Cone product reformulation for global optimization

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In this paper, we study nonconvex optimization problems involving sum of linear times convex (SLC) functions as well as conic constraints belonging to one of the five basic cones, that is, linear cone, second order cone, power cone, exponential cone, and semidefinite cone. By using the Reformulation Perspectification Technique, we can obtain a convex relaxation by forming the perspective of each convex function and linearizing all product terms with newly introduced variables. To further tighten the approximation, we can pairwise multiply the conic constraints. In this paper, we analyze all possibilities of multiplying conic constraints. Especially the results for the cases in which a power cone or an exponential cone is involved are new. Moreover, in case of an exponential cone we generate valid inequalities that can be used to further strengthen the approximation and in case of a power cone we generate additional valid inequalities. Numerical experiments on a quadratic optimization problem over exponential cone constraints and on a robust palatable diet problem over power cone constraints, demonstrate that including additional inequalities generated from the proposed pairwise multiplications improve the approximation. Moreover, when incorporated in branch and bound the global optimal solution of the original nonconvex optimization problem can often be obtained faster than BARON.

Key words: Reformulation-Linearization Technique, perspective function, conic optimization, nonconvex optimization, conjugate function, branch and bound

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1. Introduction

In this paper, we consider the following nonconvex optimization problem:

$$\min_{\boldsymbol{x}} \quad f_{00}(\boldsymbol{x}) + \sum_{i\in\mathcal{I}} \left(b_{i0} - \boldsymbol{a}_{i0}^\top \boldsymbol{x}\right) f_{i0}(\boldsymbol{x}) \tag{1a}$$

$$\text{s.t.} \quad f_{0k}(\boldsymbol{x}) + \sum_{i\in\mathcal{I}} \left(b_{ik} - \boldsymbol{a}_{ik}^\top \boldsymbol{x}\right) f_{ik}(\boldsymbol{x}) \leq 0, \qquad k \in \mathcal{K}, \tag{1b}$$

$$c_j(\boldsymbol{x}) \leq 0, \qquad j \in \mathcal{J}, \tag{1c}$$

where $\boldsymbol{x}, \boldsymbol{a}_{ik} \in \mathbb{R}^{n_x}, b_{ik} \in \mathbb{R}$, for every $i \in \mathcal{I}_0 = \mathcal{I} \cup \{0\}$, $k \in \mathcal{K}_0 = \mathcal{K} \cup \{0\}$, each function $f_{ik} : \mathbb{R}^{n_x} \rightarrow (-\infty, \infty]$ is proper, closed, and convex, and each inequality $c_j(\boldsymbol{x}) \leq 0$ is representable in one or more of the five basic cones, that is, linear cone, second-order cone, power cone, exponential cone, and semi-definite cone. Observe that (1) is nonconvex, since it contains products of linear and convex functions. A broad class of nonconvex problems can be written in the form of (1), such as concave minimization problems, which often occur due to economies of scale, and problems with a Difference of Convex (DC) objective and/or constraints, see (Bertsimas et al., 2023, Example 1).

Bertsimas et al. (2023) show how to obtain the following convex relaxation of problem (1), using the Reformulation Perspectification Technique with Branch and Bound (RPT-BB):

$$\min_{\boldsymbol{x}, \boldsymbol{U}} \quad f_{00}(\boldsymbol{x}) + \sum_{i \in \mathcal{I}} \left(b_{i0} - \boldsymbol{a}_{i0}^\top \boldsymbol{x}\right) f_{i0} \left(\frac{b_{i0}\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_{i0}}{b_{i0} - \boldsymbol{a}_{i0}^\top \boldsymbol{x}} \right) \tag{2a}$$

$$\text{s.t.} \quad f_{0k}(\boldsymbol{x}) + \sum_{i \in \mathcal{I}} \left(b_{ik} - \boldsymbol{a}_{ik}^\top \boldsymbol{x}\right) f_{ik} \left(\frac{b_{ik}\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_{ik}}{b_{ik} - \boldsymbol{a}_{ik}^\top \boldsymbol{x}} \right) \leq 0, \qquad k \in \mathcal{K}, \tag{2b}$$

$$c_j(\boldsymbol{x}) \leq 0, \qquad j \in \mathcal{J}, \tag{2c}$$

where $\boldsymbol{x}\boldsymbol{x}^\top$ is linearized by $\boldsymbol{U}$. In order to obtain good bounds on U, we can generate additional convex inequalities by pairwise multiplying the cone inequalities and subsequently convexifying the resulting inequalities. We can further link $\boldsymbol{U}$ with $\boldsymbol{x}$, via the following Linear Matrix Inequality (LMI):

$$\begin{pmatrix} \boldsymbol{U} & \boldsymbol{x} \\ \boldsymbol{x}^\top & 1 \end{pmatrix} \succeq 0. \tag{3}$$

Obtaining convexifiable constraints from pairwise multiplication of linear or quadratic inequalities is well known in Reformulation Linearization Technique (RLT), introduced by Sherali and Adams (1990). RLT consists of two steps, those are, a reformulation step and a linearization step. RLT generates redundant nonconvex quadratic constraints from pairwise multiplication of the existing linear inequalities in the reformulation step. In the linearization step, the nonconvex quadratic components are linearized by substituting each distinct product of variables by a newly introduced continuous variable. These additional generated constraints are not redundant anymore after linearization and serve as bounds on the newly introduced variables.

Linearizing the product of linear constraints is further explored in Sherali and Tuncbilek (1992) and Sherali and Tuncbilek (1995). Sturm and Zhang (2003) show how to multiply a linear inequality with a conic quadratic inequality and reformulate the resulting constraint as a conic quadratic inequality. Jiang and Li (2016) show how to obtain a conic quadratic inequality from pairwise multiplication of two conic quadratic inequalities. Yang and Burer (2016) and Anstreicher (2017) address the same multiplication by reformulating each conic quadratic inequality as an LMI and subsequently pairwise multiply them to finally obtain one additional LMI using either the Hadamard product or Kronecker product respectively. We also refer to Jiang and Li (2020) for an overview of RLT approximations for quadratic optimization problems.

Moreover, Bertsimas et al. (2023) show how to multiply a linear inequality with a general convex inequality and how to convexify the resulting inequality. However, they mention that the pairwise multiplication of two general convex inequalities does not necessarily yield a convexifiable inequality. Anstreicher (2017) shows how to obtain a convexifiable constraint from pairwise multiplication of two LMIs.

Note that the solution of Problem (2) provides a lower bound for Problem (1). Further, an upper bound can be obtained from local optimization algorithms. RPT-BB leverages both mechanisms for obtaining bounds in a systematic global optimization approach for solving nonconvex optimization problems. During BB, the gap of the RPT approximation is closed by cutting the feasible region through additionally generated hyperplanes.

In this paper, we analyze all $\binom{5}{2} = 15$ possibilities of pairwise multiplication of the five basic cone inequalities and show how to convexify the resulting constraints. Especially the results for the cases in which a power cone or an exponential cone is involved are new. Moreover, we report numerical examples showing that the cone product reformulations introduced in this paper improve the RPT approximation and as a result speed up the solution of the nonconvex optimization problem.

An example problem that arises often and can be written in problem format (1) is the following problem:

$$\begin{array}{lll} \min\limits_{\boldsymbol{x}} & \boldsymbol{x}^\top \boldsymbol{A}_0 \boldsymbol{x} + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 & \\ \text{s.t.} & \boldsymbol{x}^\top \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i^\top \boldsymbol{x} + c_i \le 0, & i \in \mathcal{I}, \\ & c_j(\boldsymbol{x}) \le 0, & j \in \mathcal{J}, \end{array} \tag{4}$$

where each matrix $\boldsymbol{A}_i$ is not necessarily positive semi-definite for every $i \in \mathcal{I}_0$. As is well known in RLT, we can obtain the following convex relaxation of (4):

$$\begin{array}{lll} \min\limits_{\boldsymbol{x},\boldsymbol{U}} & \mathrm{Tr}\left(\boldsymbol{A}_0 \boldsymbol{U}\right) + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 & \\ \text{s.t.} & \mathrm{Tr}\left(\boldsymbol{A}_i \boldsymbol{U}\right) + \boldsymbol{b}_i^\top \boldsymbol{x} + c_i \le 0, & i \in \mathcal{I}, \\ & c_j(\boldsymbol{x}) \le 0, & j \in \mathcal{J}. \end{array} \tag{5}$$

We can then reformulate the conic representable constraints $c_j(\boldsymbol{x}) \le 0$ in conic form and pairwise multiply them to obtain better bounds on $\boldsymbol{U}$ and thereby, obtain a tighter approximation.

Contributions. Our main contributions can be summarized as follows:

- In this paper, we show for all 15 possibilities of pairwise multiplications of the five basic cone constraints how to convexify the resulting constraint. Especially the results for the cases in which a power cone or an exponential cone is involved are new. In the case of a power cone inequality we generate additional valid inequalities. Further, when multiplying a power cone inequality with other cone inequalities, we show how to find the best reformulation out of the infinitely many possible ones, by using a robust optimization lens. Moreover, in the case of an exponential cone inequality we generate additional cone inequalities, utilizing the Taylor expansion of the exponential function to the first or second order, which we can then multiply with other constraints to further enhance the approximation.

- We show that there are two ways to multiply a conically representable convex constraint and a linear inequality that yield the same result. More precisely, we show that the additional inequalities generated from the pairwise multiplication of a conically representable convex constraint with a linear inequality lead to the same inequalities that one would obtain from first reformulating the conically representable convex constraint into cone constraints and then pairwise multiply them with the linear inequality.
- For constraints involving DC functions, that are conically representable, we derive additional cone constraints obtained from first order conditions. We illustrate the derived constraints on multiple small examples and also show that they improve the approximation of a nonconvex optimization problem.
- We demonstrate the effectiveness of the proposed pairwise multiplications involving a power cone and an exponential cone through numerical experiments on a nonconvex quadratic optimization problem with exponential cone constraints as well as a robust palatable diet problem, including power cone constraints. We demonstrate that the additional inequalities, which are generated from pairwise multiplications of cone inequalities outlined in this paper, enhance the approximation. Further, when incorporated in branch and bound, the computational time to find the global optimal solution is reduced, while frequently outperforming BARON.

Notation. The calligraphic letters $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$ and the corresponding capital Roman letters I, J, K, L are reserved for finite index sets and their respective cardinalities, *i.e.*, $\mathcal{I} = \{1, \dots, I\}$ etc. The subscript 0 for an index set indicates that the set additionally includes 0, *i.e.*, $\mathcal{I}_0 = \{0, \dots, I\}$ etc. Let $\mathbb{R}^{m\times n}$ denote the set of real $m \times n$ matrices, and $\mathbb{S}^n$ the set of real $n \times n$ symmetric matrices. The *domain* of a function $f : \mathbb{R}^{n_\nu} \to [-\infty, +\infty]$ is defined as $\text{dom}(f) = \{\boldsymbol{\nu} \in \mathbb{R}^{n_\nu} \mid f(\boldsymbol{\nu}) < +\infty\}$. The function f is *proper* if $f(\boldsymbol{\nu}) > -\infty$ for all $\boldsymbol{\nu} \in \mathbb{R}^{n_\nu}$ and $f(\boldsymbol{\nu}) < +\infty$ for at least one $\boldsymbol{\nu} \in \mathbb{R}^{n_\nu}$, implying that $\text{dom}(f) \neq \emptyset$. In addition, f is *closed* if f is lower semicontinuous and either $f(\boldsymbol{\nu}) > -\infty$ for all $\boldsymbol{\nu} \in \mathbb{R}^{n_\nu}$ or $f(\boldsymbol{\nu}) = -\infty$ for all $\boldsymbol{\nu} \in \mathbb{R}^{n_\nu}$. The *conjugate* of a function $f : \mathbb{R}^{n_\nu} \to [-\infty, +\infty]$ is the function $f^* : \mathbb{R}^{n_\nu} \to [-\infty, +\infty]$ defined through $f^*(\boldsymbol{w}) = \sup_{\boldsymbol{\nu}} \left\{\boldsymbol{\nu}^\top \boldsymbol{w} - f(\boldsymbol{\nu})\right\}$. The conjugate $(f^*)^*$ of f^* is called the *biconjugate* of f and is abbreviated as f^{**}. The *perspective* $h : \mathbb{R}^{n_\nu} \times \mathbb{R}_+ \to [-\infty, +\infty]$ of a proper, closed and convex function $f : \mathbb{R}^{n_\nu} \to (-\infty, +\infty)$ is defined for all $\boldsymbol{\nu} \in \mathbb{R}^{n_\nu}$ and $t \in \mathbb{R}_+$ as $h(\boldsymbol{\nu}, t) = t f(\boldsymbol{\nu}/t)$ if $t > 0$, and $h(\boldsymbol{\nu}, 0) = \delta^*_{\text{dom}(f^*)}(\boldsymbol{\nu})$, where $\delta^*_{\text{dom}(f^*)}$ denotes the recession function. For ease of exposition, we use $tf(\boldsymbol{\nu}/t)$ to denote the perspective function $h(\boldsymbol{\nu}, t)$ for the rest of this paper.

2. Overview: five basic cone inequalities and their products

In this section, we give an overview of all 15 possibilities of pairwise multiplying the five basic cone inequalities to obtain additional cone inequalities. The five basic cone inequalities are given by:

(**L**) Linear inequality:

$$b - \boldsymbol{a}^\top \boldsymbol{x} \geq 0,$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$.

(**Q**) Conic quadratic inequality:

$$b - \boldsymbol{a}^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\|,$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$.

(**P**) Power cone inequality:

$$\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}, \quad x_1, \cdots, x_m \geq 0,$$

where $n_x > m$, $\alpha_1, \cdots, \alpha_m > 0$ and $\sum_{i=1}^{m} \alpha_i = 1$, or equivalently

$$((x_1, \ldots, x_m), (x_{m+1}, \ldots, x_{n_x})) \in \mathcal{P}_{n_x}^{\boldsymbol{\alpha}} = \left\{ \boldsymbol{x} \in \mathbb{R}^{n_x} : \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}, \quad x_1, \cdots, x_m \geq 0 \right\},$$

where $\mathcal{P}_{n_x}^{\boldsymbol{\alpha}}$ denotes the power cone, with $\boldsymbol{a} = (\alpha_1, \ldots, \alpha_m)$.

(**E**) Exponential cone inequality:

$$x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right), \quad x_2 \geq 0,$$

or equivalently

$$(x_1, x_2, x_3) \in \mathcal{K}_{\exp} = \left\{ (x_1, x_2, x_3) : x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right),\ x_2 \geq 0 \right\},$$

where $\mathcal{K}_{\exp}$ denotes the exponential cone.

(**S**) Semidefinite cone inequality/LMI:

$$\boldsymbol{A}(\boldsymbol{x}) \succeq 0,$$

where $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_0 + \boldsymbol{A}_1 x_1 + \cdots + \boldsymbol{A}_{n_x} x_{n_x}$.

The results of the 15 possibilities of pairwise multiplying the five basic cone inequalities to obtain additional cone inequalities are summarized in Table 1.

In the remainder of this paper we focus on all cases involving a power cone inequality or an exponential cone inequality. We refer to Appendix A for the other cases, that have already been studied in the literature (i.e., cases 1, 2, 5, 6, 9, 12, and 15 in Table 1). We note that Case 9 is not literally in the literature, however the steps followed are from Anstreicher (2017), thus we present the case in Appendix A, although stating that it is derived in this paper in Table (1). Finally, we note that cases 3(i), 4(i) and 14(i) are from the literature, however we do state them in the main text since they are connected with other subcases.

Case	Cone-1	Cone-2	Cone-1 × Cone-2	Reference	Discussed in	Remarks
1	L	L	L	Sherali and Adams (1990)	Appendix A.1	
2	L	Q	Q	Sturm and Zhang (2003)	Appendix A.2	
3	L	P	(i) mL + P (ii) mL + P	Bertsimas et al. (2023) This paper	Section 3.2 Section 3.2	 Best reformulation
4	L	E	(i) L + E (ii) 2L + E (iii) 2L + Q + E	Bertsimas et al. (2023) This paper This paper	Section 4.2 Section 4.2 Section 4.2	No decomposition $x_3 < 0$ $x_3 \geq 0$
5	L	S	S	Anstreicher (2017)	Appendix A.3	
6	Q	Q	(i) 3Q (ii) S (iii) S	Jiang and Li (2016) Yang and Burer (2016) Anstreicher (2017)	Appendix A.4 Appendix A.4 Appendix A.4	 (i) and (iii) are at least as good as (ii)
7	Q	P	(i) mQ + 2P (ii) mQ + 2P	This paper This paper	Section 3.3 Section 3.3	 Best reformulation
8	Q	E	(i) 2Q + E (ii) 3Q + E (iii) 6Q + E	This paper This paper This paper	Section 4.3 Section 4.3 Section 4.3	No decomposition $x_3 < 0$ $x_3 \geq 0$
9	Q	S	S	This paper	Appendix A.5	
10	P	P	(i) $m_1 m_2$L + (m_1+m_2+1)P (ii) $m_1 m_2$L + (m_1+m_2+1)P	This paper This paper	Section 3.4 Section 3.4	 Best reformulation
11	P	E	(i) mL+ 2P + mE (ii) 2mL + 3P + mE (iii) 2mL + mQ + 5P + mE	This paper This paper This paper	Section 3.5 Section 3.5 Section 3.4	No decomposition $x_3 < 0$ $x_3 \geq 0$
12	P	S	mS	Anstreicher (2017)	Appendix A.6	
13	E	E	(i) L + 5E (ii) 4L + 7E (iii) 4L + 9Q + 9E (iv) 4L + 3Q + 8E	This paper This paper This paper This paper	Section 4.4 Section 4.4 Section 4.4 Section 4.4	No decomposition $x_3, x_6 < 0$ $x_3, x_6 \geq 0$ $\text{sign}(x_3) \neq \text{sign}(x_6)$
14	E	S	(i) 2S (ii) 3S (iii) 4S	Anstreicher (2017) This paper This paper	Section 4.5 Section 4.5 Section 4.5	No decomposition $x_3 < 0$ $x_3 \geq 0$
15	S	S	(i) S (ii) S	Yang and Burer (2016) Anstreicher (2017)	Appendix A.7 Appendix A.7	(i) is at least as good as (ii) only if the two cones are of the same size

Table 1 Results of multiplying two cone inequality as given in Section 2. L = Linear inequality, Q = Conic quadratic inequality, P = Power cone inequality, E = Exponential cone inequality, S = Semidefinite cone inequality. Cone-1 × Cone-2 refers to the total additional cone inequalities resulting from the multiplication of cone inequality 1 with cone inequality 2.

3. Product with a power cone inequality

In this section, we show how to obtain additional valid inequalities from a power cone inequality as given in Section 2. Moreover, we consider all cases in which we multiply one of the five basic cone inequalities with the power cone inequality and show how to obtain the best reformulation for the resulting constraint.

3.1. Generating valid inequalities from a power cone inequality

We first show that we can generate valid power cone inequalities from one power cone inequality by linearizing the product terms in the LHS of the power cone inequality. First, observe that in the LHS of the power cone inequality we can decompose the powers of the different x_i such that we get powers of products of x_i, x_j and add a power of 1 to satisfy the power cone inequality. For example, $x_1^{0.4}x_2^{0.6} = x_1^{0.3}x_1^{0.1}x_2^{0.1}x_2^{0.5} = x_1^{0.3}u_{12}^{0.1}x_2^{0.5}1^{0.1}$. In the general form we obtain the following:

$$\prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=1}^{m+1} x_i^2} \iff \prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i\leq j}^{m} (x_i x_j)^{\beta_{ij}} 1^{\delta} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}, \tag{6}$$

$$\implies \prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i\leq j}^{m} (u_{ij})^{\beta_{ij}} 1^{\delta} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2}, \tag{7}$$

$$\iff ((x_1,\dots,x_m,u_{11},\dots,u_{mm},1),(x_{m+1},\dots,x_{n_x})) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'}, \tag{8}$$

where $n_x' = n_x + m(m+1)/2 + 1$, $\boldsymbol{\alpha}' = (\varepsilon_1,\dots,\varepsilon_m,\beta_{11},\dots,\beta_{mm},\delta)$, and

$$(\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta) \in \mathcal{U} = \left\{ \sum_{i=1}^{m} \varepsilon_i + \sum_{i\leq j}^{m} \beta_{ij} + \delta = 1, \quad \varepsilon_i + \sum_{j\,:\,i\leq j}^{m} \beta_{ij} = \alpha_i,\ \forall i, \quad \varepsilon_i, \beta_{ij}, \delta \geq 0,\ \forall i,j \right\}. \tag{9}$$

Note that there are infinite ways to add such a constraint, since there are infinite possibilities to choose $\boldsymbol{\varepsilon}$, $\boldsymbol{\beta}$ and δ. One could consider (7) as a robust constraint, where $(\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta)$ are the uncertain parameters, and enforce that the constraint should hold for all $(\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta)$ in $\mathcal{U}$. Hence, the inequality becomes

$$\prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i\leq j}^{m} (u_{ij})^{\beta_{ij}} 1^{\delta} \geq \sqrt{\sum_{i=1}^{m+1} x_i^2}, \quad \forall (\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta) \in \mathcal{U}. \tag{10}$$

We can deal with the robust constraint (10), using the adversarial approach, see Bertsimas and den Hertog (2022), where at each iteration inequality (10) is added for a finite subset of scenarios (master-problem), and then one has to find the worst-case value for $(\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta)$ by minimizing the LHS of (10) (sub-problem). Utilizing a log transformation, the sub-problem to find the worst-case for $(\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta)$ is the following linear optimization problem:

$$\min_{\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta} \left\{ \sum_{i} \varepsilon_i \log x_i + \sum_{i\leq j} \beta_{ij} \log u_{ij} \;\middle|\; (\boldsymbol{\varepsilon},\boldsymbol{\beta},\delta) \in \mathcal{U} \right\}. \tag{11}$$

We refer to Appendix B for the pseudocode of the adversarial approach. Finally, note that the adversarial approach converges, since the assumptions given in Mutapcic and Boyd (2009) are satisfied.

Example 1. Consider the following toyexample

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x_1 x_2 + x_1 + x_2 \\ \text{s.t.} \quad & x_1^{1/4} x_2^{3/4} \geq 1, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{12}$$

By applying RLT we obtain the following relaxation

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & u_{12}+x_1+x_2 \\ \text{s.t.} \quad & x_1^{1/4}x_2^{3/4} \geq 1, \\ & x_1, x_2, u_{11}, u_{12}, u_{12}, u_{22} \geq 0. \end{aligned} \tag{13}$$

The solution of (13) appears to be

$$\boldsymbol{x}' = \begin{bmatrix} 0.44 \\ 1.32 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U}' = \begin{bmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with objective value 1.7548, which constitutes a lower bound on (12). The obtained $\boldsymbol{x}'$ is a feasible solution to (12), and its corresponding value is 2.3320, which constitutes an upper bound on the optimal objective value of (12). We generate the additional valid power cone inequality

$$x_2^{\varepsilon_1} x_2^{\varepsilon_2} u_{11}^{\beta_{11}} u_{12}^{\beta_{12}} u_{22}^{\beta_{22}} 1^{\delta} \geq 1, \quad \forall (\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \delta) \in \mathcal{U},$$

where $\mathcal{U}$ is defined by (9) and apply the adversarial approach. We add the following constraints to (13)

$$\begin{aligned} & x_1^{1/4} u_{22}^{3/8} 1^{3/8} \geq 1, \\ & x_2^{3/4} u_{11}^{1/8} 1^{1/8} \geq 1, \\ & x_2^{2/4} \beta_{12}^{1/4} 1^{1/4} \geq 1, \end{aligned}$$

and obtain the optimal solution

$$\boldsymbol{x}^* = \begin{bmatrix} 0.26 \\ 1.57 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U}^* = \begin{bmatrix} u^*_{11} & u^*_{12} \\ u^*_{21} & u^*_{22} \end{bmatrix} = \begin{bmatrix} 5.87 & 0.40 \\ 0.40 & 12.19 \end{bmatrix},$$

with objective value 2.3320, which constitutes a tighter lower bound on (12). The obtained $\boldsymbol{x}^*$ is again a feasible solution to (12), and its corresponding value is 2.3320. Hence $\boldsymbol{x}^*$ is an optimal solution to (12). □

3.2. Case 3 in Table 1: (L) × (P)

Consider one linear inequality and one power cone inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_i \geq 0, \quad i=1,\ldots,m, \end{cases}$$

where $\sum_{i=1}^m \alpha_i = 1, \boldsymbol{\alpha} \geq 0$.

Case 3(i) in Table 1. We multiply the linear inequality with the power cone inequality and obtain 1 additional power cone inequality:

$$
\begin{aligned}
&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \prod_{i=1}^{m} x_i^{\alpha_i} \geq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \qquad (14)\\
\iff &\prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^2}\\
\implies &\prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^2}\,.
\end{aligned}
$$

Moreover, we multiply the linear inequality with the nonnegativity constraints of the power cone and obtain m additional linear inequalities, see Appendix A.1.

We note that in the approach that we describe here, we do not follow the same treatment as in Bertsimas et al. (2023), that is multiplying the numerator and denominator of the convex function with the linear inequality, although we obtain the same result. This follows from the fact that the power cone is a cone and therefore if $\boldsymbol{x}$ belongs to the power cone, then $(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{x}$ also belongs to the power cone.

Case 3(ii) in Table 1. Note that there are infinite possibilities for linearizing the LHS of (14). More precisely, we can write the LHS of (14) as follows:

$$
\begin{aligned}
&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \prod_{i=1}^{m} x_i^{\alpha_i} \geq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \sqrt{\sum_{i=m+1}^{n_x} x_i^2}\\
\iff &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})^{\delta} \left((b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})^2\right)^{\eta} \prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i\leq j}^{m} (x_i x_j)^{\beta_{ij}} \prod_{i=1}^{m} \left((b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) x_i\right)^{\gamma_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^2}\\
\implies &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})^{\delta} \left(b_1^2 - 2b_1 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1 \boldsymbol{U} \boldsymbol{a}_1\right)^{\eta} \prod_{i=1}^{m} x_i^{\varepsilon_i} \prod_{i\leq j}^{m} (u_{ij})^{\beta_{ij}} \prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^{\gamma_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^2} \qquad (15)\\
\implies &\big((b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}, b_1^2 - 2b_1 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1 \boldsymbol{U} \boldsymbol{a}_1, x_1, \ldots, x_m, u_{11}, \ldots, u_{mm}, b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1, \ldots, b_m x_m - \boldsymbol{a}_m^\top \boldsymbol{u}_m),\\
&\quad (b_{m+1} x_{m+1} - \boldsymbol{a}_{m+1}^\top \boldsymbol{u}_{m+1}, \ldots, b_{n_x} x_{n_x} - \boldsymbol{a}_{n_x}^\top \boldsymbol{u}_{n_x})\big) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'}
\end{aligned}
$$

where $n_x' = (2 + m + m(m+1)/2 + n_x)$, $\boldsymbol{\alpha}' = (\delta, \eta, \varepsilon_1, \ldots, \varepsilon_m, \beta_{11}, \ldots, \beta_{mm}, \gamma_1, \ldots, \gamma_m)$,

$$
\delta + \eta + \sum_{i=1}^{m} \epsilon_i + \sum_{i\leq j}^{m} \beta_{ij} + \sum_{i=1}^{m} \gamma_i = 1, \quad \epsilon_i + \sum_{j:i\leq j} 2\beta_{ij} + \gamma_i = \alpha_i,\ \forall i \in [m], \quad \delta + 2\eta + \sum_{i=1}^{m} \gamma_i = 1, \qquad (16)
$$

and $\delta, \eta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma} \geq 0$. We can consider (15) as a robust constraint, where $\delta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and η are the uncertain parameters and use the adversarial approach in a similar way as described in Section 3.1 to find the worst-case for $\delta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and η.

3.3. Case 7 in Table 1: (Q) × (P)

Consider one conic quadratic inequality and one power cone inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \left\| \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \right\| \quad \text{and} \quad \begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0, \end{cases}$$

where $\boldsymbol{\alpha} \geq \boldsymbol{0}$ and $\sum_{i=1}^{m} \alpha_i = 1$.

Case 7(i) in Table 1. We multiply the LHSs and RHSs of the conic quadratic inequality and the power cone inequality with each other and obtain 1 additional power cone inequality:

$$\begin{aligned} (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}) \prod_{i=1}^{m} x_i^{\alpha_i} &\geq \left\| \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} \right\| \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \qquad (17) \\ \iff \prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{x} x_i)^{\alpha_i} &\geq \left\| (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}) \boldsymbol{x}_{[m+1]}^\top \right\|_F \\ \implies \prod_{i=1}^{m} (b_1 x_i - \boldsymbol{a}_1^\top \boldsymbol{u}_i)^{\alpha_i} &\geq \left\| \boldsymbol{D}\boldsymbol{U}_{[m+1]} + \boldsymbol{p}\boldsymbol{x}_{[m+1]}^\top \right\|_F, \end{aligned}$$

where $\boldsymbol{x}_{[m+1]} = [x_{m+1} \ \cdots \ x_{n_x}]$ and $\boldsymbol{U}_{[m+1]} = [\boldsymbol{u}_{m+1} \ \cdots \ \boldsymbol{u}_{n_x}]$. Moreover, we multiply the conic quadratic inequality with the nonnegativity constraints and obtain m additional conic quadratic inequalities, see Appendix A.2. We further multiply the LHS of the conic quadratic inequality with both sides of the power cone inequality and obtain 1 additional power cone inequality, see Section 3.2.

Case 7(ii) in Table 1. Linearizing the LHS of (17), we obtain the LHS of (15). Hence we obtain the following power cone inequality

$$\begin{aligned} &((b_1 - \boldsymbol{a}_1^\top \boldsymbol{x}, b_1^2 - 2b_1 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1 \boldsymbol{U} \boldsymbol{a}_1, x_1, \ldots, x_m, u_{11}, \ldots, u_{mm}, b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1, \ldots, b_m x_m - \boldsymbol{a}_m^\top \boldsymbol{u}_m), \\ &((\boldsymbol{D}\boldsymbol{U}_{[m+1]} + \boldsymbol{p}\boldsymbol{x}_{[m+1]}^\top)_{11}, \ldots, (U_{[m+1]} + \boldsymbol{p}\boldsymbol{x}_{[m+1]}^\top)_{L, n_x - m - 1})) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'}, \end{aligned}$$

where $n_x' = (2 + 2m + m(m+1)/2 + L(n_x - m - 1)$ and $\boldsymbol{\alpha}' = (\delta, \eta, \varepsilon_1, \ldots, \varepsilon_m, \beta_{11}, \ldots, \beta_{mm}, \gamma_1, \ldots, \gamma_m)$. We can view the above inequality as a robust constraint, where $\delta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and η are the uncertain parameters and use the adversarial approach in a similar way as described in Section 3.1 to find the worst-case for $\delta, \boldsymbol{\varepsilon}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and η.

3.4. Case 10 in Table 1: (P) × (P)

Consider two power cone inequalities

$$\begin{cases} \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \\ x_i \geq 0, \ i = 1, \ldots, m_1 \end{cases} \quad \text{and} \quad \begin{cases} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\ x_{\sigma(j)} \geq 0, \ j = 1, \ldots, m_2, \end{cases}$$

where σ is an arbitrary permutation, $n_x > m_1, m_2$, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \geq 0$ and $\sum_{i=1}^{m_1} \alpha_{1i} = \sum_{j=1}^{m_2} \alpha_{2j} = 1$.

Case 10(i) in Table 1. We multiply the left-hand sides and right-hand sides of the two power cone inequalities and obtain 1 additional power cone inequality:

$$\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \Longleftrightarrow \prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2 \sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \tag{18}$$

$$\Longrightarrow \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} u_{i,\sigma(j)}^{\theta_{ij}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} \sum_{j=m_2+1}^{n_x} u_{i,\sigma(j)}^2}, \tag{19}$$

where θ is such that

$$\sum_j \theta_{ij} = \alpha_{1i}, \quad \sum_i \theta_{ij} = \alpha_{2j}, \quad \theta_{ij} \geq 0, \quad i \in [m_1],\, j \in [m_2]. \tag{20}$$

Moreover, we multiply the nonnegativity constraints of one power cone with the nonnegativity constraints of the other power cone and obtain $m_1 m_2$ additional linear inequalities, see Appendix A.1. Finally, we multiply the nonnegativity constraints of each power cone with the power cone inequality of the other power cone and obtain $m_1 + m_2$ additional power cone inequalities, see Section 3.2.

Case 10(ii) in Table 1. Note that there are infinite possibilities for linearizing the LHS of (19). More precisely, we can write the LHS of (19) as follows:

$$\begin{aligned}
&\prod_{i=1}^{m_1} x_i^{\alpha_{1i}} \prod_{j=1}^{m_2} x_{\sigma(j)}^{\alpha_{2j}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\
\Longleftrightarrow &\prod_{i=1}^{m_1} \prod_{j=i}^{m_1} (x_i x_j)^{\alpha_{ij}} \prod_{i=1}^{m_2} \prod_{j=i}^{m_2} (x_{\sigma(i)} x_{\sigma(j)})^{\beta_{ij}} \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} (x_i x_{\sigma(j)})^{\gamma_{ij}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} x_i^2} \sqrt{\sum_{j=m_2+1}^{n_x} x_{\sigma(j)}^2} \\
\Longrightarrow &\prod_{i=1}^{m_1} \prod_{j=i}^{m_1} u_{ij}^{\alpha_{ij}} \prod_{i=1}^{m_2} \prod_{j=i}^{m_2} u_{\sigma(i)\sigma(j)}^{\beta_{ij}} \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} u_{i\sigma(j)}^{\gamma_{ij}} \geq \sqrt{\sum_{i=m_1+1}^{n_x} \sum_{j=m_2+1}^{n_x} u_{i,\sigma(j)}^2} \\
\Longrightarrow &((u_{11}, \ldots, u_{m_1 m_1}, u_{\sigma(1),\sigma(1)}, \ldots, u_{\sigma(m_2),\sigma(m_2)}, u_{1\sigma(1)}, \ldots, u_{m_1,\sigma(m_2)}), \\
&\quad (u_{m_1+1\sigma(m_2+1)}, \ldots, u_{n_x \sigma(n_x)})) \in \mathcal{P}_{n_x'}^{\boldsymbol{\alpha}'},
\end{aligned}$$

where $n_x' = m_1(m_1+1)/2 + m_2(m_2+1)/2 + m_1 m_2 + (n_x - m_1 - 1)(n_x - m_2 - 1)$ and $\boldsymbol{\alpha}' = (\alpha_{11}, \ldots, \alpha_{m_1 m_1}, \beta_{11}, \ldots, \beta_{m_2 m_2}, \gamma_{11}, \ldots, \gamma_{m_1 m_2})$. We can view the above inequality as a robust constraint, where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, are the uncertain parameters, which need to satisfy the following constraints:

$$\sum_{i=1}^{m_1} \sum_{j=i}^{m_1} \alpha_{ij} + \sum_{i=1}^{m_2} \sum_{j=i}^{m_2} \beta_{ij} + \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \gamma_{ij} = 1,$$

$$\begin{aligned}
&\sum_{j=1}^{m_1} \alpha_{ij} + \alpha_{ii} + \sum_{j=1}^{m_2} \gamma_{ij} = \alpha_{1i}, \quad i \in [m_1],\\
&\sum_{i=1}^{m_2} \beta_{ij} + \beta_{jj} + \sum_{i=1}^{m_1} \gamma_{ij} = \alpha_{2j}, \quad j \in [m_2],\\
&\alpha_{ij}, \beta_{ij}, \gamma_{ij} \geq 0, \quad i \in [m_1],\, j \in [m_2].
\end{aligned}$$

We can then use the adversarial approach in a similar way as described in Section 3.1 to find the worst-case values for $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$.

3.5. Case 11 in Table 1: (P) × (E)

Consider one power cone inequality and one exponential cone inequality

$$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} x_1 \geq x_2 \exp\left(\dfrac{x_3}{x_2}\right) \\ x_2 \geq 0, \end{cases}$$

where $\boldsymbol{\alpha} \geq \mathbf{0}$ and $\sum_{i=1}^{m} \alpha_i = 1$.

Case 11(i) in Table 1. We multiply the nonnegativity constraints of the power cone with the nonnegativity constraint of the exponential cone and obtain m additional linear inequalities, see Appendix A.1. Moreover, we multiply the nonnegativity constraints of the power cone with the exponential cone inequality and obtain m additional exponential cone inequalities, see 4(i) in Section 4.2. Finally, we multiply the nonnegativity constraint of the exponential cone as well as the LHS of the exponential cone inequality with the power cone inequality and obtain 2 additional power cone inequalities, see Section 3.2. Hence we obtain the following set of additional inequalities:

$$\begin{cases} \prod_{i=1}^{m} (x_1 x_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (x_1 x_i)^2} \\ \prod_{i=1}^{m} (x_2 x_i)^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} (x_2 x_i)^2} \\ x_1 x_1 \geq x_2 x_1 \exp\left(\frac{x_3 x_1}{x_2 x_1}\right) \\ \qquad \vdots \\ x_1 x_m \geq x_2 x_m \exp\left(\frac{x_3 x_m}{x_2 x_m}\right) \end{cases} \implies \begin{cases} \prod_{i=1}^{m} u_{1i}^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} u_{1i}^2} \\ \prod_{i=1}^{m} u_{2i}^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} u_{2i}^2} \\ x_1, \cdots, x_m \geq 0 \\ u_{11} \geq u_{21} \exp\left(\frac{u_{31}}{u_{21}}\right) \\ \qquad \vdots \\ u_{1m} \geq u_{2m} \exp\left(\frac{u_{3m}}{u_{2m}}\right) \\ u_{21}, \cdots, u_{2m} \geq 0. \end{cases}$$

Case 11(ii) in Table 1. When $x_3 < 0$ (and hence is not part of the power cone), in addition to the inequalities in Case 11(i), we multiply the linear inequality (22) obtained from the Taylor expansion with the power cone and obtain m additional linear inequalities and 1 additional power cone inequality, see Section 3.2.

Case 11(iii) in Table 1. When $x_3 \geq 0$, we multiply the linear inequality and the quadratic inequality in (23), obtained from the Taylor expansion, with the power cone and obtain m additional linear inequalities, 1 additional power cone inequality, see Section 3.2, m additional conic quadratic inequalities and 2 additional power cone inequalities, see Section 3.3, in addition to the inequalities in Case 11(i).

Remark 1. Observe that also here, we can use the adversarial approach in a similar way as described in the beginning of this section to find the best power cone reformulation, see 3(ii) in Section 3.2 and Section 3.3.

4. Product with an exponential cone inequality

In this section, we derive valid inequalities from an exponential cone inequality, by leveraging the Taylor expansion of the exponential function, which we can then pairwise multiply with other existing inequalities to tighten the approximation of problem (1). Moreover, we consider all cases in which we multiply one of the five basic cone inequalities with the exponential cone inequality as given in Section 2, except for the multiplication with the power cone inequality, which is treated in Section 3.

4.1. Generating valid inequalities from an exponential cone inequality

Sometimes, the pairwise multiplication of exponential cone inequalities may not lead to constraints that involve products of the original variables, see Section 6.1, and hence do not tighten the approximation. For this reason we derive valid inequalities from the exponential cone inequality that we can use for pairwise multiplications. We further note that even if the pairwise multiplications of the original constraints involving exponential cone inequalities yield good bounds on the new variables, we can still improve them with the derived inequalities. Our main tool in deriving those inequalities, is the Taylor expansion of the exponential function, that is

$$\exp(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots. \tag{21}$$

We can generate additional valid inequalities, depending on the sign of x_3. When $x_3 < 0$, there exists a $\xi \in \left[\frac{x_3}{x_2}, 0\right]$ such that

$$\exp\left(\frac{x_3}{x_2}\right) = 1 + \frac{x_3}{x_2} + \frac{\xi^2}{2} \geq 1 + \frac{x_3}{x_2}.$$

Therefore, we have $x_2 \exp\left(\frac{x_3}{x_2}\right) \geq x_2 + x_3$ and we derive the valid linear inequality

$$x_1 \geq x_2 + x_3. \tag{22}$$

When $x_3 \geq 0$, there exists a $\xi \in \left[0, \frac{x_3}{x_2}\right]$ such that

$$\exp\left(\frac{x_3}{x_2}\right) = 1 + \frac{x_3}{x_2} + \frac{x_3^2}{2x_2^2} + \frac{\xi^3}{6} \geq 1 + \frac{x_3}{x_2} + \frac{x_3^2}{2x_2^2}.$$

Therefore, we have $x_2 \exp\left(\frac{x_3}{x_2}\right) \geq x_2 + x_3 + \frac{x_3^2}{2x_2}$ and we derive the following valid inequalities

$$\begin{cases} x_1 \geq x_2 + x_3 + y, \\ \left\| \left(\sqrt{2}x_3, x_2 - y\right) \right\|_2 \leq x_2 + y. \end{cases} \tag{23}$$

We next show how we can obtain additional conic inequalities from pairwise multiplying the exponential cone inequality with one of the five basic cone inequalities as given in Section 2.

4.2. Case 4 in Table 1: (L) $\times$ (E)

Consider one linear inequality and one exponential cone inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0. \end{cases}$$

Case 4(i) in Table 1. Bertsimas et al. (2023) show how to multiply a linear inequality with a convex inequality by first reformulating the resulting inequality in its perspective form, and subsequently linearizing all product terms. In the case of an exponential cone inequality this boils down to the following: We multiply the linear inequality with both sides of the exponential cone inequality and obtain 1 additional exponential cone inequality:

$$\begin{aligned} &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})x_1 \geq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})x_2 \exp\left(\tfrac{(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})x_3}{(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})x_2}\right) \\ \Longrightarrow\ &b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1 \geq (b_1 x_2 - \boldsymbol{a}_1^\top \boldsymbol{u}_2) \exp\left(\tfrac{b_1 x_3 - \boldsymbol{a}_1^\top \boldsymbol{u}_3}{b_1 x_2 - \boldsymbol{a}_1^\top \boldsymbol{u}_2}\right). \end{aligned}$$

Here, the first inequality follows from multiplying both the nominator and denominator in the exponential function by the left hand side (LHS) of the linear inequality, and the second inequality follows from linearizing the product terms $\boldsymbol{x}\boldsymbol{x}^\top$ by the matrix $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$. Observe that the second inequality is jointly convex in $\boldsymbol{x}$ and $\boldsymbol{U}$, since the right hand side (RHS) is the perspective function of a convex function, which is convex, see Rockafellar (1970). Moreover, we multiply the linear inequality with the nonnegativity constraint of the exponential cone and obtain 1 additional linear inequality, see Appendix A.1.

Case 4(ii) in Table 1. When $x_3 < 0$, in addition to the additional inequalities in Case 4(i), we can multiply the linear inequality (22) obtained from (21) with the linear inequality $b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$ and obtain one additional linear inequality, see Appendix A.1.

Case 4(iii) in Table 1. When $x_3 \geq 0$, in addition to the additional inequalities in Case 4(i), we can multiply the valid inequalities (23) obtained from (21) with the linear inequality $b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0$ and obtain one additional linear inequality and one additional conic quadratic inequality, see Appendix A.1 and Appendix A.4.

4.3. Case 8 in Table 1: (Q) × (E)

Consider one conic quadratic inequality and one exponential cone inequality

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\| \quad \text{and} \quad \begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0. \end{cases}$$

Case 8(i) in Table 1. We multiply the conic quadratic inequality with the nonnegativity constraint and the LHS of the exponential cone inequality and obtain 2 additional conic quadratic inequalities, see Appendix A.2. Moreover, we multiply the LHS of the conic quadratic inequality with the exponential cone inequality and obtain 1 additional exponential cone inequality, see 4(i) in Section 4.2.

Case 8(ii) in Table 1. When $x_3 < 0$, in addition to the inequalities in Case 8(i), we multiply linear inequality (22) with the conic quadratic inequality and obtain 1 additional conic quadratic inequality, see Appendix A.2.

Case 8(iii) in Table 1. When $x_3 \geq 0$, in addition to the inequalities in Case 8(i), we multiply the linear inequality in (23) with the conic quadratic inequality and obtain 1 additional conic quadratic inequality, see Appendix A.2. Moreover, we multiply the conic quadratic inequality in (23) with the conic quadratic inequality and obtain 3 additional conic quadratic inequalities, see Appendix A.4.

4.4. Case 13 in Table 1: (E) × (E)

Consider two exponential cone inequalities

$$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} x_4 \geq x_5 \exp\left(\frac{x_6}{x_5}\right) \\ x_5 \geq 0. \end{cases}$$

Case 13(i) in Table 1. We multiply the LHSs and RHSs of the exponential cone inequalities and obtain 1 additional exponential cone inequality:

$$x_1 x_4 \geq x_2 x_5 \exp\left(x_3 x_5 / x_2 x_5 + x_6 x_2 / x_2 x_5\right) \quad \Longrightarrow \quad u_{14} \geq u_{25} \exp\left((u_{35} + u_{26})/u_{25}\right).$$

Moreover, we multiply the nonnegativity constraints of the exponential cones with each other and obtain 1 additional linear inequality, see Appendix A.1, we multiply the nonnegativity constraints of each exponential cone with the exponential cone inequality of the other cone and obtain 2 additional exponential cone inequalities and finally we multiply the LHS of each exponential cone inequality with both sides of the other exponential cone inequality and obtain 2 additional exponential cone inequalities, see Section 4.2.

Case 13(ii) in Table 1. When $x_3 < 0$ and $x_6 < 0$, for each exponential cone we obtain an additional linear inequality from (21). We multiply those two linear inequalities with each other and for each exponential cone we multiply the corresponding linear inequality from (21) with the inequalities from the other exponential cone and obtain in total 3 additional linear inequalities and 2 additional exponential cone inequalities, in addition to the inequalities in Case 13(i).

Case 13(iii) in Table 1. When $x_3 \geq 0$ and $x_6 \geq 0$, for each exponential cone, we obtain one additional linear and conic quadratic inequality from (21). We multiply each of those linear inequalities with the inequalities belonging to the other exponential cone (including the corresponding inequalities from (21)) and obtain 3 additional linear inequalities, 2 conic quadratic inequalities, and 2 exponential cone inequalities, see Appendix A.1, Appendix A.2 and Section 4.2. We also multiply each of those conic quadratic inequalities with each other and the inequalities belonging to the other exponential cone and obtain 7 additional conic quadratic inequalities and 2 exponential cone inequalities, see Appendix A.2, Appendix A.4 and Section 4.3.

Case 13(iv) in Table 1. When $x_3 < 0$ and $x_6 \geq 0$, we obtain the linear inequalities $x_1 \geq x_2 + x_3$ and $x_4 \geq x_5 + x_6 + \bar{y}$ from (21). We multiply each of those linear inequalities with the inequalities belonging to the other exponential cone and with each other and obtain 3 additional linear inequalities and 2 additional exponential cone inequalities. We also obtain the conic quadratic inequality $\left\|(\sqrt{2}x_6, x_5 - \bar{y})\right\| \leq x_5 + \bar{y}$ from (21), which we multiply with the inequalities belonging to the first exponential cone and the corresponding linear inequality $x_1 \geq x_2 + x_3$ obtained from (21). We then obtain 3 additional conic quadratic inequalities and 1 additional exponential cone inequality, in addition to the inequalities obtained from 13(i). Notice that we obtain the same number of additional constraints in case $x_3 \geq 0$ and $x_6 < 0$.

4.5. Case 14 in Table 1: (E) × (S)

Consider one exponential cone inequality and one LMI

$$\begin{cases} x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right) \\ x_2 \geq 0 \end{cases} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

Case 14(i) in Table 1. We multiply the nonnegativity constraint and the LHS of the exponential cone inequality with the LMI and obtain 2 additional LMIs:

$$\begin{cases} x_1 \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\ x_2 \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \end{cases} \implies \begin{cases} \boldsymbol{A}(\boldsymbol{u}_1) \succeq 0 \\ \boldsymbol{A}(\boldsymbol{u}_2) \succeq 0. \end{cases}$$

Note that the pairwise multiplication of a linear inequality with an LMI is already studied in Anstreicher (2017).

Case 14(ii) in Table 1. When $x_3 < 0$, in addition to the inequalities in Case 14(i), we multiply the linear inequality (22) with the LMI and obtain 1 additional LMI.

Case 14(iii) in Table 1. When $x_3 \geq 0$, we multiply the inequalities in (23) with the LMI and obtain 2 additional LMIs, see Appendix A.3 and Appendix A.5.

Remark 2. Note that for each case, we have only detailed how to pairwise multiply two generic inequalities from any of the five basic cones. However, one can also multiply each inequality by itself to derive additional inequalities.

5. Justification and enhancements

In this section, we investigate additional constraint multiplications and describe several ways to improve the approximation of nonconvex problem (1). First, we have a result on the best order of the multiplication of a linear inequality with a conically representable constraint. Further, we identify the best linearization for quadratic inequalities and finally, for DC problems, we derive additional conic constraints, by leveraging first order conditions.

5.1. Justification for first reformulating into conic constraints

It might be the case that one of the constraints is not in conic form, but since it is conically representable we can reformulate it such that it satisfies problem format (1). The question that arises then is which of the following options is better:

- Option 1: Multiply all linear constraints directly with this convex constraint that is not reformulated in conic form, following the methodology from Bertsimas et al. (2023).
- Option 2: Reformulate the conically representable constraints in conic form and then multiply this constraint with all linear constraints.

We will prove that both options lead to the same approximation. We use the definition of a conically representable constraint from Serrano (2015), that is, a constraint $f(\boldsymbol{x}) \leq 0$, where $f : \mathbb{R}^n \to \mathbb{R}$, is conically representable if its feasible set can be written as

$$\{\boldsymbol{x} \mid f(\boldsymbol{x}) \leq 0\} = \{\boldsymbol{x} \mid \exists \boldsymbol{u} \in \mathbb{R}^m,\ S(\boldsymbol{x}, \boldsymbol{u}) = 0,\ T(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{K}\}, \tag{24}$$

where $S:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{k_1}$ and $T:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{k_2}$ are affine mappings and $\mathcal{K}$ is a cone. We have the following result.

LEMMA 1. *Suppose the convex constraint $f(\boldsymbol{x})\le 0$ is conically representable, and suppose we multiply this constraint with a linear constraint $b-\boldsymbol{a}^\top\boldsymbol{x}\ge 0$. Then, the additional inequalities generated from the pairwise multiplication of the linear inequality with the convex constraint are equivalent for Options 1 and 2.*

Proof. Let S,T be the affine mappings that define the conic representation of the feasible set $\{\boldsymbol{x} \mid f(\boldsymbol{x})\le 0\}$ and let $\mathcal{K}$ be the corresponding cone. Let us denote the linear function $b-\boldsymbol{a}^\top\boldsymbol{x}$ by $\ell(\boldsymbol{x})$ and denote the linear function that results after linearizing $\boldsymbol{x}\ell(\boldsymbol{x})$ by $\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})$, where $\boldsymbol{U}=\boldsymbol{x}\boldsymbol{x}^\top$, and $\tilde{\ell}_i(\boldsymbol{x},\boldsymbol{U})=bx_i-a^\top\boldsymbol{U}_i$. Then after RPT we obtain the constraint

$$\ell(\boldsymbol{x})f\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})}\right)\le 0. \tag{25}$$

On the other hand, we can first derive the affine mappings S and T that define the conic representation of the feasible set of the original constraint, and then multiply it by $\ell(\boldsymbol{x})$, and apply RPT. Then we obtain the set

$$\left\{(\boldsymbol{x},\boldsymbol{U}) \;\middle|\; \exists\boldsymbol{u}\in\mathbb{R}^m,\ S\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})},\frac{\boldsymbol{u}}{\ell(\boldsymbol{x})}\right)=0,\ T\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})},\frac{\boldsymbol{u}}{\ell(\boldsymbol{x})}\right)\in\mathcal{K}\right\}. \tag{26}$$

Moreover we find

$$\begin{aligned}
&\left\{(\boldsymbol{x},\boldsymbol{U}) \;\middle|\; \ell(\boldsymbol{x})f\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})}\right)\le 0\right\}\\
&\quad=\left\{(\boldsymbol{x},\boldsymbol{U}) \;\middle|\; \exists\boldsymbol{u}\in\mathbb{R}^m,\ S\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})},\boldsymbol{u}\right)=0,\ T\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})},\boldsymbol{u}\right)\in\mathcal{K}\right\}\\
&\quad=\left\{(\boldsymbol{x},\boldsymbol{U}) \;\middle|\; \exists\boldsymbol{u}\in\mathbb{R}^m,\ S\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})},\frac{\boldsymbol{u}}{\ell(\boldsymbol{x})}\right)=0,\ T\left(\frac{\tilde{\boldsymbol{\ell}}(\boldsymbol{x},\boldsymbol{U})}{\ell(\boldsymbol{x})},\frac{\boldsymbol{u}}{\ell(\boldsymbol{x})}\right)\in\mathcal{K}\right\},
\end{aligned}$$

which concludes the proof. □

5.2. The best linearization for the quadratic case

In this section we describe that as for power cone inequalities, also for quadratic inequalities there are multiple choices for linearization. We first give an example that shows that different choices may lead to different solutions.

EXAMPLE 2. Consider the following toy example

$$\begin{aligned}
\max_{\boldsymbol{x}}\quad & x_1^2+x_2\\
\text{s.t.}\quad & x_1^2+x_2^2\le 1,\\
& x_1,x_2\ge 0.
\end{aligned} \tag{27}$$

The optimal solution of this problem is $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$ and the optimal value is $\frac{5}{4}$. By applying RLT we obtain the following relaxation

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & u_{11} + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 1, \\ & u_{11} + u_{22} \leq 1, \\ & x_1, x_2, u_{11}, u_{22} \geq 0. \end{aligned} \tag{28}$$

The solution of (28) appears to be $u_{11} = x_2 = 1$, $u_{22} = x_1 = 0$, with optimal value 2. This solution is suboptimal for the original problem (27). However, if we add the inequality that occurs when we partially linearize, i.e., the inequality $u_{11} + x_2^2 \leq 1$, then we do obtain the optimal solution of (27). It can easily be verified that if we add the LMI (3) to (28) then we also get the optimal solution to (27). □

The question hence arises whether we should linearize all quadratic terms, or only a part of these terms, such that the remaining part is convex. The following lemma shows that linearizing all quadratic terms in combination with adding the LMI (3) yields the tightest approximation. Hence, when LMI (3) is included, then the full linearization always yields the best approximation.

LEMMA 2. *Consider the quadratic inequality*

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + c \leq 0,$$

where $\boldsymbol{A}$ *is not necessarily positive semidefinite. Then the best linearization of the above quadratic inequality is obtained by linearizing all quadratic terms if LMI* (3) *is included in the constraints.*

Proof. We search for the best value of a semidefinite matrix $\boldsymbol{B}$ such that linearizing $\boldsymbol{x}^\top(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x}$ and keeping $\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x}$ yields the tightest approximation. In other words we consider the following inequality

$$\max_{\boldsymbol{B}\succeq \boldsymbol{0}} \left\{ \boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} + \operatorname{Tr}\left((\boldsymbol{A}-\boldsymbol{B})\boldsymbol{U}\right) + \boldsymbol{b}^\top \boldsymbol{x} + c \right\} \leq 0,$$

which is equivalent to

$$\max_{\boldsymbol{B}\succeq \boldsymbol{0}} \left\{ \operatorname{Tr}\left(\boldsymbol{B}(\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{U})\right) \right\} + \operatorname{Tr}\left(\boldsymbol{A}\boldsymbol{U}\right) + \boldsymbol{b}^\top \boldsymbol{x} + c \leq 0.$$

Taking the dual of the maximization problem we obtain that this is equivalent to

$$\operatorname{Tr}\left(\boldsymbol{A}\boldsymbol{U}\right) + \boldsymbol{b}^\top \boldsymbol{x} + c \leq 0, \qquad \begin{pmatrix} \boldsymbol{U} & \boldsymbol{x} \\ \boldsymbol{x}^\top & 1 \end{pmatrix} \succeq 0,$$

where the LMI is the same as (3). Therefore, the LMI can be interpreted (from its dual) as obtaining the best $\boldsymbol{B}$, and we do not need to add different LMIs. □

5.3. First-order conditions for DC problems

Derivation. In this section, we consider the following DC constraint

$$c_0(\boldsymbol{x}) - c_1(\boldsymbol{x}) \leq 0,$$

where $c_0, c_1 : \mathbb{R}^{n_x} \to (-\infty, +\infty]$ are proper, closed, and convex functions. Rockafellar (1970) showed that, using the biconjugate reformulation, the above inequality can be equivalently written as

$$\begin{cases} c_0(\boldsymbol{x}) - \boldsymbol{x}^\top \boldsymbol{y} + c_1^*(\boldsymbol{y}) \leq 0, \\ \boldsymbol{y} \in \mathrm{dom}(c_1^*), \end{cases}$$

as long as the infimum is attained, see also (Bertsimas et al., 2023, Example 1). Note that the obtained problem in this case is in the format of problem (1). Now suppose $c_1(\boldsymbol{x})$ is differentiable, then we have

$$\boldsymbol{y} = \nabla_{\boldsymbol{x}} c_1(\boldsymbol{x}). \tag{29}$$

We can leverage this extra equation to get a better approximation, as illustrated in the following examples.

Examples.

Example 3. Suppose $c_1(x) = -\log(x)$. Then (29) becomes $y = -1/x$, or $xy = -1$. We introduce the variable v to linearize the product xy and obtain the equality $v = -1$. □

Example 4. Suppose $c_1(x) = x\log(x)$. Then (29) becomes $y = 1 + \log(x)$. Hence, we can add the following convex inequalities to problem (1): $y \leq 1 + \log(x)$, $v \geq x + x\log(x)$, where $v = xy$. □

Example 5. Suppose $c_1(\boldsymbol{x}) = \log\left(\sum_j \exp(x_j)\right)$. Then (29) becomes

$$y_i = \frac{\exp(x_i)}{\sum_j \exp(x_j)} \iff \log(y_i) + \log\left(\sum_j \exp(x_j)\right) = x_i \iff y_i\log(y_i) + y_i\log\left(\sum_j \exp(x_j)\right) = x_i y_i.$$

We linearize $y_i x_j$ with v_{ij} and obtain the following convex inequality

$$y_i\log(y_i) + y_i\log\left(\sum_j \exp\left(\frac{v_{ij}}{y_i}\right)\right) \leq v_{ii}, \quad \forall i \tag{30}$$

which we can include in problem (1). Note that $y_i\log\left(\sum_j \exp\left(\frac{v_{ij}}{y_i}\right)\right)$ is a perspective function of the convex function $\log\left(\sum_j \exp(v_{ij})\right)$. □

Showing the benefit in a nonconvex optimization problem. We consider the following problem:

$$\max_{\boldsymbol{x}} \quad \log\left(\sum_{i=1}^{n_x} \exp(x_i)\right) \tag{31a}$$

$$\text{s.t.} \quad x_1 \exp(x_i) \le \rho, \qquad i \in [n_x], \tag{31b}$$

$$x_1 \ge 0. \tag{31c}$$

Using the biconjugate of the convex objective, we obtain the following equivalent problem:

$$\max_{\boldsymbol{x},\boldsymbol{y}} \quad \boldsymbol{x}^\top \boldsymbol{y} + \sum_{i=1}^{n_x} w_i \tag{32a}$$

$$\text{s.t.} \quad x_1 \exp(x_i) \le \rho, \qquad i \in [n_x], \tag{32b}$$

$$y_i \exp\left(\frac{w_i}{y_i}\right) \le 1, \qquad i \in [n_x], \tag{32c}$$

$$\sum_{i=1}^{n_x} y_i = 1, \tag{32d}$$

$$x_1, \boldsymbol{y} \ge \boldsymbol{0}. \tag{32e}$$

We can consider the valid inequalities (30) derived from the first order conditions, see Example 5. We next compare the obtained upper bounds for problem (32), when we use the decomposition of the exponential cone as well as the first order conditions. The results are illustrated in Table 2. We refer to Appendix C.1 for the formulations of the three different approximations. From

Table 2 Comparison of the obtained upper bounds for problem (32), including the LMI, with and without decomposing the exponential cone (dec) as well as with and without including the first order conditions (foc). n_x is the dimension of the variables. In all instances we fix $\rho = 1$.

n_x	w/o dec-foc	w dec w/o foc	w/o dec w foc	w dec-foc
10	243,922	196,411	29,335	20,314
20	435,374	429,614	26,559	10,820
30	452,940	243,679	28,396	9,909
40	542,497	487,828	20,532	11,032
50	317,332	255,552	30,662	15,259

Table 2 we observe that when we include the inequalities obtained from the decomposition of the exponential cone, the upper bound improves. We further notice a more significant improvement in the upper bound when including the inequalities obtained from the first order conditions. We therefore observe that in the considered problem, the valid inequalities obtained from the first order conditions improve the approximation significantly.

6. Numerical experiments

In this section, we demonstrate empirically the benefit of the cone product reformulations introduced in this paper. More precisely, we consider a quadratic optimization problem over exponential cone constraints, demonstrating the value of the proposed methodology for the exponential cone as well as a robust palatable diet problem showing the benefit of the proposed methodology for the power cone.

All numerical experiments are performed on an Intel i9 2.3GHz CPU core with 16 GB RAM. All computations for RPT-BB and SCIP are conducted with MOSEK version 9.2.45 (MOSEK ApS, 2020), Gurobi version 9.0.2 Gurobi Optimization (2019), and implemented using Julia 1.5.3 and the Julia package `JuMP.jl` version 0.21.6, and all computations for BARON are conducted with BARON version 20.10.16 Sahinidis (1996) implemented using the Python package `pyomo` version 6.4.1.

6.1. Quadratic optimization with exponential cone constraints

In this section we consider the following problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^\top \boldsymbol{A}_0 \boldsymbol{x} + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 && \text{(33a)} \\ \text{s.t.} \quad & \log\left(\sum_{i=1}^{n_x} \exp(-x_i)\right) \leq \alpha, && \text{(33b)} \\ & \sum_{i=1}^{n_x} \exp(x_i) \leq \beta. && \text{(33c)} \end{aligned}$$

Using the conic representation of constraints (33b), (33c), problem (33) is equivalent to the following problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^\top \boldsymbol{A}_0 \boldsymbol{x} + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 && \text{(34a)} \\ \text{s.t.} \quad & \sum_{i=1}^{n_x} z_i \leq 1, && \text{(34b)} \\ & \exp(-x_i - \alpha) \leq z_i, \qquad i \in [n_x], && \text{(34c)} \\ & \sum_{i} t_i \leq \beta, && \text{(34d)} \\ & \exp(x_i) \leq t_i. && \text{(34e)} \end{aligned}$$

Using the decomposition of the exponential cone, we obtain the additional constraints

$$\beta \geq x_i + 1, \quad z_i \geq -x_i - \alpha + 1, \quad i \in [n_x].$$

We note that without the decomposition of the exponential cone, the relaxation obtained from multiplying constraints and linearizing products with new variables is unbounded, since $\boldsymbol{x}\boldsymbol{x}^\top$ does not appear in the constraints. However, we can link them by considering the decomposition of the exponential cone and as a result obtain tighter bounds. In Table 3 we solve the problem to optimality using RPT-BB, with and without the LMI, while also comparing with BARON. The formulation after multiplying all constraints and the data generation for each instance are summarized in Appendix C.2 and Appendix D respectively.

Instance	n_x	w/o LMI			w LMI			BARON	
		Opt	Time(s)	Hyp	Opt	Time(s)	Hyp	Opt	Time(s)
1	5	-102	0.07	0	-102	0.1	0	-102	1
2	10	-175.8	0.1	0	-175.8	0.1	0	-175.8	0.5
3	10	-1885.4	0.2	0	-1885.4	0.7	0	-1885.4	220.2
4	20	-8172.8	4.4	1	-8172.8	25.1	1	-8172.8	3600^*
5	50	-37306.6	101.4	3.2	-37306.6	2100.2	3.1	-37306.6	3600^*
6	100	-326577.3	75.4	1	-	3600^*	0	-326577.3	3600^*

Table 3 Comparison of RPT-BB, when including/ not including the LMI, for problem instances 1,2 and, 3,4,5,6 which reflect the average of 10 randomly generated instances. Opt represents the optimal value, Hyp represents the total number of hyperplanes generated during branch and bound, Time represents the computation time in seconds and n_x represents the problem dimension. We set the maximum time limit equal to 3600 seconds, hence if the computation time equals 3600^*, the optimum cannot be found within 3600 seconds and all approaches return the best value they can obtain within 3600 seconds.

From Table 3, we observe that for all instances we were able to solve the problem to optimality with branch and bound in less computational time than BARON, when decomposing the exponential cone. Moreover, we observe that in instances 4, 5, and 6 corresponding to 20, 50, and 100 variables respectively, when using the proposed valid inequalities the problem could be solved to optimality in seconds, while BARON located the global optimal solution but could not prove optimality within one hour. Finally, we note that in instance 6 which involves 100 variables, the problem could not be solved at the root node after one hour when including the LMI.

6.2. Robust palatable diet problem

In this section, we consider the palatable diet problem where there is uncertainty in the coefficients of one nutrient. The palatable diet problem is an important part of the World Food Programme's (WFP) food supply chain. The problem is to maximize palatability, while satisfying diet requirements. The main variables are the ration variables r_k, i.e. the amount of ingredient k in the ration. Further, the palatability is defined as a function $\hat{h}(\boldsymbol{r})$, which we assume is quadratic, that is $\hat{h}(\boldsymbol{r}) = \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d$. Utilizing the dataset from Maragno et al. (2021), consisting of observations $(\boldsymbol{r}_i, \hat{h}(\boldsymbol{r}_i))$, we find the values of $\boldsymbol{A}, \boldsymbol{b}$ and d that fit them best by regression. Moreover, we include diet constraints, ensuring that the total nutritional value of a certain nutrient l is not below the required nutritional value η_l for that nutrient, that is $\sum_{k\in\mathcal{K}}\beta_{kl}r_k \geq \eta_l$. Finally, we also have a budget constraint, that is $\sum_{k\in\mathcal{K}} c_k r_k \leq W$. The problem formulation is as follows:

$$\begin{aligned}
\max_{\boldsymbol{r}} \quad & \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d && && \text{(35a)}\\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} c_k r_k \leq W, && && \text{(35b)}\\
& \sum_{k\in\mathcal{K}} \beta_{kl} r_k \geq \eta_l && l\in\mathcal{L}, && \text{(35c)}\\
& \boldsymbol{r} \geq \boldsymbol{0}. && && \text{(35d)}
\end{aligned}$$

It is often the case that the nutrient coefficients are uncertain. Assuming uncertainty in the coefficients of nutrient m, we obtain the following robust constraint:

$$(\boldsymbol{\beta}_m + \boldsymbol{z})^\top \boldsymbol{r} \geq \eta_m, \ \forall \boldsymbol{z} \in \mathcal{U},$$

where $\mathcal{U} = \{\boldsymbol{z} : \|\boldsymbol{z}\|_p \leq \rho\}$, for $p \geq 1$. In this case, the robust counterpart is as follows (Bertsimas and den Hertog (2022))

$$\boldsymbol{\beta}_m^\top \boldsymbol{r} - \rho\|\boldsymbol{r}\|_q \geq \eta_m,$$

where $1/p + 1/q = 1$. The constraint can be written as $\|\boldsymbol{r}\|_q \leq \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m)$ and, by using auxiliary variables $\boldsymbol{t}$, can be reformulated as the following set of linear and power cone inequalities:

$$\begin{cases} \sum_k t_k = \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m), \\ t_k^{1/q}(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m))^{1-1/q} \geq |r_k|, \quad k\in\mathcal{K}. \end{cases}$$

Hence, the final problem formulation is as follows:

$$\begin{aligned}
\max_{\boldsymbol{r},\boldsymbol{t}} \quad & \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d && && \text{(36a)}\\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} c_k r_k \leq W, && && \text{(36b)}\\
& \sum_{k} \beta_{kl} r_k \geq \eta_l, && l\in\mathcal{L}/\{m\}, && \text{(36c)}\\
& \sum_{k} t_k = \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m), && && \text{(36d)}\\
& t_k^{1/q}(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m))^{1-1/q} \geq |r_k|, && k\in\mathcal{K}, && \text{(36e)}\\
& \boldsymbol{r} \geq \boldsymbol{0}. && && \text{(36f)}
\end{aligned}$$

We compare the optimal solution and computational time of problem (36) with and without the multiplication of power cone inequalities with each other. We also compare the results with BARON. The results for ThiamineB1 and NicacinB3 as the robust nutrient, are illustrated in Table 4. The

Table 4 Comparison of optimal solutions for problem (36), including the LMI, with and without the proposed additions. Opt represents the optimal value, Hyp represents the total number of hyperplanes generated during branch and bound, and Time represents the computation time in seconds. We set the maximum time limit equal to 3600 seconds, hence if the computation time equals 3600*, the optimum cannot be found within 3600 seconds and all approaches return the best value they can obtain within 3600 seconds. We fix $p=3$, $\rho=0.1$, $W=5$.

Rob Nutr	w/o additions			w additions			BARON	
	Opt	Time(s)	Hyp	Opt	Time(s)	Hyp	Opt	Time(s)
ThiamineB1	269.1	111.2	9	269.1	16	0	269.1	1.9
NicacinB3	213	44.5	4	213	39.6	1	212.2	3600*

nutrient coefficients β_{kl} are from Peters et al. (2022) and the costs c_k are from de Moor et al. (2023). The problem formulation after multiplying all constraints is provided in Appendix C.3.

From Table 4 we observe that for ThiamineB1 as the robust nutrient, all methods find the global optimal solution, with BARON achieving the best computational time. We also notice that including the multiplication of power cone inequalities improves the approximation and as a result the computational time decreases from 111.20 to 16.01 seconds. In case the robust nutrient is NicacinB3, we observe that adding the power cone multiplications improves the computational time, while also finding the global optimal solution. In this case BARON could not solve the problem within one hour and returned a solution with slightly smaller objective value.

7. Discussion and conclusion

In this paper, we studied in detail the pairwise multiplications of cone inequalities. In particular, we showed how we can pairwise multiply one of the five basic cone constraints with exponential and power cone inequalities and obtain convex constraints. Moreover, we derived valid inequalities from exponential and power cone inequalities, which can further strengthen the approximation. In addition, for DC problems we derived valid inequalities from first order conditions. In the numerical experiments, we provided empirical evidence, suggesting that the cone product reformulations introduced in this paper improve the approximation, while often leading to smaller computational times than BARON. In future work, it would be interesting to investigate adaptations of the proposed methodology, including partial constraint multiplications as well as partial generation of product variables.

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Appendix

A. Multiplication of cone inequalities from the literature

In this appendix, we provide all multiplications of cone inequalities from Table 1 that are from the literature.

A.1. Case 1 in Table 1: (L) × (L)

Consider two linear inequalities

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq 0.$$

Multiplying the two linear inequalities yields 1 additional linear inequality (Sherali and Alameddine, 1992):

$$\begin{aligned}(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \geq 0 &\iff b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{a}_2 \geq 0 \\ &\implies b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2 \geq 0.\end{aligned}$$

A.2. Case 2 in Table 1: (L) × (Q)

Consider one linear inequality and one conic quadratic inequality

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d}\right\|.$$

Multiplying the linear inequality with both sides of the conic quadratic inequality yields 1 additional conic quadratic inequality (Sturm and Zhang (2003)) :

$$\begin{aligned}&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d}\right\| \leq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \\ \iff &\left\|(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(\boldsymbol{D}\boldsymbol{x} + \boldsymbol{d})\right\| \leq (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \\ \implies &\left\|b_1 \boldsymbol{D}\boldsymbol{x} + b_1 \boldsymbol{d} - \boldsymbol{D}\boldsymbol{U}\boldsymbol{a}_1 - \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{d}\right\| \leq b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2.\end{aligned}$$

A.3. Case 5 in Table 1: (L) × (S)

Consider one linear inequality and one LMI respectively

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq 0 \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

We apply RPT to the multiplication of these inequalities, and obtain one additional LMI:

$$\begin{aligned}
&(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\
\Longleftrightarrow\; &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_0 + (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_1 x_1 + \cdots + (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_{n_x} x_{n_x} \succeq 0 \\
\Longrightarrow\; &(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{A}_0 + (b_1 x_1 - \boldsymbol{a}_1^\top \boldsymbol{u}_1)\boldsymbol{A}_1 + \cdots + (b_1 x_{n_x} - \boldsymbol{a}_1^\top \boldsymbol{u}_{n_x})\boldsymbol{A}_{n_x} \succeq 0.
\end{aligned}$$

A.4. Case 6 in Table 1 (Q) × (Q)

Consider two conic quadratic inequalities

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1\right\| \quad \text{and} \quad b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2\right\|. \tag{37}$$

We multiply the left-hand side of the first conic quadratic inequality with both sides of the second conic quadratic inequality and the left-hand side of the second conic quadratic inequality with both sides of the first conic quadratic inequality to obtain 2 additional conic quadratic inequalities, see Appendix A.2. Moreover, we multiply the left-hand sides and right-hand sides of the conic quadratic inequalities with each other and obtain 1 additional conic quadratic inequality:

$$(b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) \geq \left\|\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1\right\|\left\|\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2\right\| \tag{38}$$

$$\Longleftrightarrow \quad b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{a}_2 \geq \left\|(\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1)(\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2)^\top\right\|_F \tag{39}$$

$$\Longrightarrow \quad b_1 b_2 - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} + \boldsymbol{a}_1^\top \boldsymbol{U} \boldsymbol{a}_2 \geq \left\|\boldsymbol{D}_1 \boldsymbol{U} \boldsymbol{D}_2^\top + \boldsymbol{p}_1 \boldsymbol{x}^\top \boldsymbol{D}^\top + \boldsymbol{D}\boldsymbol{x}\boldsymbol{p}_2^\top + \boldsymbol{p}_1 \boldsymbol{p}_2^\top\right\|_F. \tag{40}$$

This is Case 6(i) in Table 1.

In the literature also two LMIs are proposed. First observe that the two conic quadratic inequalities (37) can be written as

$$b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1\right\| \iff \begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1 \boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0$$

and

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2\right\| \iff \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2 \boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0.$$

We now assume that, without loss of generality, the matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are of the same size. Indeed, suppose that $\boldsymbol{D}_1$ has less rows than $\boldsymbol{D}_2$, then we can extend matrix $\boldsymbol{D}_1$ by zero rows or by copying scaled versions of some of the original rows. Using Lemma 3, see Appendix E, for the Kronecker product, and linearizing each element of the product, we obtain

$$\begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \otimes \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0$$

$$\Longrightarrow \left[\begin{array}{cc|cc|c|cc} \alpha & \boldsymbol{\gamma}^\top & \delta_1 & \boldsymbol{\eta}_1^\top & \cdots & \delta_r & \boldsymbol{\eta}_r^\top \\ \boldsymbol{\gamma} & \alpha I & \boldsymbol{\eta}_1 & \delta_1 I & & \boldsymbol{\eta}_r & \delta_r I \\ \hline \delta_1 & \boldsymbol{\eta}_1^\top & \alpha & \boldsymbol{\gamma}^\top & & & \\ \boldsymbol{\eta}_1 & \delta_1 I & \boldsymbol{\gamma} & \alpha I & & & \\ \hline \vdots & & & & \ddots & & \\ \hline \delta_r & \boldsymbol{\eta}_r^\top & & & & \alpha & \boldsymbol{\gamma}^\top \\ \boldsymbol{\eta}_r & \delta_r I & & & & \boldsymbol{\gamma} & \alpha I \end{array}\right] \succeq 0,$$

where

$$\begin{aligned} \alpha &= b_1 b_2 - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} + a_1^\top U a_2 \\ \boldsymbol{\gamma} &= b_1(\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2) - (a_1^\top \boldsymbol{x})\boldsymbol{p}_2 - \boldsymbol{D}_2\boldsymbol{U}\boldsymbol{a}_1 \\ \delta_i &= b_2(\boldsymbol{d}_{1i}^\top \boldsymbol{x}) + p_{1i}(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x}) - \boldsymbol{d}_{1i}^\top \boldsymbol{U}\boldsymbol{a}_1, \quad i = 1, \ldots, r \\ \boldsymbol{\eta}_i &= (\boldsymbol{d}_{1i}^\top \boldsymbol{x} + p_{1i})\boldsymbol{p}_2 + p_{1i}\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{D}_2\boldsymbol{U}\boldsymbol{d}_{1i}, \quad i = 1, \ldots, r, \end{aligned}$$

and $\boldsymbol{d}_{1i}$ and $\boldsymbol{d}_{2i}$ denote the i-th row of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. This is Case 6(ii) in Table 1. Another LMI is proposed by Jiang and Li (2016), using the Hadamard product instead of the Kronecker product. It follows for the Hadamard product that

$$\begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & (\boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1)^\top \\ \boldsymbol{D}_1\boldsymbol{x} + \boldsymbol{p}_1 & (b_1 - \boldsymbol{a}_1^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \circ \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2)^\top \\ \boldsymbol{D}_2\boldsymbol{x} + \boldsymbol{p}_2 & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0,$$

which implies

$$\begin{bmatrix} \alpha & \boldsymbol{\beta}^\top \\ \boldsymbol{\beta} & \alpha \boldsymbol{I} \end{bmatrix} \succeq 0, \tag{41}$$

where

$$\alpha = b_1 b_2 - b_2 \boldsymbol{a}_1^\top \boldsymbol{x} - b_1 \boldsymbol{a}_2^\top \boldsymbol{x} + a_1^\top \boldsymbol{U} a_2 \tag{42}$$

$$\beta_i = \boldsymbol{d}_{1i} U \boldsymbol{d}_{1i} + p_{1i} p_{2i} + p_{2i} \boldsymbol{d}_{1i} \boldsymbol{x} + p_{1i} \boldsymbol{d}_{2i} \boldsymbol{x}, \quad i = 1, \ldots, r, \tag{43}$$

and $\boldsymbol{d}_{1i}$ and $\boldsymbol{d}_{2i}$ is the i-th row of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. Notice that the matrix in the left-hand side of (41) has an arrow structure, and hence LMI (41) is equivalent with the following conic quadratic inequality:

$$\|\beta\|_2 \le \alpha. \tag{44}$$

It can easily be verified that (44) is a weaker inequality than (40). This is Case 6(iii) in Table 1.

A.5. Case 9 in Table 1: (Q) × (S)

Consider one conic quadratic inequality and one LMI

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\| \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0. \tag{45}$$

First observe that the conic quadratic inequality can be formulated as an LMI, Anstreicher (2017)

$$b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} \geq \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\| \iff \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p})^\top \\ \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \succeq 0.$$

We now multiply these inequalities. Using the fact that the Kronecker product of two positive semidefinite matrices is also positive semidefinite, see Anstreicher (2017) and Lemma 3, see Appendix E, we obtain 1 additional LMI:

$$\begin{aligned}
\left(b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} - \left\|\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p}\right\|\right) \boldsymbol{A}(\boldsymbol{x}) \succeq 0 &\implies \begin{bmatrix} b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{p})^\top \\ \boldsymbol{D}\boldsymbol{x} + \boldsymbol{p} & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{I} \end{bmatrix} \otimes \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\
&\iff \begin{bmatrix} (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) & (\boldsymbol{d}_1^\top \boldsymbol{x} + \boldsymbol{p}_1)\boldsymbol{A}(\boldsymbol{x}) & \cdots & (\boldsymbol{d}_r^\top \boldsymbol{x} + \boldsymbol{p}_r)\boldsymbol{A}(\boldsymbol{x}) \\ (d_1^\top \boldsymbol{x} + \boldsymbol{p}_1)\boldsymbol{A}(\boldsymbol{x}) & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) & & \\ \vdots & & \ddots & \\ (d_r^\top \boldsymbol{x} + \boldsymbol{p}_r)\boldsymbol{A}(\boldsymbol{x}) & & & (b_2 - \boldsymbol{a}_2^\top \boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) \end{bmatrix} \succeq 0 \\
&\implies \begin{bmatrix} \boldsymbol{A}(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) & \boldsymbol{A}(p_1\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_1) & \cdots & \boldsymbol{A}(p_r\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_r) \\ \boldsymbol{A}(p_1\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_1) & \boldsymbol{A}(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) & & \\ \vdots & & \ddots & \\ \boldsymbol{A}(p_r\boldsymbol{x} + \boldsymbol{U}\boldsymbol{d}_r) & & & \boldsymbol{A}(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) \end{bmatrix} \succeq 0,
\end{aligned} \tag{46}$$

where $\boldsymbol{d}_i$ is the i-th row of $\boldsymbol{D}$. We could also directly multiply the left-hand side of the conic quadratic inequality with the LMI and obtain 1 additional LMI:

$$A(b_2\boldsymbol{x} - \boldsymbol{U}\boldsymbol{a}_2) \succeq 0,$$

which is also implied by (46).

A.6. Case 12 in Table 1: (P) × (S)

Consider one power cone inequality and one LMI

$$\begin{cases} \prod_{i=1}^{m} x_i^{\alpha_i} \geq \sqrt{\sum_{i=m+1}^{n_x} x_i^2} \\ x_1, \cdots, x_m \geq 0 \end{cases} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) \succeq 0.$$

We multiply the nonnegativity constraints of the power cone with the LMI and obtain m additional LMIs:

$$x_i \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \quad \implies \quad \boldsymbol{A}(\boldsymbol{u}_i) \succeq 0, \qquad i = 1, \cdots, m.$$

A.7. Case 15 in Table 1: (S) $\times$ (S)

Consider two LMIs

$$\begin{cases} \boldsymbol{A}(\boldsymbol{x}) \succeq 0 \\ \boldsymbol{B}(\boldsymbol{x}) \succeq 0. \end{cases}$$

If $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are of different sizes, it follows from Lemma 3, see Appendix E, that the Kronecker product of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ is positive semidefinite, that is $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x}) \succeq 0$. Notice that each element in the Kronecker product is the multiplication of two affine functions of $\boldsymbol{x}$. After linearizing the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x})$ with the matrix $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{U})$, which is linear in both $\boldsymbol{x}$ and $\boldsymbol{U}$, we obtain Case 15(i) of Table 1. If $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are of the same size, it follows from Lemma 3, see Appendix E, that the Hadamard product of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ is positive semidefinite, that is $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x}) \succeq 0$. Notice that each element in the Hadamard product is the multiplication of two affine functions of $\boldsymbol{x}$. After linearizing the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x})$ with the matrix $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{U})$, which is linear in both $\boldsymbol{x}$ and $\boldsymbol{U}$, we obtain Case 15(ii) of Table 1.

B. Adversarial approach for best power cone reformulation

In Algorithm 1 we include generic pseudocode for the adversarial approach, utilized for finding the best reformulation when multiplying a cone inequality with the power cone. The function $g(\boldsymbol{x}, \boldsymbol{U})$ refers to the right-hand side of the constraint obtained after the multiplication of a cone inequality with a power cone inequality.

Algorithm 1 Adversarial approach for best reformulation

Input: $\boldsymbol{\theta}^0$: Initial guess for the uncertain parameters.

Output: $(\boldsymbol{x}^*, \boldsymbol{U}^*)$: Optimal solutions of the best reformulated problem.

1: Initialize $\mathcal{V} = \{\boldsymbol{\theta}^0\}$.
2: Solve the master problem with input $\mathcal{V}$ and obtain optimal solutions $(\boldsymbol{x}^*, \boldsymbol{U}^*)$.
3: Solve the sub-problem with input $(\boldsymbol{x}^*, \boldsymbol{U}^*)$ and obtain optimal solution $\boldsymbol{\theta}^*$ with cost c^*.
4: **if** $c^* < \log(g(\boldsymbol{x}^*, \boldsymbol{U}^*))$ **then**
5: $\quad \mathcal{V} = \mathcal{V} \cup \{\boldsymbol{\theta}^*\}$.
6: $\quad$ Go to Step 2.
7: **else**
8: $\quad$ Return the optimal solutions $(\boldsymbol{x}^*, \boldsymbol{U}^*)$.
9: **end if**

C. RPT formulations of numerical experiments

In this section, we include the formulations obtained when multiplying all constraints in the problems encountered in the numerical experiments.

C.1. RPT formulation of Section 5.3

We linearize $\boldsymbol{x}\boldsymbol{x}^\top$ with $\boldsymbol{X}$, $\boldsymbol{y}\boldsymbol{y}^\top$ with $\boldsymbol{Y}$, $\boldsymbol{x}\boldsymbol{y}^\top$ with $\boldsymbol{U}$, and $\boldsymbol{x}\boldsymbol{w}^\top$ with $\boldsymbol{Q}$. After multiplying all constraints, we obtain the following problem:

$$
\begin{aligned}
\max_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{X},\\ \boldsymbol{Y},\boldsymbol{U}\boldsymbol{Q},\\ \boldsymbol{W},\boldsymbol{Q}}} \quad & \mathrm{Tr}(\boldsymbol{U}) + \sum_i w_i && && \text{(47a)}\\
\text{s.t.} \quad & x_1 \exp\left(\frac{X_{1i}}{x_1}\right) \le \rho, && i \in [n], && \text{(47b)}\\
& y_i \exp\left(\frac{w_i}{y_i}\right) \le 1, && i \in [n], && \text{(47c)}\\
& U_{1i} \exp\left(\frac{Q_{1i}}{U_{1i}}\right) \le x_1, && i \in [n], && \text{(47d)}\\
& Y_{ij} \exp\left(\frac{P_{ji}}{Y_{ij}}\right) \le y_j, && i,j \in [n], && \text{(47e)}\\
& Y_{ij} \exp\left(\frac{P_{ij}+P_{ji}}{Y_{ij}}\right) \le 1, && i \le j \in [n], && \text{(47f)}\\
& \sum_i y_i = 1, && && \text{(47g)}\\
& \sum_i U_i = \boldsymbol{x}, && && \text{(47h)}\\
& \sum_i Y_i = \boldsymbol{y}, && && \text{(47i)}\\
& \sum_i P_i^\top = \boldsymbol{w}, && && \text{(47j)}\\
& x_1, \boldsymbol{u}_1, \boldsymbol{X}_1, \boldsymbol{Y}, \boldsymbol{y} \ge \boldsymbol{0}, && && \text{(47k)}\\
& \begin{pmatrix} \boldsymbol{X} & \boldsymbol{U} & \boldsymbol{Q} & \boldsymbol{x}\\ \boldsymbol{U}^\top & \boldsymbol{Y} & \boldsymbol{P} & \boldsymbol{y}\\ \boldsymbol{Q}^\top & \boldsymbol{P}^\top & \boldsymbol{W} & \boldsymbol{w}\\ \boldsymbol{x}^\top & \boldsymbol{y}^\top & \boldsymbol{w}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}. && && \text{(47l)}
\end{aligned}
$$

We can further decompose the exponential cone inequality for $\boldsymbol{y}$ and obtain the following additional inequalities

$$
\begin{aligned}
& 1 \ge y_i + w_i, && i \in [n],\\
& x_1 \ge U_{1i} + Q_{1i}, && i \in [n],\\
& \boldsymbol{y} \ge \boldsymbol{Y}_i + \boldsymbol{P}_i, && i \in [n],\\
& Y_{ij} + W_{ij} + P_{ij} + P_{ji} - y_i - y_j - w_i - w_j + 1 \ge 0, && i,j \in [n].
\end{aligned}
$$

We can reformulate the first order conditions into cone inequalities and then multiply them with the rest to obtain more cone inequalities. We linearize $\boldsymbol{x}\boldsymbol{t}^\top$, $\boldsymbol{y}\boldsymbol{t}^\top$, $\boldsymbol{w}\boldsymbol{t}^\top$ with $\boldsymbol{H}, \boldsymbol{G}$ and $\boldsymbol{R}$ respectively. We have the following:

$$
\begin{aligned}
& t_i + r_i \le U_{ii}, && i \in [n], \\
& y_i \exp\left(\frac{-t_i}{y_i}\right) \le 1, && i \in [n], \\
& \sum_j q_{ij} \le 1, && i \in [n], \\
& y_i \exp\left(\frac{U_{ji} - r_i}{y_i}\right) \le V_{ij}, && i,j \in [n], \\
& \sum_j V_{ij} \le y_i, && i \in [n], \\
& U_{1i} \exp\left(\frac{-H_{1i}}{U_{1i}}\right) \le 1, && i \in [n], \\
& Y_{ij} \exp\left(\frac{-G_{ji}}{Y_{ij}}\right) \le 1, && i,j \in [n], \\
& Y_{ij} \exp\left(\frac{P_{ji} - G_{ij}}{Y_{ij}}\right) \le 1, && i,j \in [n], \\
& Y_{ij} \exp\left(\frac{-G_{ji} - G_{ij}}{Y_{ij}}\right) \le 1, \\
& \sum_i \boldsymbol{G}_i^\top = \boldsymbol{t}, \\
& \begin{pmatrix} \boldsymbol{X} & \boldsymbol{U} & \boldsymbol{Q} & \boldsymbol{H} & \boldsymbol{x} \\ \boldsymbol{U}^\top & \boldsymbol{Y} & \boldsymbol{P} & \boldsymbol{G} & \boldsymbol{y} \\ \boldsymbol{Q}^\top & \boldsymbol{P}^\top & \boldsymbol{W} & \boldsymbol{R} & \boldsymbol{w} \\ \boldsymbol{H}^\top & \boldsymbol{G}^\top & \boldsymbol{R}^\top & \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{x}^\top & \boldsymbol{y}^\top & \boldsymbol{w}^\top & \boldsymbol{t}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}.
\end{aligned}
$$

We can further decompose the exponential cone inequalities obtained from first order conditions and obtain the following:

$$
\begin{aligned}
& 1 \ge y_i - t_i, && i \in [n], \\
& x_1 \ge U_{1i} - H_{1i}, && i \in [n], \\
& \boldsymbol{y} \ge \boldsymbol{Y}_i - \boldsymbol{G}_i, && i \in [n], \\
& (y_i - Y_{ij} + G_{ij}) \exp\left(\frac{w_i - P_{ji} + R_{ij}}{y_i - Y_{ij} + G_{ij}}\right) \le 1 - y_j + t_j, && i,j \in [n], \\
& Y_{ij} + P_{ji} - G_{ij} - R_{ij} - y_i - y_j - w_i + t_j + 1 \ge 0, && i,j \in [n], \\
& Y_{ij} + T_{ij} - G_{ij} - G_{ji} - y_i - y_j + t_i + t_j + 1 \ge 0, && i,j \in [n].
\end{aligned}
$$

C.2. RPT formulation of Section 6.1

We linearize $\boldsymbol{x}\boldsymbol{x}^\top$ with $\boldsymbol{X}$, $\boldsymbol{z}\boldsymbol{z}^\top$ with $\boldsymbol{Z}$, $\boldsymbol{x}\boldsymbol{z}^\top$ with $\boldsymbol{V}$, $\boldsymbol{x}\boldsymbol{t}^\top$ with $\boldsymbol{W}$ and $\boldsymbol{z}\boldsymbol{t}^\top$ with $\boldsymbol{Q}$. When multiplying the constraints in problem (33), without any additions, we obtain the following problem:

$$
\begin{aligned}
\min_{\substack{\boldsymbol{x},\boldsymbol{z},\boldsymbol{X},\\ \boldsymbol{V},\boldsymbol{Z}}} \quad & \operatorname{Tr}(\boldsymbol{A}_0\boldsymbol{X}) + \boldsymbol{b}_0^\top \boldsymbol{x} + c_0 && && \text{(48a)}\\
\text{s.t.} \quad & \sum_{i=1}^{n_x} z_i \le 1, && && \text{(48b)}\\
& \exp(-x_i - a) \le z_i, && i \in [n_x], && \text{(48c)}\\
& \sum_{i=1}^{n_x} t_i \le \beta, && && \text{(48d)}\\
& \exp(x_i) \le t_i, && i \in [n_x], && \text{(48e)}\\
& \left(1 - \sum_{j=1}^{n_x} z_j\right) \exp\left(\frac{-x_i - \alpha + \sum_{j=1}^{n_x} V_{ij} + \alpha \sum_{j=1}^{n_x} z_j}{1 - \sum_{j=1}^{n_x} z_j}\right) \le z_i - \sum_{j=1}^{n_x} Z_{ij} && i \in [n_x], && \text{(48f)}\\
& \beta - \sum_{j=1}^{n_x} t_j - \beta \sum_{j=1}^{n_x} z_j + \sum_{i,j=1}^{n_x} Q_{ij} \ge 0, && && \text{(48g)}\\
& \left(1 - \sum_{j=1}^{n_x} z_j\right) \exp\left(\frac{x_i - \sum_{j=1}^{n_x} V_{ij}}{1 - \sum_{j=1}^{n_x} z_j}\right) \le t_i - \sum_{j=1}^{n_x} Q_{ji}, && i \in [n_x], && \text{(48h)}\\
& \sum_{i,j=1}^{n_x} Z_{ij} - 2\sum_{i=1}^{n_x} z_i + 1 \ge 0, && && \text{(48i)}\\
& \left(\beta - \sum_{j=1}^{n_x} t_j\right) \exp\left(\frac{-\beta x_i - \alpha\beta + \sum_{j=1}^{n_x} W_{ij} + \alpha \sum_{j=1}^{n_x} t_j}{\beta - \sum_{j=1}^{n_x} t_j}\right) \le z_i\beta - \sum_{j=1}^{n_x} Q_{ij}, && i \in [n_x], && \text{(48j)}\\
& \left(\beta - \sum_{j=1}^{n_x} t_j\right) \exp\left(\frac{\beta x_i - \sum_{j=1}^{n_x} W_{ij}}{\beta - \sum_{j=1}^{n_x} t_j}\right) \le \beta t_i - \sum_{j=1}^{n_x} T_{ij}, && i \in [n_x], && \text{(48k)}\\
& \sum_{i,j=1}^{n_x} T_{ij} - 2\sum_{j=1}^{n_x} t_j + \beta^2 \ge 0, && && \text{(48l)}\\
& z_j \exp\left(\frac{V_{ij}}{z_j}\right) \le Q_{ji}, && i,j \in [n_x], && \text{(48m)}\\
& \exp(-x_i - x_j - 2\alpha) \le Z_{ij}, && i \le j \in [n_x], && \text{(48n)}\\
& \exp(-x_i - a + x_j) \le Q_{ij}, && i,j \in [n_x], && \text{(48o)}\\
& z_j \exp\left(\frac{-V_{ij} - a z_j}{z_j}\right) \le Z_{ij}, && i,j \in [n_x], && \text{(48p)}\\
& t_j \exp\left(\frac{-W_{ij} - a t_j}{t_j}\right) \le Q_{ij}, && i,j \in [n_x], && \text{(48q)}\\
& t_j \exp\left(\frac{W_{ij}}{t_j}\right) \le T_{ij}, && i,j \in [n_x], && \text{(48r)}\\
& \exp(x_i + x_j) \le T_{ij}, && i \le j \in [n_x], && \text{(48s)}\\
& && && \text{(48t)}
\end{aligned}
$$

Further, from the decomposition of the exponential cone we obtain the following additional constraints:

$$
\begin{aligned}
& -x_i - \alpha + 1 \le z_i, && i \in [n_x], && \text{(49a)}\\
& x_i + 1 \le t_i, && i \in [n_x], && \text{(49b)}\\
& z_i - \sum_{j=1}^{n_x} Z_{ij} + x_i + \alpha - 1 - \sum_{j=1}^{n_x} V_{ij} - \alpha \sum_{j=1}^{n_x} z_j + \sum_{j=1}^{n_x} z_j \ge 0, && i \in [n_x], && \text{(49c)}\\
& (z_j + x_j + \alpha - 1) \exp\left(\frac{-V_{ij} - X_{ij} - \alpha x_i + x_i - \alpha(z_j + x_j + \alpha - 1)}{z_j + x_j + \alpha - 1}\right) \le Z_{ij} + V_{ji} + \alpha z_i - z_i, && i,j \in [n_x], && \text{(49d)}
\end{aligned}
$$

$$(z_j + x_j + \alpha - 1)\exp\left(\frac{V_{ij} + X_{ij} + \alpha x_i - x_i}{z_j + x_j + \alpha - 1}\right) \le Q_{ji} + W_{ji} + \alpha t_i - t_i, \qquad i, j \in [n_x], \tag{49e}$$

$$(\alpha - 1)(z_j + x_j + \alpha - 1) + Z_{ij} + V_{ji} + (\alpha - 1)z_i + V_{ij} + X_{ij} + (\alpha - 1)x_i \ge 0, \qquad i, j \in [n_x], \tag{49f}$$

$$t_i - \sum_{j=1}^{n_x} Q_{ji} - x_i + \sum_{j=1}^{n_x} V_{ij} - 1 + \sum_{j=1}^{n_x} z_j \ge 0, \qquad i \in [n_x], \tag{49g}$$

$$(t_j - x_j - 1)\exp\left(\frac{-W_{ij} - \alpha t_j + X_{ij} + \alpha(x_j + 1) + x_i}{t_j - x_j - 1}\right) \le Q_{ij} - V_{ji} - z_i, \qquad i, j \in [n_x], \tag{49h}$$

$$(t_j - x_j - 1)\exp\left(\frac{W_{ij} - X_{ij} - x_i}{t_j - x_j - 1}\right) \le T_{ij} - Q_{ji} - t_i, \qquad i, j \in [n_x], \tag{49i}$$

$$T_{ij} - W_{ij} - t_j - W_{ji} + X_{ij} + x_j - t_i + x_i + 1 \ge 0, \qquad i, j \in [n_x], \tag{49j}$$

$$Q_{ji} + W_{ji} + \alpha t_i - t_i - V_{ij} - X_{ij} - \alpha x_i + x_i - z_j - x_j - \alpha + 1 \ge 0, \qquad i, j \in [n_x], \tag{49k}$$

$$\beta z_j - \sum_i Q_{ji} + \beta x_j - \sum_i W_{ji} - \alpha \sum_i t_i + \alpha\beta - \beta + \sum_i t_i \ge 0, \qquad j \in [n_x], \tag{49l}$$

$$\beta t_j - \sum_i T_{ij} - \beta x_j + \sum_i W_{ji} - \beta + \sum_i t_i \ge 0, \quad j \in [n_x]. \tag{49m}$$

C.3. RPT formulation of Section 6.2

We linearize $\boldsymbol{r}\boldsymbol{r}^\top$, $\boldsymbol{t}\boldsymbol{t}^\top$, $\boldsymbol{r}\boldsymbol{t}^\top$ with $\boldsymbol{R}, \boldsymbol{T}$ and $\boldsymbol{V}$ respectively. We multiply all constraints in problem (36) to obtain the following problem:

$$\max_{\substack{\boldsymbol{r},\boldsymbol{t},\boldsymbol{R},\\ \boldsymbol{T},\boldsymbol{V}}} \quad \boldsymbol{r}^\top \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}^\top \boldsymbol{r} + d \tag{50a}$$

$$\text{s.t.} \quad \boldsymbol{c}^\top \boldsymbol{r} \le W, \tag{50b}$$

$$\boldsymbol{\beta}_l^\top \boldsymbol{r} \ge \eta_l, \quad l \in \mathcal{L}/\{m\}, \tag{50c}$$

$$\sum_k t_k = \frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m), \tag{50d}$$

$$t_k^{1/q}\left(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{r} - \eta_m)\right)^{1-1/q} \ge r_k, \quad k \in \mathcal{K}, \tag{50e}$$

$$\boldsymbol{\beta}_l^\top \boldsymbol{R}_k \ge \eta_l r_k, \quad l \in \mathcal{L}/\{m\},\ k \in \mathcal{K}, \tag{50f}$$

$$\left(\boldsymbol{\beta}_l^\top \boldsymbol{V}_k - \eta_l t_k\right)^{\frac{1}{q}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_1^\top \boldsymbol{R}\boldsymbol{\beta}_l - \eta_l \boldsymbol{\beta}_1^\top \boldsymbol{r} - \eta_1 \boldsymbol{\beta}_l^\top \boldsymbol{r} + \eta_1\eta_l\right)\right)^{1-1/q} \ge \boldsymbol{\beta}_l^\top \boldsymbol{R}_k - \eta_l r_k,\ l \in \mathcal{L}/\{m\},\ k \in \mathcal{K}, \tag{50g}$$

$$\boldsymbol{c}^\top \boldsymbol{R}_k \le W r_k, \quad k \in \mathcal{K}, \tag{50h}$$

$$W^2 - 2W\boldsymbol{c}^\top \boldsymbol{r} + \boldsymbol{c}^\top \boldsymbol{R}\boldsymbol{c} \ge 0, \tag{50i}$$

$$W\boldsymbol{\beta}_l^\top \boldsymbol{r} + \eta_l \boldsymbol{c}^\top \boldsymbol{r} - \boldsymbol{c}^\top \boldsymbol{R}\boldsymbol{\beta}_l - W\eta_l \ge 0, \qquad l \in \mathcal{L}/\{m\}, \tag{50j}$$

$$\left(Wt_k - \boldsymbol{c}^\top \boldsymbol{V}_k\right)^{\frac{1}{q}} \left(\frac{1}{\rho}\left(W\boldsymbol{\beta}_m^\top \boldsymbol{r} - W\eta_m + \eta_m \boldsymbol{c}^\top \boldsymbol{r} - \boldsymbol{c}^\top \boldsymbol{R}\boldsymbol{\beta}_m\right)\right)^{1-1/q} \ge W r_k - \boldsymbol{c}^\top \boldsymbol{R}_k, \quad k \in \mathcal{K}, \tag{50k}$$

$$\sum_k \boldsymbol{V}_k = \frac{1}{\rho}(\boldsymbol{R}\boldsymbol{\beta}_m - \eta_m \boldsymbol{r}), \tag{50l}$$

$$\sum_k \boldsymbol{T}_k = \frac{1}{\rho}(\boldsymbol{V}^\top \boldsymbol{\beta}_m - \eta_m \boldsymbol{t}), \tag{50m}$$

$$V_{k'k}^{1/q}(\frac{1}{\rho}(\boldsymbol{\beta}_m^\top \boldsymbol{R}_{k'} - \eta_m r_{k'}))^{1-1/q} \geq R_{kk'}, \quad k,k' \in \mathcal{K}, \tag{50n}$$

$$T_{kk'}^{\theta_{11}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_m^\top \boldsymbol{V}_k - \eta_m t_k\right)\right)^{\theta_{12}} \left(\frac{1}{\rho}\left(\boldsymbol{\beta}_m^\top \boldsymbol{V}_{k'} - \eta_m t_{k'}\right)\right)^{\theta_{21}} \cdot$$

$$\cdot \left(\frac{1}{\rho^2}\left(\boldsymbol{\beta}_m^\top \boldsymbol{R}\boldsymbol{\beta}_m - 2\eta_m \boldsymbol{\beta}_m^\top \boldsymbol{r} + \eta_m^2\right)\right)^{\theta_{22}} \geq R_{kk'}, \quad k,k' \in \mathcal{K}, \tag{50o}$$

$$\boldsymbol{r}, \boldsymbol{R} \geq \boldsymbol{0}, \tag{50p}$$

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{V} & \boldsymbol{r} \\ \boldsymbol{V}^\top & \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{r}^\top & \boldsymbol{t}^\top & 1 \end{pmatrix} \succeq \boldsymbol{0}. \tag{50q}$$

For the multiplication of the power cone constraints we generate a feasible $\boldsymbol{\theta}$ that satisfies the following:

$$\theta_{11} + \theta_{21} = 1/q, \ \theta_{12} + \theta_{22} = 1 - 1/q,$$

$$\theta_{11} + \theta_{12} = 1/q, \ \theta_{21} + \theta_{22} = 1 - 1/q.$$

D. Data generation of Section 6.1

In problem instances 1 the objective is defined as $f(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^{20}(x_i+5)^2$ and in problem instance 2 it is defined as $f(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^{20}(x_i+7)^2$. In problem instances 3, 4, 5, and 6, the matrix $\boldsymbol{A}_0$ is generated as $\boldsymbol{L}^T\boldsymbol{L}$, where $\boldsymbol{L} \in \mathbb{R}^{n\times n}$, with $L_{ij} \sim [0,1]$, and further $\boldsymbol{b}_0 = \boldsymbol{0}$, $c_0 = 0$. We summarize all parameters describing each instance in Table 5.

Table 5 Problem (33) parameters for each instance. n_x refers to the number of variables and α, β to the constraint parameters.

Instance	n_x	α	β
1	5	2	20
2	5	2	20
3	10	2	3
4	20	3	4
5	50	3	4
6	100	13	20

E. Technical lemmas

LEMMA 3. *Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite matrices. Then also the Kronecker product is positive semidefinite, i.e.,*

$$\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x}) \succeq 0. \tag{51}$$

Moreover, if $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are positive semidefinite matrices of the same size, then the Hadamard product is also positive semidefinite, i.e.,*

$$\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x}) \succeq 0. \tag{52}$$

Proof The first statement follows from Theorem 4.2.12 of Horn and Johnson (1991). The last statement follows from the Schur Product Theorem (Schur, 1911). □

The following lemma shows that the Kronecker product relaxation is at least as tight as the Hadamard product relaxation.

LEMMA 4. *Suppose* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are of the same size. The relaxation obtained after linearization of the quadratic terms in the Kronecker product (51) is at least as tight as the relaxation obtained after linearization of the quadratic terms in the Hadamard product (52).*

Proof. Let us denote the matrix obtained after linearization of the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \otimes \boldsymbol{B}(\boldsymbol{x})$ by $\boldsymbol{C}(\boldsymbol{U}, \boldsymbol{x})$. Let us denote the matrix obtained after linearization of the quadratic terms in $\boldsymbol{A}(\boldsymbol{x}) \circ \boldsymbol{B}(\boldsymbol{x})$ by $\boldsymbol{D}(\boldsymbol{U}, \boldsymbol{x})$. It can easily be checked that $\boldsymbol{D}(\boldsymbol{U}, \boldsymbol{x})$ is a minor of $\boldsymbol{C}(\boldsymbol{U}, \boldsymbol{x})$ obtained by the elements in rows and columns $1, n+2, 2n+3, \ldots, n^2$. Since $C(\boldsymbol{U}, \boldsymbol{x})$ is positive semidefinite, and each minor of a positive semidefinite matrix is positive semidefinite, we have that $\boldsymbol{D}(\boldsymbol{U}, \boldsymbol{x})$ is positive semidefinite. □