CONVEX ENVELOPES OF BOUNDED MONOMIALS ON TWO-VARIABLE CONES

PIETRO BELOTTI

ABSTRACT. We consider an n-variate monomial function that is restricted both in value by lower and upper bounds and in domain by two homogeneous linear inequalities. Such functions are building blocks of several problems found in practical applications, and that fall under the class of Mixed Integer Nonlinear Optimization. We show that the upper envelope of the function in the given domain, for $n \geq 2$ is given by a conic inequality. We also present the lower envelope for $n = 2$. To assess the applicability of branching rules based on homogeneous linear inequalities, we also derive the volume of the convex hull for $n = 2$.

1. INTRODUCTION

Consider the function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ defined as $f(\boldsymbol{x}) = \prod_{i \in N} x_i^{a_i}$ where $N = \{1, 2, \ldots, n\}$, $n > 1$, $a_i > 0 \forall i \in N$. We are interested in the convex envelope of f on subsets of $\mathbb{R}^n_+$ where the value of f itself is bounded in the range $[\ell, u]$, with $0 < \ell < u < +\infty$. Specifically, for a given $\mathcal{D} \subseteq \mathbb{R}^n_+$ we seek the convex hull of the set $F(\mathcal{D}) = \{(\boldsymbol{x}, z) \in (X \cap \mathcal{D}) \times \mathbb{R} : z = f(\boldsymbol{x})\}$, where

$$X = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \ell \leq f(\boldsymbol{x}) \leq u\}.$$

Finding or approximating the convex hull of $F(\mathcal{D})$ is useful in solving optimization problems whose objective function or constraints contain polynomials with arbitrary real exponents. In particular, those monomials with variables restricted to the first orthant are of interest in the optimization with *posynomial* functions, or, more in general, in geometric programming [10, 11]. Some algorithms for posynomial optimization and for providing relaxations and lower bounds for monomials on positive variables have been proposed [15, 29].

Monomials also appear in optimization problems with general nonlinear constraints and discrete variables. Mixed-Integer Nonlinear Optimization (MINLO) solvers such as Antigone [18], Baron [22], Couenne [5], SCIP [9], and FICO-Xpress [1] employ a *reformulation* scheme where expressions in factorable programs are decomposed to smaller expressions that can be targeted by an operator-specific convexification algorithm [23, 28]. This allows for exploiting branch-and-bound algorithms to compute the global optimum of a MINLO problem [12]. For instance, the generic polynomial constraint

$$\sum_{j \in J} c_j \prod_{i \in I_j} x_i^{a_{ji}} \leq c_0, \tag{1}$$

Date: 1 March 2023.

Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano
Orcid: 0000-0001-6591-6886
Email: pietro.belotti@polimi.it.

where $c_j \in \mathbb{R}$, $a_{ji} \in \mathbb{R}$, and $c_0 \in \mathbb{R}$, is in general nonconvex, and is decomposed by introducing *auxiliary* variables y_j and t_{ji} as follows:

$$\sum_{j \in J} c_j y_j \leq c_0 \tag{2}$$

$$y_j = \prod_{i \in I_j} t_{ji} \quad \forall j \in J \tag{3}$$

$$t_{ji} = x_i^{a_{ji}} \quad \forall i \in I_j, \forall j \in J. \tag{4}$$

Exact solvers like those mentioned above find or approximate the convex hull of the sets defined by the constraints associated with each of the auxiliary variables. A convex relaxation of the feasible set for (2)-(4), which yields a valid lower bound for the MINLO problem, is

$$R = R' \cap \left(\bigcap_{j \in J} R''_j\right) \cap \left(\bigcap_{j \in J, i \in I_j} R'''_{ji}\right),$$

where

$$\begin{array}{rcl} R' & = & \{(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{t}) : \sum_{j \in J} c_j y_j \leq b\}, \\ R''_j & \supseteq & \operatorname{conv}\left(\left\{(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{t}) : y_j = \prod_{i \in I_j} t_{ji}\right\}\right), \\ R'''_{ji} & \supseteq & \operatorname{conv}\left(\left\{(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{t}) : t_{ji} = x_i^{a_{ji}}\right\}\right). \end{array}$$

Clearly R' is defined by a linear constraint and is added to the reformulation as is. The reformulation results in a lower bound on the optimal objective value that is tighter when R''_j and R'''_{ji} are tight approximations of the corresponding convex hulls. Note that even in the case R'' and R''' are equal to, rather than supersets of, the corresponding convex hulls, R is still in general not the convex hull of the set described by (2)-(4). The tightness of such relaxations strongly depends on tight bounds on both the original variables x_i and the auxiliaries y_j and t_{ji}. Several *bound reduction* techniques such as feasibility-based (see e.g. Neumeier [19]) and optimality-based [28] help find tight bounds on all auxiliary variables. Before generating an LP relaxation for (1), most solvers apply bound reduction to find a tight bound interval $[\ell_j, u_j]$ on y_j and $[\ell_{ji}, u_{ji}]$ on t_{ji}. This motivates the search for convex envelopes of f over $X \cap \mathcal{D}$.

The equality signs in (3) and (4) can be relaxed depending on the sign of c_j, which may render (4) convex depending on a_{ji}: if $a_{ji} \geq 1$, then the constraint $t_{ji} \geq x_i^{a_{ji}}$ is convex, and viceversa if $a_{ji} \leq 1$ then $t_{ji} \leq x_i^{a_{ji}}$ is convex.

However, (3) is nonconvex regardless of the sign, although it admits a polyhedral convex hull [21]. McCormick [16] provides a valid relaxation for the product of two variables,

$$B = \{(x_1, x_2, z) \in \mathbb{R}^3_+ : z = x_1 x_2, (x_1, x_2) \in [\ell_1, u_1] \times [\ell_2, u_2]\}, \tag{5}$$

which consists of the four so-called *McCormick inequalities*:

$$z \geq \ell_2 x_1 + \ell_1 x_2 - \ell_2 \ell_1 \tag{6}$$

$$z \geq u_2 x_1 + u_1 x_2 - u_2 u_1 \tag{7}$$

$$z \leq \ell_2 x_1 + u_1 x_2 - \ell_2 u_1 \tag{8}$$

$$z \leq u_2 x_1 + \ell_1 x_2 - u_2 \ell_1. \tag{9}$$

These inequalities, in fact, form the convex hull of B [3]. While these results hold without sign constraints on x_1 and x_2, we use the above definition of B as we focus on sets entirely contained in $\mathbb{R}^n_+$. Meyer and Floudas [17] present inequalities of the convex hull in the trilinear case ($n = 3$).

1.1. **The case for bounded monomials.** Bound reduction that uses the bounds on the initial variables $\boldsymbol{x}$ and the linear constraint (2) can obtain, as an implication, lower and upper bound on each element of the sum in (1). In the bilinear case, i.e., $n = 2$, $a_1 = a_2 = 1$, the convex hull of

$$B' = \{(x_1, x_2, z) \in B : \ell \leq z \leq u\}$$

is tighter than (6)-(9) if the bounds on z are tighter than those on x_1 and x_2, i.e., $\ell > \ell_1\ell_2$ or $u < u_1u_2$.

Belotti et al. [7] introduce inequalities that are not implied by the McCormick inequalities if at least one of the bounds on z is tighter, and that result in a tighter relaxation when solving quadratically constrained MINLO problems. Anstreicher et al. [4] show that the convex hull of B' for "tight" values of ℓ or u is the union of distinct regions, each partially delimited by a different second-order cone, i.e., a set of the form $\boldsymbol{b}^\top \boldsymbol{x} + b_0 \geq \|G\boldsymbol{x} + \boldsymbol{g}\|_2$. Nguyen et al. [20] obtain envelopes of monomials $x_1^{a_1} x_2^{a_2}$ for the cases $a_1 = 1 \leq a_2$ (convex hull and upper envelope) and $a_1, a_2 \geq 1$ (lower envelope).

This article mainly focuses on the convex hull of $F(\mathcal{D})$ where

$$\mathcal{D} = W_{ij} := \{\boldsymbol{x} \in \mathbb{R}^n_+ : px_i \leq x_j \leq qx_i\}, \tag{10}$$

with $0 < p < q$, for two indices $i, j \in N$. Therefore we depart from the most usual setting where lower and upper bounds on $\boldsymbol{x}$ are given, and instead constrain the $\boldsymbol{x}$ variables with two homogeneous (i.e. zero right-hand side) linear inequalities, which restrict the $\boldsymbol{x}$ space to a *wedge*.

Using linear inequalities instead of a bounding box as the feasible set for $\boldsymbol{x}$ deviates from the common setting exploited by MINLO solvers. An advantage of this approach is that we may obtain a tighter relaxation by avoiding the decomposition of the factorable term $\prod_{i \in N} x_i^{a_i}$ into $y = \prod_{i \in N} t_i$ and $t_i = x_i^{a_i}$.

While we assume $a_i > 0 \forall i \in N$, note that we can reduce monomials with one or more terms $x_i^{a_i}$ with $a_i < 0$ to this case by another reformulation step: introduce an auxiliary variable y_i such that $y_i = \frac{1}{x_i}$ and replace $x_i^{a_i}$ with $y_i^{-a_i}$. This requires $x_i > 0 \forall \boldsymbol{x} \in X \cap \mathcal{D}$, which is implied by $\ell > 0$, and it introduces an extra gap due to the additional reformulation step $\{(x_i, y_i) \in \mathbb{R}^2_+ : x_i y_i = 1\}$, whose convex hull is $\{(x_i, y_i) \in \mathbb{R}^2_+ : x_i y_i \geq 1\}$ if $x_i \in [0, +\infty]$, otherwise it is $\{(x_i, y_i) \in \mathbb{R}^2_+ : x_i y_i \geq 1, x_i + u\ell y_i \leq \ell + u\}$.

Definition 1. *Given $T \subseteq \mathbb{R}^n$ and a function $f : T \to \mathbb{R}$, the* epigraph *of f in T is epi$(f, T) = \{(\boldsymbol{x}, z) \in T \times \mathbb{R} : z \geq f(\boldsymbol{x})\}$. The* hypograph *of f in T is hyp$(f, T) = \{(\boldsymbol{x}, z) \in T \times \mathbb{R} : z \leq f(\boldsymbol{x})\}$.*

Definition 2. *Given $T \subseteq \mathbb{R}^n$ and a function $f : T \to \mathbb{R}$, the* lower envelope *$E_L(f, T)$ (resp.* upper envelope *$E_U(f, T)$) of f over T is the convex hull of the epigraph (resp. hypograph) of f in T.*

In Section 2 we provide some trivial results on the convex hull of $F(\mathbb{R}^n_+)$. In Section 3 we present the upper envelope of f over $X \cap W_{ij}$ for any $n \geq 2$. In Section 4 we present the lower envelope of f over $X \cap W_{ij}$ for $n = 2$, yielding the convex hull of $F(W_{12})$. In Section 5 we compute the volume of the convex hull of $F(W_{12})$ for $n = 2$; concluding remarks are in Section 6.

2. Convex hull of $F(\mathbb{R}^n_+)$

Define $\beta = \sum_{i\in N} a_i$, then for $z_0, \gamma \in \mathbb{R}$ consider the cone

$$\mathcal{K} = \left\{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : (z - z_0)^\beta \leq \gamma \prod_{i\in N} x_i^{a_i}\right\}.$$

The vertex of $\mathcal{K}$ is $(\mathbf{0}, z_0)$. Also define $F(\mathcal{D})^{\leq} := \{(\boldsymbol{x}, z) \in (X \cap \mathcal{D}) \times \mathbb{R} : z \leq f(\boldsymbol{x})\}$; note that it is obviously not the hypograph of f but rather a relaxation of the link between z and f. If $\beta = 1$ then $F(\mathcal{D})^{\leq}$ is a convex cone intersected with $S := \{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : \ell \leq z \leq u\} = \mathbb{R}^n_+ \times [\ell, u]$. Similar to the bilinear case, where $n = 2$ and $a_1 = a_2 = 1$, we look for conditions under which $\mathcal{K}$ defines a tight relaxation of $F(\mathcal{D})$. Let us find z_0 and γ such that

$$\begin{array}{l} \{(\boldsymbol{x}, z) \in F(\mathcal{D})^{\leq} : z = \ell\} = \{(\boldsymbol{x}, z) \in \mathcal{K} : z = \ell\}; \\ \{(\boldsymbol{x}, z) \in F(\mathcal{D})^{\leq} : z = u\} = \{(\boldsymbol{x}, z) \in \mathcal{K} : z = u\}. \end{array}$$

This is equivalent to finding z_0, γ such that

$$\begin{array}{ll} \ell \leq \prod_{i\in N} x_i^{a_i} \Leftrightarrow \frac{1}{\gamma}(\ell - z_0)^\beta \leq \prod_{i\in N} x_i^{a_i} & \forall \boldsymbol{x} \in \mathbb{R}^n_+; \\ u \leq \prod_{i\in N} x_i^{a_i} \Leftrightarrow \frac{1}{\gamma}(u - z_0)^\beta \leq \prod_{i\in N} x_i^{a_i} & \forall \boldsymbol{x} \in \mathbb{R}^n_+ \end{array} \tag{11}$$

and hence $\ell = \frac{1}{\gamma}(\ell - z_0)^\beta, u = \frac{1}{\gamma}(u - z_0)^\beta$, which implies $z_0 \leq \ell$. Because $u > 0$ and $u - z_0 > 0$, we divide the first equation by the second one and obtain $\left(\frac{\ell}{u}\right)^{1/\beta} = \frac{(\ell - z_0)}{(u - z_0)}$. This yields

$$z_0 = \frac{u^{\frac{1}{\beta}}\ell - \ell^{\frac{1}{\beta}}u}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}}, \quad \gamma = \left(\frac{u - \ell}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}}\right)^\beta. \tag{12}$$

Note that $\beta \geq 1 \Leftrightarrow (z_0 \leq 0, \gamma \geq 1)$. Also, if $\beta = 1$, $z_0 = 0$, $\gamma = 1$, and it is easy to verify that $\mathcal{K} \cap S \equiv F(\mathbb{R}^n_+)^{\leq}$, while $\ell = 0$ implies $z_0 = 0$, i.e., the vertex of $\mathcal{K}$ is the origin.

Lemma 1. *For $a_i > 0 \forall i \in N$, $\ell \geq 0$, $T = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \prod_{i\in N} x_i^{a_i} \geq \ell\}$ is convex.*

Proof. If $\ell = 0$, $T = \mathbb{R}^n_+$. Assume now $\ell > 0$; then $x_i > 0, i \in N$, for all $\boldsymbol{x} \in T$. Consider $\boldsymbol{x}', \boldsymbol{x}'' \in T$. For any $\mu \in [0, 1]$, we prove $\prod_{i\in N}(\mu x_i' + (1 - \mu)x_i'')^{a_i} \geq \ell$, i.e.,

$$\sum_{i\in N} a_i \log(\mu x_i' + (1 - \mu)x_i'') \geq \log \ell.$$

Concavity of the log function and positivity of all a_i's imply

$$\begin{array}{rl} & \sum_{i\in N} a_i \log(\mu x_i' + (1 - \mu)x_i'') \geq \\ \geq & \sum_{i\in N} a_i(\mu \log x_i' + (1 - \mu)\log x_i'') = \\ = & \mu \sum_{i\in N} a_i \log x_i' + (1 - \mu)\sum_{i\in N} a_i \log x_i'' \geq \\ \geq & \mu \log \ell + (1 - \mu)\log \ell = \log \ell, \end{array}$$

which proves convexity of T. □

Lemma 2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a convex (resp. concave) monotone non-decreasing function. Then for any ℓ, z, u such that $\ell < z < u$,*

$$\frac{g(z) - g(\ell)}{z - \ell} \leq \frac{g(u) - g(z)}{u - z} \qquad \left(\textit{resp. } \frac{g(z) - g(\ell)}{z - \ell} \geq \frac{g(u) - g(z)}{u - z}\right).$$

Proof. Consider $\mu \in (0,1)$ such that $z = (1-\mu)\ell + \mu u$. Then $0 < \mu < 1$ and for g concave we obtain $g(z) \geq (1-\mu)g(\ell) + \mu g(u)$, or

$$\begin{array}{rcl} (1-\mu)g(z) + \mu g(z) & \geq & (1-\mu)g(\ell) + \mu g(u) \\ (1-\mu)z + \mu z & = & (1-\mu)\ell + \mu u \\[1ex] (1-\mu)(g(z) - g(\ell)) & \geq & \mu(g(u) - g(z)) \\ (1-\mu)(z-\ell) & = & \mu(u-z). \end{array}$$

Dividing the third inequality by the last equation yields $\frac{g(z)-g(\ell)}{z-\ell} \geq \frac{g(u)-g(z)}{u-z}$. Similar considerations hold, *mutatis mutandis*, for g convex. □

Lemma 3. *$F(\mathbb{R}^n_+) \subseteq \mathcal{K}$ if and only if $\beta \geq 1$.*

Proof. $F(\mathbb{R}^n_+) \subseteq \mathcal{K}$ if and only if $(\ell \leq z \leq u) \wedge (z = \prod_{i\in N} x_i^{a_i}) \Rightarrow (z - z_0)^\beta \leq \gamma \prod_{i \in N} x_i^{a_i}$. Rewrite the left-hand side as $(z-z_0)^\beta \leq \gamma z$. Replacing z_0 and γ we get

$$\left(z - \frac{u^{1/\beta}\ell - \ell^{1/\beta}u}{u^{1/\beta} - \ell^{1/\beta}}\right)^\beta \leq \left(\frac{u-\ell}{u^{1/\beta} - \ell^{1/\beta}}\right)^\beta z,$$

from which we eliminate the (positive) common denominator:

$$\begin{array}{rcl} & \left(zu^{\frac{1}{\beta}} - z\ell^{\frac{1}{\beta}} - u^{\frac{1}{\beta}}\ell + \ell^{\frac{1}{\beta}}u\right)^\beta & = \\ = & \left(u^{\frac{1}{\beta}}(z-\ell) + \ell^{\frac{1}{\beta}}(u-z)\right)^\beta & \leq \quad (u-\ell)^\beta z. \end{array}$$

Because $\beta > 0$ and all terms in both left- and right-hand side are nonnegative, the above is equivalent to

$$u^{\frac{1}{\beta}}(z-\ell) + \ell^{\frac{1}{\beta}}(u-z) \leq (u-\ell)z^{\frac{1}{\beta}}. \tag{13}$$

Rewrite the right-hand side as $(u-z)z^{\frac{1}{\beta}} + (z-\ell)z^{\frac{1}{\beta}}$ and (13) as $(u-z)(z^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}) \geq (z-\ell)(u^{\frac{1}{\beta}} - z^{\frac{1}{\beta}})$, or

$$\frac{z^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}}{z-\ell} \geq \frac{u^{\frac{1}{\beta}} - z^{\frac{1}{\beta}}}{u-z},$$

which, for Lemma 2, is true if and only if $\beta \geq 1$ for concavity of $g(x) = x^{\frac{1}{\beta}}$. □

The structure of both upper envelope and lower envelope of f over $X \cap \mathcal{D}$ discussed in this and the following section changes radically at $\beta = 1$. For instance, $F(\mathbb{R}^n_+)^{\leq}$ is convex for $\beta \leq 1$ and nonconvex for $\beta > 1$. Therefore we split the treatment within each section depending on the ranges of β.

Proposition 1. *If $\beta \geq 1$, then conv$(F(\mathbb{R}^n_+)) = \mathcal{K} \cap S$.*

Proof. $F(\mathbb{R}^n_+) \subseteq \mathcal{K}$ from Lemma 3; this implies that $\mathrm{conv}(F(\mathbb{R}^n_+)) \subseteq \mathcal{K}$ as $\mathcal{K}$ is convex. In addition, $F(\mathbb{R}^n_+) \subseteq S$ and convexity of S imply $\mathrm{conv}(F(\mathbb{R}^n_+)) \subseteq \mathcal{K} \cap S$. We just need to prove $\mathrm{conv}(F(\mathbb{R}^n_+)) \supseteq \mathcal{K} \cap S$.

Consider $(\tilde{\boldsymbol{x}}, \tilde{z}) \in \mathcal{K} \cap S$, i.e., $(\tilde{z} - z_0)^\beta \leq \gamma \prod_{i\in N} \tilde{x}_i^{a_i}$, $\ell \leq \tilde{z} \leq u$. Let us construct $\boldsymbol{x}'$ and $\boldsymbol{x}''$ such that $(\tilde{\boldsymbol{x}}, \tilde{z} - z_0)$ is a convex combination of $(\boldsymbol{x}', \ell - z_0)$ and $(\boldsymbol{x}'', u - z_0)$. Let $(\boldsymbol{x}', \ell - z_0) = s'(\tilde{\boldsymbol{x}}, \tilde{z} - z_0)$ and $(\boldsymbol{x}'', u - z_0) = s''(\tilde{\boldsymbol{x}}, \tilde{z} - z_0)$, i.e., $(\boldsymbol{x}', \ell)$ and $(\boldsymbol{x}'', u)$ lie on the same ray of $\mathcal{K}$ as $(\tilde{\boldsymbol{x}}, \tilde{z})$. By construction of s' and s'',

$$s' = \frac{\ell - z_0}{\tilde{z} - z_0} \leq 1; \qquad s'' = \frac{u - z_0}{\tilde{z} - z_0} \geq 1.$$

In addition, $(\boldsymbol{x}', \ell) \in \mathcal{K}$ and $(\boldsymbol{x}'', u) \in \mathcal{K}$ because $(\tilde{\boldsymbol{x}}, \tilde{z}) \in \mathcal{K}$. Specifically, they belong to $\{(\boldsymbol{x}, z) \in \mathcal{K} : z = \ell\}$ and $\{(\boldsymbol{x}, z) \in \mathcal{K} : z = u\}$, respectively, but by (11) this implies $(\boldsymbol{x}', \ell) \in F_\ell^{\leq} := \{(\boldsymbol{x}, z) \in F(\mathbb{R}_+^n)^{\leq} : z = \ell\}$ and $(\boldsymbol{x}'', u) \in F_u^{\leq} := \{(\boldsymbol{x}, z) \in F(\mathbb{R}_+^n)^{\leq} : z = u\}$. Because both $F_\ell^{\leq}$ and $F_u^{\leq}$ are convex by Lemma 1, there exist two extreme points $\boldsymbol{x}'_a$ and $\boldsymbol{x}'_b$ of $F(\mathbb{R}_+^n)$ of which $\boldsymbol{x}'$ is a convex combination, such that $f(\boldsymbol{x}'_a) = f(\boldsymbol{x}'_b) = \ell$, and similar for $\boldsymbol{x}''$, thus implying $(\tilde{\boldsymbol{x}}, \tilde{z})$ is a convex combination of points of $F(\mathbb{R}_+^n)$. □

Proposition 2. *If $\beta \leq 1$, then conv$(F(\mathbb{R}_+^n)) = F(\mathbb{R}_+^n)^{\leq}$.*

Proof. $F(\mathbb{R}_+^n) \subseteq F(\mathbb{R}_+^n)^{\leq}$ and $F(\mathbb{R}_+^n)^{\leq}$ is convex, therefore $\operatorname{conv}(F(\mathbb{R}_+^n)) \subseteq F(\mathbb{R}_+^n)^{\leq}$. To prove $\operatorname{conv}(F(\mathbb{R}_+^n)) \supseteq F(\mathbb{R}_+^n)^{\leq}$, consider $(\tilde{\boldsymbol{x}}, \tilde{z}) \in F(\mathbb{R}_+^n)^{\leq}$. If $f(\tilde{\boldsymbol{x}}) = \tilde{z}$, obviously the result holds. If $\tilde{z} < f(\tilde{\boldsymbol{x}})$, similar to the proof of Proposition 1, convexity of $T = \{(\boldsymbol{x}, z) \in F(\mathbb{R}_+^n)^{\leq} : z = \tilde{z}\}$ implies there exist two extreme points of T, i.e., two elements of $\{(\boldsymbol{x}, z) \in F(\mathbb{R}_+^n) : z = \tilde{z}\}$, of which $(\tilde{\boldsymbol{x}}, \tilde{z})$ is a convex combination. □

3. Upper envelope over $X \cap W_{ij}$

We established that the convex hull of $F(\mathbb{R}_+^n)$ is defined by $\mathcal{K} \cap S$ if $\beta \geq 1$ and $F(\mathbb{R}_+^n)^{\leq}$ otherwise. From now on, we consider $\mathcal{D} = W_{ij}$ defined in (10):

$$W_{ij} = \{\boldsymbol{x} \in \mathbb{R}_+^n : px_i \leq x_j \leq qx_i\}.$$

Below we prove that the above result on the upper envelope is substantially unchanged, save an extra inequality for $n = 2$.

Lemma 4. *For any $0 < p \leq q < +\infty$, the set $X \cap W_{ij}$ is bounded for $n = 2$ and unbounded for $n > 2$.*

Proof. For $n = 2$, $x_1^{a_1} x_2^{a_2} \leq u$ and $x_2 \geq px_1$ imply $p^{a_2} x_1^{a_1 + a_2} \leq x_1^{a_1} x_2^{a_2} \leq u$, i.e., $x_1 \leq \omega_1 := (up^{-a_2})^{\frac{1}{a_1 + a_2}}$. Similarly, $x_1^{a_1} x_2^{a_2} \leq u$ and $x_2 \leq qx_1$ imply $q^{-a_1} x_2^{a_1 + a_2} \leq x_1^{a_1} x_2^{a_2} \leq u$, i.e., $x_2 \leq \omega_2 := (uq^{a_1})^{\frac{1}{a_1 + a_2}}$. Therefore $X \cap W_{ij}$ is contained in the bounding box $[0, \omega_1] \times [0, \omega_2] \times [\ell, u]$. Note that x_1 and x_2 may have tighter lower bounds than 0, but this is out of the scope of this proof.

For $n > 2$, choose $h \in N \setminus \{i, j\}$ and $r \in [p, q]$, then let $\boldsymbol{x} \in \mathbb{R}_+^n$ be such that

- $x_h = M^{\frac{1}{a_h}}$ where M is an arbitrarily large number;
- $x_i = \left(\frac{\ell}{Mr^{a_j}}\right)^{\frac{1}{a_i + a_j}}$;
- $x_j = rx_i$;
- $x_k = 1 \, \forall k \notin \{i, j, h\}$.

Then $\boldsymbol{x} \in X \cap W_{ij}$ since $\prod_{k \in N} x_k^{a_k} = x_h^{a_h} x_i^{a_i} x_j^{a_j} = \ell$ and $px_i \leq rx_i = x_j \leq qx_i$, for arbitrarily large M. □

Proposition 3. *If $\beta \geq 1$ and $n > 2$, the upper envelope of f over $X \cap W_{ij}$ is*

$$H = \{(\boldsymbol{x}, z) \in \mathbb{R}_+^n \times \mathbb{R} :$$

$$(z - z_0)^\beta \leq \gamma \prod_{k \in N} x_k^{a_k} \tag{14}$$

$$z \leq u \tag{15}$$

$$px_i \leq x_j \leq qx_i \tag{16}$$

$$\prod_{k \in N} x_k^{a_k} \geq \ell\}. \tag{17}$$

Proof. H is convex: (14) is a convex cone, (15) and (16) are linear, and (17) defines a convex set for Lemma 1. Obviously $\text{hyp}(f, X \cap W_{ij}) \subseteq H$ since any $(\boldsymbol{x}, z) \in \text{hyp}(f, X \cap W_{ij})$ satisfies (14) by Lemma 3 and (15)-(17) by construction. Therefore $E_U(f, X \cap W_{ij}) \subseteq H$. We prove now that $E_U(f, X \cap W_{ij}) \supseteq H$.

Consider $(\tilde{\boldsymbol{x}}, \tilde{z}) \in H$. If $f(\tilde{\boldsymbol{x}}) \leq u$ and $\tilde{z} \leq f(\tilde{\boldsymbol{x}})$, then $(\tilde{\boldsymbol{x}}, \tilde{z}) \in \text{hyp}(f, X \cap W_{ij})$ and the result holds. Otherwise, there are two cases: $\tilde{z} > f(\tilde{\boldsymbol{x}})$ and $f(\tilde{\boldsymbol{x}}) > u$, which are mutually exclusive since $\tilde{z} > f(\tilde{\boldsymbol{x}}) > u$ violates (15).

Case 1: $\tilde{z} > f(\tilde{\boldsymbol{x}})$. Define $f'_u(\boldsymbol{x}) = z_0 + \left(\gamma \prod_{k \in N} x_k^{a_k}\right)^{\frac{1}{\beta}}$. Since $f(\tilde{\boldsymbol{x}}) \leq u$, we can construct two points $\boldsymbol{x}' = s'\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}'' = s''\tilde{\boldsymbol{x}}$, with $s' \leq 1 \leq s''$, such that $f(\boldsymbol{x}') = \ell$ and $f(\boldsymbol{x}'') = u$. Because $f(s\boldsymbol{x}) = s^\beta f(\boldsymbol{x}) \forall s \geq 0$, both s' and s'' exist. Moreover, note that $f'_u(s\boldsymbol{x}) = s f'_u(\boldsymbol{x}) \forall s \geq 0$ as f'_u is a conic function. By construction of f'_u in Section 2, $f'_u(\boldsymbol{x}') = f(\boldsymbol{x}') = \ell$ and $f'_u(\boldsymbol{x}'') = f(\boldsymbol{x}'') = u$. Then $(\tilde{\boldsymbol{x}}, f'_u(\tilde{\boldsymbol{x}})) \in E_U(f, X \cap W_{ij})$ as it is a convex combination of $(\boldsymbol{x}', \ell)$ and $(\boldsymbol{x}'', u)$, and since $f(\tilde{\boldsymbol{x}}) < \tilde{z} \leq f'_u(\tilde{\boldsymbol{x}})$, $(\tilde{\boldsymbol{x}}, \tilde{z})$ is a convex combination of $(\tilde{\boldsymbol{x}}, f'_u(\tilde{\boldsymbol{x}}))$ and $(\tilde{\boldsymbol{x}}, f(\tilde{\boldsymbol{x}})) \in F(W_{ij})$, thus proving that $(\tilde{\boldsymbol{x}}, \tilde{z}) \in E_U(f, X \cap W_{ij})$. Note that this part of the proof is also valid for $n = 2$.

Case 2: $f(\tilde{\boldsymbol{x}}) > u$. We find $\boldsymbol{x}'$, $\boldsymbol{x}'' \in F(W_{ij})$ such that $\tilde{\boldsymbol{x}}$ is a convex combination of $\boldsymbol{x}'$ and $\boldsymbol{x}''$. Define $r = \tilde{x}_j / \tilde{x}_i > 0$; note that $p \leq r \leq q$. Since $n > 2$, select $h \in N \setminus \{i, j\}$, then define the parametric point $\boldsymbol{x}(t)$ as follows:

$$\begin{aligned} x_i(t) &= \tilde{x}_i - t \\ x_j(t) &= r x_i = \tilde{x}_j - rt = r(\tilde{x}_i - t) \\ x_h(t) &= \tilde{x}_h + t \\ x_k(t) &= \tilde{x}_k \, \forall k \in N \setminus \{i, j, h\}. \end{aligned}$$

Then $g(t) := f(\boldsymbol{x}(t)) = r^{a_j} (\tilde{x}_i - t)^{a_i + a_j} (\tilde{x}_h + t)^{a_h} \prod_{k \notin \{i,j,h\}} \tilde{x}_k$ is a continuous function such that $f(\boldsymbol{x}(0)) > u > 0$ and $f(\boldsymbol{x}(t)) = 0$ for $t = -\tilde{x}_h < 0$ and for $t = \tilde{x}_i > 0$. Hence there are two values t', t'' such that $-\tilde{x}_h < t' < 0 < t'' < \tilde{x}_i$ and $f(\boldsymbol{x}(t')) = f(\boldsymbol{x}(t'')) = u$. Because both $(\boldsymbol{x}(t'), u)$ and $(\boldsymbol{x}(t''), u)$ are in $F(W_{ij})$ and $f(\tilde{\boldsymbol{x}}) \leq \tilde{z}$, both $(\boldsymbol{x}(t'), \tilde{z})$ and $(\boldsymbol{x}(t''), \tilde{z})$ are in $\text{hyp}(f, X \cap W_{ij})$, implying that $(\tilde{\boldsymbol{x}}, \tilde{z})$ is a convex combination of two points of $\text{hyp}(f, X \cap W_{ij})$. This argument also holds for $\beta \leq 1$. □

Proposition 4. *If $\beta \leq 1$ and $n > 2$, the upper envelope of f over $X \cap W_{ij}$ is*

$$H = \{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} :$$

$$z \leq \textstyle\prod_{k \in N} x_k^{a_k} \tag{18}$$

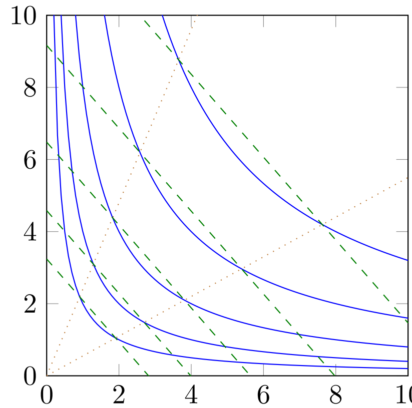
$$z \leq u \tag{19}$$

$$p x_i \leq x_j \leq q x_i \tag{20}$$

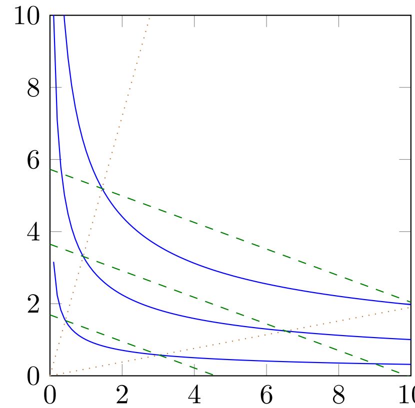
$$\textstyle\prod_{k \in N} x_k^{a_k} \geq \ell\}. \tag{21}$$

Proof. First, $\text{hyp}(f, X \cap W_{ij}) \subseteq H$ because any $(\boldsymbol{x}, z) \in \text{hyp}(f, X \cap W_{ij})$ satisfies (18)-(21). H is also convex: (18) is convex because $f(\boldsymbol{x})$ is concave for $\beta \leq 1$ and (21) is convex for Lemma 1. Therefore we also have $E_U(f, X \cap W_{ij}) \subseteq H$. We now prove $H \subseteq E_U(f, X \cap W_{ij})$.

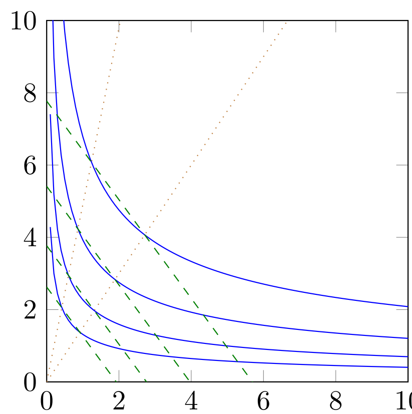
Consider $(\tilde{\boldsymbol{x}}, \tilde{z}) \in H$. If $\tilde{z} = f(\tilde{\boldsymbol{x}})$, then obviously $(\tilde{\boldsymbol{x}}, \tilde{z}) \in X \cap W_{ij}$ and the result holds. Let $\tilde{z} < f(\tilde{\boldsymbol{x}})$. If $f(\tilde{\boldsymbol{x}}) \leq u$, then $(\tilde{\boldsymbol{x}}, f(\tilde{\boldsymbol{x}})) \in F(W_{ij})$ and $(\tilde{\boldsymbol{x}}, \tilde{z})$ belongs to its hypograph and hence to its upper envelope. If $f(\tilde{\boldsymbol{x}}) > u$, then similar to Case 2 of Proposition 3, one can obtain two vectors in $\text{hyp}(f, X \cap W_{ij})$ of which $(\tilde{\boldsymbol{x}}, \tilde{z})$ is a convex combination. □



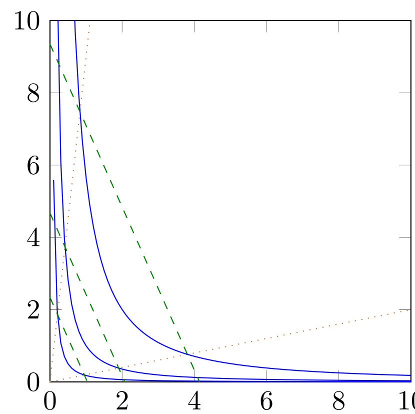
(a) $\xi \in \{2, 4, 8, 16, 32\}$, $(a_1, a_2) = (1, 1)$, $(p, q) = (0.55, 2.4)$



(b) $\xi \in \{1, 2, 3\}$, $(a_1, a_2) = (0.3, 0.6)$, $(p, q) = (0.19, 3.6)$



(c) $\xi \in \{8, 2^9, 2^{15}, 2^{21}\}$, $(a_1, a_2) = (3.9, 7.6)$, $(p, q) = (1.5, 4.9)$



(d) $\xi \in \{0.5, 1, 2\}$, $(a_1, a_2) = (0.6, 0.4)$, $(p, q) = (0.2, 9)$

FIGURE 1. Solid lines are the level curves $\{\boldsymbol{x} \in \mathbb{R}^2_+ : x_1^{a_1} x_2^{a_2} = \xi\}$; the dotted lines $x_2 = px$ and $x_2 = qx_1$ intersect each level curve in one point each. The dashed lines through the intersections of the level curves and the dotted lines are parallel to one another.

4. LOWER ENVELOPE OVER $X \cap W_{ij}$

We build on a property of the monomial function f for general $n \geq 2$ to derive a few results leading to the lower envelope of f over $X \cap W_{ij}$ for $n = 2$. Define the level set $C_\xi = \{\boldsymbol{x} \in \mathbb{R}^n_+ : f(\boldsymbol{x}) = \xi\}$ and the two subspaces $P_{ij} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_j = px_i\}$ and $Q_{ij} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_j = qx_i\}$. There is a bijection from $C_\xi \cap P_{ij}$ to $C_\xi \cap Q_{ij}$ such that all pairs of points are joined by parallel lines.

Let us illustrate the property on bivariate functions first. It is easy to show that for $n = 2$ and $a_1 = a_2 = 1$, $C_\xi \cap P_{ij} = \{\check{\boldsymbol{x}}\}$ and $C_\xi \cap Q_{ij} = \{\hat{\boldsymbol{x}}\}$ where

$$\check{\boldsymbol{x}} = \left(\sqrt{p\xi}, \sqrt{\frac{1}{p}\xi}\right), \quad \hat{\boldsymbol{x}} = \left(\sqrt{q\xi}, \sqrt{\frac{1}{q}\xi}\right).$$

The line through $\check{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$ is given by the linear equation

$$\det\left(\begin{bmatrix} x_1 & x_2 & 1 \\ \sqrt{p\xi} & \sqrt{\frac{1}{p}\xi} & 1 \\ \sqrt{q\xi} & \sqrt{\frac{1}{q}\xi} & 1 \end{bmatrix}\right) = 0,$$

and its slope $\frac{1/\sqrt{p}-1/\sqrt{q}}{\sqrt{p}-\sqrt{q}} = -(pq)^{-\frac{1}{2}}$ is independent of ξ. This also holds for general positive exponents $(a_1, a_2) \neq (1, 1)$, in which case the coefficients of x_1 and x_2 are proportional to $\xi^{\frac{1}{\beta}}$ rather than $\sqrt{\xi}$. Figure 1 shows examples for different values of a_1, a_2, p, q, and ξ. We prove that this generalizes to $n \geq 2$.

Lemma 5. *Given $\boldsymbol{a} \in \mathbb{R}^n_+$, $i, j \in N$, $\xi \in \mathbb{R}$, $p, q \in \mathbb{R}_+$ with $i \neq j$ and $0 < p < q$, there exist $d_i < 0$ and $d_j > 0$ such that for any $\check{\boldsymbol{x}}$ that satisfies $\check{x}_j = p\check{x}_i$ and $\prod_{i\in N} \check{x}_i^{a_i} = \xi$, there exists a unique solution $(\bar{s}, \bar{\boldsymbol{x}}) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ to the nonlinear system*

$$\bar{x}_j = q\bar{x}_i \tag{22}$$

$$(\bar{x}_i, \bar{x}_j) = (\check{x}_i + \bar{s}d_i, \check{x}_j + \bar{s}d_j) \tag{23}$$

$$\bar{x}_k = \check{x}_k \quad \forall k \notin \{i, j\} \tag{24}$$

$$\textstyle\prod_{i\in N} \bar{x}_i^{a_i} = \xi. \tag{25}$$

Proof. We prove that there exist η_1 and η_2 such that $(\bar{x}_i, \bar{x}_j) = (\eta_i\check{x}_i, \eta_j\check{x}_j)$. First, (25) requires $\prod_{k\in N} \bar{x}_k^{a_k} = \eta_i^{a_i}\eta_j^{a_j} \prod_{k\in N} \check{x}_k^{a_k} = \xi$, i.e., $\eta_i^{a_i}\eta_j^{a_j} = 1$. Given that $\check{x}_j = p\check{x}_i$ and $\eta_j\check{x}_j = q\eta_i\check{x}_i$, solving both for $\frac{\check{x}_i}{\check{x}_j}$ yields $\frac{1}{p} = \frac{\eta_j}{q\eta_i}$, or

$$\frac{\eta_j}{\eta_i} = \frac{q}{p}. \tag{26}$$

This yields $\eta_i = \left(\frac{q}{p}\right)^{-\frac{a_j}{a_i+a_j}} < 1$ and $\eta_j = \eta_i\frac{q}{p} = \left(\frac{q}{p}\right)^{\frac{a_i}{a_i+a_j}} > 1$.

In order to obtain d_i and d_j, observe that

$$\begin{aligned} \check{x}_i + \bar{s}d_i &= \eta_i\check{x}_i \\ \check{x}_j + \bar{s}d_j &= \eta_j\check{x}_j. \end{aligned}$$

Solving both for $\bar{s}$, we have $\frac{1}{d_i}\check{x}_i(\eta_i - 1) = \frac{1}{d_j}\check{x}_j(\eta_j - 1)$, and since $\frac{\check{x}_i}{\check{x}_j} = \frac{1}{p}$ we get

$$\frac{d_i}{d_j}\frac{\eta_j - 1}{\eta_i - 1} = \frac{1}{p}. \tag{27}$$

For convenience, define fractions $\varphi_i = \frac{a_i}{a_i+a_j}$ and $\varphi_j = \frac{a_j}{a_i+a_j}$. Note that

$$\frac{\eta_j - 1}{\eta_i - 1} = \frac{q^{\varphi_i} - p^{\varphi_i}}{p^{\varphi_i}} \cdot \frac{p^{-\varphi_j}}{q^{-\varphi_j} - p^{-\varphi_j}} = \frac{q^{\varphi_i} - p^{\varphi_i}}{q^{-\varphi_j} - p^{-\varphi_j}} \cdot \frac{1}{p}.$$

Because d_i, d_j define a direction, rather than normalizing them (which would lead to complications in the next section) we determine d_i, d_j as denominator and numerator of the fraction above:

$$\begin{array}{rcccl} d_i & = & q^{-\varphi_j} - p^{-\varphi_j} & = & q^{-a_j/(a_i+a_j)} - p^{-a_j/(a_i+a_j)} \\ d_j & = & q^{\varphi_i} - p^{\varphi_i} & = & q^{a_i/(a_i+a_j)} - p^{a_i/(a_i+a_j)}. \end{array}$$

Note that (i) $d_i < 0 < d_j$ (ii) both d_i and d_j only depend on p, q, a_i, and a_j, and not on $\check{x}_i$ or $\check{x}_j$; (iii) they are always defined. Hence, there exist unique d_i, d_j such that for any $\check{\boldsymbol{x}}$ such that $\check{x}_j = p\check{x}_i$, the parametric nonlinear system (22)–(25) admits the solution $(\bar{s}, \bar{\boldsymbol{x}}) \in \mathbb{R}_+ \times \mathbb{R}^n_+$. Non-negativity of $\bar{\boldsymbol{x}}$, in particular of $\bar{x}_i$ and $\bar{x}_j$, follows from $0 < \eta_i < 1$ and $\eta_j > 1$, while $\bar{s} \geq 0$ since $d_i < 0$ and $\check{x}_i + \bar{s}d_i = \eta_i\check{x}_i \leq \check{x}_i$. □

Although $\bar{s}$ does depend on $\check{\boldsymbol{x}}$, the key fact is that direction (d_i, d_j) defines a bijection from P_{ij} to Q_{ij} by joining pairs of points that have same value of the function $\prod_{i\in N} x_i^{a_i}$. This suggests that on a direction orthogonal to (d_i, d_j), i.e., $(d_j, -d_i)$, we can define a lower-bounding function that matches the values on P_{ij} and Q_{ij} and that, for $n = 2$, is the lower envelope of $f(\boldsymbol{x})$.

Proposition 5. *The function* $f'_\ell(\boldsymbol{x}) = \lambda(d_jx_i - d_ix_j)^{a_i+a_j}\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}$*, with* $\lambda = p^{a_j} / (d_j - d_ip)^{a_i+a_j}$,

(a) matches the value of $f(\boldsymbol{x})$ *for* $\boldsymbol{x} \in P_{ij} \cup Q_{ij}$*;*
(b) is a minorant of $f(\boldsymbol{x})$ *for* $\boldsymbol{x} \in W_{ij}$*;*
(c) is, for $n = 2$ *and* $\beta = a_1 + a_2 \geq 1$*, the lower envelope of* $f(\boldsymbol{x})$ *in* W_{ij}*.*

Proof. **(a)** In order to prove $f'_\ell(\boldsymbol{x}) = f(\boldsymbol{x}) \forall \boldsymbol{x} \in P_{ij} \cup Q_{ij}$, it suffices to prove that

$$\lambda(d_jx_i - d_ix_j)^{a_i+a_j} = x_i^{a_i}x_j^{a_j}\ \forall x_i, x_j : x_j = px_i \vee x_j = qx_i.$$

We prove this for $\boldsymbol{x} \in P_{ij}$ first. For $x_j = px_i$ we have

$$\begin{array}{rcl} \lambda(d_jx_i - d_i(px_i))^{a_i+a_j} & = & x_i^{a_i}(px_i)^{a_j} \\ \lambda(d_j - pd_i)^{a_i+a_j}x_i^{a_i+a_j} & = & p^{a_j}x_i^{a_i+a_j} \\ \lambda(d_j - pd_i)^{a_i+a_j} & = & p^{a_j}, \end{array}$$

which yields $\lambda = \frac{p^{a_j}}{(d_j-pd_i)^{a_i+a_j}}$ in the statement. For $x_j = qx_i$ we obtain $\lambda' = \frac{q^{a_j}}{(d_j-qd_i)^{a_i+a_j}}$. To prove that $\lambda = \lambda'$, observe that from (27) we obtain $d_j(\eta_i - 1) = pd_i(\eta_j - 1)$, which implies

$$d_j - pd_i = d_j\eta_i - pd_i\eta_j.$$

From (26) we obtain $p\eta_j = q\eta_i$, which means $d_j - pd_i = (d_j - qd_i)\eta_i$. Also, $\eta_i^{a_i}\eta_j^{a_j} = 1$ implies $\eta_i^{a_i} = \eta_j^{-a_j}$. Therefore,

$$\begin{array}{rcl} \lambda' & = & \frac{q^{a_j}}{(d_j-qd_i)^{a_i+a_j}} \\ & = & \frac{q^{a_j}}{(d_j-pd_i)^{a_i+a_j}\eta_i^{a_i+a_j}} \\ & = & \frac{q^{a_j}\eta_j^{a_j}}{(d_j-pd_i)^{a_i+a_j}\eta_i^{a_j}}, \end{array}$$

but (26) implies $\frac{q^{a_j}\eta_j^{a_j}}{\eta_i^{a_j}} = p^{a_j}$, hence the two values of λ match.

(b) First, $\tilde{\boldsymbol{x}}$ is a convex combination $\mu\check{\boldsymbol{x}} + (1-\mu)\hat{\boldsymbol{x}}$ of two points $\check{\boldsymbol{x}} \in P_{ij}$ and $\hat{\boldsymbol{x}} \in Q_{ij}$ such that $f'_\ell(\check{\boldsymbol{x}}) = f'_\ell(\hat{\boldsymbol{x}}) = f'_\ell(\tilde{\boldsymbol{x}})$. We construct $\check{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$ as follows: $\check{x}_h = \hat{x}_h = \tilde{x}_h$ for $i \neq h \neq j$; then we find $\check{x}_i$ and $\check{x}_j$ by solving the linear system $d_j\check{x}_i - d_i\check{x}_j =$

$d_j\tilde{x}_i - d_i\tilde{x}_j; \check{x}_j = p\check{x}_i$, and similar for $\hat{\boldsymbol{x}}$. It is easily proven that both systems admit a non-negative solution. Hence we have to prove that $f(\tilde{\boldsymbol{x}}) \geq f(\check{\boldsymbol{x}}) = f(\hat{\boldsymbol{x}})$. Note that $\tilde{x}_i = \mu\check{x}_i + (1-\mu)\eta_i\check{x}_i$ and similar for $\tilde{x}_j$, therefore

$$\begin{aligned} f(\tilde{\boldsymbol{x}}) &= \prod_{k\in N\setminus\{i,j\}} \tilde{x}_k^{a_k} \cdot \tilde{x}_i^{a_i}\tilde{x}_j^{a_j} \\ &= \prod_{k\in N\setminus\{i,j\}} \check{x}_k^{a_k} \cdot \check{x}_i^{a_i}\check{x}_j^{a_j} \cdot (\mu + (1-\mu)\eta_i)^{a_i}(\mu + (1-\mu)\eta_j)^{a_j}, \end{aligned}$$

and we simply need to prove that $(\mu + (1-\mu)\eta_i)^{a_i}(\mu + (1-\mu)\eta_j)^{a_j} \geq 1$. By concavity of the logarithm,

$$\begin{aligned} & \log\left((\mu + (1-\mu)\eta_i)^{a_i}(\mu + (1-\mu)\eta_j)^{a_j}\right) \\ = \;& a_i\log(\mu + (1-\mu)\eta_i) + a_j\log(\mu + (1-\mu)\eta_j) \\ \geq \;& a_i(\mu\log(1) + (1-\mu)\log\eta_i) + a_j(\mu\log(1) + (1-\mu)\log\eta_j) \\ = \;& (1-\mu)(a_i\log\eta_i + a_j\log\eta_j) \\ = \;& (1-\mu)\log(\eta_i^{a_i}\eta_j^{a_j}) = 0, \end{aligned}$$

which proves this part.

(c) For $n = 2$, the function $f'_\ell(\boldsymbol{x}) = \lambda(d_jx_i - d_ix_j)^\beta$ is convex since $\beta \geq 1$. Building on (a) and (b), suppose another convex function $g : \mathbb{R}^2 \to \mathbb{R}$ exists such that $g(\boldsymbol{x}) \leq f(\boldsymbol{x})$ for $\boldsymbol{x} \in W_{12}$ and there exists $\tilde{\boldsymbol{x}}$ such that $g(\tilde{\boldsymbol{x}}) > f'_\ell(\tilde{\boldsymbol{x}})$. Clearly $\tilde{\boldsymbol{x}} \notin (P_{ij} \cup Q_{ij})$ as otherwise (a) would imply $g(\tilde{\boldsymbol{x}}) > f(\tilde{\boldsymbol{x}})$. Similar to the argument in (b), consider then $\check{\boldsymbol{x}} \in P_{ij}$ and $\hat{\boldsymbol{x}} \in Q_{ij}$ such that $d_2\check{x}_1 - d_1\check{x}_2 = d_2\tilde{x}_1 - d_1\tilde{x}_2 = d_2\hat{x}_1 - d_1\hat{x}_2$, i.e., the three points are aligned on a level curve of $f'_\ell(\boldsymbol{x})$. Obviously $\tilde{\boldsymbol{x}}$ is a convex combination (with weight denoted by μ) of $\check{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$ and $f(\check{\boldsymbol{x}}) = f(\hat{\boldsymbol{x}}) = f'_\ell(\check{\boldsymbol{x}}) = f'_\ell(\hat{\boldsymbol{x}}) = f'_\ell(\tilde{\boldsymbol{x}}) < g(\tilde{\boldsymbol{x}})$. This implies that $g(\tilde{\boldsymbol{x}}) > \mu g(\check{\boldsymbol{x}}) + (1-\mu)g(\hat{\boldsymbol{x}})$ and hence g is not convex, a contradiction. □

Note that case (c) of the previous proposition is valid over the whole W_{12}; we are interested in the lower envelope over $X \cap W_{12}$, which will take an extra step. We first prove a similar result for $\beta \leq 1$.

Proposition 6. *For $\beta \leq 1$, the function*

$$f''_\ell(\boldsymbol{x}) = \zeta(d_jx_i - d_ix_j)^{\frac{a_i+a_j}{\beta}} \left(\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}\right)^{\frac{1}{\beta}} + z_0,$$

with $\zeta = \lambda^{1/\beta}\frac{u-\ell}{u^{1/\beta}-\ell^{1/\beta}}$ and z_0 as defined in (12)*,*

(a) matches $f(\boldsymbol{x})$ at $(P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u)$;
(b) is a minorant of f in $X \cap W_{ij}$;
(c) is the lower envelope of f over conv$((P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u))$ for $n = 2$.

Proof. **(a)** From Proposition 5 (a-b), $\lambda(d_jx_i - d_ix_j)^{a_i+a_j}\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}$ is a minorant of f within the same domain and matches f in $P_{ij} \cup Q_{ij}$. Hence, for $\boldsymbol{x} \in P_{ij} \cup Q_{ij}$ such that $f(\boldsymbol{x}) = \ell$ we have $\lambda(d_jx_i - d_ix_j)^{a_i+a_j}\prod_{k\in N\setminus\{i,j\}} x_k^{a_k} = \ell$, i.e., $(d_jx_i - d_ix_j)^{\frac{a_i+a_j}{\beta}}\left(\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}\right)^{\frac{1}{\beta}} = \left(\frac{\ell}{\lambda}\right)^{\frac{1}{\beta}}$, and similar for u. Therefore

$$\begin{aligned} \zeta\left(\tfrac{\ell}{\lambda}\right)^{\frac{1}{\beta}} + \zeta_0 &= \ell \\ \zeta\left(\tfrac{u}{\lambda}\right)^{\frac{1}{\beta}} + \zeta_0 &= u, \end{aligned}$$

which is a linear system in ζ, ζ_0 with solutions

$$\begin{aligned} \zeta &= \frac{u-\ell}{\left(\frac{u}{\lambda}\right)^{\frac{1}{\beta}} - \left(\frac{\ell}{\lambda}\right)^{\frac{1}{\beta}}} = \lambda^{\frac{1}{\beta}}\frac{u-\ell}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}} = (\gamma\lambda)^{\frac{1}{\beta}} \\ \zeta_0 &= \ell - \zeta\left(\tfrac{\ell}{\lambda}\right)^{1/\beta} = \ell - \frac{(u-\ell)\ell^{1/\beta}}{u^{1/\beta}-\ell^{1/\beta}} = \frac{u^{1/\beta}\ell - \ell^{1/\beta}u}{u^{1/\beta}-\ell^{1/\beta}} = z_0. \end{aligned}$$

Therefore, $f''_\ell(\boldsymbol{x}) = f'_\ell(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\boldsymbol{x}$ such that $x_j/x_i \in \{p,q\}, f(\boldsymbol{x}) \in \{\ell, u\}$.

(b) By Lemma 3, if $\beta \leq 1$ then $f_u(\boldsymbol{x}) = z_0 + \left(\gamma \prod_{k\in N} x_k^{a_k}\right)^{\frac{1}{\beta}}$ is a minorant of $f(\boldsymbol{x})$ for $f(\boldsymbol{x}) \in [\ell, u]$. Therefore it suffices to prove that $f''_\ell(\boldsymbol{x}) \leq f_u(\boldsymbol{x})$ in $X \cap W_{ij}$.

$$\begin{array}{rlcr} & z_0 + \zeta(d_j x_i - d_i x_j)^{\frac{a_i+a_j}{\beta}} \left(\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}\right)^{\frac{1}{\beta}} & \leq & z_0 + \left(\gamma \prod_{k\in N} x_k^{a_k}\right)^{\frac{1}{\beta}} \\ \Leftrightarrow & (\gamma\lambda)^{\frac{1}{\beta}} (d_j x_i - d_i x_j)^{\frac{a_i+a_j}{\beta}} \left(\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}\right)^{\frac{1}{\beta}} & \leq & \left(\gamma \prod_{k\in N} x_k^{a_k}\right)^{\frac{1}{\beta}} \\ \Leftrightarrow & \lambda^{\frac{1}{\beta}} (d_j x_i - d_i x_j)^{\frac{a_i+a_j}{\beta}} & \leq & \left(x_i^{a_i} x_j^{a_j}\right)^{\frac{1}{\beta}}. \end{array}$$

The last inequality is equivalent to $\lambda(d_j x_i - d_i x_j)^{a_i+a_j} \leq x_i^{a_i} x_j^{a_j}$, which holds in $X \cap W_{ij}$ for Proposition 5. Although it is applied here to the two-variable monomial $x_i^{a_i} x_j^{a_j}$ rather than $f(\boldsymbol{x})$, the relationship still holds because d_i, d_j only depend on p, q, a_i, a_j, and λ depends on d_i, d_j, p, q, a_i, a_j.

(c) For $n = 2$, $f''_\ell(\boldsymbol{x})$ reduces to the linear function $\zeta(d_2 x_1 - d_1 x_2) + z_0$. By (a) and (b), f''_ℓ matches the concave function $f(\boldsymbol{x})$ at all four points of the set $T = (P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u)$. For linearity of f''_ℓ, any function g defined on $\operatorname{conv}(T)$ such that $g(\tilde{\boldsymbol{x}}) > f''_\ell(\tilde{\boldsymbol{x}})$ for some $\tilde{\boldsymbol{x}} \in \operatorname{conv}(T)$ is concave w.r.t. two or more elements of T, and hence cannot be the (convex) lower envelope of f over $\operatorname{conv}(T)$. □

Lemma 6. *For $n = 2$, the convex hull of $X \cap W_{12}$ is*

$$\begin{aligned} Y = \ \{ & \boldsymbol{x} \in \mathbb{R}^2_+ : \\ & p x_1 \leq x_2 \leq q x_1, \\ & x_1^{a_1} x_2^{a_2} \geq \ell, \\ & d_2 x_1 - d_1 x_2 \leq (u/\lambda)^{\frac{1}{\beta}} \}. \end{aligned}$$

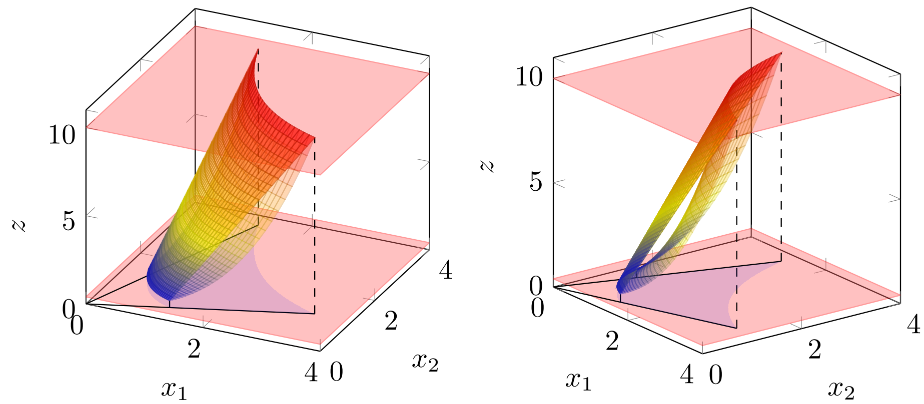
Proof. $\lambda(d_2 x_1 - d_1 x_2)^\beta \leq f(\boldsymbol{x})$ and $f(\boldsymbol{x}) \leq u$ imply $d_2 x_1 - d_1 x_2 \leq (u/\lambda)^{\frac{1}{\beta}}$ and consequently $X \cap W_{12} \subseteq Y$. By convexity of Y we also have $\operatorname{conv}(X \cap W_{12}) \subseteq Y$. In order to prove $\operatorname{conv}(X \cap W_{12}) \supseteq Y$, consider $\tilde{\boldsymbol{x}} \in Y$. If $\tilde{\boldsymbol{x}} \in X \cap W_{12}$, the result holds. Otherwise $f(\tilde{\boldsymbol{x}}) > u \geq \lambda(d_2 \tilde{x}_1 - d_1 \tilde{x}_2)^\beta$. Consider two vectors $\check{\boldsymbol{x}} \in P_{12}$ and $\hat{\boldsymbol{x}} \in Q_{12}$ such that $d_2 \check{x}_1 - d_1 \check{x}_2 = d_2 \hat{x}_1 - d_1 \hat{x}_2 = d_2 \tilde{x}_1 - d_1 \tilde{x}_2$. Such points satisfy the two systems

$$\begin{cases} d_2 \check{x}_1 - d_1 \check{x}_2 = d_2 \tilde{x}_1 - d_1 \tilde{x}_2 \\ p \check{x}_1 - \check{x}_2 = 0 \end{cases} \qquad \begin{cases} d_2 \hat{x}_1 - d_1 \hat{x}_2 = d_2 \tilde{x}_1 - d_1 \tilde{x}_2 \\ q \hat{x}_1 - \hat{x}_2 = 0, \end{cases}$$

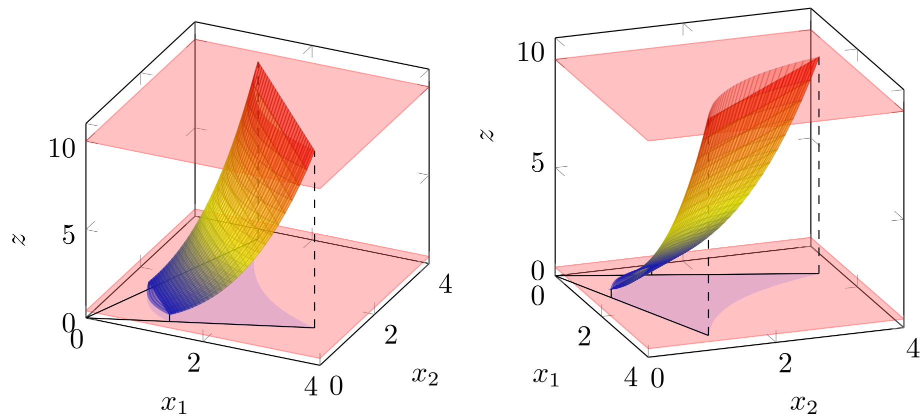
both of which always yield a solution as $d_2, p, q > 0$ and $d_1 < 0$. Because $\check{\boldsymbol{x}} \in P_{12}$, $\hat{\boldsymbol{x}} \in Q_{12}$, and $f'_\ell(\boldsymbol{x}) = \lambda(d_2 x_1 - d_1 x_2)^\beta = \lambda(d_2 \tilde{x}_1 - d_1 \tilde{x}_2)^\beta \leq u$ for both $\check{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$, these belong to $X \cap W_{12}$ and $\tilde{\boldsymbol{x}}$ is their convex combination since $\frac{\check{x}_2}{\check{x}_1} = p \leq \frac{\tilde{x}_2}{\tilde{x}_1} \leq q = \frac{\hat{x}_2}{\hat{x}_1}$, proving that $\tilde{\boldsymbol{x}} \in \operatorname{conv}(X \cap W_{12})$. □

The last condition defining Y, i.e., $d_2 x_1 - d_1 x_2 \leq (u/\lambda)^{\frac{1}{\beta}}$, is equivalent to setting an upper bound of u on the lower-bounding function $f'_\ell(\boldsymbol{x})$ for $\beta \geq 1$, as it is equivalent to $\lambda(d_2 x_1 - d_1 x_2)^\beta \leq u$. It is also equivalent to setting an upper bound of u on the lower-bounding function $f''_\ell(\boldsymbol{x})$ for the case $\beta \leq 1$: $f''_\ell(\boldsymbol{x}) = \zeta(d_2 x_1 - d_1 x_2) + z_0 \leq u$ if $d_2 x_1 - d_1 x_2 \leq \frac{u - z_0}{\zeta}$, but

$$\frac{u - z_0}{\zeta} = \frac{u - z_0}{\lambda^{\frac{1}{\beta}} \frac{u-\ell}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}}} = \frac{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}}{\lambda^{\frac{1}{\beta}}(u - \ell)} \left(u - \frac{u^{\frac{1}{\beta}}\ell - \ell^{\frac{1}{\beta}} u}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}} \right) = \frac{u^{\frac{1}{\beta}}}{\lambda^{\frac{1}{\beta}}}.$$



(a) The upper envelope $f'_u(x_1, x_2) = z_0 + (\gamma x_1^{a_1} x_2^{a_2})^{\frac{1}{\beta}}$.



(b) The lower envelope $f'_\ell(x_1, x_2) = \lambda(d_2 x_1 - d_1 x_2)^\beta$.

Figure 2. Upper envelope $f'_u(\boldsymbol{x})$ (top) and lower envelope $f'_\ell(\boldsymbol{x})$ of function $f(\boldsymbol{x}) = x_1^{a_1} x_2^{a_2}$ for $\beta \geq 1$, both shown under different angles. The domain $\{(x_1, x_2) \in \mathbb{R}^2_+ : px_1 \leq x_2 \leq qx_1, f(\boldsymbol{x}) \in [\ell, u]\}$ is the shaded area on the x_1, x_2 plane. The parameters are as follows: $(p, q) = (0.35, 3)$, $(\ell, u) = (0.4, 10)$, $(a_1, a_2) = (1.7, 1.5)$.

Because f is defined over $\mathbb{R}^n_+$, Lemma 6 defines the projection onto the $\boldsymbol{x}$-space of the lower and upper envelopes of f over $X \cap W_{12}$ and consequently the projection of $\text{conv}(F(W_{12}))$. The argument used in the above proof is worth underlining and will be used below: a vector $\tilde{\boldsymbol{x}} \in Y$ such that $f(\tilde{\boldsymbol{x}}) > u \geq f'_\ell(\tilde{\boldsymbol{x}})$ is a member of $\text{conv}(X \cap W_{12})$.

We are now ready to provide the *upper* envelope of f over $X \cap W_{ij}$ for $n = 2$, which was left uncovered from the previous section. Figures 2 and 3 show upper and lower envelopes for sample values of the parameters. The proof is, up to Case 1 included, similar to the proof of Proposition 3 for $n > 2$.

Theorem 1. *For $n = 2$, the upper envelope $E_U(f, X \cap W_{12})$ is*

$$H = \{(\boldsymbol{x}, z) \in Y \times \mathbb{R} : \; z \le f_u(\boldsymbol{x}) \; z \le u\},$$

where $f_u(\boldsymbol{x}) = f'_u(\boldsymbol{x}) = z_0 + (\gamma x_1^{a_1} x_2^{a_2})^{\frac{1}{\beta}}$ if $\beta \ge 1$ and $f_u(\boldsymbol{x}) = f(\boldsymbol{x})$ otherwise.

Proof. Any $(\boldsymbol{x}, z) \in \text{hyp}(f, X \cap W_{ij})$ satisfies $\boldsymbol{x} \in Y$ for Lemma 6 and $z \le u$ as $\boldsymbol{x} \in X$. Also, $(\boldsymbol{x}, z)$ satisfies $z \le f_u(\boldsymbol{x})$: by construction for $\beta \le 1$, and by Lemma 3 for $\beta \ge 1$. Hence $\text{hyp}(f, X \cap W_{ij}) \subseteq H$. Because H is convex, $E_U(f, X \cap W_{ij}) \subseteq H$. We now prove that $H \subseteq E_U(f, X \cap W_{ij})$. Consider $(\tilde{\boldsymbol{x}}, \tilde{z}) \in H$. Two cases arise: either $\tilde{x}_1^{a_1}\tilde{x}_2^{a_2} \le u$ or $\tilde{x}_1^{a_1}\tilde{x}_2^{a_2} > u$.
Case 1: $\tilde{x}_1^{a_1}\tilde{x}_2^{a_2} \le u$. If $\beta \le 1$ or $\tilde{z} \le \tilde{x}_1^{a_1}\tilde{x}_2^{a_2}$, then obviously $(\tilde{\boldsymbol{x}}, \tilde{z}) \in \text{hyp}(f, X \cap W_{12})$ and the result holds. Otherwise, see Case 1, proof of Proposition 3.
Case 2: $\tilde{x}_1^{a_1}\tilde{x}_2^{a_2} > u$. There exist $\check{\boldsymbol{x}} = s'\tilde{\boldsymbol{x}}$ such that $f(\check{\boldsymbol{x}}) = f_u(\check{\boldsymbol{x}}) = u$ (again, by construction for $\beta \le 1$ and by Lemma 3 for $\beta \ge 1$) and $\hat{\boldsymbol{x}} = s''\tilde{\boldsymbol{x}}$ such that $\lambda(d_2\hat{x}_1 - d_1\hat{x}_2)^\beta = f'_\ell(\hat{\boldsymbol{x}}) = u \le f(\hat{\boldsymbol{x}})$. Because $\tilde{\boldsymbol{x}}$ satisfies $\lambda(d_2\tilde{x}_1 - d_1\tilde{x}_2)^\beta \le u$ and f'_ℓ is monotone increasing along the direction $\{s\tilde{\boldsymbol{x}} : s \ge 0\}$, s'' is unique and $s'' \ge 1$. Similarly, s' is unique and $s' < 1$. Then $\tilde{\boldsymbol{x}}$ is a convex combination of $\check{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$.

Whereas $(\check{\boldsymbol{x}}, z) \in \text{hyp}(f, X \cap W_{ij})$ for $z \le u$, $\hat{\boldsymbol{x}}$ is in turn a convex combination of two points in $X \cap W_{ij}$: $\check{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$ such that $f(\check{\boldsymbol{x}}) = f(\hat{\boldsymbol{x}}) = u$, $x_2^p = px_1^p$, and $x_2^q = qx_1^q$. Properties of f'_ℓ show that $f'_\ell(\check{\boldsymbol{x}}) = f'_\ell(\hat{\boldsymbol{x}}) = f'_\ell(\hat{\boldsymbol{x}}) = u$, which implies $\check{\boldsymbol{x}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{x}}$ lie on the same level line of f'_ℓ and $d_2x_1^p - d_1x_2^p = d_2x_1^q - d_1x_2^q = d_2\check{x}_1 - d_1\check{x}_2 = \left(\frac{u}{\lambda}\right)^{\frac{1}{\beta}}$. Taking into account z, the above implies implies $(\tilde{\boldsymbol{x}}, u)$ is a convex combination of $(\check{\boldsymbol{x}}, u)$, $(\check{\boldsymbol{x}}, u)$, and $(\hat{\boldsymbol{x}}, u)$, all elements of $X \cap W_{12}$. But then $(\check{\boldsymbol{x}}, \tilde{z})$, $(\check{\boldsymbol{x}}, \tilde{z})$, and $(\hat{\boldsymbol{x}}, \tilde{z})$ all belong to $\text{hyp}(f, X \cap W_{ij})$ by definition of hypograph, and because $\tilde{z} \le u$, $(\tilde{\boldsymbol{x}}, \tilde{z})$ is their convex combination hence $(\tilde{\boldsymbol{x}}, \tilde{z}) \in E_U(f, X \cap W_{ij})$. □

Theorem 2. *For $n = 2$, the lower envelope $E_L(f, X \cap W_{12})$ is*

$$H = \{(\boldsymbol{x}, z) \in Y \times \mathbb{R} : \; z \ge f_\ell(\boldsymbol{x}), \; z \ge \ell\},$$

where $f_\ell(\boldsymbol{x}) = f'_\ell(\boldsymbol{x})$ if $\beta \ge 1$ and $f_\ell(\boldsymbol{x}) = f''_\ell(\boldsymbol{x})$ if $\beta \le 1$.

Proof. Any $(\tilde{\boldsymbol{x}}, \tilde{z}) \in \text{epi}(f, X \cap W_{12})$ satisfies $\boldsymbol{x} \in Y$ and $z \ge \ell$ by definition. It also satisfies $z \ge f'_\ell(\boldsymbol{x})$ by Proposition 5 and $z \ge f''_\ell(\boldsymbol{x})$ by Proposition 6. Hence $\text{epi}(f, X \cap W_{12}) \subseteq H$. By convexity of H we conclude $E_L(f, X \cap W_{12}) \subseteq H$. We prove now $H \subseteq E_L(f, X \cap W_{12})$.

Consider $(\tilde{\boldsymbol{x}}, \tilde{z}) \in H$. If $\tilde{\boldsymbol{x}} \in \text{conv}((C_\ell \cup C_u) \cap (P_{12} \cup Q_{12}))$, then for $\beta \le 1$ the vector $(\tilde{\boldsymbol{x}}, f''_\ell(\boldsymbol{x}))$ is a convex combination of the four vectors $(\dot{\boldsymbol{x}}, f''_\ell(\dot{\boldsymbol{x}})) = (\dot{\boldsymbol{x}}, f(\dot{\boldsymbol{x}}))$ for $\dot{\boldsymbol{x}} \in (C_\ell \cup C_u) \cap (P_{12} \cup Q_{12})$, given that f''_ℓ is linear (see Proposition 6). For $\beta \ge 1$, f'_ℓ matches f within $P_{12} \cup Q_{12}$ (see Proposition 5), hence there exist $\hat{\boldsymbol{x}}, \check{\boldsymbol{x}}$ such that $(\tilde{\boldsymbol{x}}, f'_\ell(\boldsymbol{x}))$ is a convex combination of $(\hat{\boldsymbol{x}}, f'_\ell(\hat{\boldsymbol{x}}))$ and $(\check{\boldsymbol{x}}, f'_\ell(\check{\boldsymbol{x}}))$, both in $\text{epi}(f, X \cap W_{12})$. In both cases, $z \ge f_\ell(\boldsymbol{x})$ and therefore $(\tilde{\boldsymbol{x}}, \tilde{z}) \in E_L(f, X \cap W_{12})$.

If $\tilde{\boldsymbol{x}} \in Y \setminus \text{conv}((C_\ell \cup C_u) \cap (P_{12} \cup Q_{12}))$, then $f(\boldsymbol{x}) \ge \ell \ge f_\ell(\boldsymbol{x})$. A similar argument to the above leads to three points $\check{\boldsymbol{x}} \in C_\ell \cap P_{12}$, $\hat{\boldsymbol{x}} \in C_\ell \cap Q_{12}$ (these two sets contain a single vector each), and $\dot{\boldsymbol{x}}$, such that $f(\dot{\boldsymbol{x}}) = f(\check{\boldsymbol{x}}) = f(\hat{\boldsymbol{x}}) = \ell$ and that form $(\tilde{\boldsymbol{x}}, \ell)$ as their convex combination, where $\tilde{z} \ge \ell$, yielding $(\tilde{\boldsymbol{x}}, \tilde{z}) \in E_L(f, X \cap$

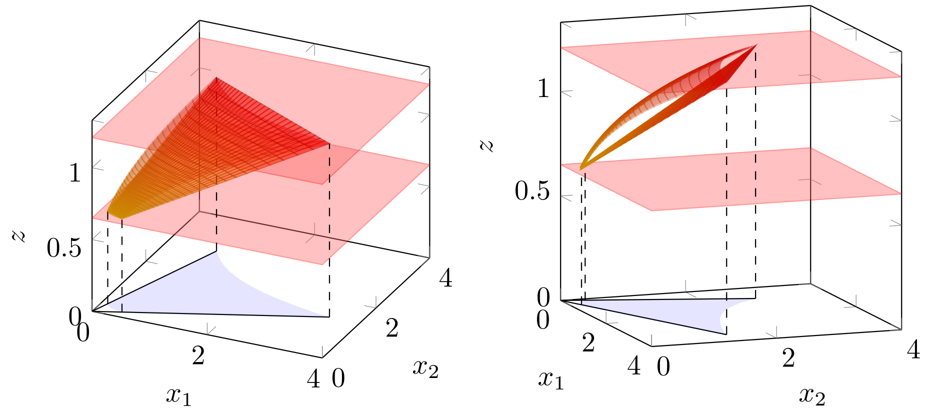


FIGURE 3. Lower envelope $f''_\ell(\boldsymbol{x})$ of function $f(\boldsymbol{x}) = x_1^{a_1} x_2^{a_2}$ shown under different angles. The domain $\{(x_1, x_2) \in \mathbb{R}^2_+ : px_1 \leq x_2 \leq qx_1, f(\boldsymbol{x}) \in [\ell, u]\}$ is the shaded area on the x, y plane. The parameters are as follows: $(p, q) = (0.4, 3.3)$, $(\ell, u) = (0.65, 1.21)$, $(a_1, a_2) = (0.1, 0.2)$.

W_{12}). This concludes the proof; note that $Y \setminus \text{int}(\text{conv}((C_\ell \cup C_u) \cap (P_{12} \cup Q_{12})))$ is the convex hull of $C_\ell \cap W_{12}$. □

Given a function f and its lower and upper envelopes f''_ℓ and f''_u over a set $\mathcal{D}$, the convex hull of $\{(\boldsymbol{x}, z) \in \mathbb{R}^{n+1} : z = f(\boldsymbol{x})\}$ is given by the intersection of lower and upper envelope [27]. The convex hull of $X \cap W_{ij}$ for $n = 2$ is hence the intersection of the upper and lower envelope described above.

Unlike the convex hull for $n = 2$, the one for $n > 2$ has one key feature: its projection onto the $\boldsymbol{x}$ variables is unbounded. For this reason the extra linear inequality $d_2 x_1 - d_1 x_2 \leq \left(\frac{u}{\lambda}\right)^{\frac{1}{\beta}}$ is only valid for $n = 2$.

Corollary 1. *The convex hull of $F(W_{12})$ for $n = 2$ and $\beta \geq 1$ is*

$$\begin{aligned} \text{conv}(F(W_{12})) = \; & \{(\boldsymbol{x}, z) \in Y \times \mathbb{R} : \\ & \max\{\ell, \lambda(d_2 x_1 - d_1 x_2)^\beta\} \leq z \leq \min\{u, z_0 + (\gamma x_1^{a_1} x_2^{a_2})^{\frac{1}{\beta}}\}\}. \end{aligned}$$

The convex hull of $F(W_{12})$ for $n = 2$ and $\beta \leq 1$ is

$$\begin{aligned} \text{conv}(F(W_{12})) = \; & \{(\boldsymbol{x}, z) \in Y \times \mathbb{R} : \\ & \max\{\ell, \zeta(d_2 x_1 - d_1 x_2) + z_0\} \leq z \leq \min\{u, x_1^{a_1} x_2^{a_2}\}\}. \end{aligned}$$

For $\beta \geq 1$, the upper envelope f'_u matches f only on the level sets of f, i.e., $C_\ell \cup C_u = \{\boldsymbol{x} \in \mathbb{R}^2_+ : f(x) \in \{\ell, u\}\}$, and the lower envelope f'_ℓ matches f on the set $P \cup Q = \{\boldsymbol{x} \in \mathbb{R}^2_+ : x_2 = px_1 \vee x_2 = qx_1\}$. For $\beta \leq 1$, the upper envelope f''_u matches f in $(P \cup Q) \cup (C_\ell \cup C_u)$, while the lower envelope f''_ℓ matches f in four points: those of the set $(P_{12} \cup Q_{12}) \cap (C_\ell \cup C_u)$.

Table 1 summarizes the results from this and the previous two sections. The lower envelope for W_{ij} is an open problem for $n > 2$.

5. VOLUME OF THE CONVEX HULL

Computing the volume of the convex hull of $f(\boldsymbol{x})$ within $X \cap W_{12}$ finds a practical application in choosing branching rules in a MINLO solver. Branching decisions

	$\beta \geq 1$	$\beta \leq 1$
	$n \geq 2$	
$\text{conv}(F(\mathbb{R}^n_+))$	$\boldsymbol{x} \in \mathbb{R}^n_+$ $(z - z_0)^\beta \leq \gamma \prod_{k\in N} x_k^{a_k}$ $z \in [\ell, u]$	$\boldsymbol{x} \in \mathbb{R}^n_+$ $z \leq \prod_{k\in N} x_k^{a_k}$ $z \in [\ell, u]$
$E_U(f, X \cap W_{ij})$	$\boldsymbol{x} \in \mathbb{R}^n_+$ $(z - z_0)^\beta \leq \gamma \prod_{k\in N} x_k^{a_k}$ $z \leq u$ $px_i \leq x_j \leq qx_i$ $\prod_{k\in N} x_k^{a_k} \geq \ell$	$\boldsymbol{x} \in \mathbb{R}^n_+$ $z \leq \prod_{k\in N} x_k^{a_k}$ $z \leq u$ $px_i \leq x_j \leq qx_i$ $\prod_{k\in N} x_k^{a_k} \geq \ell$
	$n = 2$	
$E_L(f, X \cap W_{ij})$	$px_1 \leq x_2 \leq qx_1$ $x_1^{a_1} x_2^{a_2} \geq \ell$ $d_2x_1 - d_1x_2 \leq (u/\lambda)^{\frac{1}{\beta}}$ $z \geq \lambda(d_2x_1 - d_1x_2)^\beta$ $z \geq \ell$	$px_1 \leq x_2 \leq qx_1$ $x_1^{a_1} x_2^{a_2} \geq \ell$ $d_2x_1 - d_1x_2 \leq (u/\lambda)^{\frac{1}{\beta}}$ $z \geq \zeta(d_2x_1 - d_1x_2) + z_0$ $z \geq \ell$
$E_U(f, X \cap W_{ij})$	$px_1 \leq x_2 \leq qx_1$ $x_1^{a_1} x_2^{a_2} \geq \ell$ $d_2x_1 - d_1x_2 \leq (u/\lambda)^{\frac{1}{\beta}}$ $(z - z_0)^\beta \leq \gamma x_1^{a_1} x_2^{a_2}$ $z \leq u$	$px_1 \leq x_2 \leq qx_1$ $x_1^{a_1} x_2^{a_2} \geq \ell$ $d_2x_1 - d_1x_2 \leq (u/\lambda)^{\frac{1}{\beta}}$ $z \leq x_1^{a_1} x_2^{a_2}$ $z \leq u$

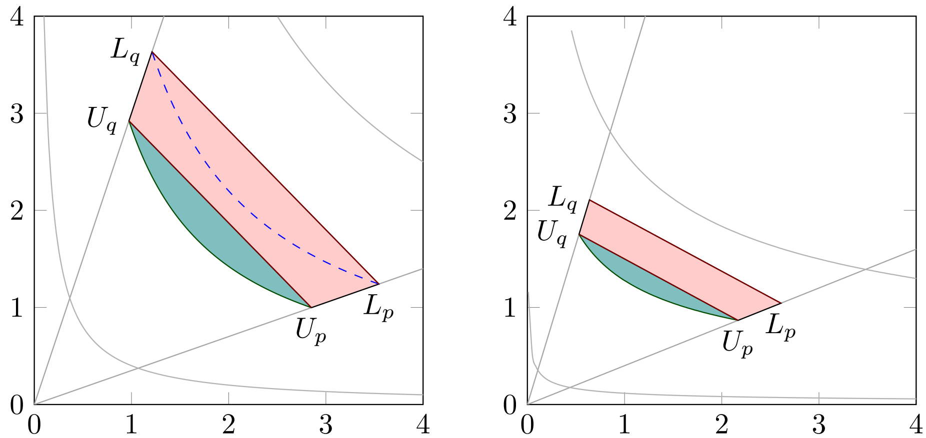
TABLE 1. Summary of results on lower and upper envelopes of f over X and over $X \cap W_{ij}$.

affect the efficiency of any BB algorithm in a strong and often unpredictable way. Even for the most standard branching decision $(x_k \leq b') \vee (x_k \geq b'')$, there is no supporting evidence of any single best branching policy for choosing the branching variable x_k [2, 8] or the branching point b', b'' [6].

The volume of the convex hull of nonlinear operators has been considered for bi- and tri-linear terms $x_i x_j$ and $x_i x_j x_k$ [13, 14, 24, 26], as a measure for quality of relaxations and for evaluating branching rules [25]–see also Anstreicher et al. [4].

Mixed Integer Linear Optimization (MILO) algorithms only concern themselves with choosing a branching variable at each node, but this is not the case for MINLO problems. Once a branching variable x_k is chosen, *balanced branching* attempts to create balanced BB subtrees by selecting a branching point such that the convex hulls of the feasible set of each of the two resulting subproblems have equal volume, in the hope of creating two balanced BB subtrees [6, 22]. An alternative branching criterion is that the resulting total volume of the convex hulls of the two subproblems is minimum. Rather than computing the volume of said convex hulls of the new feasible regions, it is more practical to compute it for the operators containing the branched-on variable.

We assume $n = 2$ from now on. The feasible region for a two-variable monomial term $z = x_1^{a_1} x_2^{a_2}$ considered here is defined by bounds $z \in [\ell, u]$ on the monomial itself and by the p, q parameters defining the homogeneous inequalities $px_1 \leq x_2 \leq qx_1$ delimiting x_1 and x_2. Branching on z and on x_1/x_2 allows for maintaining tight relaxations by using the convex envelopes described in this article. Therefore



(a) $(p,q) = (0.35, 3)$, $(a_1, a_2) = (1.7, 1.5)$, $(\ell, u) = (0.4, 10)$, $z = 4.4$

(b) $(p,q) = (0.4, 3.3)$, $(a_1, a_2) = (0.1, 0.2)$, $(\ell, u) = (0.65, 1.21)$, $z = 1.05$

FIGURE 4. Cross-section (two shades) of the convex envelope of $f(\boldsymbol{x}) = x_1^{a_1} x_2^{a_2}$ for $z \in [\ell, u]$ in the region $W_{12} = \{(x_1, x_2) \in \mathbb{R}_+^2 : px_1 \leq x_2 \leq qx_1\}$, for $\beta \geq 1$ (left) and $\beta \leq 1$. For $\beta \geq 1$, the level curve of $f(\boldsymbol{x})$ is the dashed arc between L_p and L_q. The arc between U_p and U_q is the level curve of the upper envelope $\hat{f}(\boldsymbol{x}) = z_0 + (\gamma x_1^{a_1} x_2^{a_2})^{\frac{1}{\beta}}$ (left) and the level set for $f(\boldsymbol{x})$ (right). Finally, the segment $L_p L_q$ is the level curve of the lower envelope $f'_\ell(\boldsymbol{x}) = \lambda(d_2 x_1 - d_1 x_2)^\beta = z$ (left) and $f''_\ell(\boldsymbol{x}) = \zeta(d_2 x_1 - d_1 x_2) + z_0 = z$ (right), for the given value of z.

one can consider two branching rules: given $r \in (p, q)$, either branch using $(px_1 \leq x_2 \leq rx_1) \vee (rx_1 \leq x_2 \leq qx_1)$ or using $(z \leq \nu) \vee (z \geq \nu)$.

Two questions arise: once the monomial term has been chosen for branching, do we branch on z or on $\frac{x_2}{x_1}$? Also, what is the branching point ν or r that minimizes the total volume in the two subproblems or, alternatively, finds the most balanced pair of subproblems?

Given an initial set defined by p, q, ℓ, u, a_1, a_2, the volume of the convex envelope denoted as $V(p, q, \ell, u)$, a branching rule on x_j/x_i using r as a ratio results in a total volume of $V_r = V(p, r, \ell, u) + V(r, q, \ell, u)$, while a branching rule on z with branching value ν results in a total volume of $V_b(\nu) = V(p, q, \ell, \nu) + V(p, q, \nu, u)$. If the minimum total volume is sought, one must select between two minimizers: $r \in \operatorname{argmin}\{V_r(r) : r \in [p + \epsilon, q - \epsilon]\}$ and $\nu \in \operatorname{argmin}\{V_b(\nu) : \nu \in [\ell + \epsilon, u - \epsilon]\}$. Here ϵ is used to obtain tighter bound interval on ν and r, which ensures that the feasible regions of both new subproblems are strict subsets of the subproblem being branched on. Balanced branching requires instead $V(p, r, \ell, u) = V(r, q, \ell, u)$ and $V(p, q, \ell, \nu) = V(p, q, \nu, u)$ respectively. Therefore, an analytical form of the volume of the convex hull described in Corollary 1 would be useful.

Both envelopes of f over $X \cap W_{12}$, i.e., $\max\{\ell, f_\ell(\boldsymbol{x})\}$ (lower) and $\min\{u, f_u(\boldsymbol{x})\}$ (upper), are non-smooth as f_ℓ and f_u have non-null gradient in all of W_{12} (except for f'_ℓ at $\boldsymbol{x}=0$). The domain $\mathcal{D}$ makes it additionally awkward to consider each envelope separately.

Consider any z in the range $[\ell,u]$ of z. The volume differential dV at z is the product of dz and the area of the cross-section of the convex envelope at z. Integrating the volume differential yields the volume of the convex hull. The main advantage of this method is that the structure of the cross-section only depends on z and the original parameters of the problem.

Figure 4 illustrates the cross-section as a shaded region, which we subdivide into two areas (indicated by different shades) for ease of computation. The shape of the cross-section is roughly the same for $\beta \geq 1$ and $\beta \leq 1$. The segment L_pL_q is the level curve of the lower envelope $f'_\ell(\boldsymbol{x})$ (for $\beta \geq 1$) and $f''_\ell(\boldsymbol{x})$ (for $\beta \leq 1$), while the arc between U_p and U_q is the level curve of the upper envelope $f_u(\boldsymbol{x})$. Together with the lines $x_2 = px_1$ and $x_2 = qx_1$ defining P_{12} and Q_{12}, these delimit the cross-section. The dashed arc between L_p and L_q is the level curve of $f(\boldsymbol{x})$ at z for $\beta \geq 1$ (see Figure 4a) and is only shown for completeness; it will not be used in our derivation.

As proven in Section 4, the extremes U_p, U_q of the arc also form a segment that is parallel to L_pL_q. The area of the cross-section is hence the sum of two areas: that of the lightly-shaded trapezoid with vertices L_p, L_q, U_q, U_p and the dark-shaded area between the arc and the segment both delimited by U_p, U_q.

Trapezoid. The area of the trapezoid $L_pL_qU_qU_p$ is $\frac{1}{2}\left(\Delta_{\rm U}+\Delta_{\rm L}\right)\delta$, where $\Delta_{\rm U}$ and $\Delta_{\rm L}$ are the lengths of U_pU_q and L_pL_q, respectively, and δ is the distance between these two parallel segments. We need x_1, x_2 coordinates of all four points: $L_p = (x_1^{{\rm L}p}, x_2^{{\rm L}p})$, $L_q = (x_1^{{\rm L}q}, x_2^{{\rm L}q})$, $U_p = (x_1^{{\rm U}p}, x_2^{{\rm U}p})$, $U_q = (x_1^{{\rm U}q}, x_2^{{\rm U}q})$. Then $\Delta_{\rm L}$ and $\Delta_{\rm U}$ are $\|\boldsymbol{x}^{{\rm L}p} - \boldsymbol{x}^{{\rm L}q}\|_2$ and $\|\boldsymbol{x}^{{\rm U}p} - \boldsymbol{x}^{{\rm U}q}\|_2$ respectively.

$$
\begin{aligned}
\Delta_{\rm L} &= \left((x_1^{{\rm L}p} - x_1^{{\rm L}q})^2 + (px_1^{{\rm L}p} - qx_1^{{\rm L}q})^2)\right)^{\frac{1}{2}} \\
&= \left((1+p^2)(x_1^{{\rm L}p})^2 + (1+q^2)(x_1^{{\rm L}q})^2 - 2(1+pq)x_1^{{\rm L}p}x_1^{{\rm L}q}\right)^{\frac{1}{2}} \\
&= x_1^{{\rm L}p}\left(1+p^2+\eta_1^2(1+q^2) - 2\eta_1(1+pq)\right)^{\frac{1}{2}} \\
&= x_1^{{\rm L}p}\left(1+p^2+\eta_1^2+\eta_1^2q^2 - 2\eta_1 - 2\eta_1pq\right)^{\frac{1}{2}} \\
&= x_1^{{\rm L}p}\left((1-\eta_1)^2+p^2+\eta_2^2p^2 - 2\eta_2p^2\right)^{\frac{1}{2}} \\
&= x_1^{{\rm L}p}\left((1-\eta_1)^2+p^2(1-\eta_2)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Define $\tau = \sqrt{(1-\eta_1)^2 + p^2(1-\eta_2)^2}$. Then $\Delta_{\rm L} = \tau x_1^{{\rm L}p}$ and $\Delta_{\rm U} = \tau x_1^{{\rm U}p}$.

In order to compute δ, consider the equations of the line through $\boldsymbol{x}^{{\rm U}p}$ and $\boldsymbol{x}^{{\rm U}q}$ and the one through $\boldsymbol{x}^{{\rm L}p}$ and $\boldsymbol{x}^{{\rm L}q}$. They have the same slope,

$$\sigma = \frac{d_2}{d_1} = p\frac{\eta_2 - 1}{\eta_1 - 1} < 0.$$

Then the two equations are $x_2 = \sigma x_1 + (x_2^{{\rm U}p} - \sigma x_1^{{\rm U}p})$ and $x_2 = \sigma x_1 + (x_2^{{\rm L}p} - \sigma x_1^{{\rm L}p})$. The former equation is used below to compute the area of the remaining part of the cross-section. The distance between two lines with equal slope $x_2 = \sigma x_1 + r'$ and $x_2 = \sigma x_1 + r''$ is $\frac{|r''-r'|}{\sqrt{1+\sigma^2}}$, therefore

$$\delta = \tfrac{|(x_2^{{\rm U}p} - \sigma x_1^{{\rm U}p}) - (x_2^{{\rm L}p} - \sigma x_1^{{\rm L}p})|}{\sqrt{1+\sigma^2}} = \tfrac{|(p-\sigma)x_1^{{\rm U}p} - (p-\sigma)x_1^{{\rm L}p}|}{\sqrt{1+\sigma^2}} = \tfrac{p-\sigma}{\sqrt{1+\sigma^2}}(x_1^{{\rm L}p} - x_1^{{\rm U}p}).$$

Hence the area of the trapezoid is

$$A_1(z) = \frac{1}{2}\delta(\Delta_{\rm L} + \Delta_{\rm U}) = \frac{\tau(p-\sigma)}{2\sqrt{1+\sigma^2}}((x_1^{{\rm L}p})^2 - (x_1^{{\rm U}p})^2).$$

Note that $\sqrt{1+\sigma^2} = \sqrt{1 + p^2\left(\frac{\eta_2-1}{\eta_1-1}\right)^2} = \frac{1}{1-\eta_1}\sqrt{(\eta_1-1)^2 + p^2(\eta_2-1)^2}$ and that $p - \sigma = p - p\frac{\eta_2-1}{\eta_1-1} = p\frac{\eta_2-\eta_1}{\eta_1-1}$, therefore

$$\frac{\tau(p-\sigma)}{2\sqrt{1+\sigma^2}} = \frac{p}{2}(\eta_2 - \eta_1).$$

In the remainder of this section, the calculation for the cross-section area is divided for the two cases $\beta \geq 1$ and $\beta \leq 1$.
Case 1: $\beta \geq 1$. The coordinates $(x_1^{{\rm L}p}, x_2^{{\rm L}p})$ of L_p satisfy $\lambda(d_2x_1^{{\rm L}p} - d_1x_2^{{\rm L}p})^\beta = x_1^{a_1}x_2^{a_2} = z$ (because f'_ℓ matches f on $P \cup Q$), whereas those of U_p satisfy $z_0 + \left(\gamma(x_1^{{\rm U}p})^{a_1}(x_2^{{\rm U}p})^{a_2}\right)^{\frac{1}{\beta}} = z$; both satisfy $x_2 = px_1$. Therefore

$$\begin{aligned} (x_1^{{\rm L}p})^{a_1}(px_1^{{\rm L}p})^{a_2} = z \quad &\Leftrightarrow \quad x_1^{{\rm L}p} = (p^{-a_2}z)^{\frac{1}{\beta}} \\ z_0 + \left(\gamma p^{a_2}(x_1^{{\rm U}p})^\beta\right)^{\frac{1}{\beta}} = z \quad &\Leftrightarrow \quad x_1^{{\rm U}p} = \frac{z-z_0}{(\gamma p^{a_2})^{\frac{1}{\beta}}}, \end{aligned}$$

and similarly for other points we obtain

$$\begin{array}{llll} x_1^{{\rm L}p} = (p^{-a_2}z)^{\frac{1}{\beta}}, & x_2^{{\rm L}p} = px_1^{{\rm L}p}, & x_1^{{\rm U}p} = \frac{z-z_0}{(\gamma p^{a_2})^{\frac{1}{\beta}}}, & x_2^{{\rm U}p} = px_2^{{\rm U}p}, \\ x_1^{{\rm L}q} = (q^{-a_2}z)^{\frac{1}{\beta}}, & x_2^{{\rm L}q} = qx_1^{{\rm L}q}, & x_1^{{\rm U}q} = \frac{z-z_0}{(\gamma q^{a_2})^{\frac{1}{\beta}}}, & x_2^{{\rm U}q} = qx_2^{{\rm U}q}. \end{array} \tag{28}$$

The area of the trapezoid is hence $A_1(z) = \frac{1}{2}p(\eta_2 - \eta_1)\left(z^{\frac{2}{\beta}} - (z-z_0)^2/\gamma^{\frac{2}{\beta}}\right)$.

The area defined by arc and segment between U_p and U_q is equal to the integral of the difference between two functions: the linear function through U_p and U_q, $x_2 = \sigma x_1 + (x_2^{{\rm U}p} - \sigma x_1^{{\rm U}p})$, and the function $x_2 = \left((\frac{z-z_0}{\gamma})^\beta x_1^{-a_1}\right)^{\frac{1}{a_2}}$ which arises from solving $z_0 + \gamma(x_1^{a_1}x_2^{a_2})^{\frac{1}{\beta}} = z$ for x_2.

The difference is $\sigma x_1 + (p-\sigma)x_1^{{\rm U}p} - (\frac{z-z_0}{\gamma})^{\frac{\beta}{a_2}} x_1^{-\frac{a_1}{a_2}}$, whose primitive is

$$\begin{cases} \frac{1}{2}\sigma x_1^2 + (p-\sigma)x_1^{{\rm U}p}x_1 - \frac{a_2}{a_2-a_1}(\frac{z-z_0}{\gamma})^{\frac{\beta}{a_2}} x_1^{\frac{a_2-a_1}{a_2}} & \text{if } a_1 \neq a_2 \\ \frac{1}{2}\sigma x_1^2 + (p-\sigma)x_1^{{\rm U}p}x_1 - (\frac{z-z_0}{\gamma})^{\frac{\beta}{a_2}} \log x_1 & \text{otherwise,} \end{cases}$$

which yields, for $a_1 \neq a_2$,

$$\begin{aligned} A_2(z) = \; & \tfrac{1}{2}\sigma((x_1^{{\rm U}p})^2 - (x_1^{{\rm U}q})^2) + (p-\sigma)(x_1^{{\rm U}p} - x_1^{{\rm U}q}) - \\ & \tfrac{a_2}{a_2-a_1}(\tfrac{z-z_0}{\gamma})^{\frac{\beta}{a_2}}((x_1^{{\rm U}p})^{\frac{a_2-a_1}{a_2}} - (x_1^{{\rm U}q})^{\frac{a_2-a_1}{a_2}}), \end{aligned}$$

and $A_2(z) = \frac{1}{2}\sigma((x_1^{{\rm U}p})^2 - (x_1^{{\rm U}q})^2) + (p-\sigma)(x_1^{{\rm U}p} - x_1^{{\rm U}q}) - (\frac{z-z_0}{\gamma})^{\frac{\beta}{a_2}}(\log x_1^{{\rm U}p} - \log x_1^{{\rm U}q})$ otherwise.

The total area of the cross-section is then a polynomial function with terms $z^{\frac{2}{\beta}}$, $(z-z_0)^2$, and $(z-z_0)^{\frac{\beta}{a_2}}$. For simplicity we write it as $A(z) = b_1z^{\frac{2}{\beta}} + b_2(z - z_0)^2 + b_3(z-z_0)^{\frac{\beta}{a_2}}$ for opportune values of b_1, b_2, b_3 which depend on a_1, a_2, p, q, ℓ, u.

Therefore the volume of the convex hull for $n = 2$ is $\mathrm{Vol}(\mathrm{conv}(X \cap W_{12})) = \int_\ell^u A(z)dz$, i.e.,

$$\begin{aligned} & \left[b_1 \tfrac{\beta}{\beta+2} z^{1+\frac{2}{\beta}} + \tfrac{1}{3} b_2 (z-z_0)^3 + \tfrac{a_2}{a_2+\beta} b_3 (z-z_0)^{1+\frac{\beta}{a_2}}\right]_\ell^u \\ = \; & b_1 \tfrac{\beta}{\beta+2} (u^{1+\frac{2}{\beta}} - \ell^{1+\frac{2}{\beta}}) + \tfrac{1}{3} b_2 \left((u-z_0)^3 - (\ell-z_0)^3\right) + \\ & \tfrac{a_2}{a_2+\beta} b_3 \left((u-z_0)^{1+\frac{\beta}{a_2}} - (\ell-z_0)^{1+\frac{\beta}{a_2}}\right), \end{aligned}$$

where b_3 is defined differently depending on whether $a_1 = a_2$ or not.
Case 2: $\beta \leq 1$. For L_p one must solve the system $x_2^{lp} = p x_1^{lp}$, $z = \zeta(d_2 x_1^{lp} - d_1 x_2^{lp}) + z_0$ for x_1^{lp}, which yields $x_1^{lp} = \frac{z-z_0}{(d_2-d_1 p)\zeta} = \frac{(z-z_0)\left(u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}\right)}{(u-\ell) q^{\frac{a_2}{\beta}}}$. For U_p and U_q, note that f constitutes the upper envelope, hence $(x_1^{\mathrm{U}p})^{a_1} (x_2^{\mathrm{U}p})^{a_2} = z$ and $x_2^{\mathrm{U}p} = p x_1^{\mathrm{U}p}$ yields $x_1^{\mathrm{U}p} = z^{\frac{1}{\beta}} p^{-a_2}$. Then the area of the trapezoid specializes to

$$A_1(z) = \frac{p}{2}(\eta_2 - \eta_1)\left(\left(\frac{(z-z_0)\left(u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}\right)}{(u-\ell) q^{\frac{a_2}{\beta}}}\right)^2 - (z^{\frac{1}{\beta}} p^{-a_2})^2\right).$$

Finally, the curve delimiting $X \cap W_{12}$ from below has equation $x_2 = z^{\frac{1}{a_2}} x_1^{-\frac{a_1}{a_2}}$, while the line between $\boldsymbol{x}^{\mathrm{U}q}$ and $\boldsymbol{x}^{\mathrm{U}p}$ is $x_2 = x_2^{\mathrm{U}p} + \sigma(x_1 - x_1^{\mathrm{U}p}) = \sigma x_1 + x_1^{\mathrm{U}p}(p-\sigma) = \sigma x_1 + (p-\sigma) z^{\frac{1}{\beta}} p^{-a_2}$. The area of $\mathrm{conv}(C_\ell \cap W_{12})$ is then the integral, between $x_1^{\mathrm{U}q}$ and $x_1^{\mathrm{U}p}$, of the function $\sigma x_1 + (p-\sigma) z^{\frac{1}{\beta}} p^{-a_2} - z^{\frac{1}{a_2}} x_1^{-\frac{a_1}{a_2}}$, whose primitive is similar, in structure, to that for the case $\beta \geq 1$, and thus we omit it here. The volume of the convex hull for $\beta \leq 1$ is therefore an integral of a polynomial in z with the same exponents as for $\beta \geq 1$, but with different coefficients due to different coordinates of L_p, L_q, U_p, U_q.

6. Concluding remarks and open questions

We considered the monomial $f(\boldsymbol{x}) = \prod_{k \in N} x_k^{a_k}$, with positive exponents, for $\boldsymbol{x} \in \mathbb{R}_+^n$. We proved that its upper envelope, for general $n \geq 2$, is a conic function when the domain of f restrict its value between $\ell \geq 0$ and u. The result holds even when two variables x_i, x_j, apart from being non-negative, are also constrained to form a linear cone $p x_i \leq x_j \leq q x_i$ pointed at the origin.

For $n = 2$ and when maintaining the linear cone $p x_1 \leq x_2 \leq q x_1$, we also find the lower envelope of $f(\boldsymbol{x})$ and thus are able to describe the convex hull of $F(W_{12})$ and its volume.

A seemingly easy extension that we have not considered here is the case with $n = 2, a_1, a_2 < 0$, which makes f convex and perhaps yields a convex hull similar to that of the case $\beta \leq 1$.

As discussed in Section 1, these results are applicable to either problems whose model has constraints such as $p x_i \leq x_j \leq q x_i$ or algorithms that enforce such constraints as cutting planes or branching operations. If such *wedge* constraints are not a natural way to describe a problem or an algorithm, the obvious tradeoff is between two approaches for MINLO:

- the classic approach, where variables have initial lower and upper bounds and branching rules are of the form $x_k \leq b \vee x_k \geq b$, but the convex hull of $\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1^{a_1} x_2^{a_2} \in [\ell, u], x_1 \in [\ell_1, u_1], x_2 \in [\ell_2, u_2]\}$ is not known and therefore a decomposition like the one described in Section 1 is needed;

- an approach where the conic constraint $px_1 \leq x_2 \leq qx_1$ is enforced and the convex hull is known.

Such tradeoff can make one or the other relaxation tighter. Computational tests are needed to assess the quality of the convex hull above, especially because the partitioning of $\mathbb{R}^2_+$ through linear inequalities is not as standard as variable bounds.

References

[1] The FICO-Xpress Optimizer, 2023. URL `https://www.fico.com/fico-xpress-optimization/docs/latest/overview.html`.

[2] T. Achterberg, T. Koch, and A. Martin. Branching rules revisited. *OR Letters*, 33(1):42–54, 2005.

[3] F. A. Al-Khayyal and J. E. Falk. Jointly constrained biconvex programming. *Mathematics of Operations Research*, 8:273–286, 1983.

[4] K. M. Anstreicher, S. Burer, and K. Park. Convex hull representations for bounded products of variables. *Journal of Global Optimization*, 80:757–778, 2021.

[5] P. Belotti. COUENNE: a user's manual. Technical report, Lehigh University, 2009. URL `https://github.com/coin-or/Couenne/raw/master/doc/couenne-user-manual.pdf`.

[6] P. Belotti, J. Lee, L. Liberti, F. Margot, and A. Wächter. Branching and bounds tightening techniques for non-convex MINLP. *Optimization Methods & Software*, 24(4-5):597–634, 2009.

[7] P. Belotti, A. J. Miller, and M. Namazifar. Valid inequalities and convex hulls for multilinear functions. *Electronic Notes in Discrete Mathematics*, 36:805–812, 2010. ISSN 1571-0653. doi: 10.1016/j.endm.2010.05.102. ISCO 2010 - International Symposium on Combinatorial Optimization.

[8] M. Bénichou, J. M. Gauthier, P. Girodet, G. Hentges, G. Ribière, and O. Vincent. Experiments in mixed-integer linear programming. *Mathematical Programming*, 1:76–94, 1971.

[9] K. Bestuzheva, M. Besançon, W.-K. Chen, A. Chmiela, T. Donkiewicz, J. van Doornmalen, L. Eifler, O. Gaul, G. Gamrath, A. Gleixner, et al. The SCIP optimization suite 8.0. *arXiv preprint arXiv:2112.08872*, 2021.

[10] S. Boyd, S.-J. Kim, L. Vandenberghe, and A. Hassibi. A tutorial on geometric programming. *Optimization and Engineering*, 8(1):67–127, 2007.

[11] J. G. Ecker. Geometric programming: methods, computations and applications. *SIAM review*, 22(3):338–362, 1980.

[12] R. Horst and H. Tuy. *Global Optimization*. Springer-Verlag, New York, 1993.

[13] J. Lee, D. Skipper, and E. Speakman. Algorithmic and modeling insights via volumetric comparison of polyhedral relaxations. *Mathematical Programming*, 170(1):121–140, 2018.

[14] J. Lee, D. Skipper, and E. Speakman. Gaining or losing perspective. In *World Congress on Global Optimization*, pages 387–397. Springer, 2019.

[15] H.-C. Lu, H.-L. Li, C. E. Gounaris, and C. A. Floudas. Convex relaxation for solving posynomial programs. *Journal of Global Optimization*, 46(1):147–154, 2010.

[16] G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I — Convex underestimating problems. *Mathematical Programming*, 10:146–175, 1976.

[17] C. A. Meyer and C. A. Floudas. Trilinear monomials with mixed sign domains: Facets of the convex and concave envelopes. *Journal of Global Optimization*, 29(2), 2004.

[18] R. Misener and C. A. Floudas. Antigone: algorithms for continuous/integer global optimization of nonlinear equations. *Journal of Global Optimization*, 59 (2-3):503–526, 2014.

[19] A. Neumeier. *Interval methods for systems of equations.* Cambridge Univ. Press, Cambridge, 1990.

[20] T. T. Nguyen, J.-P. P. Richard, and M. Tawarmalani. Deriving convex hulls through lifting and projection. *Mathematical Programming*, 169(2):377–415, 2018.

[21] A. D. Rikun. A convex envelope formula for multilinear functions. *Journal of Global Optimization*, 10:425–437, 1997.

[22] N. Sahinidis. Baron: a general purpose global optimization software package. *Journal of Global Optimization*, 8:201–205, 1996.

[23] E. M. B. Smith and C. C. Pantelides. A symbolic reformulation/spatial branch-and-bound algorithm for the global optimisation of nonconvex MINLPs. *Computers & Chem. Eng.*, 23:457–478, 1999.

[24] E. Speakman and J. Lee. Quantifying double McCormick. *Mathematics of Operations Research*, 42(4):1230–1253, 2017.

[25] E. Speakman and J. Lee. On branching-point selection for trilinear monomials in spatial branch-and-bound: the hull relaxation. *Journal of Global Optimization*, 72(2):129–153, 2018.

[26] E. Speakman, H. Yu, and J. Lee. Experimental validation of volume-based comparison for double-McCormick relaxations. In *International Conference on AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems*, pages 229–243. Springer, 2017.

[27] M. Tawarmalani and J.-P. P. Richard. Decomposition techniques in convexification of inequalities. Technical report, Technical report, 2013.

[28] M. Tawarmalani and N. V. Sahinidis. *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software, and Applications.* Kluwer Academic Publishers, Boston MA, 2002.

[29] J.-F. Tsai and M.-H. Lin. An efficient global approach for posynomial geometric programming problems. *INFORMS Journal on Computing*, 23(3):483–492, 2011.