

Superlinear Convergence of an Interior Point Algorithm on Linear Semi-definite Feasibility Problems with Application to Linear Matrix Inequalities

Chee-Khian Sim*

School of Mathematics and Physics
University of Portsmouth
Lion Gate Building, Lion Terrace
Portsmouth, PO1 3HF
United Kingdom

08 January 2024

Abstract

In the literature, besides the assumption of strict complementarity, superlinear convergence of implementable polynomial-time interior point algorithms using known search directions, namely, the HKM direction, its dual or the NT direction, to solve semi-definite programs (SDPs) is shown by (i) assuming that the given SDP is nondegenerate and making modifications to these algorithms [10], or (ii) considering special classes of SDPs, such as the class of linear semi-definite feasibility problems (LSDFPs) and requiring the initial iterate to the algorithm to satisfy certain conditions [26, 27]. Otherwise, these algorithms are not easy to implement even though they are shown to have polynomial iteration complexities and superlinear convergence [14]. The conditions in [26, 27] that the initial iterate to the algorithm is required to satisfy to have superlinear convergence when solving LSDFPs however are not practical. In this paper, we propose a practical initial iterate to an implementable infeasible interior point algorithm that guarantees superlinear convergence when the algorithm is used to solve the homogeneous feasibility model of an LSDFP.

Keywords. Linear semi-definite feasibility problem; strict feasibility; homogeneous feasibility model; interior point method; superlinear convergence.

1 Introduction

Many problems in diverse areas, such as optimal control, estimation and signal processing, communications and networks, statistics, and finance, can be modelled well as semi-definite programs (SDPs) [5]. Finding effective and efficient ways to solve this class of problems is hence practically important. Interior point methods (IPMs) have been proven to be successful in solving SDPs - see for example [1, 12, 15, 33]. Research on interior point methods is active with recent papers, such as [2, 8, 24, 25]¹, proposing contemporary interior point algorithms to solve SDPs. An important subclass of the class of SDPs is the class of linear semi-definite

*Email address: chee-khian.sim@port.ac.uk

¹Note that papers [2, 8, 25] consider symmetric cone problems, which generalize SDPs.

feasibility problems (LSDFPs), which also have applicability in diverse areas. This class of problems includes linear matrix inequalities (LMIs).

Among different IPMs, primal-dual path following interior point algorithms are the most successful and most widely studied. In this paper, we focus on an infeasible predictor-corrector primal-dual path following interior point algorithm to solve the homogeneous feasibility model [21] of an LSDFP. Polynomial iteration complexity of the algorithm to solve a primal-dual SDP pair has been shown in [21]. In this paper, we consider the local convergence behavior of the algorithm.

It proves not an easy task to show superlinear convergence of an implementable interior point algorithm that has polynomial iteration complexity on an SDP with minimal assumptions on the problem and no modifications to the algorithm. This is especially so for the HKM (and its dual) or the NT search direction used in these algorithms, although superlinear convergence is quickly established for the AHO search direction [11, 13], when researchers started focusing their attention on interior point algorithms to solve SDPs. In the literature when the HKM (and its dual) or the NT search direction is used in an interior point algorithm, such as [10], in addition to strict complementarity assumption, nondegeneracy assumption at an optimal solution and modifications to the algorithm, such as solving the corrector-step linear system in an iteration repeatedly instead of only once (“narrowing” the central path neighborhood²), need to be imposed for superlinear convergence of the interior point algorithm. In [14], without assuming nondegeneracy, the feasible interior point algorithm considered in the paper is shown to have polynomial iteration complexity and superlinear convergence. However, the algorithm is not easy to implement. The idea behind the algorithm considered in [14] to have superlinear convergence when solving an SDP is to force the centrality measure of the k^{th} iterate to converge to zero as k tends to infinity. This can be enforced in practice by for example, repeatedly solving the corrector-step linear system in an iteration, as in [10], so as to “narrow” the neighborhood of the central path in which iterates lie. Hence, either a polynomial-time interior point algorithm has to be modified or the algorithm is hard to implement for guaranteed superlinear convergence of these algorithms. Otherwise, special structure needs to be imposed on the SDP, such as considering linear semi-definite feasibility problems (LSDFPs), for superlinear convergence [26, 27]³. For the latter, we also require additionally that a suitable initial iterate is chosen. However, the required initial iterate to guarantee superlinear convergence given in [26, 27] is not practical. In this paper, we improve on the results in [26, 27] by proposing an initial iterate that can be obtained in practice (by solving a primal-dual SDP pair), and using this initial iterate to an implementable interior point algorithm on the homogeneous feasibility model of the LSDFP, we have superlinear convergence of the algorithm. The novelty in showing this superlinear convergence result is the nontrivial application of what is known in the literature [26, 27] to show the result. Two key steps that are needed before we can use relevant results in the literature [26, 27] are (i) applying the interior point algorithm on the homogeneous feasibility model of the LSDFP, instead of the LSDFP itself; (ii) suitably reformulating the homogeneous feasibility model of the LSDFP as a semi-definite linear complementary problem (SDLCP) before we can apply these results.

In Section 2, we describe the homogeneous feasibility model of an LSDFP, and in Section

²We note that this idea is used in [19] to show superlinear convergence of an interior point algorithm on a wide class of conic optimization problems. Also, this is related to the sufficient conditions on superlinear convergence behavior of iterates generated by the algorithm as studied in [22, 23].

³In [26, 27], the search direction used in the interior point algorithm considered is the (dual) HKM and the NT direction respectively.

3, we express the homogeneous feasibility model as a semi-definite linear complementarity problem (SDLCP). The latter allows us to apply results in the literature in Section 4 to show superlinear convergence of an implementable interior point algorithm on the homogeneous feasibility model. We conclude the paper with Section 5.

1.1 Notations

The space of symmetric $n \times n$ matrices is denoted by S^n . The cone of positive semi-definite (resp., positive definite) symmetric matrices is denoted by S_+^n (resp. S_{++}^n). The identity matrix is denoted by $I_{n \times n}$, where n stands for the size of the matrix. We omit the subscript when the size of the identity matrix is clear from the context. The matrix $E^{ij} \in \mathfrak{R}^{n_1 \times n_2}$ is defined to have $1/2$ in its (i, j) and (j, i) entry, and zero everywhere else.

Given a matrix $G \in \mathfrak{R}^{n_1 \times n_2}$, $\|G\|_F := \sqrt{\text{Tr}(GG^T)}$ is the Frobenius norm of G , where $\text{Tr}(\cdot)$ is the trace of a square matrix. G_{ij} is the entry of G in the i th row and the j th column of G . On the other hand, given a vector $x \in \mathfrak{R}^n$, $\|x\|$ refers to its Euclidean norm, and x_i is its i th entry.

Given $X \in S^n$, $\text{svec}(X)$ is defined to be

$$\text{svec}(X) := (X_{11}, \sqrt{2}X_{21}, \dots, \sqrt{2}X_{n1}, X_{22}, \sqrt{2}X_{32}, \dots, X_{n-1,n-1}, \sqrt{2}X_{n,n-1}, X_{nn})^T \in \mathfrak{R}^{\tilde{n}},$$

where $\tilde{n} = n(n+1)/2$. $\text{svec}(\cdot)$ sets up a one-to-one correspondence between S^n and $\mathfrak{R}^{\tilde{n}}$.

2 A Linear Semi-definite Feasibility Problem and its Homogeneous Feasibility Model

Given $C, A_i \in S^n$, $i = 1, \dots, m$, and $b = (b_1, \dots, b_m)^T \in \mathfrak{R}^m$. A (primal) semi-definite program (SDP) is given by

$$\begin{aligned} \min \quad & \text{Tr}(CX) \\ \text{subject to} \quad & \text{Tr}(A_i X) = b_i, \quad i = 1, \dots, m, \\ & X \in S_+^n. \end{aligned} \tag{1}$$

The dual of (1) is given by

$$\begin{aligned} \max \quad & b^T y \\ \text{subject to} \quad & \sum_{i=1}^m y_i A_i + Y = C, \\ & Y \in S_+^n. \end{aligned} \tag{2}$$

Here, $y = (y_1, \dots, y_m)^T \in \mathfrak{R}^m$. We consider the case when (1) and (2) are feasible in this paper.

A linear semi-definite feasibility problem (LSDFP) is a primal-dual SDP pair (1)-(2) with $C = 0$ or $b = 0$. WLOG, from now onwards, we consider the LSDFP with $C = 0$. Also, we assume that $b_i \neq 0$, for some $i = 1, \dots, m$, to avoid trivial considerations.

Remark 2.1 A (nonstrict) linear matrix inequality (LMI) has the form

$$\sum_{j=1}^l z_j B_j + B_0 \in S_+^n, \tag{3}$$

where $z_j \in \mathfrak{R}, j = 1, \dots, l$, are the decision variables, and $B_j \in S^n, j = 0, \dots, l$, are given. Solving LMIs has wide applicability in diverse areas, as discussed for example in [4]. It is easy to see that (3) can be written as the primal SDP (1) with $C = 0$, for suitable $A_i, b_i, i = 1, \dots, m$, where we wish to find $X \in S_+^n$ that solves the primal SDP (1) with $C = 0$. Hence, by solving the primal SDP (1) with $C = 0$, we are able to solve (3)⁴.

We impose the following assumptions on the LSDFP throughout this paper:

Assumption 2.1 (a) *There exists $(y, Y) \in \mathfrak{R}^m \times S_{++}^n$ that is feasible to the dual SDP (2) with $C = 0$.*

(b) *A_1, \dots, A_m are linearly independent.*

Assumption 2.1(b) is without loss of generality. On the other hand, Assumption 2.1(a) ensures that the duality gap between the primal SDP (1) and the dual SDP (2) is zero - see for example [5].

Remark 2.2 *We can find $(y, Y) \in \mathfrak{R}^m \times S_{++}^n$ in Assumption 2.1(a) by using a primal-dual path following interior point algorithm to solve the following primal-dual SDP pair⁵:*

$$\begin{aligned} \min \quad & 0 \\ \text{subject to} \quad & \text{Tr}(A_i X) = 0, \quad i = 1, \dots, m, \\ & X \in S_+^n, \end{aligned}$$

$$\begin{aligned} \max \quad & 0 \\ \text{subject to} \quad & \sum_{i=1}^m y_i A_i + Y = 0, \\ & Y \in S_+^n. \end{aligned}$$

From [27] (see also [23, 26]), we have polynomial iteration complexity and superlinear convergence of an infeasible primal-dual path following interior point algorithm on the above primal-dual SDP pair to obtain its solution. Hence, we are able to get a strictly feasible solution to the dual SDP (2) with $C = 0$ efficiently and fast.

In this paper, we apply an infeasible interior point algorithm using the (dual) HKM search direction or the NT search direction to solve the homogeneous feasibility model of the given LSDFP. Since the algorithm is an interior point algorithm, the initial iterate (X_0, y_0, Y_0) to the algorithm needs to satisfy $X_0, Y_0 \in S_{++}^n$. To prove superlinear convergence of the algorithm on the homogeneous feasibility model of the LSDFP, we further require the initial iterate (X_0, y_0, Y_0) to have (y_0, Y_0) feasible to the dual SDP (2) with $C = 0$. Although this is an additional requirement on the initial iterate, it can be satisfied in practice, as discussed in Remark 2.2. This superlinear convergence result improves on what is known in the literature [26, 27], where the initial iterate, (X_0, y_0, Y_0) , is required to satisfy impractical conditions for superlinear convergence.

⁴It is known that implementable interior point algorithms with polynomial iteration complexities can be used to solve LMIs [4, 5], but to the best of our knowledge, it is unknown whether these algorithms can solve (nonstrict) LMIs with proven good local convergence behavior, which we show in this paper.

⁵In fact, by applying the primal-dual path following interior point algorithm on the primal-dual SDP pair, we will know whether Assumption 2.1(a) holds by checking the solution the algorithm converges to.

We now introduce the homogeneous feasibility model of the primal-dual SDP pair (1)-(2), which first appeared in [21]. We consider the general setting when C is not required to be zero when describing the model. The model is given by the following homogeneous system:

$$\text{Tr}(A_i X) = b_i \tau, \quad i = 1, \dots, m, \quad (4)$$

$$\sum_{i=1}^m y_i A_i + Y = \tau C, \quad (5)$$

$$\kappa = b^T y - \text{Tr}(CX), \quad (6)$$

$$X \in S_+^n, y \in \mathfrak{R}^m, Y \in S_+^n, \tau \geq 0, \kappa \geq 0. \quad (7)$$

The model has been incorporated in the software package SeDuMi, according to [7], to solve SDPs, and also in the commercial software package MOSEK, according to [6] (see also [3]), to solve conic optimization problems, which include SDPs.

Observe that (4)-(6) implies that

$$\text{Tr}(XY) + \tau \kappa = 0, \quad (8)$$

from which we obtain, using (7),

$$XY = 0,$$

$$\tau \kappa = 0.$$

Furthermore, observe that a solution to the homogeneous system is readily available and is given by $(X, y, Y, \tau, \kappa) = (0, 0, 0, 0, 0)$. However, we cannot derive optimal solutions to (1) and (2) from this. If there exists a solution $(X^*, y^*, Y^*, \tau^*, \kappa^*)$ to (4)-(7) such that $\kappa^* = 0$ and $\tau^* > 0$, then $(X^*/\tau^*, y^*/\tau^*, Y^*/\tau^*)$ is an optimal solution to the primal-dual SDP pair (1)-(2) with zero duality gap. Conversely, if (X^*, y^*, Y^*) is an optimal solution to the primal-dual SDP pair (1)-(2) with zero duality gap, then $(X^*, y^*, Y^*, 1, 0)$ solves the system (4)-(7). Interior point algorithms when applied to the homogeneous feasibility model can be used to find optimal solutions to the primal-dual SDP pair (1)-(2) by finding a solution $(X^*, y^*, Y^*, \tau^*, \kappa^*)$ to (4)-(7) such that $\kappa^* = 0$ and $\tau^* > 0$. Such an interior point algorithm is discussed in [21], which is also given in Section 4 below where it is specialized to the case when $C = 0$. These interior point algorithms necessarily have to be of the infeasible type in that the initial iterate and subsequent iterates generated by the algorithm cannot satisfy (4)-(6). This is so because if an iterate $(X_k, y_k, Y_k, \tau_k, \kappa_k) \in S_{++}^n \times \mathfrak{R}^m \times S_{++}^n \times \mathfrak{R}_{++} \times \mathfrak{R}_{++}$ for some $k \geq 0$ satisfies (4)-(6), then (8) holds, which is impossible since $X_k, Y_k \in S_{++}^n$ and $\tau_k, \kappa_k > 0$.

Remark 2.3 *When specialized to the case when $C = 0$, (4)-(7) are given by:*

$$\text{Tr}(A_i X) = b_i \tau, \quad i = 1, \dots, m, \quad (9)$$

$$\sum_{i=1}^m y_i A_i + Y = 0, \quad (10)$$

$$\kappa = b^T y, \quad (11)$$

$$X \in S_+^n, y \in \mathfrak{R}^m, Y \in S_+^n, \tau \geq 0, \kappa \geq 0. \quad (12)$$

3 Homogeneous Feasibility Model as a Semi-definite Linear Complementarity Problem

Recall that a semi-definite linear complementarity problem (SDLCP), introduced in [12], is given by:

$$\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = q, \quad (13)$$

$$X^1 Y^1 = 0, \quad (14)$$

$$X^1, Y^1 \in S_+^{n_1}, \quad (15)$$

where $\mathcal{A}_1, \mathcal{B}_1 : S^{n_1} \rightarrow \mathfrak{R}^{\tilde{n}_1}$ are linear operators, $q \in \mathfrak{R}^{\tilde{n}_1}$, and $\tilde{n}_1 = n_1(n_1 + 1)/2$. The following assumptions are imposed on the SDLCP, which we show in Proposition 3.2 to hold for the SDLCP representation of the homogeneous feasibility model of the LSDFP. This SDLCP representation will be derived later in the section.

Assumption 3.1 (a) System (13)-(15) is monotone. That is, $\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = 0$ for $X^1, Y^1 \in S^{n_1} \Rightarrow \text{Tr}(X^1 Y^1) \geq 0$.

(b) There exists at least one solution to SDLCP (13)-(15).

(c) $\{\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) ; X^1, Y^1 \in S^{n_1}\} = \mathfrak{R}^{\tilde{n}_1}$.

When the SDLCP (13)-(15) is studied in some papers in the literature, instead of Assumption 3.1(b), the following assumption is imposed:

Assumption 3.2 There exist $X^1, Y^1 \in S_{++}^{n_1}$ such that $\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = q$.

Assumptions 3.1(a), (c) and 3.2 are imposed in [12] where the paper studies feasible interior point algorithms on the SDLCP (13)-(15), while [10, 26] assume Assumption 3.1 in the study of infeasible interior point algorithms on the SDLCP (13)-(15). In the study of infeasible interior point algorithms on the SDLCP (13)-(15), it is not necessary to impose Assumption 3.2. A reason is that we do not need a strictly feasible initial iterate to the algorithm. In this paper, we solve an LSDFP by applying an interior point algorithm on the homogeneous feasibility model (9)-(12). This interior point algorithm is necessarily infeasible as discussed near the end of Section 2. When we express the homogeneous feasibility model as an SDLCP (to be discussed next), the interior point algorithm that is applied to the homogeneous feasibility model can be equivalently applied to the corresponding SDLCP, and by necessity, the algorithm on the SDLCP is infeasible just like that for the homogeneous feasibility model. Therefore, having Assumption 3.2 imposed on the SDLCP is not suitable and in fact can never hold in our case as there cannot exist an $(X, Y) \in S_{++}^n \times S_{++}^m$ such that (13) holds for the SDLCP obtained from the homogeneous feasibility model. We therefore have Assumption 3.1(b) in its place. In fact, we will see later that when the LSDFP satisfies Assumption 2.1, the SDLCP representation of the homogeneous feasibility model (9)-(12) satisfies Assumption 3.1.

We now proceed to express the homogeneous feasibility model (9)-(12) as an SDLCP.

We can write (9) as

$$\begin{bmatrix} \text{svec}(A_1)^T & -b_1 \\ \vdots & \vdots \\ \text{svec}(A_m)^T & -b_m \end{bmatrix} \begin{bmatrix} \text{svec}(X) \\ \tau \end{bmatrix} = 0. \quad (16)$$

On the other hand, combining (10) and (11) into one equation, we obtain

$$\begin{bmatrix} -b_1 & \dots & -b_m \\ \text{svec}(A_1) & \dots & \text{svec}(A_m) \end{bmatrix} y + \begin{bmatrix} \kappa \\ \text{svec}(Y) \end{bmatrix} = 0. \quad (17)$$

Let us now rewrite (16) and (17) in a more compact form. Let

$$\mathcal{A} := \begin{bmatrix} \text{svec}(A_1)^T \\ \vdots \\ \text{svec}(A_m)^T \end{bmatrix} \in \mathfrak{R}^{m \times \tilde{n}},$$

and recall that $b = (b_1, \dots, b_m)^T$. Then, (16) and (17) can be written as

$$[\mathcal{A} \quad -b] \begin{bmatrix} \text{svec}(X) \\ \tau \end{bmatrix} = 0 \quad (18)$$

and

$$\begin{bmatrix} -b^T \\ \mathcal{A}^T \end{bmatrix} y + \begin{bmatrix} \kappa \\ \text{svec}(Y) \end{bmatrix} = 0 \quad (19)$$

respectively.

Let $[d_1 \text{svec}(B_1)^T]^T$, $d_1 \neq 0$, be orthogonal to the set

$$\left\{ \begin{bmatrix} -b_1 \\ \text{svec}(A_1) \end{bmatrix}, \dots, \begin{bmatrix} -b_m \\ \text{svec}(A_m) \end{bmatrix} \right\}.$$

Since $d_1 \neq 0$ and $b_i \neq 0$ for some $i = 1, \dots, m$, B_1 is necessarily nonzero.

Furthermore, let the following set of linearly independent vectors in $\mathfrak{R}^{\tilde{n}}$

$$\{\text{svec}(B_2), \dots, \text{svec}(B_{\tilde{n}+1-m})\}$$

spans the orthogonal subspace to the space spanned by

$$\{\text{svec}(A_1), \dots, \text{svec}(A_m)\},$$

where $\tilde{n} = n(n+1)/2$.

Remark 3.1 *We have*

$$\left\{ \begin{bmatrix} -b_1 \\ \text{svec}(A_1) \end{bmatrix}, \dots, \begin{bmatrix} -b_m \\ \text{svec}(A_m) \end{bmatrix}, \begin{bmatrix} d_1 \\ \text{svec}(B_1) \end{bmatrix}, \begin{bmatrix} 0 \\ \text{svec}(B_2) \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \text{svec}(B_{\tilde{n}+1-m}) \end{bmatrix} \right\}$$

spans $\mathfrak{R}^{\tilde{n}+1}$, with elements in

$$\left\{ \begin{bmatrix} -b_1 \\ \text{svec}(A_1) \end{bmatrix}, \dots, \begin{bmatrix} -b_m \\ \text{svec}(A_m) \end{bmatrix} \right\}$$

orthogonal to elements in

$$\left\{ \begin{bmatrix} d_1 \\ \text{svec}(B_1) \end{bmatrix}, \begin{bmatrix} 0 \\ \text{svec}(B_2) \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \text{svec}(B_{\tilde{n}+1-m}) \end{bmatrix} \right\}.$$

Let

$$\mathcal{B} := \begin{bmatrix} \text{svec}(B_1)^T \\ \text{svec}(B_2)^T \\ \vdots \\ \text{svec}(B_{\tilde{n}+1-m})^T \end{bmatrix} \in \mathfrak{R}^{(\tilde{n}+1-m) \times \tilde{n}}, \quad d := \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathfrak{R}^{\tilde{n}+1-m}. \quad (20)$$

Then, (19) holds if and only if

$$[d \quad \mathcal{B}] \begin{bmatrix} \kappa \\ \text{svec}(Y) \end{bmatrix} = 0 \quad (21)$$

since

$$[d \quad \mathcal{B}] \begin{bmatrix} -b^T \\ \mathcal{A}^T \end{bmatrix} = 0.$$

Remark 3.2 Note that for (y, Y) feasible to the dual SDP (2) with $C = 0$, we have the entries on the left-hand side of (21) are equal to zero, except possibly the first entry. This follows from the way d is constructed, and that for such (y, Y) , the left-hand side of (17) (or (19)) has all its entries equal to zero, except possibly the first entry. By choosing B_1 appropriately⁶, we can then apply a result in the literature, namely, Theorem 5.1 in [26], to show superlinear convergence of the interior point algorithm considered in this paper on the homogeneous feasibility model of the LSDFP. This is possible when the initial iterate to the algorithm is from a strictly feasible solution to the dual SDP (2) with $C = 0$.

The above development implies that (9)-(11) can be rewritten as

$$\begin{bmatrix} \mathcal{A} & -b & 0 & 0 \\ 0 & 0 & d & \mathcal{B} \end{bmatrix} \begin{bmatrix} \text{svec}(X) \\ \tau \\ \kappa \\ \text{svec}(Y) \end{bmatrix} = 0. \quad (22)$$

We have (22) together with

$$X \in S_+^n, Y \in S_+^n, \tau \geq 0, \kappa \geq 0, \quad (23)$$

is another way the homogeneous feasibility model of the given LSDFP can be represented.

By switching the position of d and \mathcal{B} in (22), we can rewrite it as

$$\begin{bmatrix} \mathcal{A} & -b & 0 & 0 \\ 0 & 0 & \mathcal{B} & d \end{bmatrix} \begin{bmatrix} \text{svec}(X) \\ \tau \\ \text{svec}(Y) \\ \kappa \end{bmatrix} = 0. \quad (24)$$

It is easy to convince ourselves that (24), (23) is also a representation of the homogeneous feasibility model of the LSDFP, just like (9)-(12), and (22), (23).

We are now ready to express the homogeneous feasibility model of the LSDFP as SDLCP (13)-(15) by letting $n_1 = n + 1, q = 0$, and $\mathcal{A}_1, \mathcal{B}_1$ such that

$$\begin{aligned} (\mathcal{A}_1(X^1))_i &:= \text{Tr} \left(\begin{bmatrix} A_i & 0 \\ 0 & -b_i \end{bmatrix} X^1 \right), \quad i = 1, \dots, m, \\ (\mathcal{A}_1(X^1))_i &:= \text{Tr}(E^{i-m, n+1} X^1), \quad i = m + 1, \dots, m + n, \\ (\mathcal{A}_1(X^1))_i &:= 0, \quad i = m + n + 1, \dots, \tilde{n}_1, \end{aligned} \quad (25)$$

and

$$\begin{aligned} (\mathcal{B}_1(Y^1))_j &:= 0, \quad j = 1, \dots, m + n, \\ (\mathcal{B}_1(Y^1))_j &:= \text{Tr} \left(\begin{bmatrix} B_{j-(m+n)} & 0 \\ 0 & d_{j-(m+n)} \end{bmatrix} Y^1 \right), \quad j = m + n + 1, \dots, \tilde{n}_1, \end{aligned} \quad (26)$$

for $X^1, Y^1 \in S^{n_1}$.

Remark 3.3 Note that the above SDLCP representation of the homogeneous feasibility model has the structure of an LSDFP. Recall that an LSDFP written as the SDLCP (13)-(15) is such that in (13), $(\mathcal{A}_1(X^1))_i = 0$ for $i = m_1 + 1, \dots, \tilde{n}_1$, $(\mathcal{B}_1(Y^1))_j = 0$ for $j = 1, \dots, m_1$, and

⁶Details are provided in Appendix A.

$q_i = 0$ for $i = 1, \dots, m_1$ or $q_i = 0$ for $i = m_1 + 1, \dots, \tilde{n}_1$. Here, $q = (q_1, \dots, q_{\tilde{n}_1})^T$. From (25), (26), we see that for our SDLCP representation, it has this structure with $m_1 = m + n$ and $q = 0$. Having $q = 0$, in particular, $q_i = 0$ for $i = 1, \dots, m_1$, is the property satisfied by our SDLCP representation that does not hold for the LSDFP considered in this paper. In fact, having this property is the key that allows us to apply results in the literature, namely [26], to show the main result (Theorem 4.1) in this paper on superlinear convergence.

The following proposition relates a solution of the homogeneous feasibility model of the LSDFP to that of its SDLCP representation:

Proposition 3.1 *We have if (X, Y, τ, κ) satisfies (24), (23), then*

$$X^1 = \begin{bmatrix} X & 0 \\ 0 & \tau \end{bmatrix}, \quad Y^1 = \begin{bmatrix} Y & * \\ * & \kappa \end{bmatrix}, \quad (27)$$

where the “*” entries in Y^1 are such that $Y^1 \in S_+^{n_1}$, satisfies (13)-(15), with $n_1 = n + 1$, $q = 0$, $\mathcal{A}_1, \mathcal{B}_1$ are given by (25), (26) respectively. On the other hand, if (X^1, Y^1) satisfies (13)-(15), where $n_1 = n + 1$, $q = 0$, $\mathcal{A}_1, \mathcal{B}_1$ are given by (25), (26) respectively, then X^1, Y^1 are given by (27), and (X, Y, τ, κ) satisfies (24), (23).

Proof: The proposition is clear based on how \mathcal{A}_1 and \mathcal{B}_1 are defined in (25) and (26) respectively. ■

We have another proposition below that shows that if the LSDFP satisfies Assumption 2.1, then the SDLCP representation of its homogeneous feasibility model satisfies Assumption 3.1:

Proposition 3.2 *The SDLCP (13)-(15) with $n_1 = n + 1$, $q = 0$ and $\mathcal{A}_1, \mathcal{B}_1$ given by (25), (26) respectively, satisfies Assumption 3.1 when the LSDFP satisfies Assumption 2.1.*

Proof: We first make two observations. Firstly, if (X, Y, τ, κ) satisfies (24), then $\text{Tr}(XY) + \tau\kappa = 0$, by Remark 3.1. Secondly, the matrix on the left-hand side of (24) has full row rank by Assumption 2.1(b), $\{\text{svec}(B_2), \dots, \text{svec}(B_{\tilde{n}+1-m})\}$ linearly independent, and $[\text{svec}(B_1)^T \ d_1]^T$ linearly independent from $\{[\text{svec}(B_i)^T \ 0]^T ; i = 2, \dots, \tilde{n} + 1 - m\}$.

Given the SDLCP (13)-(15) with $n_1 = n + 1$, $q = 0$ and $\mathcal{A}_1, \mathcal{B}_1$ given by (25), (26) respectively. Suppose

$$\mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1) = 0$$

for some $(X^1, Y^1) \in S^{n_1} \times S^{n_1}$. Then X^1 and Y^1 are given by

$$X^1 = \begin{bmatrix} X & 0 \\ 0 & \tau \end{bmatrix}, \quad Y^1 = \begin{bmatrix} Y & * \\ * & \kappa \end{bmatrix},$$

where (X, Y, τ, κ) satisfies (24). Hence, by the first observation above, we have $\text{Tr}(X^1 Y^1) = \text{Tr}(XY) + \tau\kappa = 0$. Therefore, Assumption 3.1(a) holds. Furthermore, Assumption 3.1(b) holds since a solution to the given SDLCP is $X^1 = 0, Y^1 = 0$. Finally, the second observation above means that the matrix

$$\begin{bmatrix} \mathcal{A} & -b & 0 & 0 \\ 0 & 0 & \mathcal{B} & d \end{bmatrix}$$

has full row rank. This implies that the matrix $(\mathcal{A}_1 \ \mathcal{B}_1)$, where \mathcal{A}_1 and \mathcal{B}_1 are defined by (25) and (26) respectively, has full row rank, and hence Assumption 3.1(c) holds as well. ■

4 An Infeasible Interior Point Algorithm on the Homogeneous Feasibility Model

We describe in this section an infeasible path-following interior point algorithm, Algorithm 4.1, on the homogeneous feasibility model (9)-(12) (or (22), (23) or (24), (23)). It generates iterates that follow an infeasible central path in a (narrow) neighborhood. This algorithm is a predictor-corrector type algorithm on the homogeneous feasibility model, and is first considered in [21].

Consider the following (narrow) neighborhood of the central path:

$$\begin{aligned} \mathcal{N}(\beta, \mu) &:= \{(X, y, Y, \tau, \kappa) \in S_{++}^n \times \Re^m \times S_{++}^n \times \Re_{++} \times \Re_{++} ; \\ &(\|Y^{1/2}XY^{1/2} - \mu I\|_F^2 + (\tau\kappa - \mu)^2)^{1/2} \leq \beta\mu, \mu = (\text{Tr}(XY) + \tau\kappa)/(n+1)\}. \end{aligned}$$

In the algorithm, Algorithm 4.1, which is described below, iterates generated by the algorithm always stay within a neighborhood of the central path. We consider the dual Helmborg-Kojima-Monteiro (HKM) search direction in the algorithm, even though our results also hold for the Nesterov-Todd (NT) search direction - see Remark 4.3. Among different search directions used in interior point algorithms on SDLCPs/SDPs, the Alizadeh-Haeberly-Overton (AHO) [1], Helmborg-Kojima-Monteiro (HKM) [9, 12, 15] and Nesterov-Todd (NT) [17, 18] search directions are better known, with the latter two being implemented in SDP solvers, such as SeDuMi [29] and SDPT3 [30]. It is worth noting that the HKM direction has also been implemented in the SDPA package to solve SDPs [31, 32].

Before describing the algorithm, let us consider the following system of equations for $(\Delta X, \Delta y, \Delta Y, \Delta \tau, \Delta \kappa) \in S^n \times \Re^m \times S^n \times \Re \times \Re$ which plays an important role in the algorithm:

$$Y^{1/2}(X\Delta Y + \Delta XY)Y^{-1/2} + Y^{-1/2}(\Delta YX + Y\Delta X)Y^{1/2} = 2(\sigma\mu I - Y^{1/2}XY^{1/2}), \quad (28)$$

$$\kappa\Delta\tau + \tau\Delta\kappa = \sigma\mu - \tau\kappa, \quad (29)$$

$$\text{Tr}(A_i\Delta X) - b_i\Delta\tau = -\bar{r}_i, \quad i = 1, \dots, m, \quad (30)$$

$$\sum_{i=1}^m \Delta y_i A_i + \Delta Y = -\bar{s}, \quad (31)$$

$$\Delta\kappa - b^T \Delta y = -\bar{\gamma}, \quad (32)$$

where $\mu = (\text{Tr}(XY) + \tau\kappa)/(n+1)$. Note that (28) and (29) arise upon linearizing the perturbed equations to $XY = 0$ and $\tau\kappa = 0$ when we use the dual HKM search direction. Furthermore, it can be shown that in Algorithm 4.1, $(\Delta X, \Delta y, \Delta Y, \Delta \tau, \Delta \kappa)$ in (28)-(32) is always uniquely determined - see for example [21].

The infeasible predictor-corrector path-following interior point algorithm on the homogeneous feasibility model (9) - (12) is described as follows:

Algorithm 4.1 (See Algorithm 5.1 of [21]) Given $\beta_1 < \beta_2$ with $\beta_2^2/(2(1-\beta_2)^2) \leq \beta_1 < \beta_2 < \beta_2/(1-\beta_2) < 1$. Choose $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$ with $(n+1)\mu_0 = \text{Tr}(X_0 Y_0) + \tau_0 \kappa_0$. For $k = 0, 1, \dots$, perform (a1) through (a5):

(a1) Set $X = X_k$, $y = y_k$, $Y = Y_k$, $\tau = \tau_k$, $\kappa = \kappa_k$, and define

$$r_i := \text{Tr}(A_i X) - b_i \tau, \quad i = 1, \dots, m,$$

$$s := \sum_{i=1}^m y_i A_i + Y,$$

$$\gamma := \kappa - b^T y.$$

(a2) If $\max\{(\text{Tr}(XY) + \tau\kappa)/\tau^2, |r_1/\tau|, \dots, |r_m/\tau|, \|s/\tau\|\} \leq \epsilon$, then report $(X/\tau, y/\tau, Y/\tau)$ as an approximate solution to the LSDFP, and terminate. If τ is sufficiently small, terminate with no optimal solutions to the LSDFP.

(a3) **[Predictor Step]** Find the solution $(\Delta X_p, \Delta y_p, \Delta Y_p, \Delta \tau_p, \Delta \kappa_p)$ of the linear system (28)-(32), with $\sigma = 0$, $\bar{r}_i = r_i, i = 1, \dots, m$, $\bar{s} = s$ and $\bar{\gamma} = \gamma$.

Define

$$\bar{X} = X + \bar{\alpha}\Delta X_p, \quad \bar{y} = y + \bar{\alpha}\Delta y_p, \quad \bar{Y} = Y + \bar{\alpha}\Delta Y_p, \quad \bar{\tau} = \tau + \bar{\alpha}\Delta \tau_p, \quad \bar{\kappa} = \kappa + \bar{\alpha}\Delta \kappa_p,$$

where the step length $\bar{\alpha}$ satisfies

$$\alpha_1 \leq \bar{\alpha} \leq \alpha_2. \quad (33)$$

Here,

$$\alpha_1 = \frac{2}{\sqrt{1 + 4\delta/(\beta_2 - \beta_1) + 1}}, \quad (34)$$

$$\delta = \frac{1}{\mu} \left\| \begin{bmatrix} Y & 0 \\ 0 & \kappa \end{bmatrix}^{1/2} \begin{bmatrix} \Delta X_p & 0 \\ 0 & \Delta \tau_p \end{bmatrix} \begin{bmatrix} \Delta Y_p & 0 \\ 0 & \Delta \kappa_p \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & \kappa \end{bmatrix}^{-1/2} \right\|_F, \quad (35)$$

where

$$\mu = \frac{\text{Tr}(XY) + \tau\kappa}{n + 1},$$

and

$$\alpha_2 = \max\{\tilde{\alpha} \in [0, 1] ; (X + \alpha\Delta X_p, y + \alpha\Delta y_p, Y + \alpha\Delta Y_p, \tau + \alpha\Delta \tau_p, \kappa + \alpha\Delta \kappa_p) \in \mathcal{N}(\beta_2, (1 - \alpha)\mu) \forall \alpha \in [0, \tilde{\alpha}]\}.$$

(a4) **[Corrector Step]** Find the solution $(\Delta X_c, \Delta y_c, \Delta Y_c, \Delta \tau_c, \Delta \kappa_c)$ of the linear system (28)-(32), with $\sigma = 1 - \bar{\alpha}$, $\bar{r}_i = 0, i = 1, \dots, m$, $\bar{s} = 0$ and $\bar{\gamma} = 0$. Set

$$X_+ = \bar{X} + \Delta X_c, \quad y_+ = \bar{y} + \Delta y_c, \quad Y_+ = \bar{Y} + \Delta Y_c, \quad \tau_+ = \bar{\tau} + \Delta \tau_c, \quad \kappa_+ = \bar{\kappa} + \Delta \kappa_c, \\ \mu_+ = (1 - \bar{\alpha})\mu.$$

(a5) Set

$$X_{k+1} = X_+, \quad y_{k+1} = y_+, \quad Y_{k+1} = Y_+, \quad \tau_{k+1} = \tau_+, \quad \kappa_{k+1} = \kappa_+, \\ \mu_{k+1} = \mu_+.$$

Polynomial iteration complexity for a general version of Algorithm 4.1 used to solve (4)-(7) is shown in [21] where the HKM direction is considered. The result there also holds for the dual HKM direction. Hence, Algorithm 4.1 has polynomial iteration complexity when it is used to solve (9)-(12). Furthermore, we remark that Algorithm 4.1 can easily be adapted to solve the LSDFP instead of its homogeneous feasibility model. An advantage of applying the algorithm on its homogeneous feasibility model is that we have guaranteed superlinear convergence of iterates generated by the algorithm, as shown in Theorem 4.1.

For $k = 0, 1, \dots$, define

$$\begin{aligned} (\tau_k)_i &:= \text{Tr}(A_i X_k) - b_i \tau_k, \quad i = 1, \dots, m, \\ s_k &:= \sum_{i=1}^m (y_k)_i A_i + Y_k, \\ \gamma_k &:= \kappa_k - b^T y_k. \end{aligned}$$

Remark 4.1 For all $k \geq 0$, we have

$$(X_k, y_k, Y_k, \tau_k, \kappa_k) \in \mathcal{N}(\beta_1, \mu_k).$$

Remark 4.1 holds by [21] - see also [23, 26].

Remark 4.2 Throughout this paper, we consider Algorithm 4.1 with initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$ such that $(y_0, Y_0) \in \mathfrak{R}^m \times S_{++}^n$ is feasible to the dual SDP (2) with $C = 0$. Therefore, in the algorithm, we have $s_0 = 0$ while $(r_0)_i, i = 1, \dots, m$, and γ_0 are generally nonzero.

Algorithm 4.1 can be written in an equivalent way as Algorithm 4.2, which we present below. We use Algorithm 4.2 to solve the SDLCP representation of the homogeneous feasibility model, (13)-(15), where $n_1 = n + 1$, $q = 0$ and $\mathcal{A}_1, \mathcal{B}_1$ are given by (25), (26) respectively. Before describing the algorithm, we first define an analogous (narrow) neighborhood of the central path of the SDLCP representation:

$$\mathcal{N}_1(\beta, \mu^1) := \{(X^1, Y^1) \in S_{++}^{n_1} \times S_{++}^{n_1}; \|(Y^1)^{1/2} X^1 (Y^1)^{1/2} - \mu^1 I\|_F \leq \beta \mu^1, \mu^1 = \text{Tr}(X^1 Y^1)/n_1\}.$$

We again have a system of equations for $(\Delta X^1, \Delta Y^1) \in S^{n_1} \times S^{n_1}$ that plays an important role in the algorithm, just like the system of equations (28)-(32) for Algorithm 4.1:

$$\begin{aligned} (Y^1)^{1/2}(X^1 \Delta Y^1 + \Delta X^1 Y^1)(Y^1)^{-1/2} + (Y^1)^{-1/2}(\Delta Y^1 X^1 + Y^1 \Delta X^1)(Y^1)^{1/2} \\ = 2(\sigma \mu^1 I - (Y^1)^{1/2} X^1 (Y^1)^{1/2}), \end{aligned} \quad (36)$$

$$\mathcal{A}_1(\Delta X^1) + \mathcal{B}_1(\Delta Y^1) = -\bar{r}. \quad (37)$$

Note that in Algorithm 4.2, it can be shown that $(\Delta X^1, \Delta Y^1)$ obtained by solving (36), (37) always exists and is unique [23, 26].

Below, we describe the algorithm:

Algorithm 4.2 (See Algorithm 4.1 of [26]; Algorithm 2.1 of [23]) Given $\beta_1 < \beta_2$ with $\beta_2^2/(2(1-\beta_2)^2) \leq \beta_1 < \beta_2 < \beta_2/(1-\beta_2) < 1$. Choose $(X_0^1, Y_0^1) \in S_{++}^{n_1} \times S_{++}^{n_1}$ such that

$$X_0^1 = \begin{bmatrix} X_0 & 0 \\ 0 & \tau_0 \end{bmatrix}, \quad Y_0^1 = \begin{bmatrix} Y_0 & 0 \\ 0 & \kappa_0 \end{bmatrix}, \quad (38)$$

where $X_0, Y_0, \tau_0, \kappa_0$ are from the initial iterate in Algorithm 4.1. For $k = 0, 1, \dots$, perform (a1) through (a5):

(a1) Set $X^1 = X_k^1$, $Y^1 = Y_k^1$, and define

$$r := \mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1).$$

(a2) If $\max\{\text{Tr}(X^1 Y^1)/(X^1)_{n_1, n_1}^2, \|r/X_{n_1, n_1}^1\|\} \leq \epsilon$, then terminate with the solution (X^1, Y^1) .
If X_{n_1, n_1}^1 is sufficiently small, terminate with no optimal solutions to the LSDFP.

(a3) **[Predictor Step]** Find the solution $(\Delta X_p^1, \Delta Y_p^1)$ of the linear system (36), (37), with $\sigma = 0, \bar{r} = r$.

Define

$$\bar{X}^1 = X^1 + \bar{\alpha} \Delta X_p^1, \quad \bar{Y}^1 = Y^1 + \bar{\alpha} \Delta Y_p^1,$$

where the step length $\bar{\alpha}$ satisfies

$$\alpha_1 \leq \bar{\alpha} \leq \alpha_2. \quad (39)$$

Here,

$$\alpha_1 = \frac{2}{\sqrt{1 + 4\delta/(\beta_2 - \beta_1) + 1}}, \quad (40)$$

$$\delta = \frac{1}{\mu^1} \|(Y^1)^{1/2} \Delta X_p^1 \Delta Y_p^1 (Y^1)^{-1/2}\|_F, \quad (41)$$

where

$$\mu^1 = \frac{\text{Tr}(X^1 Y^1)}{n_1},$$

and

$$\alpha_2 = \max\{\tilde{\alpha} \in [0, 1] ; (X^1 + \alpha \Delta X_p^1, Y^1 + \alpha \Delta Y_p^1) \in \mathcal{N}_1(\beta_2, (1 - \alpha)\mu^1) \forall \alpha \in [0, \tilde{\alpha}]\}.$$

(a4) **[Corrector Step]** Find the solution $(\Delta X_c^1, \Delta Y_c^1)$ of the linear system (36), (37), with $\sigma = 1 - \bar{\alpha}$ and $\bar{r} = 0$. Set

$$X_+^1 = \bar{X}^1 + \Delta X_c^1, \quad Y_+^1 = \bar{Y}^1 + \Delta Y_c^1, \\ \mu_+^1 = (1 - \bar{\alpha})\mu^1.$$

(a5) Set

$$X_{k+1}^1 = X_+^1, \quad Y_{k+1}^1 = Y_+^1, \\ \mu_{k+1}^1 = \mu_+^1.$$

The following proposition relates the k^{th} iterate in the two algorithms:

Proposition 4.1 For all $k \geq 0$,

$$X_k^1 = \begin{bmatrix} X_k & 0 \\ 0 & \tau_k \end{bmatrix}, \quad Y_k^1 = \begin{bmatrix} Y_k & 0 \\ 0 & \kappa_k \end{bmatrix}. \quad (42)$$

Consequently, $\mu_k^1 = \mu_k$ and $(X_k^1, Y_k^1) \in \mathcal{N}_1(\beta_1, \mu_k^1)$.

Proof: We show (42) holds by induction on $k \geq 0$. It is clear that (42) holds when $k = 0$, by the choice of X_0^1, Y_0^1 . Suppose (42) holds for $k = k_0 \geq 0$. Then, at the $(k_0 + 1)^{\text{th}}$ iteration of Algorithm 4.2, by comparing the system of equations (28)-(32) and (36), (37), it can be seen easily that

$$\Delta X_p^1 = \begin{bmatrix} \Delta X_p & 0 \\ 0 & \Delta \tau_p \end{bmatrix}, \quad \Delta Y_p^1 = \begin{bmatrix} \Delta Y_p & 0 \\ 0 & \Delta \kappa_p \end{bmatrix}$$

satisfy (36), (37) when $\sigma = 0, \bar{r} = r = \mathcal{A}_1(X^1) + \mathcal{B}_1(Y^1)$. Furthermore, the step length $\bar{\alpha}$ in the $(k_0 + 1)^{\text{th}}$ iteration of both algorithms are the same. These lead to

$$\bar{X}^1 = \begin{bmatrix} \bar{X} & 0 \\ 0 & \bar{\tau} \end{bmatrix}, \quad \bar{Y}^1 = \begin{bmatrix} \bar{Y} & 0 \\ 0 & \bar{\kappa} \end{bmatrix}. \quad (43)$$

With (43), again comparing the system of equations (28)-(32) and (36), (37), it is also easy to see that

$$\Delta X_c^1 = \begin{bmatrix} \Delta X_c & 0 \\ 0 & \Delta \tau_c \end{bmatrix}, \quad \Delta Y_c^1 = \begin{bmatrix} \Delta Y_c & 0 \\ 0 & \Delta \kappa_c \end{bmatrix}$$

satisfy (36), (37) when $\sigma = 1 - \bar{\alpha}$ and $\bar{r} = 0$. Hence, we conclude that (42) holds for $k = k_0 + 1$. Therefore, by induction, (42) holds for all $k \geq 0$. Furthermore, we have

$$\mu_k^1 = \frac{\text{Tr}(X_k^1 Y_k^1)}{n_1} = \frac{\text{Tr}(X_k Y_k) + \tau_k \kappa_k}{n + 1} = \mu_k.$$

Finally, by (42), comparing the definition of the neighborhood $\mathcal{N}(\beta, \mu)$ and the neighborhood $\mathcal{N}_1(\beta, \mu^1)$, and that $\mu_k = \mu_k^1$, we see that since $(X_k, y_k, Y_k, \tau_k, \kappa_k) \in \mathcal{N}(\beta_1, \mu_k)$ (Remark 4.1), we have $(X_k^1, Y_k^1) \in \mathcal{N}_1(\beta_1, \mu_k^1)$. \blacksquare

4.1 Superlinear Convergence

We show in this subsection that Algorithm 4.1 applied to the homogeneous feasibility model when the initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$ is such that $(y_0, Y_0) \in \Re^m \times S_{++}^n$ is feasible to the dual SDP (2) with $C = 0$ leads to superlinear convergence of iterates generated by the algorithm, besides polynomial iteration complexity. First, we state an additional assumption, Assumption 4.1, which is the assumption of strict complementarity, on the primal-dual SDP pair (1)-(2) with $C = 0$ that is needed for this result to hold. Note that strict complementarity assumption is generally considered the minimal requirement for superlinear convergence of interior point algorithms, as investigated for example in [16].

Assumption 4.1 *There exists an optimal solution (X^*, y^*, Y^*) to the primal-dual SDP pair (1)-(2) with $C = 0$ such that $X^* + Y^* \in S_{++}^n$.*

A consequence of the above assumption on the homogeneous feasibility model (9)-(12) is that it has a solution $(X^*, y^*, Y^*, \tau^*, \kappa^*)$ with $X^* + Y^* \in S_{++}^n$ and $\tau^* + \kappa^* > 0$. This then implies that its SDLCP representation has a solution $(X^{1,*}, Y^{1,*})$ such that $X^{1,*} + Y^{1,*} \in S_{++}^{n_1}$, that is, the SDLCP representation has a strictly complementary solution.

We consider local superlinear convergence using Algorithm 4.1 in the sense of

$$\frac{\mu_{k+1}}{\mu_k} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (44)$$

Consideration of superlinear convergence in the form (44) is typical in the interior point literature, such as [10, 14, 23]. It is closely related to local convergence behavior of iterates, as studied for example in [20].

The following result, which is the main result in this paper, ends this subsection:

Theorem 4.1 *Given an initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0) \in \mathcal{N}(\beta_1, \mu_0)$ to Algorithm 4.1, with $(y_0, Y_0) \in \mathbb{R}^m \times S_{++}^n$ feasible to dual SDP (2) with $C = 0$. Iterates generated by Algorithm 4.1 converge superlinearly in the sense of (44).*

Proof: Let us show the result in the theorem by considering iterates generated by Algorithm 4.2 instead. These iterates are related to those generated by Algorithm 4.1 in a close way as shown in Proposition 4.1. We note that Algorithm 4.2 is Algorithm 4.1 in [26], and Assumptions 2.1 and 3.1 in [26] are satisfied for the SDLCP representation of the homogeneous feasibility model (9)-(12) (Proposition 3.2 and strict complementarity). Hence, results in [26] are applicable to our SDLCP representation. The SDLCP representation that Algorithm 4.2 is solving has the structure of an LSDFP (Remark 3.3), and therefore Theorem 5.1⁷ in [26] can be applied on our SDLCP representation provided that Condition (52) in the theorem ((46) in Appendix A) is satisfied.

Our choice of initial iterate $(X_0, y_0, Y_0, \tau_0, \kappa_0)$ to Algorithm 4.1 leads to an initial iterate, (X_0^1, Y_0^1) , to Algorithm 4.2 that satisfies $\mathcal{B}_1(Y_0^1) = 0$ except possibly the $(m+n+1)^{th}$ entry⁸ of $\mathcal{B}_1(Y_0^1)$. Therefore, Condition (52) of Theorem 5.1 in [26] is satisfied, and by the theorem, we have superlinear convergence in the sense that

$$\frac{\mu_{k+1}^1}{\mu_k^1} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

This implies by Proposition 4.1, where we have $\mu_k^1 = \mu_k$, superlinear convergence in the sense of (44) using Algorithm 4.1 to solve the homogeneous feasibility model (9)-(12) for the given initial iterate. ■

Remark 4.3 *Similar result as Theorem 4.1 also holds when the HKM search direction or the NT search direction is used in Algorithm 4.1 instead of the dual HKM search direction. The equivalent algorithm on the SDLCP representation for the NT search direction is Algorithm 1 in [27]. We can then apply Theorem 4 or Corollary 1 in [27] to conclude superlinear convergence of iterates when the initial iterate to Algorithm 4.1 with the NT search direction is from a strictly feasible solution to the dual SDP (2) with $C = 0$. The process to show this is analogous to what we have discussed and we will not repeat it here again.*

5 Conclusion

In this paper, we show superlinear convergence of an implementable polynomial-time infeasible predictor-corrector primal-dual path following interior point algorithm (Algorithm 4.1) on the homogeneous feasibility model of an LSDFP for a suitable choice of initial iterate to the algorithm. This initial iterate can be obtained efficiently and fast by solving a primal-dual SDP pair using a primal-dual path following interior point algorithm.

⁷This theorem is reproduced in our context as Theorem A.1 in Appendix A.

⁸Remark 3.2 and Appendix A.

References

- [1] F. Alizadeh, J. A. Haeberly, and M. Overton. Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results. *SIAM Journal on Optimization*, 8:746–768, 1998.
- [2] B. Alzalg. A primal-dual interior-point method based on various selections of displacement step for symmetric optimization. *Computational Optimization and Applications*, 72:363–390, 2019.
- [3] E. D. Anderson, C. Roos, and T. Terlaky. On implementing a primal-dual interior-point method for conic quadratic optimization. *Mathematical Programming, Series B*, 95:249–277, 2003.
- [4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics. SIAM, Philadelphia, 1994.
- [5] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.
- [6] J. Dahl and E. D. Anderson. A primal-dual interior-point algorithm for nonsymmetric exponential-cone optimization. *Mathematical Programming, Series A*, 194:341–370, 2022.
- [7] E. de Klerk, T. Terlaky, and K. Roos. Self-dual embeddings. In *Handbook of Semidefinite Programming (Eds: H. Wolkowicz, R. Saigal, L. Vandenberghe)*, volume 27 of *International Series in Operations Research & Management Science*, pages 111–138. Springer, Boston, 2000.
- [8] L. Faybusovich and C. Zhou. Long-step path-following algorithm for solving symmetric programming problems with nonlinear objective functions. *Computational Optimization and Applications*, 72:769–795, 2019.
- [9] C. Helmberg, F. Rendl, R. J. Vanderbei, and H. Wolkowicz. An interior-point method for semidefinite programming. *SIAM Journal on Optimization*, 6:342–361, 1996.
- [10] M. Kojima, M. Shida, and S. Shindoh. Local convergence of predictor-corrector infeasible-interior-point algorithms for SDPs and SDLCPs. *Mathematical Programming, Series A*, 80:129–160, 1998.
- [11] M. Kojima, M. Shida, and S. Shindoh. A predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem using the Alizadeh-Haeberly-Overton search direction. *SIAM Journal on Optimization*, 9:444–465, 1999.
- [12] M. Kojima, S. Shindoh, and S. Hara. Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. *SIAM Journal on Optimization*, 7:86–125, 1997.
- [13] Z. Lu and R. D. C. Monteiro. A note on the local convergence of a predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem based on the Alizadeh-Haeberly-Overton search direction. *SIAM Journal on Optimization*, 15:1147–1154, 2005.

- [14] Z.-Q. Luo, J. F. Sturm, and S. Zhang. Superlinear convergence of a symmetric primal-dual path following algorithm for semidefinite programming. *SIAM Journal on Optimization*, 8:59–81, 1998.
- [15] R. D. C. Monteiro. Primal-dual path following algorithms for semidefinite programming. *SIAM Journal on Optimization*, 7:663–678, 1997.
- [16] R. D. C. Monteiro and S. J. Wright. Local convergence of interior-point algorithms for degenerate monotone LCP. *Computational Optimization and Applications*, 3:131–155, 1994.
- [17] Y. Nesterov and M. Todd. Self-scaled barriers and interior-point methods for convex programming. *Mathematics of Operations Research*, 22:1–42, 1997.
- [18] Y. Nesterov and M. Todd. Primal-dual interior-point methods for self-scaled cones. *SIAM Journal on Optimization*, 8:324–364, 1998.
- [19] Y. Nesterov and L. Tunçel. Local superlinear convergence of polynomial-time interior-point methods for hyperbolicity cone optimization problems. *SIAM Journal on Optimization*, 26:139–170, 2016.
- [20] F. A. Potra. Q-superlinear convergence of the iterates in primal-dual interior-point methods. *Mathematical Programming, Series A*, 91:99–115, 2001.
- [21] F. A. Potra and R. Sheng. On homogeneous interior-point algorithms for semidefinite programming. *Optimization Methods and Software*, 9:161–184, 1998.
- [22] F. A. Potra and R. Sheng. Superlinear convergence of interior-point algorithms for semidefinite programming. *Journal of Optimization Theory and Applications*, 99:103–119, 1998.
- [23] F. A. Potra and R. Sheng. A superlinearly convergent primal-dual infeasible-interior-point algorithm for semidefinite programming. *SIAM Journal on Optimization*, 8:1007–1028, 1998.
- [24] S. Pougkakiotis and J. Gondzio. An interior point-proximal method of multipliers for linear positive semi-definite programming. *Journal of Optimization Theory and Applications*, 192:97–129, 2022.
- [25] P. R. Rigó and Z. Darvay. Infeasible interior-point method for symmetric optimization using a positive-asymptotic barrier. *Computational Optimization and Applications*, 71:483–508, 2018.
- [26] C.-K. Sim. Superlinear convergence of an infeasible predictor-corrector path-following interior point algorithm for a semidefinite linear complementarity problem using the Helmberg-Kojima-Monteiro direction. *SIAM Journal on Optimization*, 21:102–126, 2011.
- [27] C.-K. Sim. Interior point method on semi-definite linear complementarity problems using the Nesterov-Todd (NT) search direction: Polynomial complexity and local convergence. *Computational Optimization and Applications*, 74:583–621, 2019.
- [28] C.-K. Sim and G. Zhao. Asymptotic behavior of Helmberg-Kojima-Monteiro (HKM) paths in interior point methods for monotone semidefinite linear complementarity problem: General theory. *Journal of Optimization Theory and Applications*, 137:11–25, 2008.

- [29] J. F. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones (Updated for Version 1.05). 1998-2001.
- [30] K.-C. Toh, M. J. Todd, and R. H. Tütüncü. On the implementation and usage of SDPT3 - a Matlab software package for semidefinite-quadratic-linear programming, Version 4.0. In *Handbook on Semidefinite, Conic and Polynomial Optimization* (Eds: M. F. Anjos, J. B. Lasserre), volume 166 of *International Series in Operations Research & Management Science*, pages 715–754. Springer, 2012.
- [31] M. Yamashita, K. Fujisawa, M. Fukuda, K. Kobayashi, K. Nakta, and M. Nakata. Latest developments in the SDPA family for solving large-scale SDPs. In *Handbook on Semidefinite, Conic and Polynomial Optimization* (Eds: M. F. Anjos, J. B. Lasserre), volume 166 of *International Series in Operations Research & Management Science*, pages 687–713. Springer, 2012.
- [32] M. Yamashita, K. Fujisawa, and M. Kojima. Implementation and evaluation of SDPA 6.0 (SemiDefinite Programming Algorithm 6.0). *Optimization Methods and Software*, 18:491–505, 2003.
- [33] Y. Zhang. On extending some primal-dual interior-point algorithms from linear programming to semidefinite programming. *SIAM Journal on Optimization*, 8:365–386, 1998.

A Appendix

For the sake of completeness, let us write down Theorem 5.1 in [26] in our context:

Theorem A.1 (see Theorem 5.1 in [26]) *Assume that there exists a solution $(X^{1,*}, Y^{1,*}) \in S_+^{n_1} \times S_+^{n_1}$ to an LSDFP (expressed as SDLCP (13)-(15)) such that $X^{1,*} + Y^{1,*} \in S_{+++}^{n_1}$. Let (X_k^1, Y_k^1) be iterates generated by Algorithm 4.2 on the LSDFP with the initial iterate (X_0^1, Y_0^1) satisfying*

$$\left(\begin{array}{c} \text{svec} \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & ((\mathcal{A}_1)_{j_1+j_2+1})_{22} & 0 \\ 0 & 0 & 0 \end{array} \right)^T \\ \vdots \\ \text{svec} \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & ((\mathcal{A}_1)_{m_1})_{22} & 0 \\ 0 & 0 & 0 \end{array} \right)^T \end{array} \right) \text{svec}(X_0^1) = 0 \quad (45)$$

or

$$\left(\begin{array}{c} \text{svec} \left(\begin{array}{ccc} ((\mathcal{B}_1)_{k_1+k_2+1})_{11} & 0 & ((\mathcal{B}_1)_{k_1+k_2+1})_{13} \\ 0 & 0 & 0 \\ ((\mathcal{B}_1)_{k_1+k_2+1})_{13}^T & 0 & ((\mathcal{B}_1)_{k_1+k_2+1})_{33} \end{array} \right)^T \\ \vdots \\ \text{svec} \left(\begin{array}{ccc} ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{11} & 0 & ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{13} \\ 0 & 0 & 0 \\ ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{13}^T & 0 & ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{33} \end{array} \right)^T \end{array} \right) \text{svec}(Y_0^1) = 0, \quad (46)$$

where $q_i = 0$ for $i = m_1 + 1, \dots, \tilde{n}_1$ in (13) when (45) holds, and $q_i = 0$ for $i = 1, \dots, m_1$ in (13) when (46) holds. Here, $\tilde{n}_1 = n_1(n_1 + 1)/2$. Then the iterates converge superlinearly.

To put things in perspective, under the assumption that there exists a solution $(X^{1,*}, Y^{1,*}) \in S_+^{m_1} \times S_+^{m_1}$ to the LSDFP (expressed as SDLCP (13)-(15)) such that $X^{1,*} + Y^{1,*} \in S_{++}^{n_1}$, we can assume that $X^{1,*}$ and $Y^{1,*}$ are given by

$$X^{1,*} = \begin{pmatrix} \Lambda_{11}^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_{k_0}^* \end{pmatrix}, \quad Y^{1,*} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{22}^* & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\Lambda_{11}^* = \text{Diag}(\lambda_1^*, \dots, \lambda_{k_0-1}^*) \in S_{++}^{k_0-1}$ and $\Lambda_{22}^* = \text{Diag}(\lambda_{k_0+1}^*, \dots, \lambda_{n_1}^*) \in S_{++}^{n_1-k_0}$. Here, $\lambda_1^*, \dots, \lambda_{n_1}^*$ are real numbers greater than zero. In this appendix, we always partition a matrix

$S \in S^{n_1}$ in the way $X^{1,*}$ and $Y^{1,*}$ are partitioned, i.e., S is partitioned as $\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & S_{33} \end{pmatrix}$,

where $S_{11} \in S^{k_0-1}$, $S_{22} \in S^{n_1-k_0}$, $S_{33} \in \Re$ and $S_{12} \in \Re^{(k_0-1) \times (n_1-k_0)}$, $S_{13} \in \Re^{k_0-1}$, $S_{23} \in \Re^{n_1-k_0}$. Using this partition, we perform block Gaussian eliminations and row operations on $\mathcal{A}_1, \mathcal{B}_1 : S^{n_1} \rightarrow \Re^{\tilde{n}_1}$ in (13) so that they can be written in matrix form as:

$$\mathcal{A}_1 = \begin{pmatrix} \bar{\mathcal{A}}_1 \\ 0 \end{pmatrix}, \quad \mathcal{B}_1 = \begin{pmatrix} 0 \\ \bar{\mathcal{B}}_1 \end{pmatrix},$$

where

$$\bar{\mathcal{A}}_1 = \begin{pmatrix} \text{svec} \begin{pmatrix} ((\mathcal{A}_1)_1)_{11} & ((\mathcal{A}_1)_1)_{12} & ((\mathcal{A}_1)_1)_{13} \\ ((\mathcal{A}_1)_1)_{12}^T & ((\mathcal{A}_1)_1)_{22} & ((\mathcal{A}_1)_1)_{23} \\ ((\mathcal{A}_1)_1)_{13}^T & ((\mathcal{A}_1)_1)_{23}^T & ((\mathcal{A}_1)_1)_{33} \end{pmatrix} \\ \vdots \\ \text{svec} \begin{pmatrix} ((\mathcal{A}_1)_{j_1})_{11} & ((\mathcal{A}_1)_{j_1})_{12} & ((\mathcal{A}_1)_{j_1})_{13} \\ ((\mathcal{A}_1)_{j_1})_{12}^T & ((\mathcal{A}_1)_{j_1})_{22} & ((\mathcal{A}_1)_{j_1})_{23} \\ ((\mathcal{A}_1)_{j_1})_{13}^T & ((\mathcal{A}_1)_{j_1})_{23}^T & ((\mathcal{A}_1)_{j_1})_{33} \end{pmatrix} \\ \text{svec} \begin{pmatrix} 0 & ((\mathcal{A}_1)_{j_1+1})_{12} & 0 \\ ((\mathcal{A}_1)_{j_1+1})_{12}^T & ((\mathcal{A}_1)_{j_1+1})_{22} & ((\mathcal{A}_1)_{j_1+1})_{23} \\ 0 & ((\mathcal{A}_1)_{j_1+1})_{23}^T & 0 \end{pmatrix} \\ \vdots \\ \text{svec} \begin{pmatrix} 0 & ((\mathcal{A}_1)_{j_1+j_2})_{12} & 0 \\ ((\mathcal{A}_1)_{j_1+j_2})_{12}^T & ((\mathcal{A}_1)_{j_1+j_2})_{22} & ((\mathcal{A}_1)_{j_1+j_2})_{23} \\ 0 & ((\mathcal{A}_1)_{j_1+j_2})_{23}^T & 0 \end{pmatrix} \\ \text{svec} \begin{pmatrix} 0 & 0 & 0 \\ 0 & ((\mathcal{A}_1)_{j_1+j_2+1})_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \vdots \\ \text{svec} \begin{pmatrix} 0 & 0 & 0 \\ 0 & ((\mathcal{A}_1)_{m_1})_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

and

$$\bar{\mathcal{B}}_1 = \begin{pmatrix} \text{svec} \begin{pmatrix} ((\mathcal{B}_1)_1)_{11} & ((\mathcal{B}_1)_1)_{12} & ((\mathcal{B}_1)_1)_{13} \\ ((\mathcal{B}_1)_1)_{12}^T & ((\mathcal{B}_1)_1)_{22} & ((\mathcal{B}_1)_1)_{23} \\ ((\mathcal{B}_1)_1)_{13}^T & ((\mathcal{B}_1)_1)_{23}^T & ((\mathcal{B}_1)_1)_{33} \end{pmatrix} \\ \vdots \\ \text{svec} \begin{pmatrix} ((\mathcal{B}_1)_{k_1})_{11} & ((\mathcal{B}_1)_{k_1})_{12} & ((\mathcal{B}_1)_{k_1})_{13} \\ ((\mathcal{B}_1)_{k_1})_{12}^T & ((\mathcal{B}_1)_{k_1})_{22} & ((\mathcal{B}_1)_{k_1})_{23} \\ ((\mathcal{B}_1)_{k_1})_{13}^T & ((\mathcal{B}_1)_{k_1})_{23}^T & ((\mathcal{B}_1)_{k_1})_{33} \end{pmatrix} \\ \text{svec} \begin{pmatrix} ((\mathcal{B}_1)_{k_1+1})_{11} & ((\mathcal{B}_1)_{k_1+1})_{12} & ((\mathcal{B}_1)_{k_1+1})_{13} \\ ((\mathcal{B}_1)_{k_1+1})_{12}^T & 0 & ((\mathcal{B}_1)_{k_1+1})_{23} \\ ((\mathcal{B}_1)_{k_1+1})_{13}^T & ((\mathcal{B}_1)_{k_1+1})_{23}^T & ((\mathcal{B}_1)_{k_1+1})_{33} \end{pmatrix} \\ \vdots \\ \text{svec} \begin{pmatrix} ((\mathcal{B}_1)_{k_1+k_2})_{11} & ((\mathcal{B}_1)_{k_1+k_2})_{12} & ((\mathcal{B}_1)_{k_1+k_2})_{13} \\ ((\mathcal{B}_1)_{k_1+k_2})_{12}^T & 0 & ((\mathcal{B}_1)_{k_1+k_2})_{23} \\ ((\mathcal{B}_1)_{k_1+k_2})_{13}^T & ((\mathcal{B}_1)_{k_1+k_2})_{23}^T & ((\mathcal{B}_1)_{k_1+k_2})_{33} \end{pmatrix} \\ \text{svec} \begin{pmatrix} ((\mathcal{B}_1)_{k_1+k_2+1})_{11} & 0 & ((\mathcal{B}_1)_{k_1+k_2+1})_{13} \\ 0 & 0 & 0 \\ ((\mathcal{B}_1)_{k_1+k_2+1})_{13}^T & 0 & ((\mathcal{B}_1)_{k_1+k_2+1})_{33} \end{pmatrix} \\ \vdots \\ \text{svec} \begin{pmatrix} ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{11} & 0 & ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{13} \\ 0 & 0 & 0 \\ ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{13}^T & 0 & ((\mathcal{B}_1)_{\tilde{n}_1-m_1})_{33} \end{pmatrix} \end{pmatrix}.$$

Details on the above reformulation of \mathcal{A}_1 and \mathcal{B}_1 can be found in [26, 27, 28].

In the context of our SDLCP representation of the homogeneous feasibility model of the given LSDFP, we choose B_1 in (20) such that $(B_1)_{22} \neq 0$. In this way, when we perform block Gaussian eliminations and row operations on \mathcal{B}_1 , its final position with respect to other rows in $\bar{\mathcal{B}}_1$ remains unchanged, that is, $\text{svec} \begin{pmatrix} B_1 & 0 \\ 0 & d_1 \end{pmatrix}^T$ remains in the first row of $\bar{\mathcal{B}}_1$ or the $(m_1 + 1)^{th}$ row of \mathcal{B}_1 . We see that (46) then holds when Y_0^1 is from a strictly feasible solution to the dual SDP (2) with $C = 0$.