Sufficient Conditions for Lipschitzian Error Bounds for Complementarity Systems

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Abstract. We are concerned with Lipschitzian error bounds and Lipschitzian stability properties for solutions of a complementarity system. For this purpose, we deal with a nonsmooth slack-variable reformulation of the complementarity system, and study conditions under which the reformulation serves as a local error bound for the solution set of the complementarity system. We also discuss conditions, guaranteeing metric regularity of the reformulation mapping, and investigate relations between the latter, and Lipschitzian stability properties for solutions of the complementarity system. Some special features of nonlinear complementarity problems are also discussed.

Keywords: Lipschitzian Error Bound · Complementarity System · Noncritical Solution · Subtransversality · Mordukhovich Normal Cone · Aubin Property · Metric Regularity

1 Introduction

Given two continuously differentiable mappings $g, h : \mathbb{R}^n \to \mathbb{R}^m$, we are concerned in this paper with properties of solutions of the complementarity system

$$g(u) \geq 0, \quad h(u) \geq 0, \quad g(u)^\top h(u) = 0. \tag{1}$$

Numerous problems can be modeled by means of complementarity system (1). For instance, we refer to [11,23] for applications in economics, [7,11] for engineering applications, and [11,16,27] for applications in constrained smooth optimization.

A typical way to determine solutions of (1) numerically is to apply an appropriate Newton method [12,27,30]. For this purpose, (1) is often [5,9,10,13,14] reformulated as nonsmooth system of constrained equations,

$$\Phi(\xi) = 0, \quad \xi = (u, y, z) \in \Omega := \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^m_+, \tag{2}$$

where $\mathbb{R}^m_+$ is the nonnegative orthant, and $\Phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is given as

$$\Phi(\xi) := \begin{pmatrix} g(u) - y \\ h(u) - z \\ \psi(y, z) \end{pmatrix},$$

for a C-mapping $\psi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, by which we mean that the i-th component function of ψ depends only on the i-th component of the arguments y, z, and the zero-set of the function ψ_i is the solution set of the complementarity problem

$$y_i \geq 0, \quad z_i \geq 0, \quad y_i^\top z_i = 0.$$

Here, we deal with the nonsmooth C-mapping

$$\psi(y,z) := \min\{y,z\},$$

where min is taken component-wise, and emphasize that some of our considerations can be reestablished easily, when other C-mappings are considered.
Solutions of the constrained equation (2) solve the unconstrained equation

$$\Phi(\xi) = 0 \tag{3}$$

and vice versa, which means that the condition $\xi \in \Omega$ in (2) does not restrict the solution set of (3). At the same time, the convergence analysis of Newton-type methods, designed for the solution of (1) (cf. the references above (2)), can benefit from the additional inclusion $\xi \in \Omega$. A key assumption to guarantee fast local convergence of Newton-type methods to a solution $\xi^* = (u^*, g(u^*), h(u^*))$ of (2) is the constrained error bound condition,

$$\exists \varepsilon, c > 0: \quad \mathrm{dist}[\xi, \Phi^{-1}(0) \cap \Omega] \leq c \cdot \|\Phi(\xi)\| \quad \forall \xi \in \Omega \cap (\xi^* + \varepsilon\mathbb{B}), \tag{4}$$

where dist stands for the (Euclidean) point-to-set distance, $\|\cdot\|$ is the Euclidean norm, and $\mathbb{B}$ is the closed (Euclidean) unit-ball of appropriate dimension. We believe that it is safe to say that the importance of constrained error bounds for the convergence analysis of Newton-type methods is nowadays without doubts [1,2,3,29,35], and put the focus on the development of sufficient conditions for the constrained error bound condition (4) in this paper.
We will say [15] that Φ provides a *local Ω-error bound* at ξ^*, whenever (4) is satisfied. If (4) holds with $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ in place of Ω, then we simply say that Φ provides a *local error bound* at ξ^*. A first contribution in Sect. 2 shows that, for our concrete problem, the local Ω-error bound property coincides with the (unconstrained) local error bound property. This enables us to characterize (4) by means of recent criteria in [15,26] under the assumption that ξ^* is *nondegenerate*, by which we mean that the underlying solution u^* of the complementarity system (1) is *nondegenerate* [11], i.e.,

$$\nexists i \in \{1,\dots,m\}: \quad g_i(u^*) = h_i(u^*) = 0. \tag{5}$$

In Sect. 3, we address the general case, where (5) is possibly absent. The key idea here is an appropriate utilization of the subtransversality [21] of two sets (see Sect. 3 for a definition). In particular, this leads to new sufficient conditions for the fulfillment of (4). One of these conditions turns out as Mordukhovich's coderivative criterion [32,37] for the solution mapping $\Sigma : \mathbb{R}^m \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$,

$$\Sigma(\alpha,\beta) := \left\{ u \,\middle|\, g(u)+\alpha \geq 0,\ h(u)+\beta \geq 0,\ (g(u)+\alpha)^\top (h(u)+\beta) = 0 \right\}, \tag{6}$$

to have the Aubin property at $(0,0)$ for u^*. We will further show that the latter is equivalent to the metric regularity of the mapping Φ at ξ^* (definitions of the properties are stated in Sect. 3), which depends strongly on the choice of the mapping ψ. To our knowledge, such an equivalence has not been noticed before in the context of complementarity systems. In Sect. 4, we consider nonlinear complementarity problems as a special instance of (1), and explain that the aforementioned equivalence extends known relations on different notions of solution-regularity.

2 Lipschitzian Error Bounds and Nondegeneracy

We show that the constrained error bound condition (4) coincides with an unconstrained one. Afterwards, we recall some criteria for the unconstrained error bound under the assumption of nondegeneracy (5). Here and throughout, the point $\xi^* = (u^*, g(u^*), h(u^*))$ is an arbitrary but fixed solution of (2).

Lemma 1. *The following statements are equivalent:*

1. *Φ provides a local Ω-error bound at ξ^*.*
2. *Φ provides a local error bound at ξ^*.*

Proof. Since $\Phi^{-1}(0) \cap \Omega = \Phi^{-1}(0)$ holds for our concrete problem, the implication 2. $\Rightarrow$ 1. is trivially satisfied, and we merely have to show 1. $\Rightarrow$ 2.:
The mapping Φ is locally Lipschitz continuous. Hence, $\Phi^{-1}(0) = \Phi^{-1}(0) \cap \Omega$, combined with discussions on p. 212 [22], yield $\varepsilon, c > 0$, so that

$$\operatorname{dist}[\xi, \Phi^{-1}(0)] \leq c \cdot (\|\Phi(\xi)\| + \operatorname{dist}[\xi, \Omega]) \quad \forall \xi \in \xi^* + \varepsilon \mathbb{B}. \tag{7}$$

From the definition of the set Ω, we get for any $\xi = (u, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$:

$$\begin{aligned} \operatorname{dist}[\xi, \Omega] &= \inf\{\|(y - \bar{y}, z - \bar{z})\| \mid (\bar{y}, \bar{z}) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+\} \\ &\leq \|\min\{0, y\}\| + \|\min\{0, z\}\|. \end{aligned}$$

At the same time, elementary considerations imply

$$\|\min\{0, y\}\| + \|\min\{0, z\}\| \leq 2\|\min\{y, z\}\| \quad \forall y, z \in \mathbb{R}^m.$$

Therefore, (7) and definition of Φ give

$$\operatorname{dist}[\xi, \Phi^{-1}(0)] \leq \gamma \|\Phi(\xi)\| \quad \forall \xi \in \xi^* + \varepsilon \mathbb{B}$$

for some $\gamma > c$, where we use the fact that norms are equivalent in Euclidean spaces. Statement 2. follows. □

Now that we know that the constrained error bound coincides with the unconstrained one, it is enough for us to seek for sufficient conditions for Φ to provide a local error bound. Let us mention [4] at this place, where a combination of a constrained and an unconstrained error bound condition is used to establish local convergence properties of a Newton-type method. In our setting, the two error bound conditions coincide, leading to simplifications in the analyses of the above paper.
In the rest of this section, we employ the nondegeneracy condition (5), which entails (is even equivalent to) strict differentiability of Φ at ξ^*. Thus, we can employ recent criteria from [15,26] to guarantee the desired error bound condition. For this purpose, recall that ξ^* (nondegenerate) is a *noncritical solution* [26, Definition 1] of the unconstrained equation (3), if $\Phi^{-1}(0)$ is *regular* [37, Definition 6.4] at ξ^*, and

$$T_{\Phi^{-1}(0)}(\xi^*) = \ker \Phi'(\xi^*),$$

where $T_{\Phi^{-1}(0)}(\xi^*)$ is the *tangent cone* [37, Definition 6.1] to $\Phi^{-1}(0)$ at ξ^*, and $\ker \Phi'(\xi^*)$ is the kernel of the matrix (linear operator) $\Phi'(\xi^*)$. We will also deal with Mordukhovich's *(basic) normal cone* [32, Definition 1.1] in what follows. Recall that Mordukhovich's normal cone to $\Phi^{-1}(0)$ at ξ^* is

$$N_{\Phi^{-1}(0)}(\xi^*) = \left\{ \eta \,\middle|\, \exists \xi^k \to \xi^*, \exists t_k \searrow 0, \exists \{\zeta^k\}:\ \zeta^k \in P_{\Phi^{-1}(0)}(\xi^k) \forall k, \frac{\xi^k - \zeta^k}{t_k} \to \eta \right\},$$

where $P_{\Phi^{-1}(0)}$ is the Euclidean projector onto the nonempty closed set $\Phi^{-1}(0)$. Criteria for the error bound in the nondegenerate case are as follows.

Theorem 1. *If ξ^* is nondegenerate, then the following are equivalent:*
1. *Φ provides a local error bound at ξ^*.*
2. *ξ^* is a noncritical solution of* (3)*.*
3. *It holds that*

$$\eta \in N_{\Phi^{-1}(0)}(\xi^*), \quad \Phi'(\xi^*)\eta = 0 \quad \Longrightarrow \quad \eta = 0.$$

Proof. This is mentioned in [15, Section 4]. □

3 Lipschitzian Error Bound without Nondegeneracy

The criteria in Theorem 1 utilize tangents and normals to the set $\Phi^{-1}(0)$. In general, of course, the set $\Phi^{-1}(0)$ may be unknown in advance, but it is clear from the definition of Φ that

$$\Phi^{-1}(0) = \{(u,y,z) \mid y = g(u),\ z = h(u),\ \psi(y,z) = 0\}.$$

Thus, with the mapping $F := (g,h) : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m, u \mapsto (g(u),h(u))$, we have

$$\Phi^{-1}(0) = \mathrm{gph} F \bigcap \left(\mathbb{R}^n \times \psi^{-1}(0)\right), \tag{8}$$

where gph$F$ denotes the *graph* of F. This means that $\Phi^{-1}(0)$ is the intersection of two sets that are fully determined through the data, defining the complementarity system (1). Under appropriate assumptions (see below), the normal cone to $\Phi^{-1}(0)$ at ξ^* can be estimated from above by

$$N_{\mathrm{gph} F}(\xi^*) + N_{\mathbb{R}^n \times \psi^{-1}(0)}(\xi^*),$$

which, as one can easily check [32, Propositions 1.4, 1.12], coincides with the set

$$\left\{ \begin{pmatrix} g'(u^*)^\top \eta + h'(u^*)^\top \nu \\ a - \eta \\ b - \nu \end{pmatrix} \,\middle|\, \eta, \nu \in \mathbb{R}^m,\ (a,b) \in N_0 \right\},$$

where

$$\begin{aligned} N_0 &:= N_{\psi^{-1}(0)}(g(u^*), h(u^*)) \\ &= \left\{ (a,b) \,\middle|\, (a_i,b_i) \in \begin{cases} \mathbb{R}^2_- \cup (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+) & \text{if } i \in I_0, \\ \mathbb{R} \times \{0\} & \text{if } i \in I_1, \\ \{0\} \times \mathbb{R} & \text{otherwise} \end{cases} \right\}, \end{aligned} \tag{9}$$

with $I_0 := \{i \mid g_i(u^*) = h_i(u^*) = 0\}$ and $I_1 := \{i \mid 0 = g_i(u^*) < h_i(u^*)\}$. One such assumption [22, Proposition 3.2] is that the sets gph$F$ and $\mathbb{R}^n \times \psi^{-1}(0)$ are *subtransversal* at ξ^*, which means [21, Definition 7.2] the existence of $\varepsilon, c > 0$, satisfying

$$\operatorname{dist}\left[\xi, \operatorname{gph}F \cap \left(\mathbb{R}^n \times \psi^{-1}(0)\right)\right] \le c \cdot \left(\operatorname{dist}[\xi, \operatorname{gph}F] + \operatorname{dist}[\xi, \mathbb{R}^n \times \psi^{-1}(0)]\right)$$

for all $\xi \in \xi^* + \varepsilon\mathbb{B}$. In view of Theorem 1, considerations above lead to sufficient conditions for an error bound in the nondegenerate case. As a first contribution in this section, we show next that merely subtransversality of gph$F$ and $\mathbb{R}^n \times \psi^{-1}(0)$ is enough to characterize the error bound condition. Based on this observation, sufficient conditions for subtransversality [21] can be considered to guarantee the error bound. This is what we deal with afterwards.

Theorem 2. *The following statements are equivalent:*
1. *Φ provides a local error bound at ξ^*.*
2. gph$F$ *and* $\mathbb{R}^n \times \psi^{-1}(0)$ *are subtransversal at* ξ^*.

Proof. An application of [21, Theorem 7.12 (a)–(b)] and short computations yield an equivalence between statement 2., and the existence of $\varepsilon, c > 0$ with

$$\operatorname{dist}[\xi, \Phi^{-1}(0)] \le c \cdot \left(\|(g(u) - y, h(u) - z)\| + \operatorname{dist}[(y,z), \psi^{-1}(0)]\right) \qquad (10)$$

for all $\xi = (u,y,z) \in \xi^* + \varepsilon\mathbb{B}$. Hence, for some possibly different $c > 0$, Robinson's result on error bounds for polyhedral mappings [36, Proposition 1] entails

$$\operatorname{dist}[\xi, \Phi^{-1}(0)] \le c \cdot (\|(g(u) - y, h(u) - z)\| + \|\psi(y,z)\|) \qquad (11)$$

for all $\xi = (u,y,z)$ near ξ^*. Thanks to norm equivalence in Euclidean spaces, and definition of Φ, statement 1. follows. Conversely, we can argue again that statement 1. implies (11) for some $c > 0$ and all $\xi = (u,y,z)$ near ξ^*. Thanks to local Lipschitz continuity of ψ, we get (10) for any ξ near ξ^*, and some possibly different $c > 0$. But as mentioned above, this is nothing else than statement 2. □

Known sufficient conditions for subtransversality can now be used to guarantee the error bound. One of these conditions is formulated below.

Proposition 1. *The following implies that Φ provides a local error bound at ξ^*:*

$$g'(u^*)^\top \eta + h'(u^*)^\top \nu = 0, \quad (\eta, \nu) \in N_0 \quad \Longrightarrow \quad \eta = \nu = 0. \qquad (12)$$

Proof. From [21, Theorems 7.9, 7.12, and 8.13 (a)], it is known that the condition

$$N_{\operatorname{gph}F}(\xi^*) \bigcap \left(-N_{\mathbb{R}^n \times \psi^{-1}(0)}(\xi^*)\right) = \{0\} \qquad (13)$$

is sufficient for the subtransversality of gph$F$ and $\mathbb{R}^n \times \psi^{-1}(0)$ at ξ^*. Hence, thanks to Theorem 2, we get sufficiency of (13) for the error bound. Thanks to [32, Propositions 1.4, 1.12], we can rewrite (13) as

$$\left\{ \left. \begin{pmatrix} g'(u^*)^\top \eta + h'(u^*)^\top \nu \\ -\eta \\ -\nu \end{pmatrix} \right| \eta, \nu \in \mathbb{R}^m \right\} \bigcap \left\{ \left. \begin{pmatrix} 0 \\ -a \\ -b \end{pmatrix} \right| (a,b) \in N_0 \right\} = \{0\},$$

which, in turn, is the same as (12). □

It is well-known [17,18,19,20,33,34] that (12) is nothing else than Mordukhovich's coderivative criterion [37] for the *metric regularity* of the set-valued mapping

$$C(\cdot) := F(\cdot) - \psi^{-1}(0)$$

at u^* for $(0,0)$, which means the existence of $\varepsilon, c > 0$, satisfying

$$\operatorname{dist}[u, C^{-1}(\alpha,\beta)] \leq c \cdot \operatorname{dist}[(\alpha,\beta), C(u)] \tag{14}$$

for all $u \in u^* + \varepsilon\mathbb{B}$, and all $(\alpha,\beta) \in \varepsilon\mathbb{B}$. At the same time, metric regularity of C at u^* for $(0,0)$ is equivalent [37, Theorem 9.43] for the mapping C^{-1} to have the *Aubin property* at $(0,0)$ for u^*, i.e., for some $\varepsilon, c > 0$,

$$C^{-1}(\alpha,\beta) \cap (u^* + \varepsilon\mathbb{B}) \subset C^{-1}(\alpha',\beta') + c\|(\alpha-\alpha', \beta-\beta')\|\mathbb{B} \tag{15}$$

holds for all $(\alpha,\beta), (\alpha',\beta') \in \varepsilon\mathbb{B}$. From definition of C, F, and Σ in (6), we get

$$\begin{aligned} C^{-1}(\alpha,\beta) &= \{u \mid (\alpha,\beta) \in F(u) - \psi^{-1}(0)\} \\ &= \{u \mid (g(u)-\alpha, h(u)-\beta) \in \psi^{-1}(0)\} = \Sigma(-\alpha,-\beta). \end{aligned}$$

Hence, for C^{-1} to have the Aubin property at $(0,0)$ for u^*, it is necessary and sufficient that Σ has the Aubin property $(0,0)$ for u^*. These properties are independent of the mapping Φ, and rely on information about the complementarity system (1) only. In what follows, we will show that the properties just described coincide with metric regularity of the mapping Φ at ξ^*, i.e.,

$$\exists \varepsilon, c > 0: \quad \operatorname{dist}[\xi, \Phi^{-1}(\zeta)] \leq c \cdot \|\zeta - \Phi(\xi)\| \quad \forall \xi \in \xi^* + \varepsilon\mathbb{B}, \forall \zeta \in \varepsilon\mathbb{B}.$$

The subsequent lemma is the key to establish the described equivalence. It invokes Mordukhovich's *(basic) coderivative* [32, Definition 1.11] of ψ, which is defined for $x = (y,z) \in \mathbb{R}^m \times \mathbb{R}^m$ as the mapping $D^*\psi(x) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m \times \mathbb{R}^m$,

$$D^*\psi(x)(\sigma) := \left\{ \eta = (a,b) \,\middle|\, (\eta, -\sigma) \in N_{\operatorname{gph}\Phi}(x, \psi(x)) \right\}.$$

Lemma 2. *The mapping ψ is metrically regular at $x^* = (g(u^*), h(u^*))$, and*

$$N_0 = \bigcup_{\sigma \in \mathbb{R}^m} D^*\psi(x^*)(\sigma).$$

Proof. Pick $\sigma \in \mathbb{R}^m$ arbitrarily, and consider $f_\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, given as

$$f_\sigma(y,z) := \sigma^\top \psi(y,z) = \sum_{i=1}^m \sigma_i \min\{y_i, z_i\}. \tag{16}$$

Then, with $x^* = (g(u^*), h(u^*))$, we obtain from [32, Theorem 1.32] that

$$D^*\psi(x^*)(\sigma) = \partial f_\sigma(x^*), \tag{17}$$

where ∂f_σ denotes Mordukhovich's *(basic) subdifferential* of the function f_σ [32, Definition 1.18]. Setting $f_i(s,t) := \sigma_i \min\{s,t\}$ for $i = 1, \ldots, m$ and $s,t \in \mathbb{R}$, then (16) and [37, Proposition 10.5] yield

$$\partial f_\sigma(x^*) = \bigotimes_{i=1}^m \partial f_i(x_i^*), \tag{18}$$

where $x_i^* = (g_i(u^*), h_i(u^*))$. We compute the subdifferential of the functions f_i in what follows. To this end, take $i \in \{1,\dots,m\}$ arbitrarily, and notice that

$$f_i(s,t) = \sigma_i \min\{s,t\} = \sigma_i s - \sigma_i \max\{0, s-t\} \quad \forall s,t \in \mathbb{R}.$$

Combining [37, Exercise 8.8, Example 7.28, Exercise 8.31, Corollary 9.21] yields

$$\partial f_i(s,t) = \begin{cases} \operatorname{conv}\left\{ \begin{pmatrix} \sigma_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} \right\} & \text{if } s = t \text{ and } \sigma_i \le 0, \\ \left\{ \begin{pmatrix} \sigma_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} \right\} & \text{if } s = t \text{ and } \sigma_i > 0, \\ \left\{ \begin{pmatrix} \sigma_i \\ 0 \end{pmatrix} \right\} & \text{if } s < t, \\ \left\{ \begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} \right\} & \text{if } s > t. \end{cases}$$

Hence, (17)–(18) give

$$D^*\psi(x^*)(\sigma) = \left\{ (a,b) \,\middle|\, (a_i,b_i) \in \begin{cases} \operatorname{conv}\left\{ \begin{pmatrix} \sigma_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} \right\} & \text{if } i \in I_0 \text{ and } \sigma_i \le 0, \\ \left\{ \begin{pmatrix} \sigma_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} \right\} & \text{if } i \in I_0 \text{ and } \sigma_i > 0, \\ \left\{ \begin{pmatrix} \sigma_i \\ 0 \end{pmatrix} \right\} & \text{if } i \in I_1, \\ \left\{ \begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} \right\} & \text{otherwise} \end{cases} \right\}.$$

From here, the formula for N_0 in terms of coderivatives follows from (9). Moreover, and because σ was arbitrarily chosen, we get metric regularity of ψ from the representation of $D^*\psi(x^*)(\sigma)$ above, and [32, Theorem 3.3]. □

We are in a position to prove the claimed relation between metric regularity of the constraint mapping C on the one hand and that of Φ on the other.

Theorem 3. *The following statements are equivalent:*

1. *Condition* (12) *is satisfied.*
2. *Σ has the Aubin property at* $(0,0)$ *for u^*.*
3. *C is metrically regular at u^* for* $(0,0)$.
4. *Φ is metrically regular at ξ^*.*

Proof. The equivalences between statements 1.–3. are explained above Lemma 2. It remains to ensure their relation to metric regularity of Φ. For this purpose, we employ [32, Theorem 3.3] again, which states an equivalence between statement 4. and

$$0 \in D^*\Phi(\xi^*)(\sigma) \quad \Longrightarrow \quad \sigma = 0. \tag{19}$$

All that remains to be done is to show that statement 1. coincides with (19). To that end, put $F_1(u,y,z) := (g(u)-y, h(u)-z, 0)$ and $F_2(u,y,z) := (0,0,\psi(y,z))$, and find $\Phi(\xi) = \Phi(u,y,z) = F_1(u,y,z) + F_2(u,y,z)$. The mapping F_1 is continuously differentiable, while the mapping F_2 is locally Lipschitz continuous. Thus, [32, Theorem 3.9] yields for $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$:

$$
\begin{aligned}
D^*\Phi(\xi^*)(\sigma) &= F'(\xi^*)^\top \sigma + D^*F_2(\xi^*)(\sigma) \\
&= \begin{pmatrix} g'(u^*)^\top \sigma_1 + h'(u^*)^\top \sigma_2 \\ -\sigma_1 \\ -\sigma_2 \end{pmatrix} + D^*F_2(\xi^*)(\sigma).
\end{aligned}
$$

At the same time, thanks to the specific structure of F_2, we have

$$
D^*F_2(\xi^*)(\sigma) = \left\{ \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \middle| \, (a,b) \in D^*\psi\left(g(u^*), h(u^*)\right)(\sigma_3) \right\}.
$$

Hence, condition (19) (and so too statement 4.) is nothing else than

$$
\left.\begin{aligned} g'(u^*)^\top \sigma_1 + h'(u^*)^\top \sigma_2 = 0, \\ (\sigma_1, \sigma_2) \in D^*\psi\left(g(u^*), h(u^*)\right)(\sigma_3) \end{aligned}\right\} \quad \Longrightarrow \quad \sigma_1 = \sigma_2 = \sigma_3 = 0, \tag{20}
$$

and we merely have to explain that (12) and (20) are the same. Suppose that (12) (i.e., statement 1.) holds, and pick $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with $g'(u^*)^\top \sigma_1 + h'(u^*)^\top \sigma_2 = 0$ and $(\sigma_1, \sigma_2) \in D^*\psi\left(g(u^*), h(u^*)\right)(\sigma_3)$. Thanks to Lemma 2, the second of these conditions gives $(\sigma_1, \sigma_2) \in N_0$. Thus, (12) implies $\sigma_1 = \sigma_2 = 0$. But then, we get $(0,0) \in D^*\psi(g(u^*), h(u^*))(\sigma_3)$, and metric regularity of ψ (Lemma 2), and [32, Theorem 3.3], yield $\sigma_3 = 0$. In other words, the implication 1. $\Rightarrow$ 4. holds true. Conversely, assume that (20) (i.e., statement 4.) is in force, and pick $(\eta, \nu) \in N_0$ with $g'(u^*)^\top \eta + h'(u^*)^\top \nu = 0$. Thanks to Lemma 2, the first of these conditions yields $\zeta \in \mathbb{R}^m$ with $(\eta, \nu) \in D^*\psi(g(u^*), h(u^*))(\zeta)$. But then, (20) necessitates $\eta = \nu = \zeta = 0$. Hence, (12) (and so too statement 1.) follows. $\square$

The theorem states an equivalence between regularity properties of solutions of the complementarity system (1) on the one hand, and regularity properties of the equation-reformulation (3) on the other hand. Relations of this kind are of interest, e.g., in [11,12,27,30], and it is safe to say that they can not be established for arbitrary reformulations of (1).

In the remainder of the section, we want to compare (12) with another common sufficient condition for the error bound condition (and solution-stability), namely, the *piecewise MFCQ* [1,24,28]. Setting $I_2 := \{1, \ldots, m\} \setminus (I_0 \cup I_1)$, then the piecewise MFCQ requests for any partition (J_1, J_2) of $I_0 = \{i \mid g_i(u^*) = h_i(u^*)\}$ that the vectors $\{g_i'(u^*)\}_{i \in I_1 \cup J_1}$ and $\{h_i'(u^*)\}_{i \in I_2 \cup J_2}$ are linearly independent, and

$$
\exists w \in \mathbb{R}^n : \quad \begin{cases} g_i'(u^*)w = 0 \; \forall i \in I_1 \cup J_1, & h_i'(u^*)w = 0 \; \forall i \in I_2 \cup J_2, \\ g_i'(u^*)w < 0 \; \forall i \in J_2, & h_i'(u^*)w < 0 \; \forall i \in J_1. \end{cases}
$$

An application of [27, Lemma A.2] shows that piecewise MFCQ is sufficient for (12). At the same time, (12) does not imply piecewise MFCQ, as can be illustrated by the example where $n = 3$, $m = 2$, and $g(u) := (u_2 + u_3, -u_3)$ and $h(u) := (u_1 + u_2, u_2)$. Short computations confirm validity of (12) for $u^* = (0,0,0)$, whereas piecewise MFCQ can not hold there. To see that the latter is true, it is enough to consider $J_1 = \varnothing$ and $J_2 = \{1, 2\}$.

4 Discussions for NCPs

In this section, we consider a standard nonlinear complementarity problem,

$$u \geq 0, \quad f(u) \geq 0, \quad u^\top f(u) = 0, \tag{21}$$

in which $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. This is a special instance of complementarity system (1), where $m = n$, $h := f$, and g is the identity.
The goal of this section is to embed the relations from Theorem 3 between well-known notions of solution-regularity for nonlinear complementarity problems presented in [25] and [27, Section 3.2.1].
To begin with, we recall [8, Theorem 1] that the solution mapping Σ for the nonlinear complementarity problem (21) has the Aubin property at $(0,0)$ for u^* if and only if u^* is a *strongly regular* solution of (21), which means that $\Sigma(\cdot) \cap (u^* + \varepsilon\mathbb{B})$ can be considered as a single-valued Lipschitzian mapping for some $\varepsilon > 0$ in a neighborhood of $(0,0)$. Consequently, statements 1.–4. in Theorem 3 all correspond to the strong regularity of u^*. According to [27, Figure 3.1], strong regularity (thus statements 1.–4.) is further equivalent to nonsingularity of certain generalized derivatives of the nonsmooth composition mapping $\psi(\cdot, f(\cdot))$. In this paper, we do not consider the composition mapping. Instead, we deal with the mapping Φ, which utilizes slack-variables y, z, and for this, we discuss how strong regularity relates to nonsingularity of generalized derivatives, frequently used in the context of locally Lipschitz continuous mappings. To that end, recall [6,11,12,27] that the *limiting Jacobian (B-differential)* of Φ at ξ^* is

$$\partial_B \Phi(\xi^*) = \left\{ A \,\middle|\, \exists \xi^k \to \xi^* : \ \Phi \text{ is differentiable at } \xi^k \forall k, \, \Phi'(\xi^k) \to A \right\},$$

while Clarke's *generalized Jacobian* of Φ at ξ^*, denoted by $\partial_C \Phi(\xi^*)$, is the convex hull of $\partial_B \Phi(\xi^*)$. One says (cf. [27, Remark 1.65]) that Φ is *BD-regular* at ξ^* if all the matrices in $\partial_B \Phi(\xi^*)$ are nonsingular (have full rank). Similarly, Φ is *CD-regular* at ξ^* if all matrices in $\partial_C \Phi(\xi^*)$ are nonsingular. It is quite clear that the limiting and the generalized Jacobian of Φ at ξ^* are

$$\partial_B \Phi(\xi^*) = \left\{ \begin{pmatrix} I & -I & 0 \\ f'(u^*) & 0 & -I \\ 0 & A_y & A_z \end{pmatrix} \middle| \, (A_y, A_z) \in \partial_B \psi(g(u^*), h(u^*)) \right\},$$

$$\partial_C \Phi(\xi^*) = \left\{ \begin{pmatrix} I & -I & 0 \\ f'(u^*) & 0 & -I \\ 0 & A_y & A_z \end{pmatrix} \middle| \, (A_y, A_z) \in \partial_C \psi(g(u^*), h(u^*)) \right\}.$$

Thus, the BD-regularity of Φ at ξ^* corresponds to

$$\left(A_y + A_z f'(u^*)\right) w = 0, \quad (A_y, A_z) \in \partial_B \psi(g(u^*), h(u^*)) \quad \Longrightarrow \quad w = 0, \tag{22}$$

and CD-regularity can be similarly characterized by simply replacing $\partial_B \Phi(\xi^*)$ above with $\partial_C \Phi(\xi^*)$. The condition in (22), also known as *b-regularity* [11,25,27], can be strictly weaker than CD-regularity [25,27]. At the same time, [27, Corollary 3.20] yields an equivalence between CD-regularity of Φ at ξ^* and strong regularity of u^*. This leads to the relationships below.

Theorem 4. *The following statements are equivalent:*

1. *u^* is strongly regular.*
2. *Φ is metrically regular at ξ^*.*
3. *Φ is CD-regular at ξ^*.*

One more property can be added to the above theorem, namely that Φ has a Lipschitzian inverse near 0, i.e., for some $\varepsilon > 0$, the mapping $\Phi^{-1}(\cdot) \cap (\xi^* + \varepsilon\mathbb{B})$ is single-valued and Lipschitz continuous near 0. This condition implies metric regularity of Φ at ξ^*. At the same time, it is necessary [12, Proposition 7.1.19] for CD-regularity of Φ at ξ^*. We would like to mention [31], where, for the case of KKT-systems, relations between strong regularity and Lipschitz invertibility of a reformulation-mapping were established.

Conclusion

The convergence analysis of several Newton-type methods, designed, e.g., for the solution of complementarity systems (1), relies often on a constrained error bound assumption (4). In this paper, we show for a complementarity system, that the constrained error bound is actually an unconstrained one. Sufficient conditions for such error bounds in the nondegenerate case can be formulated on the basis of recent results [15,16,26]. However, another approach is needed to tackle the general non-nondegenerate (possibly degenerate) case. We show that the desired error bound condition coincides with a subtransversality condition. For the latter, sufficient conditions are known [21]. The sufficient condition considered herein is Mordukhovich's coderivative criterion for the Aubin property of the solution mapping Σ. We show that the latter further agrees with (metric) regularity of the mapping Φ, defining the equation-reformulation (3) of the complementarity system (1). For the case of nonlinear complementarity problems (21), we could embed our findings in an extensive amount of relationships between regularity-notions, frequently used [11,12,27].

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