Self-concordant Smoothing for Large-Scale Convex Composite Optimization

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Abstract

We introduce a notion of *self-concordant smoothing* for minimizing the sum of two convex functions, one of which is smooth and the other may be nonsmooth. The key highlight of our approach is in a natural property of the resulting problem's structure which provides us with a variable-metric selection method and a step-length selection rule particularly suitable for proximal Newton-type algorithms. In addition, we efficiently handle specific structures promoted by the nonsmooth function, such as ℓ_1 -regularization and group-lasso penalties. We prove the convergence of two resulting algorithms: Prox-N-SCORE, a proximal Newton algorithm and Prox-GGN-SCORE, a proximal generalized Gauss-Newton algorithm. The Prox-GGN-SCORE algorithm highlights an important approximation procedure which helps to significantly reduce most of the computational overhead associated with the inverse Hessian. This approximation is essentially useful for overparameterized machine learning models and in the mini-batch settings. Numerical examples on both synthetic and real datasets demonstrate the efficiency of our approach and its superiority over existing approaches. A Julia package implementing the proposed algorithms is available at https://github.com/adeyemiadeoye/SelfConcordantSmoothOptimization.jl.

1. Introduction

We consider the composite optimization problem

$$\min_{x \in \mathbb{R}^n} \mathcal{L}(x) \coloneqq f(x) + g(x), \tag{1}$$

where f is a smooth, convex loss function and g is a closed, proper, convex regularization function which may be nonsmooth. A common smoothing framework for solving (1) involves replacing the nonsmooth function g sequentially by its smooth approximation such that with an efficient algorithm for solving the resulting smooth optimization problem, we may approach the solution of the original problem. However, as noted in [1], the nonsmooth function g in (1) often plays a key role in describing some desirable properties specific to the application in which it appears, such as sparsifying the solution of the problem or enforcing some penalties or constraints on x, e.g., in sparse signal recovery and image processing [2, 3], compressed sensing [4, 5], model predictive control of constrained dynamical systems [6, 7, 8], neural network training [9], as well as various classification and regression problems in machine learning. In order to retain such properties about the optimization vector x in these applications, [1] proposes to keep a part of g unchanged and hence considers a *partial smoothing* where g is only partially smoothed. The particular class of problems considered in [1] are those in which the nonsmooth function q takes on the form $q(x) = \mathcal{R}(x) + \Omega(x)$, and there, it is proposed

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to smooth a part of g, say, \mathcal{R} , leaving the other part, say, Ω unchanged. Nevertheless, in many of the applications where this class of problems appears, each of \mathcal{R} and Ω is used to promote particular structures in the solution estimates, and hence smoothing one of them potentially destroys the overall desired structure. Prominent examples are found in lasso and multi-task regression problems with structured sparsity-inducing penalties. More specifically, $\mathcal{R}(x)$ is the scaled ℓ_1 -norm penalty $\beta \|x\|_1$ that encourages sparse estimates of x for $\beta > 0$, and $\Omega(x)$ additionally enforces a more specific structure on these estimates, such as groups and fused structures.

One of the main motivations in [1] for the partial smoothing technique is the possibility to derive "fast" proximal gradient methods [10, 11, 12, 13] for the resulting problem. While the fast proximal methods prove to be more efficient than methods such as the subgradient and bundle-type methods. they are first-order methods which often fall back to weak solution estimates and accuracies [14]. It is evident, from their performance on unconstrained smooth optimization problems, that incorporating second-order information into a gradient scheme often yields superior performance and better solution quality. A line of work (see, e.g., [15, 16, 17, 18, 19]) has made efforts to incorporate (approximate) second-order information into proximal gradient schemes to emulate the performance of relative second-order methods for unconstrained smooth problems. The main drawback here is the computational overhead associated with second-order methods. This drawback is often largely mitigated by choosing a special structure for the matrix of the second-order terms of f. To deal with globalization issues, some of these approaches assume specific structures and regularity of the function f. For example, the authors in [18] assume a self-concordant structure of f allowing for efficient step-size and correction techniques for proximal Newton-type and proximal quasi-Newton algorithms. However, because f oftentimes define a loss or data-misfit in real-world applications, the self-concordant assumption is not easy to check for many of these applications, and also restricts the applicability of the approach. Our self-concordant smoothing framework in this work provides a remedy to this limitation. We propose a new step-size selection technique that is suitable for Newton-type and quasi-Newton-type methods. This also exploits a self-concordant structure albeit not imposed on any of functions f and q that define the original problem.

In particular, we regularize¹ problem (1) by a second smooth function g_s and propose to keep all parts of g unchanged, but instead solve the following problem:

$$\min_{x \in \mathbb{R}^n} \mathcal{L}_s(x) \coloneqq f(x) + g_s(x;\mu) + g(x), \tag{2}$$

where² g_s is a self-concordant, epi-smoothing function for g with $\mu > 0$ (see Definition 3 below). By way of construction, g and g_s do not conflict and hence, a solution of (2) can be shown to solve (1). We do not give a special attention to the particular structure induced by g in the development of our technique. Yet in §4, we propose an approach to incorporate certain known structures into our framework, thereby making it amenable to more general structured penalty functions. In particular, for the lasso and multi-task regression problems with structured sparsity-inducing penalties, we highlight the relation between Nesterov's smoothing [11] for a class of structured problems and the smoothing framework of this work, and then synthesize the so-called "prox-decomposition" property of g with the smoothness property of g_s for easily handling the structures promoted by g in the solution.

In the following, we highlight three points that are vital to the development of our algorithmic framework in this paper:

¹In this work, we use "regularization" and "smoothing" interchangeably but use "regularization" to emphasize explicit addition of a smooth function (a smooth approximation of the *nonsmooth part* of the problem) to the *smooth part* of the problem.

²We occasionally write $g_s(x)$ instead of $g_s(x;\mu)$ to refer to the same function.

- 1. The first is to notice that for many practical problems, specifically those arising from modern machine learning systems, we often deal with overparameterized models (that is, in which number of data points is much less than the size of the optimization vector x). In this case, the pure proximal Newton method is not computationally ideal. This necessitates the use of generalized Gauss-Newton (GGN) approximations which, by our stylized "augmentation" technique, can be found to provide a practically efficient proximal algorithm for overparameterized models in which f can be expressed as a finite sum.
- 2. Secondly, we observe that the infimal convolution smoothing technique that we will introduce to construct g_s reveals a structure that is characterized by the self-concordant regularization (SCORE) framework of [20]. This provides a way to devise efficient adaptive step-size selection rule for proximal Newton-type algorithms without imposing a self-concordant structure on the original problem.
- 3. Lastly, via the notion of *epi-smoothing functions* established in [21] (a weaker notion than the *smoothable functions* of [1]), we can guarantee certain convergence notions on the epigraph of g allowing to combine the smooth regularization technique of this paper with the Moreau-infimalbased (proximal) algorithms to handle the nonsmooth function g. As is customary, this assumes we can find an efficient method to compute a closed-form solution to the minimization of the sum of g and an auxiliary function ψ_{α} . However, unless the variable-metric associated with the proximal Newton-type method has a specific structure that can be exploited for computational efficiency, the scaled proximal operator can be very difficult to compute and potentially poses a serious numerical issue. For this, the "simple" structure of the Hessian of g_s naturally provides a good candidate for the variable-metric, which allows for an efficient computation of the scaled proximal operator.

Burke and Hoheisel [21, 22] developed the notion of *epi-smoothing* for studying several epigraphical convergence (*epi-convergence*) properties for convex composite functions by combining the infimal convolution smoothing framework due to Beck and Teboulle [1] with the idea of *gradient consistency* due to Chen [23]. The key variational analysis tool used throughout their development is the *coercivity* of the class of regularization kernels studied in [1]. In particular, they establish the close connection between epi-convergence of the regularization functions and supercoercivity of the regularization kernel. Then, based on the above observations, we synthesize this idea with the notion of *self-concordant regularization* [20] to propose two proximal-type algorithms, viz., Prox-N-SCORE (Algorithm 1) and Prox-GGN-SCORE (Algorithm 2), for convex composite minimization.

Paper organization. The rest of this paper is organized as follows: In §1.1, we present some notations and background on convex analysis. In §2, we establish our self-concordant smoothing notion with some properties and results. We describe our proximal Newton-type scheme in §3, and present the **Prox-N-SCORE** and **Prox-GGN-SCORE** algorithms. In §4, we describe an approach for handling specific structures promoted by the nonsmooth function g in problem (1), and propose a practical extension of the so-called *prox-decomposition* property of g for the self-concordant smoothing framework, which has certain in-built smoothness properties. Convergence properties of the **Prox-N-SCORE** and **Prox-GGN-SCORE** algorithms are studied in §5. In §6, we present some numerical simulation results for our proposed framework with an accompanying Julia package³, and compare the results with other state-of-the-art approaches. Finally, we give a concluding remark and discuss prospects for future research in §7.

 $^{{}^{3} \}tt{https://github.com/adeyemiadeoye/SelfConcordantSmoothOptimization.jl}$

1.1. Notation and preliminaries

We denote by $\mathbb{R} := \mathbb{R} \cup \{-\infty, +\infty\}$ the set of extended real numbers. The sets $\mathbb{R}_+ := [0, +\infty[$ and $\mathbb{P} := \mathbb{R}_+ \setminus \{0\}$, respectively, denote the set of nonnegative and positive real numbers. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The *(effective) domain* of g is given by dom $g := \{x \in \mathbb{R}^n \mid g(x) < +\infty\}$ and its *epigraph* (resp., *strict epigraph*) is given by epi $g := \{(x, \gamma) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leq \gamma\}$ (resp., epis $g := \{(x, \gamma) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) < \gamma\}$). Given $\gamma \in \mathbb{P}$, the γ -sublevel set of g is $\Gamma_{\gamma}(g) := \{x \in \mathbb{R}^n : g(x) \leq \gamma\}$. The standard inner product between two vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, that is, $\langle x, y \rangle := x^\top y$, where x^\top is the transpose of x. The operation $x \odot y$ denotes the Hadamard product between two vectors $x, y \in \mathbb{R}^n$; we also denote by x^2 the product $x \odot x$.

For an $n \times n$ matrix H, we write $H \succ 0$ (resp., $H \succeq 0$) to say H is positive definite (resp., positive semidefinite). The sets S_{+}^{n} and S_{++}^{n} , respectively, denote the set of $n \times n$ symmetric positive semidefinite and symmetric positive definite matrices. The set $\{\text{diag}(v) \mid v \in \mathbb{R}^{n}\}$, where $\text{diag}: \mathbb{R}^{n} \to \mathbb{R}^{n \times n}$, defines the set of all diagonal matrices in $\mathbb{R}^{n \times n}$. Matrix I_{d} denotes the $d \times d$ identity matrix. We denote by $\text{card}(\mathcal{G})$, the cardinality of a set \mathcal{G} . For any two functions f and g, we define $(f \circ g)(\cdot) \coloneqq f(g(\cdot))$. We denote by $\mathcal{C}^{k}(\mathbb{R}^{n})$, the class of k-times continuously-differentiable functions on \mathbb{R}^{n} , $k \ge 0$. If the p-th derivatives of a function $f \in \mathcal{C}^{k}(\mathbb{R}^{n})$ is L_{f} -Lipschitz continuous on \mathbb{R}^{n} with $p \le k$, $L_{f} \ge 0$, we write $f \in \mathcal{C}_{L_{f}}^{k,p}(\mathbb{R}^{n})$. The notation $\|\cdot\|$ stands for the standard Euclidean (or 2-) norm $\|\cdot\|_{2}$. We define the weighted norm induced by $H \in \mathcal{S}_{++}^{n}$ by $\|x\|_{H} \coloneqq \langle Hx, x \rangle^{\frac{1}{2}}$, for $x \in \mathbb{R}^{n}$. The associated dual norm is $\|x\|_{H}^{*} \coloneqq \langle H^{-1}x, x \rangle^{\frac{1}{2}}$. An Euclidean ball of radius r centered at \bar{x} is defined by $\mathcal{E}_{r}(\bar{x}) \coloneqq \{x \in \mathbb{R}^{n} \mid \|x - \bar{x}\|_{H} \le r\}$. We define the spectral norm $\|A\| \equiv \|A\|_{2}$ of a matrix $A \in \mathbb{R}^{m \times n}$ as the square root of the maximum eigenvalue of $A^{\top}A$, where A^{\top} is the transpose of A.

A convex function $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if dom $g \neq \emptyset$. The function g is said to be lower semicontinuous (lsc) at y if $g(y) \leq \liminf_{x \to y} g(x)$; if it is lsc at every $y \in \text{dom } g$, then it is said to be lsc on dom g. We denote by $\Gamma_0(D)$ the set of proper convex lsc functions from $D \subseteq \mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$. Given $g \in \mathcal{C}^3(\text{dom } g)$, we respectively denote by g'(t), g''(t) and g'''(t) the first, second and third derivatives of g, at $t \in \mathbb{R}$, and by $\nabla_x g(x)$, $\nabla^2_x g(x)$, and $\nabla^3_x g(x)$ the gradient, Hessian and third-order derivative tensor of g, respectively, at $x \in \mathbb{R}^n$; if the variables with respect to which the derivatives are taken are clear from context, the subscripts are omitted. If $\nabla^2 g(x) \in \mathcal{S}^n_{++}$ for a given $x \in \mathbb{R}^n$, then the local norm $\|\cdot\|_x$ with respect to g at x is defined by $\|d\|_x \coloneqq \langle \nabla^2 g(x) d, d \rangle^{1/2}$, for $v \in \mathbb{R}^n$. The subdifferential $\partial g: \mathbb{R}^n \to 2^{\mathbb{R}^n}$ of a proper function $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $x \mapsto \{u \in \mathbb{R}^n \mid (\forall y \in \mathbb{R}^n) \ \langle y - x, u \rangle + g(x) \leq g(y) \}$, where $2^{\mathbb{R}^n}$ denotes the set of all subsets of \mathbb{R}^n . The function g is said to be subdifferentiable at $x \in \mathbb{R}^n$ if $\partial g(x) \neq \emptyset$; the subgradients of g at x are the members of $\partial g(x)$.

We define set convergence in the sense of Painlevé-Kuratowski. Let \mathbb{N} denote the set of natural numbers. Let $\{C_k\}_{k\in\mathbb{N}}$ be a sequence of subsets of \mathbb{R}^n . The outer limit of $\{C_k\}_{k\in\mathbb{N}}$ is the set

$$\limsup_{k \to \infty} C_k \coloneqq \left\{ x \mid \exists N \text{ cofinal set of } \mathbb{N}, \exists x_k \in C_k \colon \{x_k\} \xrightarrow[N]{} x, \forall k \in N \right\}.$$

and its inner limit is

$$\liminf_{k \to \infty} C_k := \left\{ x \mid \exists N \text{ cofinite set of } \mathbb{N}, \exists x_k \in C_k \colon \{x_k\} \xrightarrow[N]{} x, \forall k \in N \right\}.$$

The limit C of $\{C_k\}_{k\in\mathbb{N}}$ exists if its outer and inner limits coincide, and we write

$$C = \lim_{k \to \infty} C_k \coloneqq \limsup_{k \to \infty} C_k = \liminf_{k \to \infty} C_k$$

We say that a function $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is coercive if $\liminf_{\|x\|\to\infty} g(x) = +\infty$, and supercoercive if $\liminf_{\|x\|\to\infty} \frac{g(x)}{\|x\|} = +\infty$. The sequence $\{g_k\}$ of functions $g_k: \mathbb{R}^n \to \mathbb{R}$ is said to epi-converge to the function $g: \mathbb{R}^n \to \mathbb{R}$ if $\lim_{k\to\infty} \operatorname{epi} g_k = \operatorname{epi} g$; it is said to continuously converge to g if for all $x \in \mathbb{R}^n$ and $\{x_k\} \to x$, we have $\lim_{k\to\infty} g_k = g$; and it converges pointwise to g if for all $x \in \mathbb{R}^n$, $\lim_{k\to\infty} g_k(x) = g(x)$. Epi-convergence, continuous convergence, and pointwise convergence of $\{g_k\}$ to g are respectively denoted by $e-\lim_{k\to\infty} g_k = g$ (or $g_k \not\in g$), $c-\lim_{k\to\infty} g_k = g$ (or $g_k \not\in g$), and $p-\lim_{k\to\infty} g_k = g$ (or $g_k \not\in g$).

The conjugate (or Fenchel conjugate, or Legendre transform, or Legendre-Fenchel transform) $g^* \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of a function $g \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the mapping $y \mapsto \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - g(x) \}$, and its biconjugate is $g^{**} = (g^*)^*$.

2. Self-concordant regularization

Since the pioneering work of Nesterov and Nemirovskii [24] on interior-point methods, the notion of self-concordant functions has helped to better understand the importance of exploiting the problem's structure to improve performance of optimization algorithms. This section introduces our notion of self-concordant smoothing which provides us with structures to exploit in composite optimization problems. We begin by presenting the definition of generalized self-concordant functions given in [25].

Definition 1 (Generalized self-concordant function on \mathbb{R}). A univariate convex function $g \in \mathcal{C}^3(\operatorname{dom} g)$, with dom g open, is said to be (M_g, ν) -generalized self-concordant, with $M_g \in \mathbb{R}_+$ and $\nu \in \mathbb{P}$, if

$$\left|g'''(t)\right| \le M_q \, g''(t)^{\frac{\nu}{2}}, \qquad \forall t \in \mathbb{R}.$$

Definition 2 (Generalized self-concordant function on \mathbb{R}^n of order ν). A convex function $g \in \mathcal{C}^3(\operatorname{dom} g)$, with dom g open, is said to be (M_g, ν) -generalized self-concordant of order $\nu \in \mathbb{P}$, with $M_g \in \mathbb{R}_+$, if $\forall x \in \operatorname{dom} g$

$$\left\langle \nabla^3 g(x)[v]u, u \right\rangle \right| \le M_g \|u\|_x^2 \|v\|_x^{\nu-2} \|v\|^{3-\nu}, \qquad \forall u, v \in \mathbb{R}^n,$$

where $\nabla^3 g(x)[v] \coloneqq \lim_{t \to 0} \left\{ \left(\nabla^2 g(x + tv) - \nabla^2 g(v) \right) / t \right\}.$

Note that for an (M_g, ν) -generalized self-concordant function g defined on \mathbb{R}^n , the univariate function $\varphi \colon \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) \coloneqq g(x + tv)$ is (M_g, ν) -generalized self-concordant for every $x, v \in \text{dom } g$ and $x + tv \in \text{dom } g$. This provides an alternative definition for the generalized self-concordant function on \mathbb{R}^n .

A key observation from the above definition is the possibility to extend the theory beyond the case $\nu = 3$ and u = v originally presented in [24]. This observation, for instance, allowed the authors in [26] to introduce a *pseudo* self-concordant framework, in which $\nu = 2$, for the analysis of logistic regression. In a recent development, the authors in [27] identified a new class of pseudo self-concordant functions

and showed how these functions may be slightly modified to make them *standard* self-concordant (i.e., where $M_g = 2, \nu = 3, u = v$), while preserving desirable structures. With such generalizations, and stemming from the idea of *Newton decrement* in [24], new analytic step-size selection and correction techniques for a number of proximal algorithms were developed in [18]. It is in the same spirit that we propose new step-size selection techniques from the self-concordant smoothing framework developed in this paper. We denote by $\mathcal{F}_{M_g,\nu}$ the class of (M_g,ν) -generalized self-concordant functions, with generalized self-concordant parameters $M_g \in \mathbb{R}_+$ and $\nu \in \mathbb{P}$.

Definition 3 (Self-concordant smoothing function). We say that the parameterized function $g_s \colon \mathbb{R}^n \times \mathbb{P} \to \mathbb{R}$ is a self-concordant smoothing function for $g \in \Gamma_0(\mathbb{R}^n)$ if the following two conditions are satisfied:

SC.1
$$e = \lim_{\mu \downarrow 0} g_s(x; \mu) = g(x).$$

SC.2 $g_s(x;\mu) \in \mathcal{F}_{M_g,\nu}$.

By construction, the class of functions exhibiting the property in SC.1 inherits the gradient and/or the Jacobian consistency properties introduced in [23] and [28], respectively. In [21, Lemma 3.4], the authors show the following property for epi-convergent smoothing functions (that is, the ones for which condition SC.1 holds):

$$\limsup_{\substack{x \to \bar{x} \\ \mu \downarrow 0}} \nabla g_s(x;\mu) = \partial g(\bar{x}). \tag{3}$$

The gradient consistency property holds upon taking the convex hull on both sides of (3) (see [23, Equation 4]). However, since $g \in \Gamma_0(\mathbb{R}^n)$, g is subdifferentially regular at any point $\bar{x} \in \text{dom } g$ (see [29, Definition 7.25 and Example 7.27]), and hence, the equivalence between (3) and gradient/Jacobian consistency holds [21]. Clearly, the relation in (3) implies inclusion in both directions, and hence as shown in [21], if an algorithm seeks sequences $\{x_k\}$ such that

$$\limsup_{x_k \to \bar{x}} \nabla g_s(x_k; \mu) \to 0,$$

one finds \bar{x} is a critical point of g in the sense of satisfying the necessary optimality conditions for problem (2) (cf. (13)), provided that $g_s \underline{e} g$.

We denote by $\mathcal{S}^{\mu}_{M_{g},\nu}$ the set of self-concordant smoothing functions for a function $g \in \Gamma_{0}(\mathbb{R}^{n})$, that is, $\mathcal{S}^{\mu}_{M_{g},\nu} \coloneqq \Big\{ g_{s} \colon \mathbb{R}^{n} \times \mathbb{P} \to \mathbb{R} \mid g_{s} \underbrace{e}_{} g, \ g_{s} \in \mathcal{F}_{M_{g},\nu} \Big\}.$

2.1. Self-concordant regularization via infimal convolution

We next present some important elements of smoothing via infimal convolution which provides the Moreau-Yosida regularization process as a special case in defining the (scaled) proximal operator.

Definition 4 (Infimal convolution). Let g and h be two functions from \mathbb{R}^n to $\mathbb{R} \cup \{+\infty\}$. The infimal convolution (or "inf-convolution" or "inf-conv")⁴ of g and h is the function $g \Box h \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$(g\Box h)(x) = \inf_{w\in\mathbb{R}^n} \left\{ g(w) + h(x-w) \right\}.$$
(4)

⁴Also sometimes called "epigraphic sum" or "epi-sum", as its operation yields the (strict) *epigraphic sum* epi f + epi g [30, p. 93].

The infimal convolution of g with h is said to be *exact at* $x \in \text{dom } g$ if the infimum (4) is attained. It is *exact* if it is exact at each $x \in \text{dom } g$, in which case we write $g \square h$. Of utmost importance about the inf-conv operation in this paper is its use in the approximation of a function $g \in \Gamma_0(\mathbb{R}^n)$; that is, the approximation of g by its infimal convolution with a member $h_{\mu}(\cdot)$ of a parameterized family $\mathcal{H} := \{h_{\mu} \mid \mu \in \mathbb{P}\}$ of (regularization) kernels. In more formal terms, we recall the notion of inf-conv regularization in Definition 5 below. For $h \in \Gamma_0(\mathbb{R}^n)$ and $\mu \in \mathbb{P}$, we define the function $h_{\mu} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by the *epi-multiplication* operation⁵

$$h_{\mu}(\cdot) \coloneqq \mu h\left(\frac{\cdot}{\mu}\right), \quad \mu \in \mathbb{P}.$$
 (5)

Definition 5 (Inf-conv regularization). Let g be a function in $\Gamma_0(\mathbb{R}^n)$. Define

$$\mathcal{H} \coloneqq \left\{ (x, w) \mapsto h_{\mu}(x - w) \mid x, w \in \mathbb{R}^{n}, \mu \in \mathbb{P} \right\},\tag{6}$$

a parameterized family of regularization kernels. The inf-conv regularization process of g with $h_{\mu} \in \mathcal{H}$ is given by $(g \Box h_{\mu})(x)$, for any $x \in \mathbb{R}^{n}$.

The operation of the inf-conv regularization generalizes the Moreau-Yosida regularization process in which case, $h_{\mu}(\cdot) = \|\cdot\|^2/(2\mu)$ or, with a scaled norm, $h_{\mu}(\cdot) = \|\cdot\|_Q^2/(2\mu)$ for some $Q \in S_{++}^n$. The Moreau-Yosida regularization process provides the value function of the proximal operator associated with a function $g \in \Gamma_0(\mathbb{R}^n)$. This leads us to the definition of the scaled proximal operator.

Definition 6 (Scaled proximal operator). The scaled proximal operator of a function $g \in \Gamma_0(\mathbb{R}^n)$, written $\operatorname{prox}_{\alpha g}^Q(\cdot)$, for $\alpha \in \mathbb{P}$ and $Q \in \mathcal{S}_{++}^n$, is defined as the unique point in dom g that satisfies

$$(g\Box\psi_{\alpha})(x) = g(\operatorname{prox}_{\alpha g}^{Q}(x)) + \psi_{\alpha}(x - \operatorname{prox}_{\alpha g}^{Q}(x)),$$

where $\psi_{\alpha}(\cdot) \coloneqq \left\|\cdot\right\|_{Q}^{2}/(2\alpha)$. That is, $\operatorname{prox}_{\alpha g}^{Q}(x) \coloneqq \operatorname*{arg\,min}_{w \in \mathbb{R}^{n}} \{g(w) + \psi_{\alpha}(x-w)\}.$

A key property of the scaled proximal operator is its *nonexpansiveness*; that is, the property that (see, e.g., [31, 18])

$$\left\|\operatorname{prox}_{\alpha g}^{Q}(x) - \operatorname{prox}_{\alpha g}^{Q}(y)\right\|_{y} \leq \|x - y\|_{y}^{*},\tag{7}$$

for all $x, y \in \mathbb{R}^n$.

In the sequel, we assume that the regularization kernel function h is of the form

$$h(x) = \sum_{i=1}^{n} \phi(x_i),\tag{8}$$

where ϕ is a univariate *potential function*. We note that, by convexity, g is lsc and hence $g \in \Gamma_0(\mathbb{R}^n)$. We are left with the question of what properties we need to hold for ϕ such that $g \Box h_{\mu}$ produces g_s satisfying the self-concordant smoothing conditions SC.1 – SC.2. To this end, we impose the following conditions on ϕ :

K.1 ϕ is supercoercive.

⁵It is easy to show that $h^*_{\mu} = \mu h^*$.

K.2 $\phi \in \mathcal{F}_{M_{\phi},\nu}$.

Many functions that appear in different settings naturally exhibit the structures in conditions K.1 – K.2. For example, the ones belonging to the class of *Bregman/Legendre functions* introduced by Bauschke and Borwein [32] (see also [33] for a related characterization of the class of *Bregman functions*). In the context of proximal gradient algorithms for solving (1), the recent paper by Bauschke and Borwein [34] enlists these functions as satisfying the new descent lemma (a.k.a descent lemma without Lipschitz gradient continuity) which the paper introduced. We summarize examples of these regularization kernel functions on different domains in **Table 1**. We extract practical examples on \mathbb{R} for the smoothing of the 1-norm and the indicator functions below.

Remark 1. Suppose that dom h is a nonempty bounded subset of \mathbb{R}^n , for example, if $\phi \in \Gamma_0(\mathbb{R})$, then since we have that $g \in \Gamma_0(\mathbb{R}^n)$ is bounded below as it possesses a continuous affine minorant (in view of [35, Theorem 9.20]), the less restrictive condition that ϕ is coercive sufficiently replaces the condition K.1. In other words, the key convergence notion presented below holds similarly for the resulting function $g \Box h_{\mu}$ in this case. Particularly, we get that $g \Box h_{\mu}$ in this case is exact, finite-valued and locally Lipschitz continuous (see, e.g., [22, Proposition 3.6]) making it fit into our algorithmic framework. We keep the supercoercivity condition to emphasize other realizable properties of $g \Box h_{\mu}$ highlighted below. Our examples in **Table 1** therefore include both coercive and supercoercive functions, where in either case, we have $\phi \in \mathcal{F}_{M_{\phi},\nu}$.

Examples. For some functions g and h_{μ} , there exists a closed form solution to $g \Box h_{\mu}$. On the other hand, if one gets that $g \Box h_{\mu} = g \boxdot h_{\mu} \in \Gamma_0(\mathbb{R}^n)$, e.g., as a result of Proposition 2(i) below, then knowing in this case that

$$g\Box h_{\mu} = (g^* + h_{\mu}^*)^*, \tag{9}$$

we can efficiently estimate $g \Box h_{\mu}$ using *fast* numerical schemes (see, e.g., [36]).

Infimal convolution of $\|\cdot\|_1$ with h_{μ} . Let $g(x) = \|x\|_1$. Below, we provide the infimal convolution of the function g with h_{μ} , where h is given by (8), for two of functions ϕ of practical interest given in **Table 1** such that SC.1 – SC.2 hold.

Example 1. Let p = 1 in $\phi(t) = \frac{1}{p}\sqrt{1 + p^2|t|^2} - 1$, with dom $\phi = \mathbb{R}$. Then,

$$(g\Box h_{\mu})(x) = \frac{\mu^2 - \mu\sqrt{\mu^2 + x^2} + x^2}{\sqrt{\mu^2 + x^2}}.$$

Example 2. $\phi(t) = \frac{1}{2} \left[\sqrt{1 + 4t^2} - 1 + \log\left(\frac{\sqrt{1 + 4t^2} - 1}{2t^2}\right) \right]$, with dom $\phi = \mathbb{R}$: $(g \Box h_{\mu})(x) = \frac{\sqrt{\mu^2 + 4x^2}}{2} - \frac{\mu}{2} \left[1 + \log(2) - \log\left(\frac{2x - \sqrt{\mu^2 + 4x^2} + \mu}{x}\right) - \log\left(\frac{2x + \sqrt{\mu^2 + 4x^2} - \mu}{x}\right) \right]$.



Fig. 1: Generalized self-concordant smoothing of $g(x) \coloneqq ||x||_1$ with h_{μ} defined by (5) where $h(x) = \sum \phi(x_i)$; $\phi(t) = \sqrt{1+|t|^2} - 1$ (left) and $\phi(t) = \frac{1}{2} \left[\sqrt{1+4t^2} - 1 + \log\left(\frac{\sqrt{1+4t^2}-1}{2t^2}\right) \right]$ (right). The smooth approximation $(g\Box h_{\mu})(x)$ is shown for $\mu = 0.2, 0.5, 1.0$.

Infimal convolution of $\delta_C(x)$ with h_{μ} . In a third example, we consider $g(x) = \delta_C(x)$, where $C := \{x \in \mathbb{R}^n \mid l \leq x \leq u\}$ and

$$\delta_C(x) \coloneqq \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Example 3. Let $\phi(t) = exp(-t)$ with dom $\phi = \mathbb{R}$, and consider $g(x) = \delta_C(x)$. We have

$$(g\Box h_{\mu})(x) = \mu \exp\left(\frac{l-x}{\mu}\right)$$

The next two results characterize the functions h and h_{μ} defined by supercoercive and generalized self-concordant kernel functions.

Lemma 1. Let $\phi \in \Gamma_0(\mathbb{R})$ be a function from \mathbb{R} to $\mathbb{R} \cup \{+\infty\}$, and let the function $h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined by $h(x) := \sum_{i=1}^n \lambda_i \phi_i$ with $\phi_i := \phi(x_i), x_i \in \text{dom } \phi, \lambda_i > 0, i = 1, 2, ..., n$. Then the following properties hold:

- (i) $h \in \Gamma_0(\mathbb{R}^n)$.
- (ii) h is supercoercive if and only if ϕ is supercoercive on its domain.
- (iii) If $\phi \in \mathcal{F}_{M_{\phi},\nu}$, where $M_{\phi} \in \mathbb{R}_+$ and $\nu \ge 2$, then h(x) is well-defined on dom $h = \{\operatorname{dom} \phi\}^n$, and $h(x) \in \mathcal{F}_{M_h,\nu}$, with $M_h \coloneqq \max\{\lambda_i^{1-\frac{\nu}{2}}M_{\phi} \mid 1 \le i \le n\} \ge 0$.

Proof. (i) This statement is a direct consequence of [35, Corollary 9.4, Lemma 1.27 and Proposition 8.17].

(ii) Follows directly from the definition of supercoercivity.

$\phi(t)$	$\mathrm{dom}\phi$	M_{ϕ}	ν	Remark
$-\sqrt{1-t^2}$	[-1, +1]	2.25	4	"Hellinger"
$\frac{1}{n}\sqrt{1+p^2 t ^2}-1, \ p\in\mathbb{P}$	\mathbb{R}	2	2.6	p = 1
$\frac{7}{22\sqrt{t(1-t)}}$	[0,1]	2.02	4	"Arcsine probability density"
$\frac{1}{2}\left[\sqrt{1+4t^2}-1+\log\left(\frac{\sqrt{1+4t^2}-1}{2t^2}\right)\right]$	\mathbb{R}	$2\sqrt{2}$	3	Ostrovskii & Bach [27]
$\frac{1}{2}t^2$	\mathbb{R}	0	3	"Energy"
$\frac{1}{n} t ^{p}, \ \tilde{p} \in (1,2)$	\mathbb{R}_+	4	6	p = 1.5
$\log^{r}(1+\exp(t))$	\mathbb{R}	1	2	"Logistic"
$\exp(-t)$	\mathbb{R}	1	2	"Exponential"
$t\log t - t$	$[0, +\infty]$	1	4	"Boltzmann-Shannon"
$t\log t + (1-t)\log(1-t)$	[0, 1]	1	4	"Fermi-Dirac"
$-\frac{1}{2}\log t$	\mathbb{P}^{1}	8	3	"Burg"
$\begin{cases} \frac{1}{2}(t^2 - 4t + 3), & \text{if } t \le 1\\ -\log t, & \text{otherwise} \end{cases}$	\mathbb{R}	4	3	De Pierro & Iusem [33]

Table 1: Examples of regularization kernel functions for self-concordant smoothing, and their generalized self-concordant parameters M_{ϕ} and ν (see Definition 1).

(iii) $h(\cdot) \in \mathcal{F}_{M_h,\nu}$ with $M_h \coloneqq \max\{\lambda_i^{1-\frac{\nu}{2}}M_\phi \mid 1 \le i \le n\} \ge 0$ follows from [25, Proposition 1].

Proposition 1 (Self-concordance of h_{μ}). Suppose the conditions of Lemma 1 hold such that the function $h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by (8) is (M_h, ν) -generalized self-concordant. Let $A \in \mathbb{R}^{n \times n}$ be a diagonal matrix defined by $A := \operatorname{diag}(\frac{1}{\mu})$ such that $h(\frac{x}{\mu}) \equiv h(Ax)$ is an affine transformation of h(x). Then the following properties hold:

- (i) If $\nu \in (0,3]$, then $h_{\mu} \in \mathcal{F}_{M,\nu}$ with $M = n^{\frac{3-\nu}{2}} \mu^{\frac{\nu}{2}-2} M_h$.
- (ii) If $\nu > 3$, then $h_{\mu} \in \mathcal{F}_{M,\nu}$ with $M = \mu^{4-\frac{3\nu}{2}} M_h$.

Proof. (i) We have $||A|| = \frac{\sqrt{n}}{\mu}$. By [25, Proposition 2(a)], $h(\frac{x}{\mu}) \in \mathcal{F}_{M,\nu}$ with $M = ||A||^{3-\nu} M_h$. In view of Lemma 1(iii), the scaling $h(\frac{\cdot}{\mu}) \mapsto \mu h(\frac{\cdot}{\mu})$ gives $M \mapsto \mu^{1-\frac{\nu}{2}}M$. The result follows.

(ii) The value $\mu^2 > 0$ corresponds to the unique eigenvalues of $A^{\top}A$. By [25, Proposition 2(b)], $h(\frac{x}{\mu}) \in \mathcal{F}_{M,\nu}$ with $M = \mu^{3-\nu}M_h$. The result follows as in Item (i) above.

In addition to (3), the next result due to [22] concerns the epi-convergence of smoothing via infimal convolution under the condition of supercoercive regularization kernels in $\Gamma_0(\mathbb{R}^n)$.

Lemma 2. [22, Theorem 3.8] Let $g, h \in \Gamma_0(\mathbb{R}^n)$ with h supercoercive and $0 \in \text{dom } h$. Let h_μ be defined as in (5). Then the following hold:

(i) $e - \lim_{\mu \downarrow 0} \inf\{g^* + \mu h^*\} \ge g^*.$ (ii) $e - \lim_{\mu \downarrow 0} \{g^* + \mu h^*\} = g^*.$ (iii) $e_{\mu\downarrow0} \{g\Box h_{\mu}\} = g.$ (iv) If $h(0) \le 0$, we have $p_{\mu\downarrow0} \{g\Box h_{\mu}\} = g.$

The main argument for the notion of epi-convergence in optimization problems is that when working with functions that may take infinite values, it is necessary to extend traditional convergence notions by applying the theory of *set convergence* to epigraphs in order to adequately capture local properties of the function (through a resulting calculus of smoothing functions), which on the other hand may be challenging due to the *curse of differentiation* associated with nonsmoothness. We refer the interested reader to [29, Chapter 7] for further details on the notion of epi-convergence, and to [37, 21, 22] for extended results on epi-convergent smoothing via infimal convolution.

The following result highlights key properties of the infimal convolution of $g \in \Gamma_0(\mathbb{R}^n)$ with h_μ satisfying $h \in \mathcal{F}_{M_h,\nu}$.

Proposition 2. Let $g, h \in \Gamma_0(\mathbb{R}^n)$. Suppose further that h is (M_h, ν) -generalized self-concordant and supercoercive, and define $g_s \coloneqq g \Box h_\mu$ for all $\mu > 0$. Then the following hold:

- (i) $g\Box h_{\mu} = g \boxdot h_{\mu} \in \Gamma_0(\mathbb{R}^n).$
- (*ii*) $g_s \in \mathcal{S}^{\mu}_{M_a,\nu}$ with

$$M_g = \begin{cases} n^{\frac{3-\nu}{2}} \mu^{\frac{\nu}{2}-2} M_h, & \text{if } \nu \in (0,3], \\ \mu^{4-\frac{3\nu}{2}} M_h, & \text{if } \nu > 3. \end{cases}$$

(iii) g_s is locally Lipschitz continuous.

Proof. First, as an immediate consequence of [35, Lemma 1.28, Lemma 1.27 and Proposition 8.17], we have $h_{\mu} \in \Gamma_0(\mathbb{R}^n)$.

- (i) Follows immediately from [35, Proposition 12.14].
- (ii) By Item (i), $g_s = g \boxdot h_{\mu} \in \Gamma_0(\mathbb{R}^n)$. As a consequence of [35, Proposition 12.14], we have

$$g_s(x,\mu) = \min_{w \in \mathbb{R}^n} \left\{ g(w) + h_\mu(x-w) \right\},\,$$

and $g_s \not\in g$ (by [29, Theorem 11.34]). In view of [29, Proposition 7.2], for $x \in \text{dom } g$ and

$$w_{\mu}(x) \in \operatorname*{arg\,min}_{w \in \mathbb{R}^n} \left\{ g(w) + h_{\mu}(x-w) \right\} \neq \emptyset,$$

 $g_s \underline{e} g$ implies that $g_s(x,\mu) \to g(x)$ for at least one sequence $w_\mu(x) \to x$. Hence, we have

$$(g\Box h_{\mu})(x) = g(w_{\mu}(x)) + h_{\mu}(x - w_{\mu}(x)).$$

And, given $h \in \mathcal{F}_{M_h,\nu}$, we have by Proposition 1 that h_{μ} is (M_g, ν) -generalized self-concordant, where M_g is defined by

$$M_g = \begin{cases} n^{\frac{3-\nu}{2}} \mu^{\frac{\nu}{2}-2} M_h, & \text{if } \nu \in (0,3], \\ \mu^{4-\frac{3\nu}{2}} M_h, & \text{if } \nu > 3. \end{cases}$$

Hence, $h_{\mu} \in C^3(\operatorname{dom} g)$, and by [35, Proposition 18.7/Corollary 18.8], noting that higher-order derivatives are defined inductively in this sense, we have

$$\left| \left\langle \nabla^3 (g \Box h_\mu)(x)[v]u, u \right\rangle \right| = \left| \left\langle \nabla^3 h_\mu(x - w_\mu(x))[v]u, u \right\rangle \right|, \quad \forall u, v \in \operatorname{dom} g,$$

and similarly for the second-order derivatives. By definition, the univariate function

$$\varphi(t) \coloneqq h_{\mu}(u_1 + tv_1), \tag{10}$$

is (M_q, ν) -generalized self-concordant, for every $u_1, v_1 \in \text{dom } g$. That is, $\forall t \in \mathbb{R}$,

$$\left|\varphi'''(t)\right| \le M_g \,\varphi''(t)^{\frac{\nu}{2}},$$

which concludes the proof after setting $u_1 = x$, $v_1 = w(\frac{x}{\mu})$ and $t = -\mu$ in (10).

(iii) Following the arguments in Items (i) and (ii) above, w_{μ} (and hence g_s) is finite-valued (see also [21, Lemma 4.2]). Then the Lipschitz continuity of g_s near some $\bar{x} \in \text{dom } g$ follows from the convexity of g_s (see [29, Example 9.14]; see also [22, Proposition 3.6]).

3. A proximal Newton-type scheme

Our notion of self-concordant smoothing developed in the previous section is motivated by algorithmic purposes. Specifically, we are interested in practically efficient composite minimization algorithms that utilize the idea of the *Newton decrement* framework but without imposing the self-concordant structure on the problem's objective functions. In this section, we present proximal Newton-type algorithms that exploit the structure of self-concordant smoothing functions developed in § 2 for variable-metric selection and the computation of their step-lengths.

For the optimization problem of concern, i.e., (2), we assume the following:

- **P.1** f is convex and $f \in \mathcal{C}_{L_f}^{2,2}(\mathbb{R}^n)$.
- **P.2** $\rho_0 I_n \leq \nabla^2 f(x^*) \leq L I_n, \ \rho I_n \leq \nabla^2 g_s(x^*) \leq L_0 I_n$ at a locally optimal solution x^* of (2) with $L \geq \rho_0 > 0$ and $L_0 \geq \rho > 0$.
- **P.3** $g \in \Gamma_0(\mathbb{R}^n)$.

P.4
$$g_s \in \mathcal{S}^{\mu}_{M_a,\nu}$$
.

In particular, we consider $g_s(x;\mu) \coloneqq g \Box h_{\mu}$ such that h is a suitable regularization kernel for selfconcordant smoothing of g in the sense of §2. Proximal Newton-type algorithms for solving (2) consist in minimizing a sequence of *upper approximation* of \mathcal{L} obtained by summing the nonsmooth part $g(x_k)$ and a local quadratic model of the smooth part $q(x_k) \coloneqq f(x_k) + g_s(x_k)$ near x_k . That is, for $x \in \text{dom } \mathcal{L} \equiv \text{dom } f \cap \text{dom } g$, we iteratively define

$$\hat{q}_k(x) \coloneqq q(x_k) + \left\langle \nabla q(x_k), x - x_k \right\rangle + \frac{1}{2} \|x - x_k\|_Q^2,$$
(11a)

$$\hat{m}_k(x) \coloneqq \hat{q}_k(x) + g(x), \tag{11b}$$

where $Q \in \mathcal{S}_{++}^n$, and then solve the subproblem

$$\delta_k = \underset{d \in \mathbb{R}^n}{\arg\min} \hat{m}_k(x_k + d), \tag{12}$$

for a proximal Newton-type search direction δ_k . With proximal Newton-type algorithms comprising only their special cases, we proceed by recalling the class of *cost approximation (CA) methods* [38] which helps us to propose a new method for selecting $\{x_k\}$ from the iterates $\{\delta_k\}$. The necessary optimality conditions for (2) are defined by

$$0 \in \nabla q(x^*) + \partial g(x^*), \tag{13}$$

for $x^* \in \text{dom } \mathcal{L}$. To find points x^* satisfying (13), CA methods, as the name implies, iteratively approximate $\nabla q(x_k)$ by a cost approximating mapping $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$, taking into account the fixed approximation error term $\Phi(x_k) - \nabla q(x_k)$. That is, a point *d* is sought satisfying

$$0 \in \Phi(d) + \partial g(d) + \nabla q(x_k) - \Phi(x_k).$$
(14)

Let Φ be the gradient mapping of a continuously differentiable convex function $\psi \colon \mathbb{R}^n \to \mathbb{R}$. A CA method iteratively solves the subproblem

$$\min_{d \in \mathbb{R}^n} \left\{ \psi(d) + q(x_k) + g(d) - \psi(x_k) + \left\langle \nabla q(x_k) - \nabla \psi(x_k), d - x_k \right\rangle \right\}.$$
(15)

A step is then taken in the direction $\delta_k - x_k$, namely

$$x_{k+1} = x_k + \alpha_k (\delta_k - x_k), \tag{16}$$

where δ_k solves (15) and $\alpha_k > 0$ is a step-length typically computed via a line search such that an appropriately selected *merit function* is sufficiently decreased along the direction $\delta_k - x_k$.

Remark 2. Evaluating the merit function too many times can be practically intractable. One way to (completely) mitigate this difficulty for large-scale problems is to incorporate "predetermined steplengths" [39] into the solution scheme of (15), so that we may update x_k as $x_{k+1} \equiv \delta_k$. However, methods that use this approach do not, in general, yield monotonically decreasing sequence of objective values, and convergence is instead characterized by a metric that measures the distance from iteration points to the set of optimal solutions [39].

In view of Remark 2, we discuss next a new proximal Newton-type scheme that compromises between minimizing the objective values and decreasing the distance from iteration points to the set of optimal solutions as specified by a curvature-exploiting variable-metric.

3.1. Variable-metric and adaptive step-length selection

A very nice feature of the CA framework is that it can help, for instance, through the specific choice of Φ , to efficiently utilize the original problem's structure—a practice which is particularly useful when solving medium- to large-scale problems. This feature fits directly into our self-concordant smoothing framework. We notice that (15) gives (12) with the following choice of ψ :

$$\psi(\cdot) = \frac{1}{2} \|\cdot\|_Q^2, \qquad Q \in \mathcal{S}_{++}^n.$$
(17)

In this case, the optimality conditions and our assumptions give

$$(Q - \nabla q)(x_k) \in (Q + \partial g)(d), \tag{18}$$

which leads to

$$\delta_k = \operatorname{prox}_q^Q(x_k - Q^{-1} \nabla q(x_k)).$$
(19)

In the proximal Newton-type scheme, Q may be the Hessian of $q(x_k)$ or its approximation⁶. Although a diagonal structure of Q is often desired due to its ease of implementation in the proximal framework, we most likely throw away relevant curvature information by performing a diagonal or scalar approximation of $\nabla^2 q(x_k)$, especially when q is not assumed to be separable. Our consideration in this work entails the following characterization of the optimality conditions:

$$(H_k - \nabla q)(x_k) \in (Q_k + \partial g)(d), \tag{20}$$

where H_k may be the Hessian, $\nabla^2 q(x_k) \equiv H_f + H_g$, of q or its approximation, where $H_f \equiv \nabla^2 f(x_k)$, $H_g \equiv \nabla^2 g_s(x_k; \mu)$, and $Q_k \in S_{++}^n$. Specifically, we set $Q_k = H_g$ in (20) and propose the following step update formula:

$$x_{k+1} = \operatorname{prox}_{\alpha_k g}^{H_g}(x_k - \bar{\alpha}_k H_k^{-1} \nabla q(x_k)), \qquad (21)$$

where $\bar{\alpha}_k \in \mathbb{P}$ results from *damping* the Newton-type steps.

Algorithm 1 Prox-N-SCORE (A proximal Newton algorithm)

Require: $x_0 \in \mathbb{R}^n$, problem functions f, g, self-concordant smoothing function $g_s \in \mathcal{S}_{M_g,\nu}^{\mu}, \alpha \in (0,1]$ 1: for k = 0, ... do 2: grad_k $\leftarrow \nabla f(x_k) + \nabla g_s(x_k)$ 3: $H_g \leftarrow \nabla^2 g_s(x_k); \eta_k \leftarrow \|\nabla g_s(x_k)\|_{H_g}^*$ \triangleright Note: H_g is diagonal 4: $\bar{\alpha}_k = \frac{\alpha}{1+M_g\eta_k}$ 5: $H_k \leftarrow \nabla^2 f(x_k) + H_g$; Solve for Δ_k : $H_k \Delta_k = \text{grad}_k$ 6: $x_{k+1} \leftarrow \text{prox}_{\alpha g}^{H_g}(x_k - \bar{\alpha}_k \Delta_k)$ 7: end for

The validity of this procedure in the present scheme may be seen in the interpretation of the proximal operator $\operatorname{prox}_g(x^+)$ for some $x^+ \in \operatorname{dom} g$ as compromising between minimizing the function g and staying close to x^+ (see [40, Chapter 1]). When scaled by, say, H_g , "closeness" is quantified in terms of the metric induced by H_g , and we want the proximal steps to stay close (as much as possible) to the Newton iterates relative to, say, $\|\cdot\|_{H_g}$. To see this, we note that in view of the fixed-point characterization (15) via CA methods, we may interpret proximal Newton-type algorithms as a fixation of the error term $\nabla \psi - \nabla q$ at some point in dom $q \cap \operatorname{dom} g$. Let us fix some $\bar{x} \in \operatorname{dom} q \cap \operatorname{dom} g$ and introduce the operator $E_{\bar{x}}$ defined by

$$E_{\bar{x}}(z) \coloneqq \nabla^2 q(\bar{x}) z - \bar{\alpha} \nabla q(z), \qquad (22)$$

⁶If Q is the scaled identity matrix, then we have the proximal gradient method, if $Q = \nabla^2 q$, we have the proximal Newton method, and if Q is a quasi-Newton-type, say BFGS, approximation of the Hessian, we have a proximal quasi-Newton-type method.

where $0 < \bar{\alpha} \leq \alpha \leq 1$. Set $Q = Q_k \in S_{++}^n$ arbitrary in (17). We aim to exploit the structure in g_s (and $\nabla^2 g_s$), so we define an operator $\xi_{\bar{x}}(Q_k, \cdot)$ to quantify the error between $\nabla^2 g_s$ and Q_k as follows:

$$\xi_{\bar{x}}(Q_k, z) \coloneqq (\nabla^2 g_s(\bar{x}) - Q_k)(z - x_k).$$
(23)

We provide a local characterization of the optimality conditions for (15) in terms of $E_{\bar{x}}$ and $\xi_{\bar{x}}$ in the next result.

Proposition 3. Let the operators $E_{\bar{x}}$ and $\xi_{\bar{x}}(Q_k, \cdot)$ be defined by (22) and (23), respectively. Then the optimality conditions for (15) with $\psi(\cdot) = \frac{1}{2} \|\cdot\|_{Q_k}^2$ are locally characterized in terms of $E_{\bar{x}}$ and $\xi_{\bar{x}}(Q_k, \cdot)$ by

$$E_{\bar{x}}(x_k) + \xi_{\bar{x}}(Q_k, d) \in \nabla^2 g_s(\bar{x})d + \alpha \partial g(d).$$
(24)

More precisely, (20) holds with $Q_k = \nabla^2 g_s(\bar{x})$ whenever \bar{x} is the unique optimizer satisfying (24) at a local solution d of (15).

Proof. As a result of (3), Lemma 2, and by [29, Theorem 13.2], there exists $v_g \in \mathbb{R}^n$, in the *extended* sense of differentiability (see [29, Definition 13.1]), such that

$$\limsup_{\substack{x \to \bar{x} \\ \mu \downarrow 0}} \nabla g_s(x) = \partial g(\bar{x}) = \{ v_g \}, \tag{25a}$$

$$\emptyset \neq \partial g(d) \subset v_g + \nabla^2 g_s(\bar{x})(d - \bar{x}) + o(|d - \bar{x}|)\mathcal{E}_r(\bar{x}).$$
(25b)

Let x_k be in some neighbourhood of \bar{x} and let $\{x_k\} \to \bar{x}$ be generated by an iterative process. By assumption, the differentiable terms in (25b) are convex and the differential operators are monotone. It hen holds that

$$\partial g(d) \subset v_g + \nabla^2 g_s(\bar{x})(d - x_k) + o(|d - \bar{x}|) \mathcal{E}_r(\bar{x}), \tag{26}$$

for all x_k in the neighbourhood of \bar{x} . Since differentiability in the extended sense is necessary and sufficient for differentiability in the *classical sense* (see [29, Definition 13.1 and Theorem 13.2]), it holds for some $\mu \in \mathbb{P}$ that $v_g \equiv \nabla g_s(\bar{x})$ which is defined through:

$$\nabla g_s(d) = \nabla g_s(\bar{x}) + \nabla^2 g_s(\bar{x})(d-\bar{x}) + o(|d-\bar{x}|).$$

$$\tag{27}$$

Consequently, using (14) (with $\Phi = \nabla \psi$), and defining the Dikin ellipsoid $\mathcal{E}_r(\bar{x})$ in terms of g_s for rsmall enough, we deduce from (26), (27) that $Q_k(x_k - d) + \nabla^2 g_s(\bar{x})(x_k - d) - \bar{\alpha} \nabla q(x_k) \in \bar{\alpha} \nabla g_s(\bar{x})$ for $0 < \bar{\alpha} \leq 1$. We assert $\nabla^2 f(\bar{x})(d - \bar{x}) \in \mathcal{E}_r(\bar{x})$ at a local solution d of (15), and then deduce again from (26), (27) that $\bar{\alpha} \nabla g_s(\bar{x}) + \nabla^2 g_s(\bar{x})(d - x_k) + \nabla^2 f(\bar{x})x_k \in \alpha \partial g(d)$ holds for $0 < \bar{\alpha} \leq \alpha \leq 1$ near \bar{x} , whenever \bar{x} is the unique solution x^* of (2). As a result, using $q \coloneqq f + g_s$, we get

$$(\nabla^2 q(\bar{x}) - \bar{\alpha} \nabla q) x_k - \nabla^2 g_s(\bar{x}) x_k \in Q_k(d - x_k) + \alpha \partial g(d).$$
(28)

In terms of $E_{\bar{x}}$ and $\xi_{\bar{x}}(Q_k, \cdot)$, (28) may be written as (24), which exactly gives (20) with the choice $Q_k = \nabla^2 g_s(\bar{x})$.

As we shall see in the GGN approximation discussed below, we may exploit the properties of the function g_s in ensuring stability of the Newton-type steps via the notion of *Newton decrement*. In essence, we consider damping the Newton-type steps such that

$$\bar{\alpha}_k = \frac{\alpha_k}{1 + M_g \eta_k},\tag{29}$$

where by P.4, M_g is a generalized self-concordant parameter for g_s , and $\eta_k := \|\nabla g_s(x_k)\|_{x_k}^*$ is the dual norm associated with g_s . Note that the above choice for $\bar{\alpha}_k$, in the context of minimizing generalized self-concordant functions, assumes $\nu \geq 2$ (see e.g. [25, Equation 12]). Suppose for example $\alpha_k = 1$ is fixed and $\nu = 3$, then (28) leads to the standard damped-step proximal Newton-type method (cf. [18, 25]) in the framework of Newton decrement.

In view of (8), H_g has a desirable diagonal structure and hence can be cheaply updated from iteration to iteration. This structure provides an efficient way to compute the scaled proximal operator $\operatorname{prox}_{g}^{H_g}$, for example via a special case of the proximal calculus derived in [41] (see §6 for two practical examples). Overall, by exploiting the structure of the problem, precisely

- (i) taking adaptive steps that properly capture the curvature of the objective functions, and
- (ii) scaling the proximal operator of g by a variable-metric H_q which has a simple, diagonal structure,

we can adapt to an affine-invariant structure due to the algorithm and ensure we remain close to the Newton-type iterates towards convergence.

If we choose $H_k \equiv \nabla^2 q(x_k)$ in (21), we obtain a proximal Newton step (see Algorithm 1):

$$x_{k+1} = \operatorname{prox}_{\alpha_k g}^{H_g}(x_k - \bar{\alpha}_k \,\nabla^2 \, q(x_k)^{-1} \,\nabla \, q(x_k)).$$
(30)

However, H_k may be any approximation of the Hessian of q at x_k . In view of (24), this corresponds to replacing the Hessian term $\nabla^2 q(\bar{x})$ in (22) by the approximating matrix evaluated at \bar{x} .

Algorithm 2 Prox-GGN-SCORE (A proximal generalized Gauss-Newton algorithm)

Require: $x_0 \in \mathbb{R}^n$, problem functions f, g, self-concordant smoothing function $g_s \in \mathcal{S}^{\mu}_{M_g,\nu}$, model \mathcal{M} , input-output pairs $\{u_i, y_i\}_{i=1}^m$ with $y_i \in \mathbb{R}^{n_y}$, $\alpha \in (0, 1]$

1: for k = 0, ... do $H_g \leftarrow \nabla^2 g_s(x_k); \eta_k \leftarrow \left\| \nabla g_s(x_k) \right\|_{H_a}^*$ \triangleright Note: H_g is diagonal 2: $\bar{\alpha}_{k} \leftarrow \frac{\alpha}{1+M_{g}\eta_{k}}$ **if** $m + n_{y} \leq n$ **then** Compute δ_{k}^{ggn} via (35) 3: 4: 5:else 6: Compute δ_k^{ggn} via (34) 7: end if 8: $x_{k+1} \leftarrow \operatorname{prox}_{\alpha g}^{H_g}(x_k + \bar{\alpha}_k \delta_k^{\operatorname{ggn}})$ 9: 10: end for

3.2. A proximal generalized Gauss-Newton algorithm

In describing the proximal GGN algorithm, consider first the simple case $g \equiv 0$. Then (21) with $\bar{\alpha}_k = 1$ gives exactly the pure Newton-type direction

$$\delta_k^{\rm ggn} = -H_k^{-1} \nabla q(x_k). \tag{31}$$

Now suppose that the function f quantifies a data-misfit or loss between the outputs⁷ \hat{y}_i of a model $\mathcal{M}(\cdot; x)$ and the expected outputs y_i , for i = 1, 2, ..., m, as in a typical machine learning problem,

⁷Note that for the sake of simplicity, we assume here $y_i \in \mathbb{R}$, but it is straightforward to extend the approach that follows to cases where $y_i \in \mathbb{R}^{n_y}$, $n_y > 1$.

and that $g \neq 0$. Precisely, let $\hat{y}_i \equiv \mathcal{M}(u_i; x)$, and suppose that f can be written as

$$f(x) = \sum_{i=1}^{m} \ell(y_i, \hat{y}_i),$$
(32)

where $\ell \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a loss function. Define an "augmented" Jacobian matrix $J_k \in \mathbb{R}^{(m+1) \times n}$ by [20]

$$J_{k}^{T} \coloneqq \begin{bmatrix} \hat{y}_{1}'(x^{(1)}) & \hat{y}_{2}'(x^{(1)}) & \cdots & \hat{y}_{m}'(x^{(1)}) & g^{(1)'}(x^{(1)}) \\ \hat{y}_{1}'(x^{(2)}) & \hat{y}_{2}'(x^{(2)}) & \cdots & \hat{y}_{m}'(x^{(2)}) & g^{(2)'}(x^{(2)}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{y}_{1}'(x^{(n)}) & \hat{y}_{2}'(x^{(n)}) & \cdots & \hat{y}_{m}'(x^{(n)}) & g^{(n)'}(x^{(n)}) \end{bmatrix},$$
(33)

where $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are components of x_k , and $g^{(1)}, g^{(2)}, \ldots, g^{(n)}$ are the components of $g_s(x_k; \mu)$. Then GGN approximation of the Newton direction (31) gives

$$\delta_k^{\text{ggn}} = -(H_f + H_g)^{-1} \nabla q \approx -(J_k^\top V_k J_k + H_g)^{-1} J_k^\top u_k, \tag{34}$$

where $V_k \equiv \operatorname{diag}(v_k)$, $v_k \coloneqq [l''_{\hat{y}_1}(y_1, \hat{y}_1; x_k), \dots, l''_{\hat{y}_m}(y_m, \hat{y}_m; x_k), 0]^\top \in \mathbb{R}^{(m+1)}$, and the vector $u_k \coloneqq [l'_{\hat{y}_1}(y_1, \hat{y}_1; x_k), \dots, l'_{\hat{y}_m}(y_m, \hat{y}_m; x_k), 1]^\top \in \mathbb{R}^{m+1}$ defines an augmented "residual" term. If m + 1 < n (possibly $m \ll n$), that is, when the model is overparameterized, the following equivalent formulation of (34) provides a convenient way to compute the GGN search direction [20]:

$$\delta_k^{\rm ggn} = -H_g^{-1} J_k^{\top} (I_m + V_k J_k H_g^{-1} J_k^{\top})^{-1} u_k.$$
(35)

Note that in case the function g (and hence g_s) is scaled by some (nonnegative) constant, only the identity matrix I_m may be scaled accordingly. Following [20, Section 4], it suffices to assume stability of the GGN iterates by ensuring the stability of H_g . This is achieved, for instance, through the generalized self-concordant structure of g_s .

Now if we choose $H_k \equiv J_k^{\top} V_k J_k + H_g$ in the proximal Newton-type scheme of (21), we have the proximal GGN update (see Algorithm 2):

$$x_{k+1} = \operatorname{prox}_{\alpha_k g}^{H_g}(x_k + \bar{\alpha}_k \delta_k^{\operatorname{ggn}}),$$
(36)

where δ_k is computed via (34), or by (35) in case m + 1 is less than n, and $\bar{\alpha}_k$ is as defined in (29).

4. Structured penalties

As we have noted, more general nonsmooth regularized problems impose certain structures on the variables that must be handled explicitly by the algorithm. Such situations can be seen in some lasso and multi-task regression problems in which problem (1) takes on the form

$$\min_{x \in \mathbb{R}^n} f(x) + \underbrace{\mathcal{R}(x) + \Omega(Cx)}_{g(x)},\tag{37}$$

where, in addition to $\mathcal{R}(x)$, the function (cf. [42, 43])

$$\Omega(Cx) \coloneqq \max_{u \in \mathcal{Q}} \langle u, Cx \rangle, \tag{38}$$

characterizes a specific desired structure of the solution estimates and, for \mathbb{V} a finite-dimensional vector space such that $C \colon \mathbb{R}^n \to \mathbb{V}$ is a linear map, $\mathcal{Q} \subseteq \mathbb{V}^*$ is closed and convex, where \mathbb{V}^* is the dual space to \mathbb{V} .

For example, in the sparse group lasso problem [44, 45], $\Omega(Cx) = \gamma \sum_{j \in \mathcal{G}} \omega_j ||x_j||$ induces group level sparsity on the solution estimates and $\mathcal{R}(x) = \beta ||x||_1$ promotes the overall sparsity of the solution, so that the optimization problem is written as

$$\min_{x \in \mathbb{R}^n} f(x) + \beta \|x\|_1 + \beta_{\mathcal{G}} \sum_{j \in \mathcal{G}} \omega_j \|x_j\|,$$
(39)

where $\beta \in \mathbb{P}$, $\beta_{\mathcal{G}} \in \mathbb{P}$, $\mathcal{G} = \{j_k, \ldots, j_{n_g}\}$ is the set of variables groups with $n_g = \operatorname{card}(\mathcal{G}), x_j \in \mathbb{R}^{n_j}$ is the subvector of x corresponding to variables in group j and $\omega_j \in \mathbb{P}$ is the group penalty parameter. Another example is the graph-guided fused lasso for multi-task regression problems [46], where the function $\Omega(Cx) = \beta_{\mathcal{G}} \sum_{e=(r,s)\in E, r<s} \tau(\omega_{rs}) |x_r - \operatorname{sign}(\omega_{rs})x_s|$ encourages a fusion effect over variables x_r and x_s shared across tasks through a graph $G \equiv (V, E)$ of relatedness, where $V = \{1, \ldots, n\}$ denotes the set of nodes and E the edges; $\beta_{\mathcal{G}} \in \mathbb{P}$, $\tau(\omega_{rs})$ is a fusion penalty function, and $\omega_{rs} \in \mathbb{R}$ is the weight of the edge $e = (r, s) \in E$. Here, with $\mathcal{R}(x) = \beta ||x||_1$, $\beta \in \mathbb{P}$, the optimization problem is written as

$$\min_{x \in \mathbb{R}^n} f(x) + \beta \|x\|_1 + \beta_{\mathcal{G}} \sum_{e=(r,s) \in E, r < s} \tau(\omega_{rs}) |x_r - \operatorname{sign}(\omega_{rs})x_s|.$$
(40)

4.1. Structure reformulation for self-concordant smoothing

The key observation in problems of the form (37) is that the function $\Omega(Cx)$ belongs to the class of nonsmooth convex functions that is well-structured for Nesterov's smoothing [11] in which a smooth approximation Ω_s of Ω has the form⁸

$$\Omega_s(Cx;\mu) = \max_{u \in \mathcal{Q}} \left\{ \langle u, Cx \rangle - \mu d(u) \right\}, \quad \mu \in \mathbb{P},$$
(41)

where d is a prox-function⁹ of the set Q. Note that Nesterov's smoothing approach assumes the knowledge of the exact structure of C. In the sequel, we shall write $\Omega^{C}(x) \equiv \Omega(Cx)$ or $\Omega_{s}^{C}(x;\mu) \equiv \Omega_{s}(Cx)$, with the superscript "C" to indicate the function is structure-aware via C.

Proposition 4. Let $C \colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and let ω be a continuous convex function defined on a closed and convex set $Q \subseteq \text{dom } \omega \subseteq \mathbb{R}^n$. Further, define

$$\tilde{\Omega}(x) \coloneqq \max_{u \in \mathcal{Q}} \left\{ \langle u, Cx \rangle - \omega(u) \right\},$$

and let $d \coloneqq h^*$, where $h \colon \mathbb{R}^n \to \mathbb{R}$ satisfies $\nabla^2 h \in \mathcal{S}_{++}^n$ and is of the form (8) with ϕ satisfying K.1 – K.2 so that $h \in \mathcal{F}_{M_h,\nu}$ with $\nu \in [3,6)$ if n > 1 and with $\nu \in (0,6)$ if n = 1. Then the function

$$\Omega_s(x;\mu) = \max_{u \in \mathcal{Q}} \left\{ \langle u, Cx \rangle - \omega(u) - \mu d(u) \right\}, \quad \mu \in \mathbb{P},$$
(42)

is a self-concordant smoothing function for $\tilde{\Omega}(x)$.

⁸The reader should not confuse the barrier smoothing technique of, say, [47, 48], with the self-concordant smoothing framework of this paper. The self-concordant barrier smoothing techniques, just like Nesterov's smoothing, realize first-order and subgradient algorithms that solve problems of this exact form.

⁹A function d_1 is called a *prox-function* of a closed and convex set Q_1 if $Q_1 \subseteq \text{dom } d_1$, and d_1 is continuous and strongly convex on Q_1 with convexity parameter $\rho_1 > 0$.

Proof. We follow the approach in [1, Section 4]. First note that we can write $\tilde{\Omega}(x) = \Omega(Cx)$, where

$$\Omega \coloneqq (\omega + \delta_Q)^*.$$

Now, let $\tilde{d} \coloneqq d + \delta_Q$. In view of [25, Proposition 6], we have $d, \tilde{d} \in \mathcal{F}_{M_d,\nu_d}$ where $M_d = M_h$ and $\nu_d = 6 - \nu$. Next, define $\tilde{h} \coloneqq (\tilde{d})^*$. We have

$$(\Omega^* + \tilde{h}^*_{\mu})^*(x) = (\omega + \delta_Q + \mu \tilde{d})^*(x)$$
$$= \max_{u \in \mathcal{Q}} \left\{ \langle u, x \rangle - \omega(u) - \mu d(u) \right\}$$

which is precisely $(\tilde{\Omega} \Box h_{\mu}^*)(x)$ according to [1, Theorem 4.1(a)] (cf. (9)). Now, since $d \coloneqq h^* \in \mathcal{F}_{M_d,\nu_d}$, the result follows from Proposition 1 and Proposition 2(ii).

Under the assumptions of Proposition 4, $\Omega_s^C(x;\mu)$ provides a self-concordant smooth approximation of $\Omega(x)$ with $\mathbb{V} \equiv \mathbb{R}^n$. In this case, the prox-function d in (41) is given by h^* , the dual of $h \in \mathcal{F}_{M_h,\nu}$.

4.2. Prox-decomposition and smoothness properties

An important property of the function $g = \mathcal{R} + \Omega^C$ we want to infer here is its prox-decomposition property [49] in which the (unscaled) proximal operator of g satisfies

$$\operatorname{prox}_{q} = \operatorname{prox}_{\Omega^{C}} \circ \operatorname{prox}_{\mathcal{R}}.$$
(43)

Under our assumptions on g and h, this property extends for the inf-conv regularization (and hence the self-concordant smoothing framework)¹⁰. To see this, let $z := (\mathcal{R} \Box h_{\mu})(x)$, and note the following equivalent expression for the definition of inf-convolution (4):

$$z \coloneqq (\mathcal{R}\Box h_{\mu})(x) = \inf_{\substack{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \\ u+v=x}} \left\{ \mathcal{R}(u) + h_{\mu}(v) \right\}.$$
(44)

The next result follows, highlighting what we propose as the *inf-decomposition* property.

Proposition 5. Let $g \in \Gamma_0(\mathbb{R}^n)$ be given as the sum $g(x) = \mathcal{R}(x) + \Omega^C(x)$. Suppose that the function $h \in \Gamma_0(\mathbb{R}^n)$ is supercoercive. Then the regularization process $g_s := g \Box h_{\mu}$, for all $\mu > 0$, is given by the composition

$$g_s(x) = (\Omega^C \Box h_\mu) \circ (\mathcal{R} \Box h_\mu)(x).$$
(45)

Proof. Recall $z := (\mathcal{R} \Box h_{\mu})(x)$ and let $\mathbb{V} = \mathbb{R}^n$. We have

$$(\Omega^C \Box h_{\mu}) \circ (\mathcal{R} \Box h_{\mu})(x) = \inf_{\substack{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \\ u+v=z}} \left\{ \Omega^C(u) + h_{\mu}(v) \right\}$$
$$= \inf_{\substack{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \\ 2u+v=x}} \left\{ \mathcal{R}(u) + \Omega^C(u) + h_{\mu}(v) \right\}$$
$$= ((\mathcal{R} + \Omega^C) \Box h_{\mu})(x) = (g \Box h_{\mu})(x),$$

where the first equality directly uses the definition of inf-conv operation in (44) and the second equality is inferred from the exactness of the inf-conv regularization process by Proposition 2(i).

¹⁰Additional assumptions may be required to hold in order to *correctly* define this property in our framework, e.g., nonoverlapping groups in case of the sparse group lasso problem, in which case, \mathbb{V} is the space \mathbb{R}^n .

Given the smoothness properties of $\Omega^C \Box h_{\mu}$ and $\mathcal{R} \Box h_{\mu}$, we can apply the chain rule to obtain the derivatives of their composition $g \Box h_{\mu}$. Precisely, [50, Lemma 2.1] provides sufficient conditions for the validity of the derivatives obtained via the chain rule for composite functions, which are indeed satisfied for $g \Box h_{\mu}$ by our assumptions.

5. Convergence analysis

We analyze the convergence of Algorithms 1 and 2 under the smoothing framework of this paper. In view of the numerical examples considered in §6, we restrict our analyses to the case $2 \le \nu \le 3$. However, similar convergence properties are expected to hold for the general case $\nu > 0$, as the key bounds describing generalized self-concordant functions hold similarly for all of these cases (see, e.g., Section 2 and the concluding remark of [25]). We define the following metric term, taking the local norm $\|\cdot\|_x$ with respect to g_s :

$$d_{\nu}(x,y) \coloneqq \begin{cases} M_g \|y - x\| & \text{if } \nu = 2, \\ \left(\frac{\nu}{2} - 1\right) M_g \|y - x\|_2^{3-\nu} \|y - x\|_x^{\nu-2} & \text{if } \nu > 2. \end{cases}$$
(46)

We introduce the notations $H_g^* \equiv \nabla^2 g_s(x^*)$, $H_f^* \equiv \nabla^2 f(x^*)$ and $H^* \equiv \nabla^2 q(x^*)$. Recall also the notations $H_g \equiv \nabla^2 g_s(x_k)$, $H_f \equiv \nabla^2 f(x_k)$ and $H_k \equiv \nabla^2 q(x_k)$ at x_k . Furthermore, we define the following matrices associated with any given twice differentiable function f:

$$\Sigma_f^{x,y} \coloneqq \int_0^1 \left(\nabla^2 f(x + \tau(y - x)) - \nabla^2 f(x) \right) d\tau, \tag{47a}$$

$$\Upsilon_f^{x,y} \coloneqq \nabla^2 f(x)^{-1/2} \Sigma_f^{x,y} \nabla^2 f(x)^{-1/2}.$$
(47b)

We begin by stating some useful preliminary results. The following result provides bounds on the function g_s in (2).

Lemma 3. [25, Proposition 10] Suppose that P.3–P.4 hold. Then, given any $x, y \in \text{dom } g$, we have

$$\omega_{\nu}(-d_{\nu}(x,y))\|y-x\|_{x}^{2} \leq g_{s}(y) - g_{s}(x) - \langle \nabla g_{s}(x), y-x \rangle \leq \omega_{\nu}(d_{\nu}(x,y))\|y-x\|_{x}^{2},$$
(48)

in which, if $\nu > 2$, the right-hand side inequality holds if $d_{\nu}(x, y) < 1$, and

$$\omega_{\nu}(\tau) \coloneqq \begin{cases} \frac{\exp(\tau) - \tau - 1}{\tau^{2}} & \text{if } \nu = 2, \\ \frac{-\tau - \ln(1 - \tau)}{\tau^{2}} & \text{if } \nu = 3, \\ \frac{(1 - \tau)\ln(1 - \tau) + \tau}{\tau^{2}} & \text{if } \nu = 4, \\ \left(\frac{\nu - 2}{4 - \nu}\right) \frac{1}{\tau} \left[\frac{\nu - 2}{2(3 - \nu)\tau} \left((1 - \tau)\frac{2(3 - \nu)}{2 - \nu} - 1\right) - 1\right] & \text{otherwise.} \end{cases}$$
(49)

The next two lemmas are instrumental in our convergence analysis, and are immediate consequences of the (local) Hessian regularity of the smooth functions f and g_s in (2).

Lemma 4. [51, Lemma 1.2.4] For any given $x, y \in \text{dom } f$, we have

$$\left\|\nabla f(y) - \nabla f(x) - \nabla^2 f(x)(y-x)\right\| \le \frac{L_f}{2} \|y-x\|^2,$$
 (50)

$$\left| f(y) - f(x) - \langle \nabla f(x), y - x \rangle - \frac{1}{2} \langle \nabla^2 f(x)(y - x), y - x \rangle \right| \le \frac{L_f}{6} \|y - x\|^3.$$
 (51)

Lemma 5. [25, Lemma 2] For any given $x, y \in \text{dom } g$, $\Upsilon_{g_s}^{x,y}$ satisfies

$$\|\Upsilon_{q_s}^{x,y}\| \le R_{\nu}(d_{\nu}(x,y))d_{\nu}(x,y)$$

where, for $\tau \in [0,1)$, $R_{\nu}(\tau)$ is defined by

$$R_{\nu}(\tau) \coloneqq \begin{cases} \left(\frac{3}{2} + \frac{\tau}{3}\right) \exp(\tau) & \text{if } \nu = 2, \\ \frac{1 - (1 - \tau)^{\frac{4 - \nu}{\nu - 2}} - \left(\frac{4 - \nu}{\nu - 2}\right) \tau (1 - \tau)^{\frac{4 - \nu}{\nu - 2}}}{\left(\frac{4 - \nu}{\nu - 2}\right) \tau^2 (1 - \tau)^{\frac{4 - \nu}{\nu - 2}}} & \text{if } \nu \in (2, 3]. \end{cases}$$
(52)

Global convergence. We prove a first global result for the proximal Newton-type scheme (21). We show that the iterates of this scheme decrease the objective function values with the step-lengths specified by (29) and $\alpha_k \in (0, 1]$, and converge to an optimal solution of (1).

Let us define the following mapping:

$$G_{\alpha_k g}(x_k) \coloneqq \frac{1}{\bar{\alpha}_k} H_k \left(x_k - \operatorname{prox}_{\alpha_k g}^{H_g}(x_k - \bar{\alpha}_k H_k^{-1} \nabla q(x_k)) \right).$$
(53)

Clearly, (21) is equivalent to

$$x_{k+1} = x_k - \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k) \,. \tag{54}$$

Using (20) with $Q_k = H_g$ and the definition of the (scaled) proximal operator, $G_{\alpha_k g}(x_k)$ satisfies

$$G_{\alpha_k g}(x_k) \in \nabla q(x_k) + \partial g(x_k - \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k)).$$
(55)

Moreover, $G_{\alpha_k g}(\bar{x}) = 0$ if and only if \bar{x} solves problem (2).

Theorem 1. Suppose that P.1, P.3 and P.4 hold for (2). Let $\{x_k\}$ be the sequence generated by scheme (21) for problem (2) and satisfying $\omega_{\nu}(d_{\nu}(x_{k+1}, x_k)) \leq 0.5$, where ω_{ν} and d_{ν} are respectively defined by (49) and (46). Define $\epsilon_{\hat{g}_s}(y) \coloneqq (L_f/6) ||y - x_k||^3$, and let $\bar{\alpha}_k$ be specified by (29) with $\alpha_k \in (0, 1]$. Then $\{x_k\}$ satisfies

$$\mathcal{L}(x_{k+1}) \le \mathcal{L}(x_k) - \epsilon_{\hat{g}_s}(x_{k+1}).$$
(56)

Proof. Letting $y = x_k - \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k)$ and $x = x_k$ in Lemma 4, where $G_{\alpha_k g}$ is defined by (53), we have

$$f(x_{k+1}) \leq f(x_k) - \bar{\alpha}_k (H_k^{-1} \nabla f(x_k))^\top G_{\alpha_k g}(x_k) + \frac{\bar{\alpha}_k^2}{2} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|_{H_f}^2 + \frac{\bar{\alpha}_k^3 L_f}{6} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|^3.$$
(57)

Using $\mathcal{L}(x_{k+1}) \coloneqq f(x_{k+1}) + g(x_{k+1})$ and (57), we get

$$\mathcal{L}(x_{k+1}) \leq f(x_k) - \bar{\alpha}_k (H_k^{-1} \nabla f(x_k))^\top G_{\alpha_k g}(x_k) + \frac{\bar{\alpha}_k^2}{2} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|_{H_f}^2 + \frac{\bar{\alpha}_k^3 L_f}{6} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|^3 + g(x_k - \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k))$$

$$\overset{Lemma \ 4}{\leq} f(z) - \langle \nabla f(x_k), z - x_k \rangle - \frac{1}{2} \| z - x_k \|_{H_f}^2 + \frac{L_f}{6} \| z - x_k \|^3
- \bar{\alpha}_k (H_k^{-1} \nabla f(x_k))^\top G_{\alpha_k g}(x_k) + \frac{\bar{\alpha}_k^2}{2} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|_{H_f}^2
+ \frac{\bar{\alpha}_k^3 L_f}{6} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|^3 + g(x_k - \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k)). \tag{58}$$

In the above, we used the lower bound in Lemma 4 on f(z). By the convexity of g, we have $g(z) - g(x_{k+1}) \ge v^{\top}(z - x_{k+1})$ for all $v \in \partial g(x_{k+1})$. Now since from (55), we have $G_{\alpha_k g}(x_k) - \nabla q(x_k) \in \partial g(x_k - \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k))$, and noting that $\nabla q - \nabla f = \nabla g_s$, (58) gives

$$\mathcal{L}(x_{k+1}) \leq f(z) + g(z) - \langle \nabla f(x_k), z - x_k \rangle - \frac{1}{2} ||z - x_k||_{H_f}^2 + \frac{L_f}{6} ||z - x_k||^3
- \bar{\alpha}_k (H_k^{-1} \nabla f(x_k))^\top G_{\alpha_k g}(x_k) + \frac{\bar{\alpha}_k^2}{2} ||H_k^{-1} G_{\alpha_k g}(x_k)||_{H_f}^2 + \frac{\bar{\alpha}_k^3 L_f}{6} ||H_k^{-1} G_{\alpha_k g}(x_k)||^3
- (G_{\alpha_k g}(x_k) - \nabla q(x_k))^\top (z - x_k + \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k))
\leq \mathcal{L}(z) - \langle \nabla f(x_k), z - x_k \rangle - \frac{1}{2} ||z - x_k||_{H_f}^2 - \bar{\alpha}_k (H_k^{-1} \nabla f(x_k))^\top G_{\alpha_k g}(x_k)
+ \frac{\bar{\alpha}_k^2}{2} ||H_k^{-1} G_{\alpha_k g}(x_k)||_{H_f}^2 + \frac{L_f}{6} ||z - x_k||^3 + \frac{\bar{\alpha}_k^3 L_f}{6} ||H_k^{-1} G_{\alpha_k g}(x_k)||^3
- G_{\alpha_k g}(x_k)^\top (z - x_k) - \frac{\bar{\alpha}_k^2}{2} \langle H_k^{-1} G_{\alpha_k g}(x_k), G_{\alpha_k g}(x_k) \rangle
- \nabla q(x_k)^\top (z - x_k + \bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k))
= \mathcal{L}(z) + G_{\alpha_k g}(x_k)^\top (x_k - z) + \frac{\bar{\alpha}_k^2}{2} \langle H_k^{-1} (H_f H_k^{-1} - I_n) G_{\alpha_k g}(x_k), G_{\alpha_k g}(x_k) \rangle
+ \nabla g_s(x_k)^\top (z - x_k) + \bar{\alpha}_k (H_k^{-1} \nabla g_s(x_k))^\top G_{\alpha_k g}(x_k) - \frac{1}{2} ||z - x_k||_{H_f}^2
+ \frac{L_f}{6} ||z - x_k||^3 + \frac{\bar{\alpha}_k^3 L_f}{6} ||H_k^{-1} G_{\alpha_k g}(x_k)||^3,$$
(59)

where the second inequality results from the fact that $\langle H_k^{-1} G_{\alpha_k g}(x_k), G_{\alpha_k g}(x_k) \rangle \ge 0$ and $\bar{\alpha}_k \ge \bar{\alpha}_k^2$ for $0 < \bar{\alpha}_k \le 1$. Now set $z = x_k$ in (59) and use the following relations from (54):

$$\bar{\alpha}_k H_k^{-1} G_{\alpha_k g}(x_k) = x_k - x_{k+1}, \quad G_{\alpha_k g}(x_k) = \frac{1}{\bar{\alpha}_k} H_k(x_k - x_{k+1}).$$

We get

$$\mathcal{L}(x_{k+1}) \leq \mathcal{L}(x_k) + \frac{\bar{\alpha}_k^2}{2} \langle H_k^{-1} (H_f H_k^{-1} - I_n) G_{\alpha_k g}(x_k), G_{\alpha_k g}(x_k) \rangle + \bar{\alpha}_k (H_k^{-1} \nabla g_s(x_k))^\top G_{\alpha_k g}(x_k) + \frac{\bar{\alpha}_k^3 L_f}{6} \left\| H_k^{-1} G_{\alpha_k g}(x_k) \right\|^3 = \mathcal{L}(x_k) - \left[\langle \nabla g_s(x_k), x_{k+1} - x_k \rangle + \frac{1}{2} \langle H_g(x_{k+1} - x_k), x_{k+1} - x_k \rangle + \frac{L_f}{6} \| x_{k+1} - x_k \|^3 \right].$$
(60)

Now, let us define the following cubic-regularized upper quadratic model of g_s near x_k (cf. [52]):

$$\hat{g}_s(y) \coloneqq g_s(x_k) + \langle \nabla g_s(x_k), y - x_k \rangle + \frac{1}{2} \langle H_g(y - x_k), y - x_k \rangle + \frac{L_f}{6} \|y - x_k\|^3$$

for $y \in \mathbb{R}^n$ and L_f given by P.1. Then, using Lemma 3 with $x = x_k$, we have

$$g_s(y) - \hat{g}_s(y) \le \omega_\nu (d_\nu(y, x_k)) \|y - x_k\|_x^2 - \frac{1}{2} \langle H_g(y - x_k), y - x_k \rangle - \frac{L_f}{6} \|y - x_k\|^3.$$
(61)

Next, using (61) with $y = x_{k+1}$, (60) gives

$$\mathcal{L}(x_{k+1}) \leq \mathcal{L}(x_k) + g_s(x_{k+1}) - \hat{g}_s(x_{k+1}) \\ \leq \mathcal{L}(x_k) + \left(\omega_\nu(d_\nu(x_{k+1}, x_k)) - \frac{1}{2}\right) \|x_{k+1} - x_k\|_x^2 - \frac{L_f}{6} \|x_{k+1} - x_k\|^3,$$

which proves the result.

Corollary 1. Let $\{x_k\} \subset \mathbb{R}^n$ in Theorem 1. Then every limit point \bar{x} of $\{x_k\}$ at which (20) holds with $Q_k = H_g$ is a stationary point of the objective function \mathcal{L} in problem (1).

Proof. We infer existence of the limit point \bar{x} from discussions in the proof of Proposition 2. As a result of Theorem 1, $\{x_k\}$ necessarily converges to an optimal solution x^* satisfying (20) with $Q_k = H_g$. The result follows from the gradient consistency property specified by (3) and our characterization of the optimality conditions in (20) (cf. [53, Section 3] and [21, Section 5]).

How to choose α_k . In previous results, we did not specify a particular way to choose α_k . Our algorithms converge for any value of $\alpha_k \in (0, 1]$. Compared to the step-length selection rule proposed in [25], for instance, our approach and analysis do not directly rely on the actual value of ν in the choice of both $\bar{\alpha}_k$ and α_k . Indeed, in the context of minimizing a function $g_s \in \mathcal{F}_{M_g,\nu}$, an optimal choice for $\bar{\alpha}_k$, in view of [25], corresponds to setting

$$\alpha_k = \begin{cases} \frac{\ln(1+d_k)(1+M\eta_k)}{d_k} & \text{if } \nu = 2, \\ \frac{2(1+M_g\eta_k)}{2+M_g\eta_k} & \text{if } \nu = 3, \end{cases}$$

where $d_k := M_g \| \nabla^2 H_g^{-1} \nabla g_s(x_k) \|$ and in each case, it can be shown that $\bar{\alpha}_k \in (0, 1)$. However, choosing α_k this way does not guarantee certain theoretical bounds in the context of the framework studied in this work, especially for $\nu = 2$. We therefore propose to leave α_k as a hyperparameter that must satisfy $0 < \alpha_k \equiv \alpha \leq 1$. This however gives us the freedom to exploit specific properties about the function f, when they are known to hold. One of such properties is the global Lipschitz continuity of ∇f , where supposing the Lipschitz constant L is known, one may set

$$\alpha_k = \min\{1/L, 1\}.$$

Local convergence. We next discuss the local convergence properties of Algorithms 1 and 2. In our discussion, we take the local norm $\|\cdot\|_x$ (and its dual) with respect to g_s , and the standard Euclidean norm $\|\cdot\|$ with respect to the (local) Euclidean ball $\mathcal{B}_{r_0}(\cdot) \subset \mathcal{E}_r(\cdot)$.

Theorem 2. Suppose that P.1–P.4 hold, and let x^* be an optimal solution of (2). Let $\{x_k\}$ be the sequence of iterates generated by Algorithm 1 and define $\lambda_k := 1 + M_g \omega_{\nu} (-d_{\nu}(x^*, x_k)) ||x_k - x^*||_{x_k}$, where ω_{ν} is defined by (49). Then starting from a point $x_0 \in \mathcal{E}_r(x^*)$, if $d_{\nu}(x^*, x_k) < 1$ with d_{ν} defined by (46), the sequence $\{x_k\}$ converges to x^* according to

$$\|x_{k+1} - x^*\|_{x^*} \le \vartheta_k \|x_k - x^*\| + R_k \|x_k - x^*\|_{x^*} + \frac{L_f}{2\sqrt{\rho}} \|x_k - x^*\|^2,$$
(62)

where $\vartheta_k \coloneqq (L+L_0)(\lambda_k - \alpha_k)/(\lambda_k\sqrt{\rho}), \ \alpha_k \in (0,1], \ R_k \coloneqq R_\nu(d_\nu(x^*, x_k))d_\nu(x^*, x_k) \ with \ R_\nu \ defined \ by$ (52).

Proof. The iterative process of Algorithm 1 is given by

$$x_{k+1} = \operatorname{prox}_{\alpha_k g}^{H_g}(x_k - \bar{\alpha}_k \nabla^2 q(x_k)^{-1} \nabla q(x_k)).$$

In terms of $E_{\bar{x}}$ and $\xi_{\bar{x}}(Q_k, \cdot)$ with $Q_k \equiv H_g$, and using the definition of q, we have

$$\begin{aligned} \left\| x_{k+1} - x^* \right\|_{x^*} &= \left\| \operatorname{prox}_{\alpha_k g}^{H_g^*}(E_{x^*}(x_k) + \xi_{x^*}(Q_k, x_{k+1})) - \operatorname{prox}_{\alpha_k g}^{H_g^*}(E_{x^*}(x^*)) \right\|_{x^*} \\ &\leq \left\| E_{x^*}(x_k) - E_{x^*}(x^*) + \xi_{x^*}(Q_k, x_{k+1}) \right\|_{x^*}^* \\ &= \left\| H^* x_k - \bar{\alpha}_k \nabla q(x_k) - H^* x^* + \bar{\alpha}_k q(x^*) \right\|_{x^*}^* \\ &= \left\| \nabla q(x^*) - \nabla q(x_k) + (1 - \bar{\alpha}_k)(\nabla q(x_k) - \nabla q(x^*)) + H^*(x_k - x^*) \right\|_{x^*}^* \\ &\leq \left\| \nabla q(x_k) - \nabla q(x^*) - H^*(x_k - x^*) \right\|_{x^*}^* + (1 - \bar{\alpha}_k) \left\| \nabla q(x_k) - \nabla q(x^*) \right\|_{x^*}^* \\ &\leq \left\| \nabla f(x_k) - \nabla f(x^*) - H_f^*(x_k - x^*) \right\|_{x^*}^* + \left\| \nabla g_s(x_k) - \nabla g_s(x^*) - H_g^*(x_k - x^*) \right\|_{x^*}^* \\ &+ (1 - \bar{\alpha}_k) \left(\left\| \nabla f(x_k) - \nabla f(x^*) \right\|_{x^*}^* + \left\| \nabla g_s(x_k) - \nabla g_s(x^*) \right\|_{x^*}^* \right). \end{aligned}$$
(63)

To estimate $\|\nabla f(x_k) - \nabla f(x^*) - H_f^*(x_k - x^*)\|_{x^*}^*$, we note that for $v \in \mathbb{R}^n$, $\|v\|_{x^*}^* \equiv \|H_g^{*-\frac{1}{2}}v\|$ since we take the dual norm with respect to g_s . Now, using P.2, we get that the matrix H_g^* is positive definite and

$$\|H_g^{*-\frac{1}{2}}\| \le \frac{1}{\sqrt{\rho}}.$$
(64)

Consequently, we have

$$\begin{aligned} \left\| \nabla f(x_k) - \nabla f(x^*) - H_f^*(x_k - x^*) \right\|_{x^*}^* &= \left\| H_g^{*-\frac{1}{2}} \left(\nabla f(x_k) - \nabla f(x^*) - H_f^*(x_k - x^*) \right) \right\| \\ &\leq \left\| H_g^{*-\frac{1}{2}} \right\| \left\| \nabla f(x_k) - \nabla f(x^*) - H_f^*(x_k - x^*) \right\| \\ & \overset{Lemma \ 4}{\leq} \frac{L_f \| x_k - x^* \|^2}{2\sqrt{\rho}}. \end{aligned}$$

To estimate $\|\nabla g_s(x_k) - \nabla g_s(x^*) - H_g^*(x_k - x^*)\|_{x^*}^*$, we can apply Lemma 5 as in the proof of [25, Theorem 5], and get

$$\left\| \nabla g_s(x_k) - \nabla g_s(x^*) - H_g^*(x_k - x^*) \right\|_{x^*}^* \le R_\nu(d_\nu(x^*, x_k)) d_\nu(x^*, x_k) \left\| x_k - x^* \right\|_{x^*}.$$

Following [25, p. 195], we can derive the following inequality in a neighbourhood of the sublevel set of \mathcal{L}_s in (2) using Lemma 3 and the convexity of g_s :

$$\|\nabla g_s(x_k)\|_{x_k}^* \ge \omega_\nu (-d_\nu(x^*, x_k)) \|x_k - x^*\|_{x_k}.$$
(65)

In this regard, (29) gives

$$1 - \bar{\alpha}_k \le \frac{\lambda_k - \alpha_k}{\lambda_k}.$$
(66)

Next, by P.2, we deduce

$$\|\nabla g_s(x_k) - \nabla g_s(x^*)\| \le L_0 \|x_k - x^*\|$$

and

$$\left\|\nabla f(x_k) - \nabla f(x^*)\right\| \le L \left\|x_k - x^*\right\|.$$

Then, using (64), we get

$$\begin{aligned} \left\| \nabla g_s(x_k) - \nabla g_s(x^*) \right\|_{x^*}^* &= \left\| H_g^{*-\frac{1}{2}} \left(\nabla g_s(x_k) - \nabla g_s(x^*) \right) \right\| \\ &\leq \frac{L_0}{\sqrt{\rho}} \left\| x_k - x^* \right\|. \end{aligned}$$

Similarly,

$$\left\| \nabla f(x_k) - \nabla f(x^*) \right\|_{x^*}^* \le \frac{L}{\sqrt{\rho}} \left\| x_k - x^* \right\|.$$

Finally, putting the above estimates into (63), we obtain (62).

To prove the local convergence of Algorithm 2, we need an additional assumption about the behaviour of the Jacobian matrix J_k near x^* . As before, J_k denotes the Jacobian matrix evaluated at x_k ; likewise, V_k and u_k . At x^* , we respectively write J^* , V^* and u^* . We assume the following:

G.1 $||J_k v|| \ge \beta_1 ||v||, \beta_1 > 0$, for all x_k near x^* , and for any $v \in \mathbb{R}^n$.

For f defined by (32), condition G.1 implies that the singular values of J_k are uniformly bounded away from zero, at least locally. Let the unaugmented version of the residual vector u_k be denoted \tilde{u}_k at x_k , that is,

 $\tilde{u}_k \coloneqq [l'_{\hat{y}_1}(y_1, \hat{y}_1; x_k), \dots, l'_{\hat{y}_m}(y_m, \hat{y}_m; x_k)]^\top \in \mathbb{R}^m.$

Define the following matrix:

$$W_k^T \coloneqq \begin{bmatrix} \hat{y}_1''(x^{(1)}) & \hat{y}_2''(x^{(1)}) & \cdots & \hat{y}_m''(x^{(1)}) \\ \hat{y}_1''(x^{(2)}) & \hat{y}_2''(x^{(2)}) & \cdots & \hat{y}_m''(x^{(2)}) \\ \vdots & \vdots & & \vdots \\ \hat{y}_1''(x^{(n)}) & \hat{y}_2''(x^{(n)}) & \cdots & \hat{y}_m''(x^{(n)}) \end{bmatrix} \in \mathbb{R}^{n \times m}.$$
(67)

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We note that the "full" Hessian matrix H_k can be expressed as

$$H_k \equiv J_k^{\top} V_k J_k + (\mathbf{1} \otimes (W_k^{\top} \tilde{u}_k))^{\top} + H_g, \tag{68}$$

where $\mathbf{1} \in \mathbb{R}^{n \times 1}$ is the $n \times 1$ matrix of ones and \otimes denotes the outer product. By P.1, P.2 and the Lipschitz continuity of g_s around x^* in Proposition 2(iii), we have: for r small enough, there exists a constant $\beta_2 > 0$ such that $\|\tilde{u}_k\| \leq \beta_2$ near x^* . Furthermore by our assumptions (see, e.g., [54, Theorem 10.1]), we deduce that there exists $\beta_3 > 0$ such that $\|W_k\| \leq \beta_3$ near x^* .

The next result about the local convergence of Algorithm 2 follows. Note that for Algorithm 2, we consider the case where f in problem (2) may, in general, be expressed in the form (32).

Theorem 3. Suppose that P.1–P.4 hold, and let x^* be an optimal solution of (2) where f is defined by (32). Additionally, let G.1 hold for the Jacobian matrix J_k defined by (33). Let $\{x_k\}$ be the sequence of iterates generated by Algorithm 2, and define $\lambda_k := 1 + M_g \omega_\nu (-d_\nu(x^*, x_k)) ||x_k - x^*||_{x_k}$, where ω_ν is defined by (49). Then starting from a point $x_0 \in \mathcal{E}_r(x^*)$, if $d_\nu(x^*, x_k) < 1$ with d_ν defined by (46), the sequence $\{x_k\}$ converges to x^* according to

$$\|x_{k+1} - x^*\|_{x^*} \le \vartheta_k \|x_k - x^*\| + R_k \|x_k - x^*\|_{x^*} + \frac{L_f}{2\sqrt{\rho}} \|x_k - x^*\|^2,$$
(69)

where R_k is as defined in Theorem 2, $\vartheta_k \coloneqq (\lambda_k(L+L_0)(\lambda_k - \alpha_k) + \tilde{\beta})/\sqrt{\rho}$, $\alpha_k \in (0,1]$, and $\tilde{\beta} \coloneqq \beta_2\beta_3 > 0$.

Proof. Let $\hat{H}_k \coloneqq J_k^\top V_k J_k + H_g$, and consider the iterative process of Algorithm 2 given by

$$x_{k+1} = \operatorname{prox}_{\alpha_k g}^{H_g}(x_k - \bar{\alpha}_k \hat{H}_k^{-1} J_k^{\top} u_k).$$

We first note that $J_k^{\top} u_k$ is a compact way of writing $\nabla f(x_k) + \nabla g_s(x_k) =: \nabla q(x_k)$, where f is given by (32). Following the proof of Theorem 2, we have

$$\begin{aligned} \|x_{k+1} - x^*\|_{x^*} &= \left\| \operatorname{prox}_{\alpha_k g}^{H_g^*}(E_{x^*}(x_k) + \xi_{x^*}(Q_k, x_{k+1})) - \operatorname{prox}_{\alpha_k g}^{H_g^*}(E_{x^*}(x^*)) \right\|_{x^*} \\ &\leq \left\| \nabla q(x_k) - \nabla q(x^*) - \hat{H}_k^*(x_k - x^*) \right\|_{x^*}^* + (1 - \bar{\alpha}_k) \left\| \nabla q(x_k) - \nabla q(x^*) \right\|_{x^*}^*. \end{aligned}$$
(70)

Let W^* and \tilde{u}^* respectively denote expressions for W_k and \tilde{u} evaluated at x^* . Substituting (68) into (70) and using (64) in the estimate

$$\left\| (\mathbf{1} \otimes (W^{*^{\top}} \tilde{u}_{k}))^{\top} (x_{k} - x^{*}) \right\|_{x^{*}}^{*} \leq \left\| H_{g}^{*-\frac{1}{2}} (\mathbf{1} \otimes (W^{*^{\top}} \tilde{u}^{*}))^{\top} \right\| \left\| x_{k} - x^{*} \right\|,$$

where W_k is defined by (67), we get

$$\begin{aligned} \|x_{k+1} - x^*\|_{x^*} &\leq \|\nabla q(x_k) - \nabla q(x^*) - H^*(x_k - x^*)\|_{x^*}^* + \left\| (\mathbf{1} \otimes (W^{*^\top} \tilde{u}^*))^\top (x_k - x^*) \right\|_{x^*}^* \\ &+ (1 - \bar{\alpha}_k) \|\nabla q(x_k) - \nabla q(x^*)\|_{x^*}^* \\ &\leq \left\|\nabla f(x_k) - \nabla f(x^*) - H_f^*(x_k - x^*) \right\|_{x^*}^* + \left\|\nabla g_s(x_k) - \nabla g_s(x^*) - H_g^*(x_k - x^*) \right\|_{x^*}^* \\ &+ (1 - \bar{\alpha}_k) \left(\left\|\nabla f(x_k) - \nabla f(x^*)\right\|_{x^*}^* + \left\|\nabla g_s(x_k) - \nabla g_s(x^*)\right\|_{x^*}^* \right) + \frac{\tilde{\beta} \|x_k - x^*\|}{\sqrt{\rho}}, \end{aligned}$$
(71)

where $\tilde{\beta} = \beta_2 \beta_3$. Now, using the estimates derived in the proof of Theorem 2 in (71) above, we obtain (69).

6. Numerical experiments

In this section, we validate the efficiency of the technique introduced in this paper in numerical examples using both synthetic and real datasets from the LIBSVM repository [55]. The approach and algorithms proposed in this paper are implemented in the Julia programming language and are available online as an open-source package¹¹. We test the performance of Algorithms 1 and 2 for various fixed values of $\alpha_k \equiv \alpha \in (0,1]$ (see Fig. 2). Clearly, convergence is fastest with $\alpha_k = 1$, so we fix this value for the two algorithms in the remainder of our experiments. We compare our technique with other algorithms, namely PANOC [8], ZeroFPR [56], OWL-QN [57], proximal gradient [58], and fast proximal gradient [13] algorithms¹². In the sparse group lasso experiments, we compare our approach with the proximal gradient, block coordinate descent $(BCD)^{13}$ algorithm, and the semismooth Newton augmented Lagrangian (SSNAL) method [60] which was extended¹⁴ in [61] to solve sparse group lasso problems. BCD is known to be an efficient algorithm for general regularized problems [62], and is used as a standard approach for the sparse group lasso problem [63, 44, 45]. Since the problems considered in our experiments use the ℓ_1 and ℓ_2 regularizers, we use $\phi(t) = \frac{1}{p}\sqrt{1+p^2|t|^2}-1$ from Example 1, with p = 1 and derive q_s in problem (2) accordingly (see also Fig. 1). This provides a good (smooth) approximation for the 1- and 2-norms with an appropriate value of μ . Its gradient and Hessian are respectively

$$\nabla g_s(x) = x/\sqrt{\mu^2 + x^2}, \qquad \nabla^2 g_s(x) = \operatorname{diag}\left(\mu^2/(\mu^2 + x^2)^{\frac{3}{2}}\right).$$

For a diagonal matrix $H_g \in \mathbb{R}^{n \times n}$, the scaled proximal operator for the 1- and 2-norms are obtained using the proximal calculus derived in [41]. Let the vector $\hat{d} \in \mathbb{R}^n$ contain the diagonal entries of H_g , and let $\beta \in \mathbb{P}$:

- (i) $\operatorname{prox}_{\beta \|x\|_1}^{H_g} = \operatorname{sign}(x) \cdot \max\{|x| \beta \hat{d}, 0\}, \text{ and }$
- (ii) $\operatorname{prox}_{\beta \|x\|}^{H_g} = x \cdot \max\{1 \beta \hat{d} / \|x\|, 0\}.$

All experiments are performed on a laptop with dual (2.30GHz + 2.30GHz) Intel Core i7-11800 H CPU and 32GB RAM.

6.1. Sparse logistic regression

We consider the problem of finding a sparse solution x to the following logistic regression problem

$$\min_{x \in \mathbb{R}^n} \mathcal{L}(x) \coloneqq \underbrace{\sum_{i=1}^m \log\left(1 + \exp(-y_i \langle a_i, x \rangle)\right)}_{=:f(x)} + \beta \|x\|_1, \tag{72}$$

¹¹https://github.com/adeyemiadeoye/SelfConcordantSmoothOptimization.jl

¹²We use the open-source package ProximalAlgorithms.jl for the PANOC, ZeroFPR, and fast proximal gradient algorithms, while we use our own implementation of the OWL-QN (modification of https://gist.github.com/ yegortk/ce18975200e7dffd1759125972cd54f4) and proximal gradient methods which can also be found in our package SelfConcordantSmoothOptimization.jl.

¹³We use the BCD method of [59] which is efficiently implemented with a gap safe screening rule. The open-source implementation can be found in https://github.com/EugeneNdiaye/Gap_Safe_Rules.

¹⁴We use the freely available implementation provided by the authors in https://github.com/YangjingZhang/SparseGroupLasso.





Fig. 2: Behaviour of Prox-N-SCORE and Prox-GGN-SCORE for different fixed values of α_k in problem (72).

experiments on both randomly generated data and real datasets summarized in **Table 2**. For the synthetic data, we set $\beta = 0.2$, while for the real datasets, we set $\beta = 1$. We fix $\mu = 1$ in both Algorithms 1 and 2, and set $\alpha_k = 1/L$ for the proximal gradient algorithm, where L is estimated as $L = \lambda_{max}(A^{\top}A)$, the columns of $A \in \mathbb{R}^{n \times m}$ are the vectors a_i and λ_{max} denotes the largest eigenvalue. For the sake of fairness, we provide this value of L to each of PANOC, ZeroFPR, and fast proximal gradient algorithms for computing their step-lengths in our comparison. The results are shown in **Fig. 2**, **Fig. 3** and **Fig. 4**. In **Fig. 3**, we observe that **Prox-GGN-SCORE** reduces most of computational burden of the Newton-type method when $m + n_y < n$ and makes the method competitive with the first-order methods considered. However, as shown in both **Fig. 2** and **Fig. 3**, **Prox-GGN-SCORE** is no longer preferred when $n < m + n_y$ and, by our experiments, the algorithm can run into computational issues when $n \ll m$. In this case (particularly for all of the real datasets that we use in this example), **Prox-N-SCORE** would be preferred and, as shown in the performance profile of **Fig. 4**, outperforms other tested algorithms in most cases, especially with $\alpha = 1$.



Fig. 3: Comparison of Prox-N-SCORE and Prox-GGN-SCORE with PANOC, ZeroFPR, OWL-QN, proximal gradient, and fast proximal gradient algorithms for problem (72). Besides the default termination criterion used in the ProximalAlgorithms.jl package, we set all the algorithms to terminate when $\frac{||x_k - x_{k-1}||}{\max\{||x_{k-1}||,1\}} < 10^{-10}$. Prox-GGN-SCORE reduces most of the computational burden of Prox-N-SCORE if $m + n_y < n$ (or $m \ll n$). However, Prox-N-SCORE solves the problem faster, and is more stable, if $n < m + n_y$ (or $n \ll m$).

6.2. Sparse group lasso

In this example, we consider the sparse group lasso problem:

$$\min_{x \in \mathbb{R}^n} \mathcal{L}(x) \coloneqq \underbrace{\frac{1}{2} \|Ax - y\|^2}_{=:f(x)} + \beta \|x\|_1 + \beta_{\mathcal{G}} \sum_{j \in \mathcal{G}} \omega_j \|x_j\|, \qquad (73)$$

as described in §4. We use the common example used in the literature [64, 65], which is based on the model $y = Ax^* + 0.01\epsilon \in \mathbb{R}^{m \times 1}$, $\epsilon \sim \mathcal{N}(0, 1)$. The entries of the data matrix $A \in \mathbb{R}^{m \times n}$ are drawn from the normal distribution with pairwise correlation $\operatorname{corr}(A_i, A_j) = 0.5^{|i-j|}, \forall (i, j) \in [n]^2$. We generate datasets for different values of m and n with n satisfying $(n \mod n_g) = 0$. In this problem, we want to further highlight the faster computational time achieved by the approximation



Fig. 4: Performance profile (CPU time) for the sparse logistic regression problem (72) using the LIBSVM datasets summarized in **Table 2**. Here, τ denotes the performance ratio (CPU times in seconds) averaged over 20 independent runs with different random initializations, and $\rho(\tau)$ is the corresponding frequency. Each algorithm is stopped when $\frac{\|x_k - x_{k-1}\|}{\max\{\|x_{k-1}\|,1\}} < 10^{-6}$ or when the default tolerance is reached.

Data	m	n	Density		
mushrooms	8124	112	0.19		
phishing	11055	68	0.44		
w1a	2477	300	0.04		
w2a	3470	300	0.04		
w3a	4912	300	0.04		
w4a	7366	300	0.04		
w5a	9888	300	0.04		
w8a	49749	300	0.04		
a1a	1605	123	0.11		
a2a	2265	123	0.11		
a3a	3185	123	0.11		
a4a	4781	123	0.11		
a5a	6414	123	0.11		

Table 2: Summary of the real datasets used for sparse logistic regression.

in Prox-GGN-SCORE, so we consider only overparameterized models (i.e., with $m + n_y \leq n$). In this problem, the matrix C in the reformulation (37) is a diagonal matrix with row indices given by all

Table 3: Performance of Prox-GGN-SCORE (alg.A), SSNAL (alg.B), Prox-Grad (alg.C) and BCD (alg.D) on the sparse group lasso problem (73) for different values of m and n. nnz stands for the number of nonzero entries of x^* and of the solutions found by the algorithms. MSE stands for the mean squared error between the true solution x^* and the estimated solutions.

(m,n;nnz)	nnz				Iteration			Time [s]			MSE					
	alg.A	alg.B	alg.C	alg.D	alg.A	alg.B	alg.C	alg.D	alg.A	alg.B	alg.C	alg.D	alg.A	alg.B	alg.C	alg.D
(500, 2000; 19)	19	198	19	19	161	62	5904	9690	2.81	4.65	13.39	3.55	2.9305E-09	5.3188E-08	2.5189E-07	8.0350E-06
(500, 4000; 36)	36	39	36	36	253	140	10991	16790	8.44	39.51	51.91	11.60	1.4291E-08	4.3952E-08	1.1653E-06	1.7127E-05
(500, 5000; 45)	45	45	45	45	530	111	13919	20830	16.60	35.57	90.05	18.53	2.6339E-07	6.0898E-08	2.0121E-06	2.1641E-05
(1000, 5000; 45)	45	82	45	45	112	35	3051	9100	8.71	15.73	11.57	22.14	2.4667E-07	1.9757E-06	2.1747E-06	5.0779E-06
(1000, 7000; 65)	65	65	65	65	185	82	7012	20870	30.26	148.07	42.45	70.08	4.5689E-07	2.2847E-08	4.0172E-06	1.8038E-05
(1000, 10000; 94)	93	94	94	94	497	102	9879	29330	53.26	252.05	90.25	126.17	3.8421E-06	2.8441E-08	3.6320E-06	3.5855E-05
(1000, 12000; 112)	112	113	113	164	663	68	21178	59360	166.15	194.40	221.26	373.50	1.5750E-05	4.6965E-08	7.3285E-06	5.9521E-05

*Iteration = Number of "outer" iterations for SSNAL (alg.B).

pairs $(i, j) \in \{(i, j) | i \in j, i \in \{1, \dots, n_g\}\}$, and column indices given by $k \in \{1, \dots, n_g\}$. That is,

$$C_{(i,j),k} = \begin{cases} \beta_{\mathcal{G}} \omega_j & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

We construct x^* in a similar way as [66]: We fix $n_g = 100$ and break n randomly into groups of equal sizes with 0.1 percent of the groups selected to be *active*. The entries of the subvectors in the *nonactive* groups are set to zero, while for the active groups, $\lceil \frac{n}{n_g} \rceil \times 0.1$ of the subvector entries are drawn randomly and set to $\operatorname{sign}(\xi) \times U$ where ξ and U are uniformly distributed in [0.5, 10] and [-1, 1], respectively; the remaining entries are set to zero. For the sake of fair comparison, each data and the associated initial vector x_0 are generated in Julia, and exported for the BCD implementation in Python and also for the SSNAL implementation in MATLAB.

For Prox-GGN-SCORE, Prox-Grad and BCD, we set $\beta = \tau_1 \gamma ||A^T y||_{\infty}$, $\beta_{\mathcal{G}} = (10 - \tau_1) \gamma ||A^T y||_{\infty}$ with $\tau_1 = 0.9$ and $\gamma \in \{10^{-7}, 10^{-8}\}$. SSNAL can be made to return a solution estimate that has number of nonzero entries close to that of the true solution with a carefully tuned β and simply setting $\beta_{\mathcal{G}} = \beta$ (cf., [61, Table 1]). However, by our numerical experience, SSNAL is very sensitive to the choice of β and $\beta_{\mathcal{G}}$ if the goal is to have a reasonable convergence to the true solution with the correct withingroup sparsity in the solution estimate. Besides, different values of β and $\beta_{\mathcal{G}}$ are suitable for different problem sizes with SSNAL. After a careful tuning, and with the aim of reporting a fair comparison, we set $\beta = \tau_1 \gamma ||A^T y||_{\infty}$ and $\beta_{\mathcal{G}} = ||A^T y||_{\infty}$ with $\gamma = 10^{-5}$ and $\tau_1 \in \{4, 5, 10, 12\}$ for SSNAL. For each group j, the parameter ω_j is set to the standard value $\sqrt{n_j}$ [44, 45], where $n_j = \operatorname{card}(j)$. For fairness, the estimate $\alpha_k = 1/L$ with $L = \lambda_{max}(A^{\top}A)$ is used in the proximal gradient and SSNAL algorithms.

The smoothing parameter μ is set to 1.2 for the problem with m = 500, n = 2000, 2.0 for the problem with m = 1000, n = 12000, and to 1.6 for the remaining problems. In principle, μ is a



Fig. 5: Mean squared error (MSE) between the estimates x_k and the true coefficient x^* for Prox-GGN-SCORE, SSNAL, Prox-Grad and BCD on the sparse group lasso problem (73). The algorithms terminate after successfully solving the problem for a given tolerance or when $\frac{||x_k - x_{k-1}||}{\max\{||x_{k-1}||,1\}} < \varepsilon_{tol}$ with $\varepsilon_{tol} \in \{10^{-6}, 10^{-9}\}$.

hyperparameter that has to be chosen to scale with the size of the optimization vector x. Any $\mu > 0$ is suitable, but larger values of n may require to set $\mu \ge 1$. This is intuitive in the sense that a rather small μ when n is large results into a "weak" smoothing, and in the algorithmic scope, we only require that g and g_s do not conflict, which holds by construction for any $\mu > 0$.

In the experiments, we stop Prox-GGN-SCORE, SSNAL and proximal gradient algorithms when $\frac{\|x_k - x_{k-1}\|}{\max\{\|x_{k-1}\|,1\}} < \varepsilon_{tol}$ with $\varepsilon_{tol} \in \{10^{-6}, 10^{-9}\}$. The BCD algorithm is stopped when the default tolerance is reached. The simulation results are shown in **Table 3** and Figure 5. As shown, Prox-GGN-SCORE solves the problem faster than SSNAL, Prox-Grad and BCD algorithms in most cases with the correct number of nonzero entries in its solution estimates. In the problems considered, Prox-GGN-SCORE benefits a lot from overparameterization which would have potentially posed a serious computational issue for a typical second-order method. This makes the algorithm very competitive with first-order methods and other Newton-type methods in large-scale problems.

6.3. Sparse deconvolution

In this example, we consider the problem of estimating the unknown sparse input x to a linear system, given a noisy output signal and the system response. That is,

$$\min_{x \in \mathbb{R}^n} \mathcal{L}(x) \coloneqq \underbrace{\frac{1}{2} \|Ax - y\|^2}_{=:f(x)} + \beta \|x\|_p, \tag{74}$$

where $A \in \mathbb{R}^{n \times n}$ and $y \in \mathbb{R}^{n \times 1}$ are given data about the system which we randomly generate according to [67, Example F]. We solve with both ℓ_1 (p = 1) and ℓ_2 (p = 2) regularizers, and set $\beta = 10^{-3}$. We



Fig. 6: Sparse deconvolution via ℓ_1 -regularized least squares (74) using Prox-N-SCORE, Prox-GGN-SCORE, PANOC, ZeroFPR, proximal gradient, and fast proximal gradient algorithms with n = 1024. Each algorithm is stopped when $\frac{\|x_k - x_{k-1}\|}{\max\{\|x_{k-1}\|, 1\}} < 10^{-6}$ or when the default tolerance is reached.

set $\mu = 5 \times 10^{-2}$ in the smooth approximation g_s of g. We estimate $L = \lambda_{max}(A^{\top}A)$ and set $\alpha_k = 1/L$ in the proximal gradient algorithm. Again, for fairness, we provide this value of L to each of PANOC, ZeroFPR, and fast proximal gradient procedures in our comparison. The results of the simulations are displayed in **Fig. 6** and **Fig. 7**. While **Prox-GGN-SCORE** and **Prox-N-SCORE** sometimes have more running time in this problem, they provide better solution quality with smaller reconstruction error than the other tested algorithms, which is often more important for signal reconstruction problems.

7. Conclusions

Generalized self-concordant optimization provides very useful tools for implementing and analyzing Newton-type methods for unconstrained problems. This helps to reconcile the geometric properties of Newton-type methods with their implementations, while providing convergence guarantees. In the presence of constraints or nonsmooth terms in the objective functions, it becomes natural



Fig. 7: Sparse deconvolution via ℓ_2 -regularized least squares (74) using Prox-N-SCORE, Prox-GGN-SCORE, PANOC, ZeroFPR, proximal gradient, and fast proximal gradient algorithms with n = 1024. Each algorithm is stopped when $\frac{\|x_k - x_{k-1}\|}{\max\{\|x_{k-1}\|,1\}} < 10^{-6}$ or when the default tolerance is reached.

to extend these methods via proximal schemes. However, when the (generalized) self-concordant property is uncheckable for the objective functions, these methods are no longer applicable and the convergence guarantees becomes difficult to prove. In addition to other related computational issues, this paper addresses this with a self-concordant smoothing notion which combines and synthesizes different regularization/smoothing phenomena, namely: inf-conv regularization, self-concordant regularization (SCORE), and Moreau-Yosida regularization. This approach, leading to two algorithms in this paper (Prox-N-SCORE and Prox-GGN-SCORE), is able to utilize certain properties of generalized self-concordant functions in the selection of adaptive step-lengths and a simple variable-metric in the proximal Newton-type scheme. We prove global and local convergence guarantees for our approach. As demonstrated in numerical simulations, in most cases, our approach compares favourably against other state-of-the-art first- and second-order approaches from the literature.

In future research, it would be interesting to analyze our framework in the nonconvex setting. In particular, we believe that our notion of self-concordant smoothing could lead to interesting research directions in applications such as deep neural network training, in which our stylized approximation technique in Prox-GGN-SCORE would become very instrumental in scaling our method with respect to the problem size. Another possible future consideration is the handling of more general structured penalties with our approach, allowing to relax some of the conditions in §4.

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