td>
</tr>
<tr>
<td>$\omega_\ell \in \mathbb{R}$</td>
<td>Lower bound for the optimal objective value of the $\ell$th min-max problem of the nominal type</td>
</tr>
</tbody>
</table>

3. Mixed-Integer Linear Min-Max Problems

In this section, we focus on mixed-integer min-max problems as a special case of (BMIP). Here, we set $d = f$, i.e., in its deterministic form, we consider the bilevel problem
\[
\min_x \quad c^T x + f^T y \\
\text{s.t.} \quad x \in X, \quad y \in \arg \max_{y'} \{ f^T y' : y' \in Y(x) \}.
\]

Here, we do not need to distinguish between an optimistic and a pessimistic follower since the follower’s response always yields the worst-possible outcome for the leader. Using the lower-level optimal-value function (1), we obtain a single-level reformulation of (Min-Max) that is given by
\[
\min_{x \in X} \{ c^T x + \Phi(x) \}.
\]

The $\Gamma$-robust counterpart of the problem in which the follower hedges against at most $\Gamma$ deviations in his uncertain objective function coefficients is obtained by replacing $\Phi(x)$ with $\Phi_\Gamma(x)$ as stated in (1) and (4), i.e.,
\[
v_\Gamma := \min_{x \in X} \{ c^T x + \Phi_\Gamma(x) \} = \min_{x \in X} \left\{ c^T x + \max_{\ell \in \mathcal{L}} \{ \Phi_{\ell}(x) \} \right\}.
\]

3.1. Exact Solution Approach. We now present an equivalent reformulation of the $\Gamma$-robust counterpart ($\Gamma$-Min-Max) of the min-max problem.
Theorem 1. For all $\ell \in \mathcal{L}$, let $x^\ell$ be an optimal solution to the problem
\[
v_\ell := \min_{x \in \mathcal{X}} \left\{ c^\top x + \Phi_\ell(x) \right\}.
\] (\ell\text{-Min-Max})
Further, let $k = \arg\max_{\ell \in \mathcal{L}} \{v_\ell\}$. Then, $x^k$ solves the $\Gamma$-robust counterpart (\Gamma\text{-Min-Max}) of the min-max problem (Min-Max), i.e., $v_\ell = v_k$ holds.

Proof. Since $\ell \in \mathcal{L}$ does not affect the upper-level constraints, any solution to (\Gamma\text{-Min-Max}) is also feasible for (\ell\text{-Min-Max}) and vice versa. Let $x^*$ be an optimal solution to (\Gamma\text{-Min-Max}). For all $\ell \in \mathcal{L}$, we obtain
\[
v_\ell = c^\top x^* + \Phi_\ell(x^*) = c^\top x^* + \max_{k \in \mathcal{L}} \{\Phi_k(x^*)\} \geq c^\top x^* + \Phi_\ell(x^*) \geq v_\ell, \tag{5}
\]
where the last inequality follows from the feasibility of $x^*$ for (\ell\text{-Min-Max}). In particular, (5) for all $\ell \in \mathcal{L}$ is equivalent to $v_\ell \geq \max_{\ell \in \mathcal{L}} \{v_\ell\}$.

Now, let $k \in \mathcal{L}$ be arbitrary but fixed and let $x^k$ denote a solution to the $k$th min-max problem as stated in (\ell\text{-Min-Max}). Then, we have
\[
\max_{\ell \in \mathcal{L}} \{v_\ell\} \geq v_k = c^\top x^k + \Phi_k(x^k) \geq v_\Gamma,
\]
where the last inequality follows from the feasibility of $x^k$ for Problem (\Gamma\text{-Min-Max}). To sum up, we obtain $v_\ell = \max_{\ell \in \mathcal{L}} \{v_\ell\}$, which concludes the proof. \hfill \qedsymbol

In Theorem 1, we state that the $\Gamma$-robust counterpart (\Gamma\text{-Min-Max}) of the min-max problem (Min-Max) can be solved by solving $|\mathcal{L}|$ many problems of the nominal type, i.e., deterministic min-max problems. In particular, the number of min-max problems to be solved is polynomial in the (lower-level) problem data. Hence, the result by Bertsimas and Sim (2003) for single-level $\Gamma$-robust optimization problems can naturally be extended to the min-max bilevel setting. In particular, we show
\[
\min_{x \in \mathcal{X}} \left\{ \max_{\ell \in \mathcal{L}} \left\{ c^\top x + \Phi_\ell(x) \right\} \right\} = \max_{\ell \in \mathcal{L}} \left\{ \min_{x \in \mathcal{X}} \left\{ c^\top x + \Phi_\ell(x) \right\} \right\}.
\]
Thus, Theorem 1 seems to be—at least to some extent—related to a minimax theorem from game theory. However, existing minimax theorems such as, e.g., the one by von Neumann (1928) are not applicable in our setting.

Let us further emphasize the benefits of reformulating the $\Gamma$-robust counterpart (\Gamma\text{-Min-Max}) of the min-max problem as it is done in Theorem 1. First, note that the sub-problems (\ell\text{-Min-Max}) are independent, i.e., they do not share any variables. Hence, the sub-problems can be solved in parallel. Second, the reformulation presented in (\ell\text{-Min-Max}) allows us to employ state-of-the-art solution approaches for min-max problems such as, e.g., the one by Weninger and Fukasawa (2022) for the bilevel knapsack interdiction problem. The latter is not possible when using the formulation in (\Gamma\text{-Min-Max}) for which problem-tailored branch-and-cut approaches are necessary; see Beck et al. (2023b). In Section 5, we computationally assess the effectiveness of reformulating Problem (\Gamma\text{-Min-Max}) as it is done in Theorem 1.

3.2. $\alpha$-Approximation Algorithm. In this section, we show how Theorem 1 can be further exploited if, instead of exact solutions, $\alpha$-approximate solutions to the sub-problems (\ell\text{-Min-Max}) are considered.

Proposition 1. If there is an $\alpha$-approximation algorithm with $\alpha \geq 1$ for mixed-integer min-max problems of the form (\ell\text{-Min-Max}), then there is an $\alpha$-approximation algorithm for the $\Gamma$-robust counterpart (\Gamma\text{-Min-Max}) of the min-max problem.

Proof. For all $\ell \in \mathcal{L}$, let $x^\ell$ be an $\alpha$-approximate solution to (\ell\text{-Min-Max}). Further, let $v_{\alpha,\ell}$ denote the corresponding objective function value, i.e., $v_{\alpha,\ell} \leq \alpha v_\ell$. By Theorem 1 and since $\alpha \geq 1$, we further have
\[
v_{\alpha,\ell} \leq \alpha v_\ell \leq \alpha \max_{\ell \in \mathcal{L}} \{v_\ell\} = \alpha v_\Gamma \quad \text{for all } \ell \in \mathcal{L},
\]
which concludes the proof. □

As a direct consequence of Proposition 1, we can state the following algorithm.

**Algorithm 1** $\alpha$-Approximation Algorithm for $\Gamma$-Robust Min-Max Problems

**Input:** An instance of $(\Gamma$-Min-Max), an $\alpha$-approximation algorithm for $(\ell$-Min-Max), an index set $L$ as stated around (4)

**Output:** An $\alpha$-approximate solution to $(\Gamma$-Min-Max)

1: for all $\ell \in L$ do
2: Compute an $\alpha$-approximate solution $x^{\ell}$ to $(\ell$-Min-Max) and let $v_{\alpha,\ell}$ denote the corresponding objective function value.
3: Set $k \leftarrow \arg \max_{\ell \in L} \{v_{\alpha,\ell}\}$.
4: return $v_\alpha := v_{\alpha,k}$ and $x^* := x^k$

**Corollary 1.** Algorithm 1 is correct, i.e., it returns an $\alpha$-approximate solution to the $\Gamma$-robust counterpart $(\Gamma$-Min-Max) of the min-max problem $(\text{Min-Max})$.

3.3. Primal Heuristic and Lower Bounding Scheme. Similar as it is done in Bardossy and Raghavan (2016) for the single-level setting, we further consider a modification of Algorithm 1 in which, instead of an $\alpha$-approximation algorithm, a heuristic for the problems of the nominal type as well as a lower bounding scheme is embedded. The aim is to obtain a heuristic solution as well as an overall lower bound for the $\Gamma$-robust min-max problem $(\Gamma$-Min-Max) in the case that, e.g., there is no $\alpha$-approximation algorithm available.

**Proposition 2.** For all $\ell \in L$, let $\omega_{\ell}$ be a lower bound for the optimal objective function value of the $\ell$th min-max problem $(\ell$-Min-Max). Further, let $\omega := \max_{\ell \in L} \{\omega_{\ell}\}$. Then, $\omega$ is a lower bound for the optimal objective function value of Problem $(\text{Min-Max})$.

**Proof.** Let $k \in L$ be such that $k = \arg \max_{\ell \in L} \{\omega_{\ell}\}$. Since, by assumption, $\omega_k$ is a valid lower bound for the $k$th min-max problem as stated in $(\ell$-Min-Max), we then obtain $\omega = \omega_k \leq v_k \leq \max_{\ell \in L} \{v_\ell\} =: v^*$. □

In Proposition 2, a lower bound for the optimal objective function value of the $\ell$th min-max problem $(\ell$-Min-Max) can be obtained by solving a properly chosen relaxation such as, e.g., the high-point relaxation. Let us further emphasize that, for any $\ell \in L$, a decision of the leader that is feasible for the $\ell$th min-max problem $(\ell$-Min-Max) is also feasible for the $\Gamma$-robust min-max problem $(\Gamma$-Min-Max). A primal heuristic that exploits the latter as well as Proposition 2 is formally given in Algorithm 2.

**Algorithm 2** Primal Heuristic for $\Gamma$-Robust Min-Max Problems

**Input:** An instance of $(\Gamma$-Min-Max), a heuristic and a lower bounding scheme for $(\ell$-Min-Max), an index set $L$ as stated around (4)

**Output:** A heuristic solution $x^*$ and a lower bound $\omega$ for $(\Gamma$-Min-Max)

1: for all $\ell \in L$ do
2: Compute a heuristic solution $x^{\ell}$ to $(\ell$-Min-Max) and let $v^h_{\ell}$ denote its objective function value.
3: Compute a lower bound $\omega_{\ell}$ for $(\ell$-Min-Max).
4: Set $k_1 \leftarrow \arg \max_{\ell \in L} \{v^h_{\ell}\}$ and $k_2 \leftarrow \arg \max_{\ell \in L} \{\omega_{\ell}\}$.
5: return $x^* := x^{k_1}, v^h := v^h_{k_1}$, and $\omega := \omega_{k_2}$
Corollary 2. Algorithm 2 is correct, i.e., it returns a feasible leader’s decision \( x^* \) as well as a valid lower bound \( \omega \) for the \( \Gamma \)-robust counterpart (\( \Gamma \)-Min-Max) of the min-max problem (Min-Max).

In what follows, we provide quality guarantees for the heuristic solution \( x^* \) obtained by Algorithm 2.

Proposition 3. Let \( k_1 \in L \) be chosen as in Step 4 of Algorithm 2 and let \( \gamma \) be the optimality gap of the heuristic solution \( x^{k_1} \) to the \( k_1 \)-th min-max problem as stated in (\( \ell \)-Min-Max), i.e., \( \gamma = (v^{k_1}_h - \omega_{k_1})/\omega_{k_1} \). Then, \( x^{k_1} \) is a \((1 + \gamma)\)-approximate solution to (\( \Gamma \)-Min-Max). Moreover, any heuristic solution \( x^\ell, \ell \in L \), computed in Step 2 of Algorithm 2 is a \((1 + \gamma)\)-approximate solution to (\( \Gamma \)-Min-Max).

Proof. By assumption, we have
\[
v^{k_1}_h = (1 + \gamma)\omega_{k_1} \leq (1 + \gamma)v_{k_1} \leq (1 + \gamma)\max_{\ell \in L} \{v_{\ell}\} =: (1 + \gamma)v_T,
\]
i.e., \( x^{k_1} \) is a \((1 + \gamma)\)-approximate solution to (\( \Gamma \)-Min-Max). Moreover, for \( \ell \in L \) arbitrary but fixed, we thus have
\[
v^\ell_h \leq \max_{\ell \in L} \{v^\ell_h\} = v^{k_1}_h \leq (1 + \gamma)v_T,
\]
i.e., \( x^\ell \) is a \((1 + \gamma)\)-approximate solution to (\( \Gamma \)-Min-Max) as well. \( \square \)

Before we conclude this section, we further need to impose the following.

Assumption 3. The lower bound \( \omega_{k_1} \) determined in Steps 3 and 4 of Algorithm 2 is non-negative.

Assumption 3 is w.l.o.g. since, by adding a sufficiently large constant to the upper-level objective function, (\( \ell \)-Min-Max) can be reformulated such that the upper-level objective function value is non-negative for every bilevel feasible pair \((x, y)\). This sufficiently large constant can be derived from variable bounds on the leader’s and the follower’s variables that follow from Assumption 1 as well as from the fact that all objective function coefficients are known exactly.

Proposition 4. Let \( \gamma \) be as in Proposition 3 and suppose that Assumption 3 holds. Further, let \((x^*, v^h, \omega)\) be the output of Algorithm 2. Then, the optimality gap \( \beta = (v^h - \omega)/\omega \) of a heuristic solution \( x^* \) satisfies \( \beta \leq \gamma \).

Proof. By Proposition 2, \( \omega \) is a lower bound for (\( \Gamma \)-Min-Max). In particular, \( \omega \) is the best lower bound for (\( \ell \)-Min-Max) determined in Algorithm 2. Since \( x^{k_1} \) obtained in Step 2 of Algorithm 2 is feasible for (\( \Gamma \)-Min-Max), the heuristic solution \( x^* := x^{k_1} \) has an optimality gap of at most \( \beta = (v^{k_1}_h - \omega)/\omega = (v^h - \omega)/\omega \). We further obtain
\[
\beta = \frac{v^{k_1}_h - \omega}{\omega} \leq \frac{v^{k_1}_h - \omega_{k_1} + (\omega_{k_1} - \omega)}{\omega_{k_1}} = \gamma + \frac{\omega_{k_1} - \omega}{\omega_{k_1}} \leq \gamma.
\]
Here, the inequalities follow from \( 0 \leq \omega_{k_1} \leq \max_{\ell \in L} \{\omega_{\ell}\} =: \omega \), which is due to Assumption 3 and the selection rule in Step 4 of Algorithm 2. \( \square \)

Note that Algorithm 2 still yields a heuristic solution with optimality gap of at most \( \beta = (v^h - \omega)/\omega \) if we omit Assumption 3 in the last proposition. In this case, however, we can no longer guarantee that \( \beta \leq \gamma \) holds.
4. General Mixed-Integer Linear Bilevel Problems

Let us now return to the more general setting of $\Gamma$-robust mixed-integer linear bilevel problems as stated in $(\Gamma$-BMIP). Here, the objective function coefficients for the follower’s variables $y$ may differ in the upper- and the lower-level problem, i.e., $d = f$ does not need to hold anymore. In this section, we illustrate that the ideas of Section 3, namely that an optimal solution can be obtained by solving a polynomial number of problems of the nominal type, cannot be carried over completely for general $\Gamma$-robust bilevel problems. Nevertheless, we present a heuristic for $(\Gamma$-BMIP) that exploits the solution of a polynomial number of mixed-integer linear bilevel problems of the nominal type. The method is formally stated in Algorithm 3. Moreover, we provide quality guarantees for heuristically obtained solutions.

Algorithm 3 Primal Heuristic for $\Gamma$-Robust Mixed-Integer Linear Bilevel Problems

**Input:** An instance of $(\Gamma$-BMIP), an exact solution method for (BMIP), an index set $\mathcal{L}$ as stated around (4)

**Output:** A pair $(x^*, y^*)$ feasible for $(\Gamma$-BMIP)

1: for all $\ell \in \mathcal{L}$ do
2: Compute a solution $(x^\ell, y^\ell)$ to the bilevel problem
\[
\min_{x,y} c^\top x + d^\top y \\
\text{s.t. } x \in X, \\
\quad y \in \arg\max_{y' \in Y(x)} \left\{ -\Gamma \Delta f_\ell + \tilde{f}(\ell)^\top y' \right\}. \\
(P^{\ell})
\]
3: Set $\ell_1 \leftarrow \arg\min_{\ell \in \mathcal{L}} \left\{ c^\top x^\ell + d^\top y^\ell \right\}$ and $x^* \leftarrow x^{\ell_1}$.
4: if $\exists \ell \in \mathcal{L}$ with $x^* \neq x^\ell$ then
5: for all $\ell \in \mathcal{L} \setminus \{\ell_1\}$ do
6: Compute a solution $\hat{y}^\ell$ to the $x^*$-parameterized $\ell$th lower-level sub-problem
\[
\Phi_\ell(x^*) = -\Gamma \Delta f_\ell + \max_{y \in Y(x^*)} \left\{ \tilde{f}(\ell)^\top y \right\}
\]
and set $y^\ell \leftarrow \hat{y}^\ell$.
7: Determine
\[
\mathcal{C} := \left\{ \ell \in \mathcal{L}: \Phi_\ell(x^*) = \max_{\ell' \in \mathcal{L}} \{ \Phi_{\ell'}(x^*) \} \right\}.
\]
8: Set $\ell_2 \leftarrow \arg\min_{\ell \in \mathcal{C}} \left\{ c^\top x^* + d^\top y^\ell \right\}$ and $y^* \leftarrow y^{\ell_2}$.
9: return $(x^*, y^*)$

In Algorithm 3, we start by solving $|\mathcal{L}|$ many bilevel problems of the nominal type. Then, in Line 3, we select the bilevel sub-problem that possibly yields the best objective function value for the leader. Afterward, we check for $\Gamma$-robust bilevel feasibility. To this end, we distinguish two cases. First, if the leader plays the same decision in each bilevel sub-problem, a $\Gamma$-robust optimal response of the follower can be obtained by taking the maximum of all lower-level optimal objective function values associated with the $|\mathcal{L}|$ bilevel problems solved previously. In this paper, we consider the optimistic approach to bilevel optimization. Hence, if the follower’s choice is not unique, we select the sub-problem that yields the best-possible objective function value for the leader; see Lines 7–8. Second, if the leader does not play the same decision in each bilevel sub-problem (see Line 4 of the algorithm above), we solve additional lower-level sub-problems that are parameterized in the leader’s decision fixed in Line 3 of the algorithm; see Line 6. A $\Gamma$-robust optimal response of the follower is then obtained as before.
To show the correctness of Algorithm 3, we start with the following technical observation.

**Proposition 5.** For a given upper-level decision \( x \in X \) and \( y \in Y(x) \),
\[
f^\top y - \max_{\{S \subseteq [n_1] : |S| \leq |\Gamma|\}} \sum_{i \in S} \Delta f_i y_i = \max_{\ell \in \mathcal{L}} \left\{-\Gamma \Delta f_\ell + \bar{f}(\ell)^\top y\right\}
\]
holds.

The last proposition can be shown by applying the same reformulation techniques as in the proof of Theorem 3 by Bertsimas and Sim (2003) and by exploiting the results in Lemma 1 and 2 by Lee and Kwon (2014).

**Proposition 6.** Algorithm 3 is correct, i.e., it returns a pair \((x^*, y^*)\) that is feasible for \((\Gamma\text{-BMIP})\).

**Proof.** Let \((x^\ell, y^\ell)\) \(\ell \in \mathcal{L}\) be the family of solutions to the bilevel problems of the nominal type solved in Line 2 of Algorithm 3. Note that, by Assumption 1, a solution to \((P')\) always exists. Further, let \((x^*, y^*)\) be the output of Algorithm 3. We now show that \((x^*, y^*)\) is feasible for \((\Gamma\text{-BMIP})\). Suppose first that \(x^\ell = x^f \in X\) holds for all \(\ell \in \mathcal{L}\). By the optimality of \((x^\ell, y^\ell)\) for the \(\ell\)th min-max problem, we thus have
\[
y^\ell \in Y(x^\ell) \quad \text{and} \quad \Phi_\ell(x^*) = -\Gamma \Delta f_\ell + \bar{f}(\ell)^\top y^\ell \quad \text{for all } \ell \in \mathcal{L}.
\]
The latter particularly holds for \(y^\ell = y^f\) with \(\ell_2 \in \mathcal{L}\) chosen according to Lines 7–8 of the algorithm. Thus, we obtain
\[
f^\top y^* - \max_{\{S \subseteq [n_1] : |S| \leq |\Gamma|\}} \sum_{i \in S} \Delta f_i y_i^*
= \max_{\ell \in \mathcal{L}} \left\{-\Gamma \Delta f_\ell + \bar{f}(\ell)^\top y^*\right\}
\geq - \Gamma \Delta f_{\ell_2} + \bar{f}(\ell_2)^\top y^*
= \Phi_{\ell_2}(x^*) = \Phi_\ell(x^*).
\]
Here, the first equality is due to Proposition 5, whereas the last equality follows from Lemma 1. Hence, the pair \((x^*, y^*)\) is feasible for \((\Gamma\text{-BMIP})\). The case in which there exists an \(\ell \in \mathcal{L}\) with \(x^\ell \neq x^f\) can be shown in analogy. \(\square\)

Regarding Algorithm 3, there are several aspects that we find particularly worth mentioning. First, note that any solver for mixed-integer linear bilevel problems such as, e.g., the MiBS solver (Tahernejad et al. 2020) or the general branch-and-cut solver presented in Fischetti et al. (2017), can be used for the solution of the sub-problems in Line 2. Second, to show the feasibility of the pair \((x^*, y^*)\), it is sufficient to consider
\[
\ell_2 \leftarrow \arg \max_{\ell \in \mathcal{L}} \{\Phi_\ell(x^*)\}
\]
instead of the selection rule presented in Lines 7–8. Then, however, the upper-level objective would not be taken into account if a feasible (and ideally good) pair \((x^*, y^*)\) is reported. In particular, the latter may lead to ambiguities if the choice of \(\ell_2 \in \mathcal{L}\) is not unique. Third, let us emphasize that it may additionally be necessary to solve \(|\mathcal{L}| - 1\) many lower-level, i.e., single-level, problems to obtain a feasible pair \((x^*, y^*)\) for \((\Gamma\text{-BMIP})\). In particular, the solution to these lower-level sub-problems may not be unique. To guarantee that we indeed obtain an optimistic response of the follower, we need to incorporate a so-called refinement step after Line 6 of the algorithm. In this additional step, we solve the binary problem
\[
\min_{y \in Y(x^*)} d^\top y \quad \text{s.t.} \quad -\Gamma \Delta f_\ell + \bar{f}(\ell)^\top y \geq \Phi_\ell(x^*),
\]
which yields an optimistic optimal response of the follower for the given leader’s
decision \( x^* \). Fourth, the bilevel as well as the single-level problems solved in Steps 2
and 6 of the algorithm are independent. Hence, if the necessary capacities are
available, we can parallelize these steps. Nevertheless, even obtaining feasibility in
the general \( \Gamma \)-robust setting may be (computationally) more expensive compared
to showing optimality in the \( \Gamma \)-robust min-max case. Related to the latter, we
make the following final, and perhaps most important, observation. It may be the
case that none of the solutions to the bilevel problems of the nominal type solved
in Step 2 of the algorithm is feasible for the \( \Gamma \)-robust bilevel problem (\( \Gamma \)-BMIP).
Hence, it is possible that the optimal solution to \((\Gamma \)-BMIP\) is not among the family
of solutions \( (x^\ell, y^\ell)_{\ell \in L} \), which is why the Bertsimas–Sim result does not carry over
to general \( \Gamma \)-robust bilevel problems.\(^2\)

4.1. Quality Guarantees. In the remainder of this section, we aim to provide
quality guarantees for the pair \((x^*, y^*)\) obtained by Algorithm 3. To this end, we
start by making the following technical observations.

**Lemma 2.** Let \((x^*, y^*)\) be an optimal solution to \((\Gamma \)-BMIP\). Then, there is an
index \( k \in L \) such that

\[
f^\top y^* - \max_{\{S \subseteq [n_s]: |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y^*_i = \Phi_k(x^*) = -\Gamma \Delta f_k + \bar{f}(k)^\top y^*.
\]

**Proof.** By Lemma 1 and the optimality of \((x^*, y^*)\), we have

\[
f^\top y^* - \max_{\{S \subseteq [n_s]: |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y^*_i = \Phi_\Gamma(x^*) = \max_{\ell \in L} \{ \Phi_\ell(x^*) \}.
\]

Let \( k_1 \in L \) be such that

\[
\Phi_{k_1}(x^*) = \max_{\ell \in L} \{ \Phi_\ell(x^*) \}. \tag{6}
\]

Further, let \( k_2 \in L \) be such that

\[
f^\top y^* - \max_{\{S \subseteq [n_s]: |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y^*_i = \max_{\ell \in L} \left\{ -\Gamma \Delta f_\ell + \bar{f}(\ell)^\top y^* \right\} = -\Gamma \Delta f_{k_2} + \bar{f}(k_2)^\top y^*.
\]

Note that the first equality follows from Proposition 5. For \( i \in \{1, 2\} \), let \( y^{k_i} \) be an
optimal solution to the \( x^*\)-parameterized \( k_i \)-th lower-level sub-problem, i.e.,

\[
\Phi_{k_i}(x^*) = -\Gamma \Delta f_{k_i} + \max_{y \in Y(x^*)} \left\{ \bar{f}(k_i)^\top y \right\}.
\]

Then, we obtain

\[
-\Gamma \Delta f_{k_2} + \bar{f}(k_2)^\top y^{k_2} \leq -\Gamma \Delta f_{k_1} + \bar{f}(k_1)^\top y^{k_1}
\]

\[
= \Phi_{k_1}(x^*)
\]

\[
= f^\top y^* - \max_{\{S \subseteq [n_s]: |S| \leq \Gamma\}} \sum_{i \in S} \Delta f_i y^*_i
\]

\[
= -\Gamma \Delta f_{k_2} + \bar{f}(k_2)^\top y^*
\]

\[
\leq -\Gamma \Delta f_{k_2} + \bar{f}(k_2)^\top y^{k_2}.
\]

Here, the inequalities as well as the first equality follow from the choice of \( k_1, k_2 \in L \),
whereas the last inequality is due to the optimality of \( y^{k_2} \) for the \( k_2 \)-th lower-level
sub-problem. Consequently, we have

\[
-\Gamma \Delta f_{k_1} + \bar{f}(k_1)^\top y^{k_1} = -\Gamma \Delta f_{k_2} + \bar{f}(k_2)^\top y^* = -\Gamma \Delta f_{k_2} + \bar{f}(k_2)^\top y^{k_2},
\]

\(^2\)An instance showing this behavior can be found at https://github.com/YasmineBeck/gamma-robust-min-max-problems.
are valid lower and upper bounds, respectively, for the optimal objective function value of the bilevel problem. This means that, in (6), we could have chosen $k_2$ instead of $k_1$. Hence, we can assume w.l.o.g. that there is $k \in \mathcal{L}$ such that

$$f^\top y^* - \max_{\{S \subseteq [n_u] : |S| \leq \Gamma \}} \sum_{i \in S} \Delta f_i y_i^* = \Phi_k(x^*) = -\Gamma \Delta f_k + \tilde{f}(k)^\top y^*. \quad \square$$

In what follows, we exploit Lemma 2 to provide lower bounds for the optimal objective function value of (\(\Gamma\)-BMIP).

**Proposition 7.** There is an index $k \in \mathcal{L}$ such that the optimal objective function value of the bilevel problem

$$\min_{x,y} \quad c^\top x + d^\top y$$

subject to

$$x \in X,$$

$$y \in \arg \max_{y' \in Y(x)} \left\{ -\Gamma \Delta f_k + \tilde{f}(k)^\top y' \right\}$$

yields a valid lower bound for the optimal objective function value of (\(\Gamma\)-BMIP).

**Proof.** Let \((x^*, y^*)\) denote an optimal solution to (\(\Gamma\)-BMIP). By Lemma 2, there is an index $k \in \mathcal{L}$ such that

$$f^\top y^* - \max_{\{S \subseteq [n_u] : |S| \leq \Gamma \}} \sum_{i \in S} \Delta f_i y_i^* = \Phi_k(x^*) = -\Gamma \Delta f_k + \tilde{f}(k)^\top y^*. \quad \square$$

Further, let \((x^k, y^k)\) be an optimal solution to Problem (\(P^k\)). Since $x^* \in X$ as well as $y^* \in Y(x^*)$ hold by assumption, \((x^*, y^*)\) is feasible for Problem (\(P^k\)). Consequently, we obtain $c^\top x^* + d^\top y^* \geq c^\top x^k + d^\top y^k$, which concludes the proof. \(\square\)

Let us point out that Proposition 7 only yields an ex-post result since it requires the knowledge of an optimal solution to the \(\Gamma\)-robust bilevel problem (\(\Gamma\)-BMIP) in advance. Nevertheless, it can be exploited to obtain an overall valid lower bound for (\(\Gamma\)-BMIP), which is what we do in the following.

**Theorem 2.** Let \((x^\ell, y^\ell)\)\(_{\ell \in \mathcal{L}}\) be the family of solutions to the bilevel problems solved in Line 2 of Algorithm 3. Further, let \((x^*, y^*)\) be the output of Algorithm 3. Then,

$$L := \min_{\ell \in \mathcal{L}} \left\{ c^\top x^\ell + d^\top y^\ell \right\} \quad \text{and} \quad U := c^\top x^* + d^\top y^*$$

are valid lower and upper bounds, respectively, for the optimal objective function value of (\(\Gamma\)-BMIP) with $U - L \leq \|d\|_1$.

**Proof.** The validity of $U$ and $L$ as an upper and a lower bound for the optimal objective function value of (\(\Gamma\)-BMIP) immediately follows from Propositions 6 and 7, respectively. Hence, it remains to show that $U - L \leq \|d\|_1$ holds. To this end, let $\ell_1 \in \mathcal{L}$ be chosen according to Line 3 of Algorithm 3. Then, we have

$$L = c^\top x^{\ell_1} + d^\top y^{\ell_1} = c^\top x^* + d^\top y^{\ell_1}.$$

Further, suppose that $x^* = x^\ell$ holds for all $\ell \in \mathcal{L}$. Due to Hölder’s inequality and $y_i^* - y_i^{\ell_1} \in \{-1,0,1\}$ for all $i \in [n_y]$, we obtain

$$U - L = c^\top x^* + d^\top y^* - (c^\top x^* + d^\top y^{\ell_1}) = d^\top (y^* - y^{\ell_1}) \leq |d^\top (y^* - y^{\ell_1})| \leq \|d\|_1 \|y^* - y^{\ell_1}\|_{\infty} \leq \|d\|_1.$$

The case in which there exists an $\ell \in \mathcal{L}$ with $x^* \neq x^\ell$ can be shown in analogy. \(\square\)

To conclude this section, we show under which conditions optimality of the output \((x^*, y^*)\) of Algorithm 3 can be guaranteed.
Corollary 3. Let \((x^\ell, y^\ell)_{\ell \in L}\) be the family of solutions to the bilevel problems solved in Line 2 of Algorithm 3. Further, let \((x^*, y^*)\) be the output of Algorithm 3. If \(c^\top x^\ell + d^\top y^\ell \leq c^\top x^\ell + d^\top y^\ell\) holds for all \(\ell \in L\), the pair \((x^*, y^*)\) is optimal for \((\Gamma\text{-BMIP})\).

Proof. By Theorem 2, we obtain
\[
U - L = c^\top x^* + d^\top y^* - \min_{\ell \in L} \{c^\top x^\ell + d^\top y^\ell\} = 0,
\]
i.e., \((x^*, y^*)\) is optimal for \((\Gamma\text{-BMIP})\).

The lower and the upper bound presented in Theorem 2 can be used to compute the optimality gap of the \(\Gamma\)-robust bilevel problem at hand. However, let us emphasize that the obtained gap may not accurately reflect the “true” one since the lower bound given in Theorem 2 may be rather loose. Hence, when referring to optimality gaps in our computational study in Section 5, we interpret them as “worst-case” optimality gaps.

5. Computational Results

In this section, we computationally assess the performance of the methods presented in this paper. Before we discuss the numerical results for each method in detail, we briefly describe the generation of the test instances as well as the computational setup in Section 5.1 and 5.2, respectively. In Section 5.3, we consider the exact solution approach presented in Section 3 for the min-max setting. Afterward, in Section 5.4, we evaluate the primal heuristic for general \(\Gamma\)-robust mixed-integer linear bilevel problems as stated in Algorithm 3.

The evaluations of the proposed methods rely on (i) the running times, (ii) the number of investigated branch-and-bound nodes, (iii) the number of solved instances, as well as on (iv) optimality gaps. Moreover, as it is mentioned in Sections 3 and 4, the proposed methods can be partially parallelized. To assess the potential of parallelization, we further use so-called idealized parallel runtimes. The latter reflect the overall runtime of a solution method provided that there are sufficient capacities available to solve all sub-problems in parallel. For each instance, we compute the idealized parallel runtime after solving all sub-problems sequentially by taking the maximum of all runtimes for the sub-problems. Hence, if a problem could not be solved within a reasonable amount of time in the sequential setting, we also consider it as unsolved in the idealized setting. For the presentation of our numerical results, we particularly use figures showing empirical cumulative distribution functions (ECDFs). The ECDFs can be interpreted as the percentage of instances (y-axis) that can be solved within a certain amount of time, with a certain optimality gap, or after investigating a certain number of branch-and-bound nodes (x-axis).

5.1. Generation of Knapsack Test Instances. For our computational study, we consider modifications of the deterministic knapsack interdiction problem that has been considered in Caprara et al. (2016). The deterministic problem is formally stated as

\[
\begin{align*}
\min_{x \in \{0,1\}^n} & \quad p^\top y \\
\text{s.t.} & \quad v^\top x \leq B, \\
& \quad y \in \arg\max_{y' \in \{0,1\}^n} \{p^\top y' : w^\top y' \leq C, y'_i \leq 1 - x_i, i \in [n]\},
\end{align*}
\]

in which all parameters are assumed to be non-negative integers, i.e., \(B, C \in \mathbb{Z}_+\), and \(p, v, w \in \mathbb{Z}_{+}^n\). For each instance size \(n \in \{35, 40, 45, 50, 55, \ldots, 100\}\), 10 instances have been generated according to Martello et al. (1999).
To account for a $\Gamma$-robust follower, we adapt the deterministic instances in the following way. We assume that the deviations in the objective function coefficients take either 10% or 25% of the nominal value. The parameter $\Gamma$ is set to either 10% or 50% of the instance size $n$. In the case of a fractional value for $\Gamma$, the closest integer is considered. A detailed description of the generation of test instances can be found in Section 4.1 in Beck et al. (2023b). In summary, we consider 40 instances per size such that our overall test set contains 560 robustified knapsack interdiction instances. The latter are used to compare the performance of the exact solution approaches in the min-max setting in Section 5.3.

For the more general case evaluated in Section 5.4, we do the following. We reconsider the previously generated 560 robustified knapsack interdiction instances, maintaining the uncertainty parameterization as well as the structure of the lower-level problem. An instance of the general form ($\Gamma$-BMIP) is then obtained by adding upper-level objective function coefficients for the leader’s and the follower’s variables. The latter take uniformly distributed integer values from the interval $[0, 100]$. Hence, we also consider 560 robustified instances in the more general setting.

5.2. Computational Setup. All tests have been realized on an Intel XEON SP 6126 at 2.6 GHz (6 cores) with 32 GB RAM, which is part of the high-performance cluster “Elwetrutsch” at TU Kaiserslautern within the “Alliance of High Performance Computing Rheinland-Pfalz” (AHRP). The approaches considered in our computational study use Gurobi 10.0.0 to solve all arising optimization problems. We now comment on the implementation for the exact method in the min-max setting as well as the primal heuristic in the more general setting.

5.2.1. Mixed-Integer Linear Min-Max Problems. We assess the performance of the exact solution method for the min-max setting by comparing the following three methods.

First, we consider the formulation in Theorem 1 in which a polynomial number of problems of the nominal type need to be solved. For the solution of these deterministic problems, we consider the bkpsolver (Weninger and Fukasawa 2022), which is publicly available at https://github.com/nwoehnogaehr/bkpsolver. The method is based on a branch-and-bound framework that incorporates ideas from dynamic programming to obtain strong lower bounds. In the following, we abbreviate this method with BKP.

Second, we again consider the formulation in Theorem 1 but, instead of the bkpsolver, exploit the problem-tailored branch-and-cut method presented in Fischetti et al. (2019) for the solution of the deterministic problems. Hence, compared to the previous approach, the changes made in our algorithm only involve the choice of the black-box solver used for the solution of the min-max problems of the nominal type. In the branch-and-cut approach presented in Fischetti et al. (2019), the authors exploit so-called interdiction cuts to separate bilevel infeasible points. Hence, we refer to this approach as IC. Since the original method uses CPLEX 12.7 to solve all arising optimization problems, we have re-implemented it using Gurobi to have a fair comparison between the three considered approaches.

Finally, we consider the single-leader multi-follower approach presented in our previous work in Beck et al. (2023b), which does not exploit the result in Theorem 1 but solves ($\Gamma$-Min-Max) instead. We refer to this approach as MF in the following. The method relies on a branch-and-bound framework in which interdiction cuts tailored for the $\Gamma$-robust setting are added. In Beck et al. (2023b), various cut

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3We kindly acknowledge the support of RHRK (https://rz.rptu.de/en/).
4The code for the single-leader multi-follower approach is publicly available at https://github.com/YasmineBeck/gamma-robust-knapsack-interdiction-solver.
separation strategies are studied. In our computational study, we consider the setting in which a single most-violated cut is added at each node of the branch-and-cut search tree. To generate these cuts, all lower-level sub-problems need to be solved. Note, however, that the independence of the lower-level sub-problems allows for a parallelization of their solution. In our evaluations, we account for this feature by also considering idealized parallel runtimes for MF.

Both branch-and-cut approaches IC and MF are implemented in Python 3.7.11. To add interdiction cuts, we use Gurobi’s lazy constraint callbacks, which requires to set the parameter LazyConstraints to 1. All other parameters have been left at their default settings. For the solution of each optimization problem, we set a time limit of 1 h.

5.2.2. General Mixed-Integer Linear Bilevel Problems. We now briefly describe the implementation of the primal heuristic presented in Algorithm 3. As mentioned in Section 4, any solver for mixed-integer linear bilevel problems can be used for the solution of the problems of the nominal type in Step 2 of the algorithm. In our computational study, we employ a problem-tailored branch-and-cut approach. The method is based on IC, which we have adapted accordingly to account for general mixed-integer linear bilevel problems. Again, the method is implemented in Python 3.7.11 and we use Gurobi’s lazy constraint callbacks to add cuts by setting the parameter LazyConstraints to 1. We set a time limit of 1 h for the solution of all arising optimization problems, while the remaining parameters have been left at their default settings. Since we do not expect many ambiguities regarding lower-level solutions, and for the ease of implementation, we implement Algorithm 3 without refinement. Note, however, that including a refinement step would involve the solution of additional binary problems, which would further increase the computational burden of the presented primal heuristic.

Preliminary computational tests revealed that our problem-tailored branch-and-cut approach outperforms general-purpose solvers, which is why we refrain from using solvers such as, e.g., the MibS solver (Tahernejad et al. 2020) or the general branch-and-cut solver presented in Fischetti et al. (2017).

5.3. Evaluation of the Exact Solution Approach for the Min-Max Setting. We now evaluate the potential of reformulating the Γ-robust min-max problem (Γ-Min-Max) as it is done in Theorem 1. To this end, we apply the methods BKP, IC, and MF to the Γ-robust knapsack interdiction problem, which is a special case of a min-max problem. While BKP and IC exploit the result in Theorem 1, i.e., they solve a polynomial number of problems of the nominal type, MF solves a single problem of the form given in (Γ-Min-Max). Figure 1 shows the ECDFs w.r.t. the running times (top and middle) and the number of nodes (bottom). Let us first focus on the comparison of the branch-and-cut approaches IC and MF. It can be seen that MF dominates IC both w.r.t. sequential runtimes as well as the number of investigated nodes. Provided that the necessary capacities are available to solve all sub-problems in parallel, however, IC seems to have an advantage over MF on the easier instances. Nevertheless, MF still performs slightly better than IC on the harder instances. The latter is also reflected by the number of solved instances shown in Table 2 (521 instances solved by MF vs. 478 solved by IC). The results in Figure 1 and Table 2 indicate that, in general, the solution of |L| bilevel problems (as it is done by IC) seems to be computationally more expensive than

5The code for IC as well as for the primal heuristic presented in this paper, along with the nominal instance data used for our computational study, is publicly available at https://github.com/YasmineBeck/gamma-robust-min-max-problems.
Table 2. Mean and median runtimes (in s), mean and median number of branch-and-bound nodes as well as the number of solved instances for the approaches MF, IC, and BKP in the min-max setting.

<table>
<thead>
<tr>
<th></th>
<th>sequential runtimes</th>
<th>idealized runtimes</th>
<th>node count</th>
<th>solved</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>median</td>
<td>mean</td>
<td>median</td>
</tr>
<tr>
<td>BKP</td>
<td>1.01</td>
<td>0.50</td>
<td>0.06</td>
<td>0.04</td>
</tr>
<tr>
<td>IC</td>
<td>7154.36</td>
<td>19.37</td>
<td>271.12</td>
<td>1.65</td>
</tr>
<tr>
<td>MF</td>
<td>478.37</td>
<td>16.01</td>
<td>47.22</td>
<td>4.42</td>
</tr>
</tbody>
</table>

However, the situation changes significantly if we use BKP instead of IC for the solution of the interdiction problems of the nominal type. Based on Figure 1 and Table 2, we clearly see the advantage of using the reformulation presented in Theorem 1. In the sequential as well as in the idealized setting, BKP significantly outperforms the remaining solution approaches. It is particularly worth mentioning that BKP solves almost 40% of the instances by investigating at most one node. In Table 2, we further observe that, compared to MF, the sequential runtime of BKP is more than a factor of 400 and 32 smaller in the mean and the median, respectively. In the idealized setting, the same qualitative behavior can be observed even more clearly. Finally, let us emphasize that the application of BKP is only possible due to the extension of the main result in Bertsimas and Sim (2003) for the min-max setting as stated in Theorem 1 and that it cannot be used for the solution of the single-leader multi-follower formulation in (\(\Gamma\)-Min-Max).

To sum up, our computational results clearly show that it can be beneficial to solve an interdiction problem with a \(\Gamma\)-robust follower by solving a polynomial number of problems of the nominal type. For the solution of the problems of the nominal type, state-of-the-art solvers such as the one by Weninger and Fukasawa (2022) but also any other off-the-shelf solver can be exploited, whereas problem-tailored branch-and-cut approaches are necessary if a single problem of the form given in (\(\Gamma\)-Min-Max) needs to be solved. Hence, the result in Theorem 1 is of significant computational relevance.

5.4. Evaluation of the Primal Heuristic for General Discrete Linear \(\Gamma\)-Robust Bilevel Problems. We now evaluate the performance of the primal heuristic presented in Algorithm 3. In what follows, we exclude those 278 instances for which at least one of the \(|L|\) many bilevel sub-problems considered in Line 2 of the algorithm could not be solved to global optimality within the time limit of 1 h. Reflected by the number of excluded instances, we acknowledge that the computational burden of the presented primal heuristic is rather large. The longest running time we have observed to determine a feasible point is 60029.39 s; see Table 3. Despite this initial drawback, however, the primal heuristic offers two main advantages. First, there is the possibility to parallelize the solution of the bilevel problems of the nominal type and, if necessary, the solution of the additional lower-level sub-problems. Figure 2 (left) as well as the results in Table 3 clearly show the potential of parallelization. Second, it is noteworthy that 235 of the considered 282 instances (83.3%) satisfy the requirement in Corollary 3 such that the obtained point is indeed an optimal solution of the \(\Gamma\)-robust bilevel problem. The latter can also be derived from the ECDF w.r.t. the worst-case optimality gaps shown in
Figure 1. Log-scaled ECDF plots of the sequential (top) and the idealized parallel runtimes (middle) as well as the number of branch-and-bound nodes (bottom) for the approaches MF, IC, and BKP in the min-max setting.
Table 3. Statistics for the sequential and idealized parallel runtimes (in s) as well as for the worst-case optimality gaps (in %) in the general bilevel setting.

<table>
<thead>
<tr>
<th></th>
<th>min</th>
<th>1st quartile</th>
<th>median</th>
<th>3rd quartile</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>sequential runtime</td>
<td>1.67</td>
<td>120.15</td>
<td>647.42</td>
<td>3339.23</td>
<td>60029.39</td>
</tr>
<tr>
<td>idealized runtime</td>
<td>0.36</td>
<td>10.72</td>
<td>52.83</td>
<td>248.64</td>
<td>3548.09</td>
</tr>
<tr>
<td>optimality gap</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>69.41</td>
</tr>
</tbody>
</table>

Figure 2. Log-scaled ECDF plots of the sequential and the idealized parallel runtimes (left) as well as the linear-scaled ECDF plot of the worst-case optimality gaps in the general bilevel setting.

Figure 2 (right). Additionally, Figure 3 shows the number of instances solved to global optimality for each instance size separately. Moreover, when returning to the worst-case optimality gaps depicted in Figure 2 (right), it is worth mentioning that we obtain a gap of at most 20% for around 99.3% of the considered 282 instances. Hence, despite its rather large computational burden, the primal heuristic seems to provide favorable results in terms of solution quality. Finally, we would like to point out that for 66 of the considered 282 instances, feasibility is obtained without solving further single-level problems, i.e., the same leader’s decision is optimal for each bilevel sub-problem. For the majority of the instances, however, this is not the case such that determining a feasible point is indeed computationally more expensive in the general $\Gamma$-robust bilevel setting compared to the min-max case.

6. Conclusion

In this paper, we consider mixed-integer linear bilevel problems with a follower facing uncertainties regarding his objective function coefficients. To deal with this uncertainty, we pursue a $\Gamma$-robust approach in which the follower hedges against a subset of the uncertain parameters that adversely influence the solution to the problem. More specifically, we exploit the main result by Bertsimas and Sim (2003) and Sim (2004) for $\Gamma$-robust single-level optimization, namely, that the $\Gamma$-robust counterpart of a binary problem can be solved by solving a polynomial number of binary problems of the nominal type. We show that the Bertsimas–Sim result can be extended for $\Gamma$-robust min-max problems, which is an important special case of general bilevel problems with a $\Gamma$-robust lower level. Moreover, we discuss the situation in which the min-max problems of the nominal type are not solved exactly
but in which either an α-approximation algorithm or a primal heuristic together with a lower bounding scheme is used. For general Γ-robust bilevel problems, however, we illustrate that the previous ideas cannot be carried over completely. Nevertheless, we provide a primal heuristic that exploits the solution of a polynomial number of bilevel problems of the nominal type. To assess the performance of the presented methods, we perform a computational study on 560 instances for both the exact solution approach in the min-max setting as well as the primal heuristic for general Γ-robust bilevel problems. For the primal heuristic, we observe that the optimality gap is closed for a substantial part of the considered instances. Moreover, we find speed-up factors exceeding 400 and 32 in the mean and the median, respectively, if our result for the min-max case is exploited. The latter clearly shows that our results are of significant relevance, not only in theory but also in computational practice.

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