FUNCTIONS ASSOCIATED WITH THE NONCONVEX SECOND-ORDER CONE

LILIA BENAKKOUCHE AND BAHA ALZALG

Abstract. The nonconvex second-order cone (nonconvex SOC for short) is a nonconvex extension to the convex second-order cone, in the sense that it consists of any vector divided into two sub-vectors for which the Euclidean norm of the first sub-vector is at least as large as the Euclidean norm of the second sub-vector. This cone can be used to reformulate nonconvex quadratic programs in conic format and can arise in real-world applications. In this paper, spectral scalar and vector-valued functions associated with the nonconvex SOC are defined analogously to the corresponding functions associated with the convex second-order cone. We present several properties and key characteristics of the nonconvex SOC-related functions. The results in this paper are useful for developing and analyzing solution methods for solving optimization problems over the nonconvex SOC and their complementarity problems.

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1. Introduction

The purpose of this paper is to introduce functions associated with the nonconvex second-order cone (nonconvex SOC for short) and study their properties. The $(m+n)$th-dimensional nonconvex SOC is defined as [1]

$$\mathcal{M}_+^{m|n} \triangleq \left\{ \boldsymbol{x} = \begin{bmatrix} \hat{x} \\ \bar{x} \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : \|\hat{x}\| \geq \|\bar{x}\| \right\}, \tag{1}$$

where $\|\cdot\|$ denotes the standard Euclidean norm. In (1), when $m = 1$ and $\hat{x} \geq 0$, the nonconvex SOC reduces to the well-known convex SOC

$$\mathcal{E}_+^{n+1} \triangleq \left\{ \boldsymbol{x} = \begin{bmatrix} \hat{x} \\ \bar{x} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n : \hat{x} \geq \|\bar{x}\| \right\}.$$

The nonconvex SOC can be used to reformulate classes of nonconvex programming problems in conic format, such as nonconvex quadratic programming and nonconvex quadratically constrained quadratic programming [1]. The cone can also arise in real-world applications, such as facility location problems when some existing facilities are more likely to be closer to new facilities than other existing facilities. In [1], we introduced and

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studied the nonconvex SOC $\mathcal{M}_+^{m|n}$ from an algebraic perspective. More specifically, we extended many algebraic properties that previously existed in the frame of the convex SOC to the frame of the nonconvex SOC.

Analogous to the special case, the convex SOC $\mathcal{E}_+^{n+1}$, our definition of the vector-valued functions related to the nonconvex SOC $\mathcal{M}_+^{m|n}$ is based on the spectral decomposition of vectors $\boldsymbol{x}$ in the ambient space of $\mathcal{M}_+^{m|n}$, which we denote as $\mathcal{M}^{m|n}$. Let f be any function ranging from $\mathbb{R}$ to $\mathbb{R}$. By applying f to the eigenvalues in the spectral decomposition of $\boldsymbol{x}$ with respect to $\mathcal{M}^{m|n}$, one can assign an associated vector-valued function $f^{\mathcal{M}_+^{m|n}}$ on $\mathcal{M}^{m|n}$. If f is only appointed to a subset of $\mathbb{R}$, $f^{\mathcal{M}_+^{m|n}}$ is described in the corresponding subset of $\mathcal{M}^{m|n}$. Sun and Sun [2] referred to the vector-valued function associated with the Euclidean Jordan algebra of $\mathcal{E}_+^{n+1}$ as the "Löwner operator" in honor of Löwner Karl's significant contributions in connection with this vector-valued function.

The particular structure of the nonconvex SOC allows for a deep analysis of the properties associated with its functions. In this paper, we provide the spectral scalar functions related to the nonconvex SOC and demonstrate their properties. Then, we describe the vector-valued functions associated with the nonconvex SOC and point out some of their intriguing characteristics. We also verify and analyze the differentiability of the vector-valued functions associated with the cone. We then show that the Fréchet differentiability property of $f^{\mathcal{M}_+^{m|n}}$ is inherited by the corresponding real-valued function f. The importance of the study in this paper stems from the fact that the nonconvex SOC-related functions can be employed in developing solution methods for solving optimization problems over the nonconvex SOC and their complementarity problems.

The paper is organized as follows. In Section 2, we introduce some notations and review some notions and concepts from the algebraic study of the nonconvex SOC carried out in [1]. In Section 3, we present spectral scalar functions associated with the cone and prove some of their key properties. In Section 4, we introduce the vector-valued functions corresponding to the nonconvex SOC and prove some of their key characteristics. The conclusion is made in Section 5.

2. Review of the algebraic structure of the nonconvex SOC

In this section, we will briefly recall some key results and traits that will be useful for the rest of our work. We will not seek to be exhaustive. Developments on the subject can be found in [1]. To start with, we introduce some notations. Most notations we employ are standard, but for the entirety, we give a quick description before we start in earnest.

We bring to the attention of the readers that our notations closely follow those of [1]. We review a few basic outcomes concerning the algebraic structure of $\mathcal{M}^{m|n}$. In this paper, we look at a Euclidean vector space (in plain English, we mean a finite-dimensional real vector space equipped with a standard inner product symbolized by $\langle\cdot,\cdot\rangle$ and the induced norm expressed by $\|\cdot\|$).

When joining matrices and vectors in a row, we use the symbol ",", and when joining them in a column, we use the symbol ";". Therefore, we obtain $(\boldsymbol{x}^{\mathsf{T}}, \boldsymbol{y}^{\mathsf{T}})^{\mathsf{T}} = (\boldsymbol{x}; \boldsymbol{y})$ for the column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

Let m, n be positive integers. By $\mathcal{M}^{m|n}$, we mean the $(m+n)$th-dimensional real vector space $\mathbb{R}^m \times \mathbb{R}^n$ supplied with a standard inner product. That is,

$$\mathcal{M}^{m|n} \triangleq \{\boldsymbol{x} = (\hat{\boldsymbol{x}}; \bar{\boldsymbol{x}}) : \hat{\boldsymbol{x}} \in \mathbb{R}^m, \bar{\boldsymbol{x}} \in \mathbb{R}^n\}.$$

The open set

$$\text{int}\, \mathcal{M}^{m|n}_+ \triangleq \left\{\boldsymbol{x} \in \mathcal{M}^{m|n} : \|\hat{\boldsymbol{x}}\| > \|\bar{\boldsymbol{x}}\|\right\}$$

represents the interior of the nonconvex SOC $\mathcal{M}^{m|n}_+$.

For a real number $\alpha \in \mathbb{R}$, we express $\alpha^+ \triangleq \max\{\alpha, 0\}$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, we denote $|\boldsymbol{x}| \triangleq (|x_1|; \ldots ; |x_n|)$, i.e., $|\boldsymbol{x}|$ is the vector $\boldsymbol{x}$ with every ith coordinate is replaced by $|x_i|$.

Let $\boldsymbol{x} \in \mathcal{M}^{m|n}$. If $\hat{\boldsymbol{x}} = \mathbf{0}$, the element $\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|$ is considered to be any vector in $\mathbb{R}^m$ of Euclidean norm one. Similarly, if $\bar{\boldsymbol{x}} = \mathbf{0}$, the element $\bar{\boldsymbol{x}}/\|\bar{\boldsymbol{x}}\|$ is regarded to be any vector in $\mathbb{R}^n$ of Euclidean norm one.

In $\mathcal{M}^{m|n}$, each vector $\boldsymbol{x}$ is associated with a crane-shaped matrix, $\mathsf{Crn}(\boldsymbol{x})$, which is defined as [1]

$$\mathsf{Crn}(\boldsymbol{x}) \triangleq \begin{bmatrix} \|\hat{\boldsymbol{x}}\|\, I_m & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}}\left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix}.$$

Note that, the symmetric matrix $\mathsf{Crn}(\boldsymbol{x})$ is positive definite (and hence invertible) iff $\boldsymbol{x} \in \text{int}\, \mathcal{M}^{m|n}_+$. It is not hard to verify that

$$\frac{1}{2}\left(\mathsf{Crn}(\boldsymbol{x})\boldsymbol{y} + \mathsf{Crn}(\boldsymbol{y})\boldsymbol{x}\right) = \boldsymbol{x} \circledcirc \boldsymbol{y},\ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}^{m|n},$$

where the product $\circledcirc : \mathcal{M}^{m|n} \times \mathcal{M}^{m|n} \to \mathcal{M}^{m|n}$ is defined as [1]

$$\boldsymbol{x} \circledcirc \boldsymbol{y} \triangleq \frac{1}{2}\begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|\left\|\hat{\boldsymbol{y}}\right\| + \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\right)\left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} + \dfrac{\hat{\boldsymbol{y}}}{\left\|\hat{\boldsymbol{y}}\right\|}\right) \\ \left(\|\hat{\boldsymbol{x}}\| + \hat{\boldsymbol{x}}^{\mathsf{T}}\dfrac{\hat{\boldsymbol{y}}}{\left\|\hat{\boldsymbol{y}}\right\|}\right)\bar{\boldsymbol{y}} + \left(\left\|\hat{\boldsymbol{y}}\right\| + \hat{\boldsymbol{y}}^{\mathsf{T}}\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)\bar{\boldsymbol{x}} \end{bmatrix}. \tag{2}$$

The product "$\circledcirc$" in $\mathcal{M}^{m|n}$ is not bilinear, but it is commutative and power-associative. That is, $\boldsymbol{x}\circledcirc\boldsymbol{y} = \boldsymbol{y}\circledcirc\boldsymbol{x}$ and $\boldsymbol{x}^p\circledcirc\boldsymbol{x}^q = \boldsymbol{x}^{p+q}$ for any positive integers p and q. It is also worth noting that the product (2) is not generally associative.

The fact that we can specify the eigen-decomposition of any element $\boldsymbol{x} \in \mathcal{M}^{m|n}$ makes the algebraic structure $(\mathcal{M}^{m|n}, \circledcirc)$ so appealing. Associated with the nonconvex SOC $\mathcal{M}_+^{m|n}$, each $\boldsymbol{x} \in \mathcal{M}^{m|n}$ can be factorized as

$$\boldsymbol{x} \triangleq \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}), \tag{3}$$

where, for $i = 1, 2$, $\lambda_i(\boldsymbol{x}) \triangleq \|\hat{\boldsymbol{x}}\| + (-1)^{i+1} \|\bar{\boldsymbol{x}}\|$ are termed the eigenvalues of $\boldsymbol{x}$, and $\boldsymbol{c}_i(\boldsymbol{x}) \triangleq \frac{1}{2}(\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|; (-1)^{i+1}\bar{\boldsymbol{x}}/\|\bar{\boldsymbol{x}}\|)$ are termed the eigenvectors of $\boldsymbol{x}$. The factorization in (3) is called the spectral decomposition or spectral factorization of $\boldsymbol{x}$ [1]. The determinant and the trace of $\boldsymbol{x}$ are defined in terms of the eigenvalues as

$$\det(\boldsymbol{x}) \triangleq \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}) = \|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2, \text{ and } \operatorname{trace}(\boldsymbol{x}) \triangleq \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) = 2\|\hat{\boldsymbol{x}}\|,$$

respectively. We have the following lemma [1].

Lemma 2.1. *For any $\boldsymbol{x} \in \mathcal{M}^{m|n}$, its eigenvalues $\lambda_1(\boldsymbol{x})$ and $\lambda_2(\boldsymbol{x})$ and eigenvectors $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ have the following properties:*

(i) *$\lambda_1(\boldsymbol{x})$ and $\lambda_2(\boldsymbol{x})$ are nonnegative (respectively, positive) iff $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ (respectively, $\boldsymbol{x} \in$ int $\mathcal{M}_+^{m|n}$).*

(ii) *$\lambda_1(\boldsymbol{x})$ and $\lambda_2(\boldsymbol{x})$ are eigenvalues of $\mathsf{Crn}(\boldsymbol{x})$. Moreover, if $\lambda_1(\boldsymbol{x}) \neq \lambda_2(\boldsymbol{x})$ then each one has multiplicity one; the corresponding eigenvectors are $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$. Furthermore, the remaining $m + n - 2$ eigenvalues of $\mathsf{Crn}(\boldsymbol{x})$ are $\|\hat{\boldsymbol{x}}\|$ when $\boldsymbol{x} \neq \boldsymbol{0}$.*

(iii) *$\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ have length $1/\sqrt{2}$ and are orthogonal with respect to the multiplication "$\circledcirc$". That is, $\|\boldsymbol{c}_1(\boldsymbol{x})\| = \|\boldsymbol{c}_2(\boldsymbol{x})\| = 1/\sqrt{2}$ and $\boldsymbol{c}_1(\boldsymbol{x}) \circledcirc \boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{0}$.*

(iv) *$\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ are idempotent under the product "$\circledcirc$", i.e., $\boldsymbol{c}_i^2(\boldsymbol{x}) = \boldsymbol{c}_i(\boldsymbol{x}) \circledcirc \boldsymbol{c}_i(\boldsymbol{x}) = \boldsymbol{c}_i(\boldsymbol{x})$ for $i = 1, 2$. More generally, $\boldsymbol{c}_i^p(\boldsymbol{x}) = \boldsymbol{c}_i(\boldsymbol{x})$, for any positive integer p.*

Let $\boldsymbol{x} \in \mathcal{M}^{m|n}$. Using Lemma 2.1, one can show that

$$\boldsymbol{x}^2 = (\lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}))^2 = \lambda_1^2(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}), \tag{4}$$

and, more generally, that

$$\boldsymbol{x}^p = (\lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}))^p = \lambda_1^p(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}), \tag{5}$$

for any nonnegative integer p. For a formal proof of (5), see the proof of Lemma 4.1 in [1].

3. Spectral scalar functions associated with the nonconvex SOC

In this section, we present with proofs key properties of the spectral scalar functions associated with the nonconvex SOC.

The result in the following lemma for the nonconvex SOC generalizes that in [3, Lemma 2] for the convex SOC.

Lemma 3.1. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}^{m|n}$ have eigenvalues $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})$, and $\lambda_1(\boldsymbol{y}), \lambda_2(\boldsymbol{y})$, respectively. Then*

$$\left|\lambda_i(\boldsymbol{x}) - \lambda_i(\boldsymbol{y})\right| \leq \sqrt{2}\left\|\boldsymbol{x} - \boldsymbol{y}\right\|, \ i = 1, 2.$$

Proof. Let $x = (\hat{x}; \bar{x}), y = (\hat{y}; \bar{y}) \in \mathcal{M}^{m|n}$. Using the triangle inequality and the fact that $a + b \leq \sqrt{2(a^2 + b^2)}$, for any two real numbers a and b, we have

$$\begin{aligned} \left|\lambda_1(x) - \lambda_1(y)\right| &= \left|\|\hat{x}\| + \|\bar{x}\| - \|\hat{y}\| - \|\bar{y}\|\right| \\ &= \left|\left(\|\hat{x}\| - \|\hat{y}\|\right) + \left(\|\bar{x}\| - \|\bar{y}\|\right)\right| \\ &\leq \left|\|\hat{x}\| - \|\hat{y}\|\right| + \left|\|\bar{x}\| - \|\bar{y}\|\right| \\ &\leq \left\|\hat{x} - \hat{y}\right\| + \left\|\bar{x} - \bar{y}\right\| \\ &\leq \sqrt{2}\left(\left\|\hat{x} - \hat{y}\right\|^2 + \left\|\bar{x} - \bar{y}\right\|^2\right)^{1/2} = \sqrt{2}\left\|x - y\right\|. \end{aligned}$$

An analogous argument can be made for $\left|\lambda_2(x) - \lambda_2(y)\right|$, hence the desired result follows. ◻

Below, we prove more basic properties of the trace, determinant, and eigenvalue functions associated with the ambient space of the nonconvex SOC.

Theorem 3.2. *For any $x, y \in \mathcal{M}^{m|n}$, we have*

(i) $\left|\lambda_1(x) - \lambda_1(y)\right|^2 + \left|\lambda_2(x) - \lambda_2(y)\right|^2 \leq 2\left(\|x\| + \|y\|\right)^2$.

(ii) $\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) \leq \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y)$.

(iii) $\dfrac{\lambda_i^2(x) + \lambda_j^2(x)}{2} - \left(\dfrac{\lambda_i(x) + \lambda_j(x)}{2}\right)^2 = \left(\dfrac{\lambda_i(x) - \lambda_j(x)}{2}\right)^2$, *for $i, j = 1, 2$.*

Proof. Let $x = (\hat{x}; \bar{x}), y = (\hat{y}; \bar{y}) \in \mathcal{M}^{m|n}$ have eigenvalues $\lambda_1(x), \lambda_2(x)$, and $\lambda_1(y), \lambda_2(y)$, respectively. We prove the theorem item by item.

(i) The proof is a simple chain of obvious equalities and inequalities:

$$\begin{aligned} &\left|\lambda_1(x) - \lambda_1(y)\right|^2 + \left|\lambda_2(x) - \lambda_2(y)\right|^2 = (\lambda_1(x) - \lambda_1(y))^2 + (\lambda_2(x) - \lambda_2(y))^2 \\ &= \left(\|\hat{x}\| + \|\bar{x}\| - \|\hat{y}\| - \|\bar{y}\|\right)^2 + \left(\|\hat{x}\| - \|\bar{x}\| - \|\hat{y}\| + \|\bar{y}\|\right)^2 \\ &= \left(\|\hat{x}\| - \|\hat{y}\|\right)^2 + 2\left(\|\hat{x}\| - \|\hat{y}\|\right)\left(\|\bar{x}\| - \|\bar{y}\|\right) + \left(\|\bar{x}\| - \|\bar{y}\|\right)^2 \\ &\quad + \left(\|\hat{x}\| - \|\hat{y}\|\right)^2 + 2\left(\|\hat{x}\| - \|\hat{y}\|\right)\left(\|\bar{y}\| - \|\bar{x}\|\right) + \left(\|\bar{y}\| - \|\bar{x}\|\right)^2 \\ &= 2\left(\left(\|\hat{x}\| - \|\hat{y}\|\right)^2 + \left(\|\hat{x}\| - \|\hat{y}\|\right)\left(\|\bar{x}\| - \|\bar{y}\| + \|\bar{y}\| - \|\bar{x}\|\right) + \left(\|\bar{x}\| - \|\bar{y}\|\right)^2\right) \\ &= 2\left(\|\hat{x}\|^2 - 2\|\hat{x}\|\|\hat{y}\| + \|\hat{y}\|^2 + \|\bar{x}\|^2 - 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2\right) \\ &= 2\left(\|x\|^2 + \|y\|^2 - 2\|\hat{x}\|\|\hat{y}\| - 2\|\bar{x}\|\|\bar{y}\|\right) \\ &\leq 2\left(\|x\|^2 + \|y\|^2 + 2\langle \hat{x}, \hat{y}\rangle + 2\langle \bar{x}, \bar{y}\rangle\right) \\ &= 2\left(\|x\|^2 + \|y\|^2 + 2\langle x, y\rangle\right) \\ &= 2\left\|x + y\right\|^2 \leq 2\left(\|x\| + \|y\|\right)^2. \end{aligned}$$

Note that the first inequality follows from the Cauchy–Schwartz inequality, and the second one from the triangle inequality.

(ii) An easy computation shows that

$$\begin{array}{rcl}\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{y})+\lambda_1(\boldsymbol{y})\lambda_2(\boldsymbol{x}) &=& (\|\hat{x}\|+\|\bar{x}\|)\left(\|\hat{y}\|-\|\bar{y}\|\right)+\left(\|\hat{y}\|+\|\bar{y}\|\right)(\|\hat{x}\|-\|\bar{x}\|)\\ &=& 2\left(\|\hat{x}\|\|\hat{y}\|-\|\bar{x}\|\|\bar{y}\|\right)\\ &\leq& 2\left(\|\hat{x}\|\|\hat{y}\|+\|\bar{x}\|\|\bar{y}\|\right)\\ &=& (\|\hat{x}\|+\|\bar{x}\|)\left(\|\hat{y}\|+\|\bar{y}\|\right)+(\|\hat{x}\|-\|\bar{x}\|)\left(\|\hat{y}\|-\|\bar{y}\|\right)\\ &=& \lambda_1(\boldsymbol{x})\lambda_1(\boldsymbol{y})+\lambda_2(\boldsymbol{x})\lambda_2(\boldsymbol{y}).\end{array}$$

(iii) We verify the identity in item (iii) for $i=1$ and $j=2$, and obtain the other cases similarly. We have

$$\begin{array}{l}\dfrac{\lambda_1^2(\boldsymbol{x})+\lambda_2^2(\boldsymbol{x})}{2}-\left(\dfrac{\lambda_1(\boldsymbol{x})+\lambda_2(\boldsymbol{x})}{2}\right)^2\\ =\dfrac{(\|\hat{x}\|+\|\bar{x}\|)^2+(\|\hat{x}\|-\|\bar{x}\|)^2}{2}-\left(\dfrac{\|\hat{x}\|+\|\bar{x}\|+\|\hat{x}\|-\|\bar{x}\|}{2}\right)^2\\ =\dfrac{\|\hat{x}\|^2+2\|\hat{x}\|\|\bar{x}\|+\|\bar{x}\|^2+\|\hat{x}\|^2-2\|\hat{x}\|\|\bar{x}\|+\|\bar{x}\|^2}{2}-\|\hat{x}\|^2\\ =\|\hat{x}\|^2+\|\bar{x}\|^2-\|\hat{x}\|^2\\ =\|\bar{x}\|^2\\ =\dfrac{\|\hat{x}\|+\|\bar{x}\|-\|\hat{x}\|+\|\bar{x}\|}{2}\quad=\quad\left(\dfrac{\lambda_1(\boldsymbol{x})-\lambda_2(\boldsymbol{x})}{2}\right)^2.\end{array}$$

The proof is complete. □

Lemma 3.3. *For any $\boldsymbol{x},\boldsymbol{y}\in\mathcal{M}^{m|n}$, we have*

(i) $\operatorname{trace}(\boldsymbol{x})=\dfrac{\lambda_1^2(\boldsymbol{x})-\lambda_2^2(\boldsymbol{x})}{2\|\bar{x}\|}$, *provided that* $\bar{x}\neq\mathbf{0}$.

(ii) $(\operatorname{trace}(\boldsymbol{x}+\boldsymbol{y}))^2\leq(\operatorname{trace}(\boldsymbol{x})+\operatorname{trace}(\boldsymbol{y}))^2$.

Proof. Let $\boldsymbol{x},\boldsymbol{y}\in\mathcal{M}^{m|n}$.

(i) Note that

$$\begin{array}{rcl}\lambda_1^2(\boldsymbol{x})-\lambda_2^2(\boldsymbol{x}) &=& (\|\hat{x}\|+\|\bar{x}\|)^2-(\|\hat{x}\|-\|\bar{x}\|)^2\\ &=& \|\hat{x}\|^2+2\|\hat{x}\|\|\bar{x}\|+\|\bar{x}\|^2-\|\hat{x}\|^2+2\|\hat{x}\|\|\bar{x}\|-\|\bar{x}\|^2\\ &=& 4\|\hat{x}\|\|\bar{x}\|\;=\;2\operatorname{trace}(\boldsymbol{x})\|\bar{x}\|.\end{array}$$

(ii) It is enough to show that

$$(\operatorname{trace}(\boldsymbol{x}+\boldsymbol{y}))^2-(\operatorname{trace}(\boldsymbol{x}))^2-(\operatorname{trace}(\boldsymbol{y}))^2\leq 2\operatorname{trace}(\boldsymbol{x})\operatorname{trace}(\boldsymbol{y}).$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{array}{l}(\operatorname{trace}(\boldsymbol{x}+\boldsymbol{y}))^2-(\operatorname{trace}(\boldsymbol{x}))^2-(\operatorname{trace}(\boldsymbol{y}))^2=\left(2\|\hat{x}+\hat{y}\|\right)^2-(2\|\hat{x}\|)^2-\left(2\|\hat{y}\|\right)^2\\ =4\|\hat{x}\|^2+8\langle\hat{x},\hat{y}\rangle+4\|\hat{y}\|^2-4\|\hat{x}\|^2-4\|\hat{y}\|^2\\ =8\langle\hat{x},\hat{y}\rangle\\ \leq 8\|\hat{x}\|\|\hat{y}\|\;=\;2\operatorname{trace}(\boldsymbol{x})\operatorname{trace}(\boldsymbol{y}).\end{array}$$

□

Lemma 3.4. *For any $x \in \mathcal{M}^{m|n}$, we have*

(i) $\|x\|^2 = \frac{1}{2}\left(\lambda_1^2(x) + \lambda_2^2(x)\right)$.
(ii) $\|x\|^2 = \frac{1}{2}\mathrm{trace}(x^2)$.
(iii) $\|x\|^2 \geq \frac{1}{2}\left(\lambda_1^2(x) - \lambda_2^2(x)\right)$, *provided that* $\bar{x} \neq \mathbf{0}$.
(iv) $\|x\|^2 \leq \frac{1}{2}(\mathrm{trace}(x))^2$, *provided that* $x \in \mathcal{M}_+^{m|n}$.
(v) $\|x\|^2 \geq \det(x)$.

Proof. Let $x = (\hat{x}; \bar{x}) \in \mathcal{M}^{m|n}$. We prove the lemma item by item.

(i) We have

$$\begin{aligned}\lambda_1^2(x) + \lambda_2^2(x) &= (\|\hat{x}\| + \|\bar{x}\|)^2 + (\|\hat{x}\| - \|\bar{x}\|)^2 \\ &= \|\hat{x}\|^2 + 2\,\|\hat{x}\|\,\|\bar{x}\| + \|\bar{x}\|^2 + \|\hat{x}\|^2 - 2\,\|\hat{x}\|\,\|\bar{x}\| + \|\bar{x}\|^2 \\ &= 2\left(\|\hat{x}\|^2 + \|\bar{x}\|^2\right) \;=\; 2\,\|x\|^2\end{aligned}$$

as desired.

(ii) Note that

$$x^2 = \left(x^{\mathsf{T}} x \,\frac{\hat{x}}{\|\hat{x}\|}; 2\,\|\hat{x}\|\,\bar{x}\right).$$

Consequently

$$\mathrm{trace}(x^2) = 2\left\|x^{\mathsf{T}} x \,\frac{\hat{x}}{\|\hat{x}\|}\right\| = 2\,\|x\|^2.$$

(iii) Suppose that $\bar{x} \neq \mathbf{0}$. From item (i) in Lemma 3.3, we get

$$\lambda_1^2(x) - \lambda_2^2(x) \;=\; 2\mathrm{trace}(x)\,\|\bar{x}\| \;=\; 4\,\|\hat{x}\|\,\|\bar{x}\| \;\leq\; 2\left(\|\hat{x}\|^2 + \|\bar{x}\|^2\right) \;=\; 2\,\|x\|^2.$$

(iv) If $x \in \mathcal{M}_+^{m|n}$, then $\lambda_1(x), \lambda_2(x) \geq 0$. It follows that

$$\|x\|^2 = \frac{1}{2}(\lambda_1^2(x) + \lambda_2^2(x)) \leq \frac{1}{2}(\lambda_1(x) + \lambda_2(x))^2 = \frac{1}{2}(\mathrm{trace}(x))^2.$$

(v) We have $\|x\|^2 = \|\hat{x}\|^2 + \|\bar{x}\|^2 \geq \|\hat{x}\|^2 - \|\bar{x}\|^2 = \det(x)$ as desired.

The proof is complete. □

Theorem 3.5. *For any $x, y \in \mathcal{M}_+^{m|n}$, we have*

(i) $\mathrm{trace}(x) \geq 2\sqrt{\det(x)}$.
(ii) $\lambda_1(x) + \lambda_1(y) \leq \mathrm{trace}(x) + \mathrm{trace}(y)$.
(iii) $\lambda_1(x)\lambda_1(y) \leq \mathrm{trace}(x)\mathrm{trace}(y)$.
(iv) $\det(x + y) - \det(x) - \det(y) \leq \mathrm{trace}(x)\mathrm{trace}(y)$.
(v) $(\lambda_1(x + y))^2 - (\lambda_1(x))^2 - (\lambda_1(y))^2 \leq (\mathrm{trace}(x) + \mathrm{trace}(y))^2$.
(vi) $(\lambda_2(x + y))^2 - (\lambda_2(x))^2 - (\lambda_2(y))^2 \geq -\dfrac{1}{4}\,(\mathrm{trace}(x) + \mathrm{trace}(y))^2$.

Proof. Let $x = (\hat{x}; \bar{x}), y = (\hat{y}; \bar{y}) \in \mathcal{M}_+^{m|n}$. We prove the theorem item by item.

(i) Note that $\lambda_1(x), \lambda_2(x) \geq 0$, hence

$$\begin{array}{rcl} \text{trace}(x) & = & \lambda_1(x) + \lambda_2(x) \\ & \geq & 2\sqrt{\lambda_1(x)\lambda_2(x)} \; = \; 2\sqrt{\det(x)}. \end{array}$$

(ii) We have

$$\begin{array}{rcl} \lambda_1(x) + \lambda_1(y) & = & \|\hat{x}\| + \|\bar{x}\| + \|\hat{y}\| + \|\bar{y}\| \\ & \leq & \|\hat{x}\| + \|\hat{x}\| + \|\hat{y}\| + \|\hat{y}\| \\ & = & 2\|\hat{x}\| + 2\|\hat{y}\| \; = \; \text{trace}(x) + \text{trace}(y) \end{array}$$

as desired.

(iii) We have

$$\begin{array}{rcl} \lambda_1(x)\lambda_1(y) & = & (\|\hat{x}\| + \|\bar{x}\|)\left(\|\hat{y}\| + \|\bar{y}\|\right) \\ & = & \|\hat{x}\|\|\hat{y}\| + \|\hat{x}\|\|\bar{y}\| + \|\bar{x}\|\|\hat{y}\| + \|\bar{x}\|\|\bar{y}\| \\ & \leq & 4\|\hat{x}\|\|\hat{y}\| \; = \; \text{trace}(x)\text{trace}(y) \end{array}$$

as desired.

(iv) Applying the Cauchy-Schwartz inequality and using the fact that $x, y \in \mathcal{M}_+^{m|n}$, we get

$$\begin{array}{l} \det(x+y) - \det(x) - \det(y) \\ = \|\hat{x}+\hat{y}\|^2 - \|\bar{x}+\bar{y}\|^2 - \left(\|\hat{x}\|^2 - \|\bar{x}\|^2\right) - \left(\|\hat{y}\|^2 - \|\bar{y}\|^2\right) \\ = \|\hat{x}\|^2 + 2\langle \hat{x}, \hat{y}\rangle + \|\hat{y}\|^2 - \|\bar{x}\|^2 - 2\langle \bar{x}, \bar{y}\rangle - \|\bar{y}\|^2 - \|\hat{x}\|^2 + \|\bar{x}\|^2 - \|\hat{y}\|^2 + \|\bar{y}\|^2 \\ = 2\langle \hat{x}, \hat{y}\rangle - 2\langle \bar{x}, \bar{y}\rangle \\ \leq 2\|\hat{x}\|\|\hat{y}\| + 2\|\bar{x}\|\|\bar{y}\| \\ \leq 4\|\hat{x}\|\|\hat{y}\| \; = \; \text{trace}(x)\text{trace}(y). \end{array}$$

(v) The result follows from a straightforward series of equalities and inequalities. The first inequality below follows from the fact that $2ab \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$, the second one follows from Cauchy-Schwartz inequality, the third one follows from the fact that $x, y \in \mathcal{M}_+^{m|n}$, and the fourth one follows from item (i).

$$\begin{array}{l} (\lambda_1(x+y))^2 - (\lambda_1(x))^2 - (\lambda_1(y))^2 \\ = \left(\|\hat{x}+\hat{y}\| + \|\bar{x}+\bar{y}\|\right)^2 - (\|\hat{x}\| + \|\bar{x}\|)^2 - \left(\|\hat{y}\| + \|\bar{y}\|\right)^2 \\ = \|\hat{x}+\hat{y}\|^2 + 2\|\hat{x}+\hat{y}\|\|\bar{x}+\bar{y}\| + \|\bar{x}+\bar{y}\|^2 - \|\hat{x}\|^2 - 2\|\hat{x}\|\|\bar{x}\| - \|\bar{x}\|^2 \\ - \|\hat{y}\|^2 - 2\|\hat{y}\|\|\bar{y}\| - \|\bar{y}\|^2 \\ = 2\langle \hat{x}, \hat{y}\rangle + 2\|\hat{x}+\hat{y}\|\|\bar{x}+\bar{y}\| + 2\langle \bar{x}, \bar{y}\rangle - 2\|\hat{x}\|\|\bar{x}\| - 2\|\hat{y}\|\|\bar{y}\| \\ \leq 2\langle \hat{x}, \hat{y}\rangle + \|\hat{x}+\hat{y}\|^2 + \|\bar{x}+\bar{y}\|^2 + 2\langle \bar{x}, \bar{y}\rangle - 2\|\hat{x}\|\|\bar{x}\| - 2\|\hat{y}\|\|\bar{y}\| \\ = 4\langle \hat{x}, \hat{y}\rangle + \|\hat{x}\|^2 + \|\hat{y}\|^2 + 4\langle \bar{x}, \bar{y}\rangle + \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2\|\hat{x}\|\|\bar{x}\| - 2\|\hat{y}\|\|\bar{y}\| \\ \leq 4\|\hat{x}\|\|\hat{y}\| + \|\hat{x}\|^2 + \|\hat{y}\|^2 + 4\|\bar{x}\|\|\bar{y}\| + \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2\|\hat{x}\|\|\bar{x}\| \\ \quad -2\|\hat{y}\|\|\bar{y}\| \end{array}$$

$$
\begin{aligned}
&\le 4\|\hat{x}\|\left\|\hat{y}\right\| + \|\hat{x}\|^2 + \left\|\hat{y}\right\|^2 + 4\|\hat{x}\|\left\|\hat{y}\right\| + \|\bar{x}\|^2 + \left\|\bar{y}\right\|^2 - 2\|\bar{x}\|^2 - 2\left\|\bar{y}\right\|^2 \\
&= 8\|\hat{x}\|\left\|\hat{y}\right\| + \left(\|\hat{x}\|^2 - \|\bar{x}\|^2\right) + \left(\left\|\hat{y}\right\|^2 - \left\|\bar{y}\right\|^2\right) \\
&= 2\mathrm{trace}(x)\mathrm{trace}(y) + \det(x) + \det(y) \\
&\le 2\mathrm{trace}(x)\mathrm{trace}(y) + \frac{1}{4}\left(\mathrm{trace}(x)\right)^2 + \frac{1}{4}\left(\mathrm{trace}(y)\right)^2 \\
&\le 2\mathrm{trace}(x)\mathrm{trace}(y) + \left(\mathrm{trace}(x)\right)^2 + \left(\mathrm{trace}(y)\right)^2 \\
&= \left(\mathrm{trace}(x) + \mathrm{trace}(y)\right)^2.
\end{aligned}
$$

(vi) The first inequality below follows from the fact that $-2ab \ge -a^2 - b^2$ for any $a, b \in \mathbb{R}$, the second one follows from the fact that $x, y \in \mathcal{M}_+^{m|n}$, the third one follows from item (i), and the fourth one follows from the fact that $a^2 + b^2 \le (a+b)^2$ for any $a, b \in \mathbb{R}$.

$$
\begin{aligned}
&(\lambda_2(x+y))^2 - (\lambda_2(x))^2 - (\lambda_2(y))^2 \\
&= \left(\left\|\hat{x} + \hat{y}\right\| - \left\|\bar{x} + \bar{y}\right\|\right)^2 - \left(\|\hat{x}\| - \|\bar{x}\|\right)^2 - \left(\left\|\hat{y}\right\| - \left\|\bar{y}\right\|\right)^2 \\
&= \left\|\hat{x} + \hat{y}\right\|^2 - 2\left\|\hat{x} + \hat{y}\right\|\left\|\bar{x} + \bar{y}\right\| + \left\|\bar{x} + \bar{y}\right\|^2 - \|\hat{x}\|^2 + 2\|\hat{x}\|\|\bar{x}\| - \|\bar{x}\|^2 \\
&\quad - \left\|\hat{y}\right\|^2 + 2\left\|\hat{y}\right\|\left\|\bar{y}\right\| - \left\|\bar{y}\right\|^2 \\
&= 2\langle \hat{x}, \hat{y}\rangle - 2\left\|\hat{x} + \hat{y}\right\|\left\|\bar{x} + \bar{y}\right\| + 2\langle \bar{x}, \bar{y}\rangle + 2\|\hat{x}\|\|\bar{x}\| + 2\left\|\hat{y}\right\|\left\|\bar{y}\right\| \\
&\ge 2\langle \hat{x}, \hat{y}\rangle - \left\|\hat{x} + \hat{y}\right\|^2 - \left\|\bar{x} + \bar{y}\right\|^2 + 2\langle \bar{x}, \bar{y}\rangle + 2\|\hat{x}\|\|\bar{x}\| + 2\left\|\hat{y}\right\|\left\|\bar{y}\right\| \\
&= -\|\hat{x}\|^2 - \left\|\hat{y}\right\|^2 - \|\bar{x}\|^2 - \left\|\bar{y}\right\|^2 + 2\|\hat{x}\|\|\bar{x}\| + 2\left\|\hat{y}\right\|\left\|\bar{y}\right\| \\
&\ge -\left(\left(\|\hat{x}\|^2 - \|\bar{x}\|^2\right) + \left(\left\|\hat{y}\right\|^2 - \left\|\bar{y}\right\|^2\right)\right) \\
&= -\left(\det(x) + \det(y)\right) \\
&\ge -\frac{1}{4}\left(\left(\mathrm{trace}(x)\right)^2 + \left(\mathrm{trace}(y)\right)^2\right) \ \ge\ -\frac{1}{4}\left(\mathrm{trace}(x) + (\mathrm{trace}y)\right)^2.
\end{aligned}
$$

The proof is complete. □

The results in the following lemma for the nonconvex SOC generalize the corresponding ones in Chen [4, Lemma 3.1] for the convex SOC.

Lemma 3.6. *For any $x, y \in \mathcal{M}_+^{m|n}$, we have*

(i) $\left(\dfrac{\mathrm{trace}(x) + \mathrm{trace}(y)}{2} \pm \left\|\bar{y}\right\|\right)^2 \ge \mathrm{trace}(x)\left(\mathrm{trace}(y) \pm 2\left\|\bar{y}\right\|\right).$

(ii) $\dfrac{\left(\mathrm{trace}(x) + \mathrm{trace}(y)\right)^2}{4} - \left\|\bar{y}\right\|^2 \ge 2\mathrm{trace}(x)\sqrt{\det(y)}.$

(iii) $\dfrac{\mathrm{trace}(x)\mathrm{trace}(y)}{4} - \|\bar{x}\|\left\|\bar{y}\right\| \ge \sqrt{\det(x)\det(y)}.$

(iv) $\dfrac{\left(\mathrm{trace}(x) + \mathrm{trace}(y)\right)^2}{4} - \left\|\bar{x} + \bar{y}\right\|^2 \ge 4\sqrt{\det(x)\det(y)}.$

Proof.

(i) The desired inequality follows by noting that

$$\begin{aligned}
&\left(\frac{\text{trace}(x)+\text{trace}(y)}{2} \pm \left\|\bar{y}\right\|\right)^2 - \text{trace}(x)\left(\text{trace}(y) \pm 2\left\|\bar{y}\right\|\right)\\
&= \frac{(\text{trace}(x))^2+(\text{trace}(y))^2}{4} + \frac{\text{trace}(x)\text{trace}(y)}{2} + \left\|\bar{y}\right\|^2 \pm \text{trace}(x)\left(-\left\|\bar{y}\right\| \mp \text{trace}(y)\right)\\
&\quad \pm \text{trace}(y)\left\|\bar{y}\right\|\\
&= \frac{(\text{trace}(x))^2+(\text{trace}(y))^2}{4} - \frac{\text{trace}(x)\text{trace}(y)}{2} + \left\|\bar{y}\right\|^2 \mp \text{trace}(x)\left\|\bar{y}\right\|\\
&\quad \pm \text{trace}(y)\left\|\bar{y}\right\|\\
&= \left(\frac{\text{trace}(x)-\text{trace}(y)}{2} \mp \left\|\bar{y}\right\|\right)^2 \geq 0.
\end{aligned}$$

(ii) The first inequality below follows from the inequality of arithmetic and geometric means (AM-GM inequality) for nonnegative real numbers (which is applied on trace(x) and det(y); we must point out that det(y) is nonnegative because $y \in \mathcal{M}_+^{m|n}$), and the second one from item (i) of Theorem 3.5.

$$\begin{aligned}
&\frac{(\text{trace}(x)+\text{trace}(y))^2}{4} - \left\|\bar{y}\right\|^2\\
&= \frac{1}{4}\left((\text{trace}(x))^2 + 2\text{trace}(x)\text{trace}(y) + (\text{trace}(y))^2\right) - \left\|\bar{y}\right\|^2\\
&= \frac{(\text{trace}(x))^2}{4} + \frac{\text{trace}(x)\text{trace}(y)}{2} + \left\|\hat{y}\right\|^2 - \left\|\bar{y}\right\|^2\\
&= \frac{(\text{trace}(x))^2}{4} + \det(y) + \frac{\text{trace}(x)\text{trace}(y)}{2}\\
&\geq \text{trace}(x)\sqrt{\det(y)} + \frac{\text{trace}(x)\text{trace}(y)}{2} \;\geq\; 2\text{trace}(x)\sqrt{\det(y)}.
\end{aligned}$$

(iii) The desired inequality follows by noting that

$$\begin{aligned}
&\left(\frac{\text{trace}(x)\text{trace}(y)}{4} - \|\bar{x}\|\left\|\bar{y}\right\|\right)^2 - \det(x)\det(y)\\
&= \frac{(\text{trace}(x))^2(\text{trace}(y))^2}{16} - \frac{\text{trace}(x)\text{trace}(y)\,\|\bar{x}\|\left\|\bar{y}\right\|}{2} + \|\bar{x}\|^2\left\|\bar{y}\right\|^2\\
&\quad - \left(\|\hat{x}\|^2 - \|\bar{x}\|^2\right)\left(\left\|\hat{y}\right\|^2 - \left\|\bar{y}\right\|^2\right)\\
&= \frac{(\text{trace}(x))^2(\text{trace}(y))^2}{16} - \frac{\text{trace}(x)\text{trace}(y)\,\|\bar{x}\|\left\|\bar{y}\right\|}{2} - \|\hat{x}\|^2\left\|\hat{y}\right\|^2 + \|\hat{x}\|^2\left\|\bar{y}\right\|^2\\
&\quad + \|\bar{x}\|^2\left\|\hat{y}\right\|^2
\end{aligned}$$

$$= \frac{(\mathrm{trace}(x))^2(\mathrm{trace}(y))^2}{16} - \frac{\mathrm{trace}(x)\mathrm{trace}(y)\,\|\bar{x}\|\,\|\bar{y}\|}{2} - \frac{(\mathrm{trace}(x))^2(\mathrm{trace}(y))^2}{16}$$
$$+\frac{(\mathrm{trace}(x))^2\,\|\bar{y}\|^2}{4} + \frac{(\mathrm{trace}(y))^2\,\|\bar{x}\|^2}{4}$$
$$= \frac{(\mathrm{trace}(x))^2\,\|\bar{y}\|^2}{4} - \frac{\mathrm{trace}(x)\mathrm{trace}(y)\,\|\bar{x}\|\,\|\bar{y}\|}{2} + \frac{(\mathrm{trace}(y))^2\,\|\bar{x}\|^2}{4}$$
$$= \frac{1}{4}\left(\mathrm{trace}(x)\,\|\bar{y}\| - \mathrm{trace}(y)\,\|\bar{x}\|\right)^2 \;\geq\; 0.$$

(iv) The first inequality below is obtained from the AM-GM inequality (which is applied on $\det(x)$ and $\det(y)$; they are both nonnegative because $x, y \in \mathcal{M}_+^{m|n}$), the second one is obtained from the Cauchy-Schwartz inequality, and the last inequality is due to item (iii).

$$\frac{(\mathrm{trace}(x) + \mathrm{trace}(y))^2}{4} - \|\bar{x} + \bar{y}\|^2$$
$$= \frac{1}{4}\left((\mathrm{trace}(x))^2 + 2\mathrm{trace}(x)\mathrm{trace}(y) + (\mathrm{trace}(y))^2\right) - \|\bar{x}\|^2 - 2\langle \bar{x}, \bar{y}\rangle - \|\bar{y}\|^2$$
$$= \det(x) + \det(y) + 2\left(\frac{\mathrm{trace}(x)\mathrm{trace}(y)}{4}\right) - 2\langle \bar{x}, \bar{y}\rangle$$
$$\geq 2\sqrt{\det(x)\det(y)} + 2\left(\frac{\mathrm{trace}(x)\mathrm{trace}(y)}{4}\right) - 2\langle \bar{x}, \bar{y}\rangle$$
$$\geq 2\sqrt{\det(x)\det(y)} + 2\left(\frac{\mathrm{trace}(x)\mathrm{trace}(y)}{4}\right) - 2\,\|\bar{x}\|\,\|\bar{y}\|$$
$$= 2\sqrt{\det(x)\det(y)} + 2\left(\frac{\mathrm{trace}(x)\mathrm{trace}(y)}{4} - \|\bar{x}\|\,\|\bar{y}\|\right)$$
$$\geq 2\sqrt{\det(x)\det(y)} + 2\sqrt{\det(x)\det(y)} \;=\; 4\sqrt{\det(x)\det(y)}.$$

The proof is complete. □

4. Vector-valued functions associated with the nonconvex SOC

In this section, we define the vector-valued functions associated with the nonconvex SOC $\mathcal{M}_+^{m|n}$ and present some of their remarkable properties.

Definition 4.1. *Let n and m be positive integers. For any function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define a corresponding vector-valued function $f^{\mathcal{M}_+^{m|n}} : \mathcal{M}^{m|n} \longrightarrow \mathcal{M}^{m|n}$ associated with the nonconvex SOC $\mathcal{M}_+^{m|n}$ as*

$$f^{\mathcal{M}_+^{m|n}}(x) \triangleq f(\lambda_1(x))c_1(x) + f(\lambda_2(x))c_2(x), \tag{6}$$

where $\lambda_i(x)$ and $c_i(x)$ (for $i = 1, 2$) are the eigenvalues and eigenvectors of $x \in \mathcal{M}^{m|n}$, respectively.

For instance, based on Definition 4.1 (see also (4)), the eigenvalues of x^2 are $\lambda_1^2(x)$ and $\lambda_2^2(x)$, which are nonnegative, hence $x^2 \in \mathcal{M}_+^{m|n}$ even if $x \in \mathcal{M}^{m|n}/\mathcal{M}_+^{m|n}$. Inversely, when $x \in \mathcal{M}_+^{m|n}$, we have $\lambda_1(x), \lambda_2(x) \geq 0$, and therefore $t = \sqrt{\lambda_1(x)}c_1(x) + \sqrt{\lambda_2(x)}c_2(x)$ is well defined in $\mathcal{M}_+^{m|n}$. In addition, an intuitive but rigorous computation can show that $t^2 = x$ (see the proof of Theorem 4.1 in [1] for a formal proof). Thus $x^{1/2} = t$, i.e.,

$$x^{1/2} = (\lambda_1(x)c_1(x) + \lambda_2(x)c_2(x))^{1/2} = \sqrt{\lambda_1(x)}c_1(x) + \sqrt{\lambda_2(x)}c_2(x). \quad (7)$$

Thus, if $x \in \mathcal{M}_+^{m|n}$, then there exists a unique vector in $\mathcal{M}_+^{m|n}$, which we denote by $x^{1/2}$, such that $(x^{1/2})^2 = x$.

For any $x \in \mathcal{M}^{m|n}$, we have $x^2 \in \mathcal{M}_+^{m|n}$. Consequently, there is a unique vector $(x^2)^{1/2} \in \mathcal{M}_+^{m|n}$, which we denote by $|x|$, such that $x^2 = |x|^2$. The following lemma demonstrates that $|x|$ has the same form as in (6).

Lemma 4.2. *For any vector $x = (\hat{x}; \bar{x}) \in \mathcal{M}^{m|n}$, we have $|x| = (x^2)^{1/2} = |\lambda_1(x)|c_1(x) + |\lambda_2(x)|c_2(x)$.*

Proof. From (4) and (7), we have that $x^2 = \lambda_1^2(x)c_1(x) + \lambda_2^2(x)c_2(x)$ and that $x^{1/2} = \sqrt{\lambda_1(x)}c_1(x) + \sqrt{\lambda_2(x)}c_2(x)$. It follows that

$$\begin{aligned}
(x^2)^{1/2} &= \left(\lambda_1^2(x)c_1(x) + \lambda_2^2(x)c_2(x)\right)^{1/2} \\
&= (\lambda_1^2(x))^{1/2}c_1(x) + (\lambda_2^2(x))^{1/2}c_2(x) \\
&= |\lambda_1(x)|c_1(x) + |\lambda_2(x)|c_2(x) \quad = \quad |x|.
\end{aligned}$$

The proof is complete. □

The following proposition stated in the nonconvex SOC setting is the counterpart of [5, Proposition 3.1] in the convex SOC setting. The proposition establishes an important property of the vector-valued functions associated with $\mathcal{M}_+^{m|n}$.

Proposition 4.3. *Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ admits a power series expansion $f(\alpha) = \sum_{p=0}^{\infty} a_p \alpha^p$ for some real coefficients $a_0, a_1, \ldots$. Then the corresponding function $f^{\mathcal{M}_+^{m|n}} : \mathcal{M}^{m|n} \longrightarrow \mathcal{M}^{m|n}$ as defined in (6) has the power series expansion*

$$f^{\mathcal{M}_+^{m|n}}(x) = \sum_{p=0}^{\infty} a_p x^p, \quad \textit{for } x \in \mathcal{M}^{m|n}.$$

Proof. Note that

$$\begin{aligned}
f^{\mathcal{M}_+^{m|n}}(x) &= f(\lambda_1(x))c_1(x) + f(\lambda_2(x))c_2(x) \\
&= \left(\sum_{p=0}^{\infty} a_p \lambda_1^p(x)\right) c_1(x) + \left(\sum_{p=0}^{\infty} a_p \lambda_2^p(x)\right) c_2(x) \\
&= \sum_{p=0}^{\infty} a_p \left(\lambda_1^p(x)c_1(x) + \lambda_2^p(x)c_2(x)\right) = \sum_{p=0}^{\infty} a_p x^p,
\end{aligned}$$

where we used (5) to obtain the last equality. □

The real exponential function $\exp(\alpha)$ is commonly represented by the power series

$$\exp(\alpha) = \sum_{p=0}^{\infty} \frac{\alpha^p}{p!}, \ \forall \alpha \in \mathbb{R}.$$

We, in turn, depending on the result obtained in Proposition 4.3, define the function $\exp^{\mathcal{M}_+^{m|n}}(\cdot)$ on $\mathcal{M}^{m|n}$ in this way as in the following definition.

Definition 4.4. *The exponential vector-valued function* $\exp^{\mathcal{M}_+^{m|n}} : \mathcal{M}^{m|n} \longrightarrow \mathcal{M}^{m|n}$ *is given by*

$$\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = \sum_{p=0}^{\infty} \frac{\boldsymbol{x}^p}{p!}, \ \textit{for } \boldsymbol{x} \in \mathcal{M}^{m|n}.$$

We also define the natural logarithmic $\ln^{\mathcal{M}_+^{m|n}}(\cdot)$ on $\mathcal{M}^{m|n}$ as the unique vector $\boldsymbol{w} \in \mathcal{M}^{m|n}$ satisfying $\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{w}) = \boldsymbol{x}$ for each $\boldsymbol{x} \in \operatorname{int} \mathcal{M}_+^{m|n}$.

Carefully written definitions can occasionally obscure the main idea, although they are appealing for their tightness. In these situations, it is better to use straightforward examples to convey the notion. The following theorem employs two instances to explain and illustrate the concept of the functions associated with the nonconvex SOC. The proof of this theorem requires some technicalities.

Theorem 4.5. *Let* $\boldsymbol{x} = (\hat{\boldsymbol{x}}; \bar{\boldsymbol{x}}) \in \mathcal{M}^{m|n}$.

(i) The exponential function $\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{x})$ *can be written as*

$$\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = \exp(\|\hat{\boldsymbol{x}}\|)\left(\cosh(\|\bar{\boldsymbol{x}}\|)\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}; \sinh(\|\bar{\boldsymbol{x}}\|)\frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\right).$$

(ii) The logarithmic function $\ln^{\mathcal{M}_+^{m|n}}(\boldsymbol{x})$ *can be written as*

$$\ln^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = \left(\ln\left(\sqrt{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\right)\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}; \ln\left(\sqrt{\frac{\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|}{\|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|}}\right)\frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\right),$$

provided that $\boldsymbol{x} \in \operatorname{int} \mathcal{M}_+^{m|n}$.

Proof. Let $\boldsymbol{x} = (\hat{\boldsymbol{x}}, \bar{\boldsymbol{x}}) \in \mathcal{M}^{m|n}$ have the eigenvalues $\lambda_{1,2}(\boldsymbol{x})$ and eigenvectors $\boldsymbol{c}_{1,2}(\boldsymbol{x})$.

(i) Note that

$$\begin{aligned}
&\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = \exp(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + \exp(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x})\\
&= \exp\left(\|\hat{x}\| + \|\bar{x}\|\right)\boldsymbol{c}_1(\boldsymbol{x}) + \exp\left(\|\hat{x}\| - \|\bar{x}\|\right)\boldsymbol{c}_2(\boldsymbol{x})\\
&= \exp(\|\hat{x}\|)\exp(\|\bar{x}\|)\boldsymbol{c}_1(\boldsymbol{x}) + \exp(\|\hat{x}\|)\exp(-\|\bar{x}\|)\boldsymbol{c}_2(\boldsymbol{x})\\
&= \exp(\|\hat{x}\|)\left(\exp(\|\bar{x}\|)\boldsymbol{c}_1(\boldsymbol{x}) + \exp(-\|\bar{x}\|)\boldsymbol{c}_2(\boldsymbol{x})\right)\\
&= \exp(\|\hat{x}\|)\left(\exp(\|\bar{x}\|)\left(\frac{1}{2}\right)\left(\frac{\hat{x}}{\|\hat{x}\|};\frac{\bar{x}}{\|\bar{x}\|}\right) + \exp(-\|\bar{x}\|)\left(\frac{1}{2}\right)\left(\frac{\hat{x}}{\|\hat{x}\|};-\frac{\bar{x}}{\|\bar{x}\|}\right)\right)\\
&= \exp(\|\hat{x}\|)\left(\frac{\exp(\|\bar{x}\|)}{2}\frac{\hat{x}}{\|\hat{x}\|} + \frac{\exp(-\|\bar{x}\|)}{2}\frac{\hat{x}}{\|\hat{x}\|};\frac{\exp(\|\bar{x}\|)}{2}\frac{\bar{x}}{\|\bar{x}\|} - \frac{\exp(-\|\bar{x}\|)}{2}\frac{\bar{x}}{\|\bar{x}\|}\right)\\
&= \exp(\|\hat{x}\|)\left(\left(\frac{\exp(\|\bar{x}\|) + \exp(-\|\bar{x}\|)}{2}\right)\frac{\hat{x}}{\|\hat{x}\|};\left(\frac{\exp(\|\bar{x}\|) - \exp(-\|\bar{x}\|)}{2}\right)\frac{\bar{x}}{\|\bar{x}\|}\right)\\
&= \exp(\|\hat{x}\|)\left(\cosh(\|\bar{x}\|)\frac{\hat{x}}{\|\hat{x}\|};\sinh(\|\bar{x}\|)\frac{\bar{x}}{\|\bar{x}\|}\right).
\end{aligned}$$

This proves the first item.

(ii) The proof of this part is constructive. Let $\boldsymbol{x} = (\hat{x};\bar{x}) \in \operatorname{int}\mathcal{M}_+^{m|n}$, i.e., $\|\hat{x}\| > \|\bar{x}\|$. We show that there is a unique element $\boldsymbol{w} \in \mathcal{M}^{m|n}$ satisfying $\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{w}) = \boldsymbol{x}$, which demonstrates that $\ln^{\mathcal{M}_+^{m|n}}(\boldsymbol{x})$ is well defined.

Using item (i), $\boldsymbol{w} = (\hat{w};\bar{w}) \in \mathcal{M}^{m|n}$ satisfies $\exp^{\mathcal{M}_+^{m|n}}(\boldsymbol{w}) = \boldsymbol{x}$ iff

$$\hat{x} = \exp(\|\hat{w}\|)\left(\frac{\exp(\|\bar{w}\|) + \exp(-\|\bar{w}\|)}{2}\right)\frac{\hat{w}}{\|\hat{w}\|} \tag{8}$$

and

$$\bar{x} = \exp(\|\hat{w}\|)\left(\frac{\exp(\|\bar{w}\|) - \exp(-\|\bar{w}\|)}{2}\right)\frac{\bar{w}}{\|\bar{w}\|}. \tag{9}$$

The crux of the formation is highlighted in the following steps, which are simply implementable. To simplify and make things easier, let us replace $\exp(\|\hat{w}\|)$ with a and $\exp(\|\bar{w}\|)$ with b. From (8) and (9), we have

$$\|\hat{x}\| = a\left(\frac{b + b^{-1}}{2}\right) \quad\text{and}\quad \|\bar{x}\| = a\left(\frac{b - b^{-1}}{2}\right). \tag{10}$$

Now, we solve the two equations in (10) uniquely for a and b. Note that

$$\frac{\|\hat{x}\| + \|\bar{x}\|}{\|\hat{x}\| - \|\bar{x}\|} = \frac{a\left(\frac{b + b^{-1}}{2}\right) + a\left(\frac{b - b^{-1}}{2}\right)}{a\left(\frac{b + b^{-1}}{2}\right) - a\left(\frac{b - b^{-1}}{2}\right)} = \frac{ab}{ab^{-1}} = b^2, \text{ hence } b = \sqrt{\frac{\|\hat{x}\| + \|\bar{x}\|}{\|\hat{x}\| - \|\bar{x}\|}}.$$

From the first equation of (10), we have $a = 2\|\hat{x}\|/(b+b^{-1})$, where

$$
\begin{aligned}
b+b^{-1} &= \sqrt{\frac{\|\hat{x}\|+\|\bar{x}\|}{\|\hat{x}\|-\|\bar{x}\|}}+\sqrt{\frac{\|\hat{x}\|-\|\bar{x}\|}{\|\hat{x}\|+\|\bar{x}\|}} \\
&= \frac{\left(\sqrt{\|\hat{x}\|+\|\bar{x}\|}\right)^2+\left(\sqrt{\|\hat{x}\|-\|\bar{x}\|}\right)^2}{\sqrt{\|\hat{x}\|-\|\bar{x}\|}\sqrt{\|\hat{x}\|+\|\bar{x}\|}} = \frac{2\|\hat{x}\|}{\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}},
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
b-b^{-1} &= \sqrt{\frac{\|\hat{x}\|+\|\bar{x}\|}{\|\hat{x}\|-\|\bar{x}\|}}-\sqrt{\frac{\|\hat{x}\|-\|\bar{x}\|}{\|\hat{x}\|+\|\bar{x}\|}} \\
&= \frac{\left(\sqrt{\|\hat{x}\|+\|\bar{x}\|}\right)^2-\left(\sqrt{\|\hat{x}\|-\|\bar{x}\|}\right)^2}{\sqrt{\|\hat{x}\|-\|\bar{x}\|}\sqrt{\|\hat{x}\|+\|\bar{x}\|}} = \frac{2\|\bar{x}\|}{\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}}.
\end{aligned} \tag{12}
$$

Thus, from (11), we have

$$
a = \frac{2\|\hat{x}\|\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}}{2\|\hat{x}\|} = \sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}. \tag{13}
$$

From the expressions of a and b, it follows that

$$
\|\hat{w}\| = \ln(a) = \ln\left(\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}\right) \quad \text{and} \quad \|\bar{w}\| = \ln(b) = \ln\left(\sqrt{\frac{\|\hat{x}\|+\|\bar{x}\|}{\|\hat{x}\|-\|\bar{x}\|}}\right). \tag{14}
$$

From (8) and (9), we have

$$
\hat{w} = \frac{2a^{-1}\|\hat{w}\|}{b+b^{-1}}\hat{x} \quad \text{and} \quad \bar{w} = \frac{2a^{-1}\|\bar{w}\|}{b-b^{-1}}\bar{x}. \tag{15}
$$

Now, substituting (11), (13) and the expression of $\|\hat{w}\|$ from (14) in the first equation of (15) to get

$$
\hat{w} = \frac{2\ln\left(\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}\right)\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}}{2\|\hat{x}\|\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}}\hat{x} = \ln\left(\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}\right)\frac{\hat{x}}{\|\hat{x}\|}.
$$

Similarly, substituting (12), (13), and the expression of $\|\bar{w}\|$ from (14) in the second equation of (15) to get

$$
\bar{w} = \frac{2\ln\left(\sqrt{\frac{\|\hat{x}\|+\|\bar{x}\|}{\|\hat{x}\|-\|\bar{x}\|}}\right)\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}}{2\|\bar{x}\|\sqrt{\|\hat{x}\|^2-\|\bar{x}\|^2}}\bar{x} = \ln\left(\sqrt{\frac{\|\hat{x}\|+\|\bar{x}\|}{\|\hat{x}\|-\|\bar{x}\|}}\right)\frac{\bar{x}}{\|\bar{x}\|}.
$$

Given this, the explicit form of $\ln^{\mathcal{M}_+^{m|n}}(\boldsymbol{x})$ can be written now as

$$\begin{aligned}\ln^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) &= (\hat{\boldsymbol{w}}, \bar{\boldsymbol{w}}) \\ &= \left(\ln\left(\sqrt{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\right)\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}; \ln\left(\sqrt{\frac{\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|}{\|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|}}\right)\frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\right)\end{aligned}$$

The proof is complete. □

Note that from Theorem 4.5, the logarithmic function is well defined when $\boldsymbol{x} \in \operatorname{int} \mathcal{M}_+^{m|n}$. Note also, in view of item (ii), we can immediately write

$$\ln^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = \left(\ln\left(\sqrt{\det(\boldsymbol{x})}\right)\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}; \ln\left(\sqrt{\frac{\lambda_1(\boldsymbol{x})}{\lambda_2(\boldsymbol{x})}}\right)\frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\right), \quad \text{for } \boldsymbol{x} \in \operatorname{int} \mathcal{M}^{m|n}.$$

The remaining part of this section concludes a key result that the function $f^{\mathcal{M}_+^{m|n}}$ associated with the nonconvex SOC inherits the Fréchet-differentiability property from the corresponding real-valued function f. The work of Fukushima et al. [5] inspired the idea of determining the differentiability interpretation of $f^{\mathcal{M}_+^{m|n}}$. They established the Jacobian of the vector-valued function on the framework of the convex SOC $\mathcal{E}_+^{n+1}$. In this part, we derive a closed-form formula for the Jacobian of the vector-valued function defined in (6).

Theorem 4.6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function, and $f^{\mathcal{M}_+^{m|n}} : \mathcal{M}^{m|n} \longrightarrow \mathcal{M}^{m|n}$ be the corresponding vector-valued function as defined in (6). If f is Fréchet-differentiable, then so is $f^{\mathcal{M}_+^{m|n}}$. Moreover, the Jacobian of $f^{\mathcal{M}_+^{m|n}}(\cdot)$ at $\boldsymbol{x} = (\hat{\boldsymbol{x}}, \bar{\boldsymbol{x}}) \in \mathcal{M}^{m|n}$, with $\hat{\boldsymbol{x}} \neq \boldsymbol{0}$ and $\bar{\boldsymbol{x}} \neq \boldsymbol{0}$, is given by*

$$\mathsf{J}f^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = \begin{bmatrix} sI_m + (r-s)\dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2} & h\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\dfrac{\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|} \\ h\dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\dfrac{\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|} & dI_n + (r-d)\dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix},$$

where

$$\begin{aligned} s &= \frac{f(\lambda_1(\boldsymbol{x})) + f(\lambda_2(\boldsymbol{x}))}{\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})}, & d &= \frac{f(\lambda_1(\boldsymbol{x})) - f(\lambda_2(\boldsymbol{x}))}{\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})}, \\ r &= \frac{f'(\lambda_1(\boldsymbol{x})) + f'(\lambda_2(\boldsymbol{x}))}{2}, & h &= \frac{f'(\lambda_1(\boldsymbol{x})) - f'(\lambda_2(\boldsymbol{x}))}{2}. \end{aligned}$$

Proof. Let $\boldsymbol{x} = (\hat{\boldsymbol{x}}, \bar{\boldsymbol{x}}) \in \mathcal{M}^{m|n}$, and f be Fréchet-differentiable. We have four cases to consider and verify.

Case (i) When $\hat{\boldsymbol{x}} \neq \boldsymbol{0}$ and $\bar{\boldsymbol{x}} \neq \boldsymbol{0}$: By definition, we have $f^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) \triangleq f(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x})$. Note that

$$\mathsf{J}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{1}{\|\boldsymbol{x}\|}\left(I - \frac{\boldsymbol{x}\boldsymbol{x}^\mathsf{T}}{\|\boldsymbol{x}\|^2}\right).$$

This implies that eigenvectors $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ are Fréchet-differentiable with respect to $\boldsymbol{x}$, with Jacobian

$$\mathsf{J}\boldsymbol{c}_i(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} \frac{1}{\|\hat{\boldsymbol{x}}\|}\left(I_m - \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top}{\|\hat{\boldsymbol{x}}\|^2}\right) & O \\ O & (-1)^{i+1}\frac{1}{\|\bar{\boldsymbol{x}}\|}\left(I_n - \frac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top}{\|\bar{\boldsymbol{x}}\|^2}\right) \end{bmatrix}, \text{ for } i = 1, 2.$$

Furthermore, the eigenvalues $\lambda_1(\boldsymbol{x})$ and $\lambda_2(\boldsymbol{x})$ are also differentiable with respect to $\boldsymbol{x}$, with gradients

$$\nabla\lambda_i(\boldsymbol{x}) = \begin{bmatrix} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ (-1)^{i+1}\frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix} = 2\boldsymbol{c}_i(\boldsymbol{x}), \text{ for } i = 1, 2.$$

Since f is Fréchet-differentiable, by using the product rule and the chain rule, we obtain

$$\begin{aligned}
&\mathsf{J}f^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) \\
&= f(\lambda_1(\boldsymbol{x}))\mathsf{J}\boldsymbol{c}_1(\boldsymbol{x}) + \boldsymbol{c}_1(\boldsymbol{x})\left(\nabla f(\lambda_1(\boldsymbol{x}))\right)^\top + f(\lambda_2(\boldsymbol{x}))\mathsf{J}\boldsymbol{c}_2(\boldsymbol{x}) + \boldsymbol{c}_2(\boldsymbol{x})\left(\nabla f(\lambda_2(\boldsymbol{x}))\right)^\top \\
&= f(\lambda_1(\boldsymbol{x}))\mathsf{J}\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda_2(\boldsymbol{x}))\mathsf{J}\boldsymbol{c}_2(\boldsymbol{x}) + \boldsymbol{c}_1(\boldsymbol{x})f'(\lambda_1(\boldsymbol{x}))\left(\nabla\lambda_1(\boldsymbol{x})\right)^\top \\
&\quad + \boldsymbol{c}_2(\boldsymbol{x})f'(\lambda_2(\boldsymbol{x}))\left(\nabla\lambda_2(\boldsymbol{x})\right)^\top \\
&= \frac{f(\lambda_1(\boldsymbol{x}))}{2}\begin{bmatrix} \frac{1}{\|\hat{\boldsymbol{x}}\|}\left(I_m - \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top}{\|\hat{\boldsymbol{x}}\|^2}\right) & O \\ O & \frac{1}{\|\bar{\boldsymbol{x}}\|}\left(I_n - \frac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top}{\|\bar{\boldsymbol{x}}\|^2}\right) \end{bmatrix} + 2f'(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x})^\top \\
&\quad + \frac{f(\lambda_2(\boldsymbol{x}))}{2}\begin{bmatrix} \frac{1}{\|\hat{\boldsymbol{x}}\|}\left(I_m - \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top}{\|\hat{\boldsymbol{x}}\|^2}\right) & O \\ O & -\frac{1}{\|\bar{\boldsymbol{x}}\|}\left(I_n - \frac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top}{\|\bar{\boldsymbol{x}}\|^2}\right) \end{bmatrix} + 2f'(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})^\top \\
&= \begin{bmatrix} \frac{f(\lambda_1(\boldsymbol{x})) + f(\lambda_2(\boldsymbol{x}))}{2\|\hat{\boldsymbol{x}}\|}\left(I_m - \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top}{\|\hat{\boldsymbol{x}}\|^2}\right) & O \\ O & \frac{f(\lambda_1(\boldsymbol{x})) - f(\lambda_2(\boldsymbol{x}))}{2\|\bar{\boldsymbol{x}}\|}\left(I_n - \frac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top}{\|\bar{\boldsymbol{x}}\|^2}\right) \end{bmatrix} \\
&\quad + 2f'(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x})^\top + 2f'(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})^\top.
\end{aligned}$$

Note that $\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) = 2\|\hat{\boldsymbol{x}}\|$ and $\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}) = 2\|\bar{\boldsymbol{x}}\|$. Note also that

$$\boldsymbol{c}_i(\boldsymbol{x})\boldsymbol{c}_i(\boldsymbol{x})^\top = \frac{1}{4}\begin{bmatrix} \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top}{\|\hat{\boldsymbol{x}}\|^2} & (-1)^{i+1}\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\frac{\bar{\boldsymbol{x}}^\top}{\|\bar{\boldsymbol{x}}\|} \\ (-1)^{i+1}\frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\frac{\hat{\boldsymbol{x}}^\top}{\|\hat{\boldsymbol{x}}\|} & \frac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix}, \text{ for } i = 1, 2.$$

In applying these observations to the expression of $\mathsf{J}f^{\mathcal{M}_+^{m|n}}(\cdot)$, we get

$$
\begin{aligned}
&\mathsf{J}f^{\mathcal{M}_+^{m|n}}(\boldsymbol{x})\\
&= \begin{bmatrix} \dfrac{f(\lambda_1(\boldsymbol{x}))+f(\lambda_2(\boldsymbol{x}))}{\lambda_1(\boldsymbol{x})+\lambda_2(\boldsymbol{x})}\left(I_m - \dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2}\right) & O \\ O & \dfrac{f(\lambda_1(\boldsymbol{x}))-f(\lambda_2(\boldsymbol{x}))}{\lambda_1(\boldsymbol{x})-\lambda_2(\boldsymbol{x})}\left(I_n - \dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2}\right) \end{bmatrix}\\
&\quad + \frac{f'(\lambda_1(\boldsymbol{x}))}{2}\begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2} & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\dfrac{\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|} \\ \dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\dfrac{\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|} & \dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix} + \frac{f'(\lambda_2(\boldsymbol{x}))}{2}\begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2} & -\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\dfrac{\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|} \\ -\dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\dfrac{\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|} & \dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix}\\
&= \begin{bmatrix} \dfrac{f(\lambda_1(\boldsymbol{x}))+f(\lambda_2(\boldsymbol{x}))}{\lambda_1(\boldsymbol{x})+\lambda_2(\boldsymbol{x})}\left(I_m - \dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2}\right) & O \\ O & \dfrac{f(\lambda_1(\boldsymbol{x}))-f(\lambda_2(\boldsymbol{x}))}{\lambda_1(\boldsymbol{x})-\lambda_2(\boldsymbol{x})}\left(I_n - \dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2}\right) \end{bmatrix}\\
&\quad + \begin{bmatrix} \left(\dfrac{f'(\lambda_1(\boldsymbol{x}))+f'(\lambda_2(\boldsymbol{x}))}{2}\right)\dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2} & \left(\dfrac{f'(\lambda_1(\boldsymbol{x}))-f'(\lambda_2(\boldsymbol{x}))}{2}\right)\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\dfrac{\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|} \\ \left(\dfrac{f'(\lambda_1(\boldsymbol{x}))-f'(\lambda_2(\boldsymbol{x}))}{2}\right)\dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\dfrac{\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|} & \left(\dfrac{f'(\lambda_1(\boldsymbol{x}))+f'(\lambda_2(\boldsymbol{x}))}{2}\right)\dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix}\\
&= \begin{bmatrix} s\left(I_m - \dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2}\right) + r\dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2} & h\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\dfrac{\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|} \\ h\dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\dfrac{\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|} & d\left(I_n - \dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2}\right) + r\dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix}\\
&= \begin{bmatrix} sI_m + (r-s)\dfrac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|^2} & h\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\dfrac{\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|} \\ h\dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|}\dfrac{\hat{\boldsymbol{x}}^\mathsf{T}}{\|\hat{\boldsymbol{x}}\|} & dI_n + (r-d)\dfrac{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T}}{\|\bar{\boldsymbol{x}}\|^2} \end{bmatrix},
\end{aligned}
$$

where s, d, r and h are defined in the theorem statement. This enables the function $f^{\mathcal{M}_+^{m|n}}$ to be Fréchet-differentiable at $\boldsymbol{x}$ when $\hat{\boldsymbol{x}} \neq \mathbf{0}$ and $\bar{\boldsymbol{x}} \neq \mathbf{0}$. The desired representation of the matrix $\mathsf{J}f^{\mathcal{M}_+^{m|n}}(\boldsymbol{x})$ is established in this case.

Case (ii) When $\hat{\boldsymbol{x}} \neq \mathbf{0}$ *and* $\bar{\boldsymbol{x}} = \mathbf{0}$: For this and the remaining two cases, we verify the Fréchet-differentiability of $f^{\mathcal{M}_+^{m|n}}$ at $\boldsymbol{x}$ by perturbing $\boldsymbol{x}$ by $\Delta\boldsymbol{x} = (\Delta\hat{\boldsymbol{x}}, \Delta\bar{\boldsymbol{x}})$ and accounting for the change in $f^{\mathcal{M}_+^{m|n}}$ that results. In what follows, we define

$$\Delta\lambda_i(\boldsymbol{x}) \triangleq \|\Delta\hat{\boldsymbol{x}}\| + (-1)^{i+1}\|\Delta\bar{\boldsymbol{x}}\| \text{ and } \boldsymbol{c}_i(\boldsymbol{x}) \triangleq \frac{1}{2}\left(\frac{\Delta\hat{\boldsymbol{x}}}{\|\Delta\hat{\boldsymbol{x}}\|}; (-1)^{i+1}\frac{\Delta\bar{\boldsymbol{x}}}{\|\Delta\bar{\boldsymbol{x}}\|}\right), \text{ for } i = 1, 2.$$

In this case, when $\bar{\boldsymbol{x}} = \mathbf{0}$, we have $\lambda_1(\boldsymbol{x}) = \lambda_2(\boldsymbol{x}) = \|\hat{\boldsymbol{x}}\|$. So, in this case let us denote $\lambda(\boldsymbol{x}) \triangleq \|\hat{\boldsymbol{x}}\|$. We first consider the subcase in which $\Delta\bar{\boldsymbol{x}} \neq \mathbf{0}$. From (6), we have

$$f^{\mathcal{M}_+^{m|n}}(\boldsymbol{x}) = f(\lambda(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x})$$

and

$$f^{\mathcal{M}_+^{m|n}}(x + \Delta x) = f(\lambda(x) + \Delta \lambda_1(x))c_1(x) + f(\lambda(x) + \Delta \lambda_2(x))c_2(x).$$

From the Taylor expansion of f about $\lambda(x)$, we also have

$$f(\lambda(x)+\Delta \lambda_i(x))-f(\lambda(x)) = f'(\lambda(x))\Delta \lambda_i(x)+o(\Delta \lambda_i(x)) = f'(\lambda(x))\Delta \lambda_i(x)+o(\|\Delta x\|)\,,$$

where $o(\cdot)$ is the usual "little o" notation and $i = 1, 2,=$. Using the above identities, it immediately follows that

$$\begin{aligned}
&f^{\mathcal{M}_+^{m|n}}(x + \Delta x) - f^{\mathcal{M}_+^{m|n}}(x)\\
&= f(\lambda(x) + \Delta \lambda_1(x))c_1(x) + f(\lambda(x) + \Delta \lambda_2(x))c_2(x) - (f(\lambda(x))c_1(x) + f(\lambda(x))c_2(x))\\
&= (f(\lambda(x) + \Delta \lambda_1(x)) - f(\lambda(x)))\, c_1(x) + (f(\lambda(x) + \Delta \lambda_2(x)) - f(\lambda(x)))\, c_2(x)\\
&= f'(\lambda(x))\Delta \lambda_1(x)c_1(x) + f'(\lambda(x))\Delta \lambda_2(x)c_2(x) + o(\|\Delta x\|)\\
&= f'(\lambda(x))\,(\Delta \lambda_1(x)c_1(x) + \Delta \lambda_2(x)c_2(x)) + o(\|\Delta x\|) \;=\; f'(\lambda(x))\Delta x + o(\|\Delta x\|)\,.
\end{aligned}$$

The same reasoning still holds true when $\Delta \bar{x} = \mathbf{0}$, but the vector $\Delta \bar{x}/\,\|\Delta \bar{x}\|$ is replaced with any vector $w \in \mathbb{R}^n$ with $\|w\| = 1$.

Case (iii) When $\hat{x} = \mathbf{0}$ *and* $\bar{x} \neq \mathbf{0}$: If $\hat{x} = \mathbf{0}$, then $\lambda_1(x) = \|\bar{x}\|$ and $\lambda_2(x) = -\|\bar{x}\|$. Let us denote $\lambda(x) \triangleq \|\bar{x}\|$. First, we consider the subcase in which $\Delta \hat{x} \neq \mathbf{0}$. From (6), we have

$$f^{\mathcal{M}_+^{m|n}}(x) = f(\lambda(x))c_1(x) + f(-\lambda(x))c_2(x)$$

and

$$f^{\mathcal{M}_+^{m|n}}(x + \Delta x) = f(\lambda(x) + \Delta \lambda_1(x))c_1(x) + f(-\lambda(x) + \Delta \lambda_2(x))c_2(x).$$

Using the Taylor expansion of f about $\pm\lambda(x)$, we also have

$$f(\pm\lambda(x)+\Delta \lambda_i(x))-f(\pm\lambda(x)) = f'(\pm\lambda(x))\Delta \lambda_i(x)+o(\Delta \lambda_i(x)) = f'(\pm\lambda(x))\Delta \lambda_i(x)+o(\|\Delta x\|)\,,$$

for $i = 1, 2$. Using the above identities, it follows that

$$\begin{aligned}
&f^{\mathcal{M}_+^{m|n}}(x + \Delta x) - f^{\mathcal{M}_+^{m|n}}(x)\\
&= f(\lambda(x) + \Delta \lambda_1(x))c_1(x) + f(-\lambda(x) + \Delta \lambda_2(x))c_2(x) - (f(\lambda(x))c_1(x) + f(-\lambda(x))c_2(x))\\
&= (f(\lambda(x) + \Delta \lambda_1(x)) - f(\lambda(x)))\, c_1(x) + (f(-\lambda(x) + \Delta \lambda_2(x)) - f(-\lambda(x)))\, c_2(x)\\
&= f'(\lambda(x))\Delta \lambda_1(x)c_1(x) + f'(-\lambda(x))\Delta \lambda_2(x)c_2(x) + o(\|\Delta x\|)\\
&= f'(\lambda(x))\Delta \lambda_1(x)c_1(x) - f'(\lambda(x))\Delta \lambda_2(x)c_2(x) + o(\|\Delta x\|)\\
&= f'(\lambda(x))\,(\Delta \lambda_1(x)c_1(x) - \Delta \lambda_2(x)c_2(x)) + o(\|\Delta x\|)\\
&= f'(\lambda(x))\left(\|\Delta \bar{x}\|\,\frac{\Delta \hat{x}}{\|\Delta \hat{x}\|}; \|\Delta \hat{x}\|\,\frac{\Delta \bar{x}}{\|\Delta \bar{x}\|}\right) + o(\|\Delta x\|)\,.
\end{aligned}$$

The same reasoning still holds true when $\Delta \hat{x} = \mathbf{0}$, but the vector $\Delta \hat{x}/\,\|\Delta \hat{x}\|$ is replaced with any vector $z \in \mathbb{R}^m$ with $\|z\| = 1$.

Case (iv) When $\hat{x} = \mathbf{0}$ *and* $\bar{x} = \mathbf{0}$: If $x = \mathbf{0}$, then $\lambda_1(x) = \lambda_2(x) = 0$. This case is handled in the same manner as Cases (ii) and (iii). We begin with the most general case, in which $\Delta \hat{x} \neq \mathbf{0}$ and $\Delta \bar{x} \neq \mathbf{0}$. From (6), we have

$$f^{\mathcal{M}_+^{m|n}}(\mathbf{0}) = f(0)c_1(x) + f(0)c_2(x)$$

and

$$f^{\mathcal{M}_+^{m|n}}(\Delta\, \boldsymbol{x}) = f(\Delta\, \lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\Delta\, \lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x}).$$

Using the Taylor expansion of f about 0, we also have

$$f(\Delta\, \lambda_i(\boldsymbol{x})) - f(0) = f'(0)\Delta\, \lambda_i(\boldsymbol{x}) + o(\Delta\, \lambda_i(\boldsymbol{x})) = f'(0)\Delta\, \lambda_i(\boldsymbol{x}) + o(\|\Delta \boldsymbol{x}\|)\,, \text{ for } i = 1, 2.$$

Using the above identities, it follows that

$$\begin{aligned}
& f^{\mathcal{M}_+^{m|n}}(\Delta\, \boldsymbol{x}) - f^{\mathcal{M}_+^{m|n}}(\boldsymbol{0}) \\
&= f(\Delta\, \lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\Delta\, \lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x}) - (f(0)\boldsymbol{c}_1(\boldsymbol{x}) + f(0)\boldsymbol{c}_2(\boldsymbol{x})) \\
&= (f(\Delta\, \lambda_1(\boldsymbol{x})) - f(0))\, \boldsymbol{c}_1(\boldsymbol{x}) + (f(\Delta\, \lambda_2(\boldsymbol{x})) - f(0))\, \boldsymbol{c}_2(\boldsymbol{x}) \\
&= f'(0)\Delta\, \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + f'(0)\Delta\, \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) + o(\|\Delta \boldsymbol{x}\|) \\
&= f'(0)\, (\Delta\, \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \Delta\, \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})) + o(\|\Delta \boldsymbol{x}\|) \;=\; f'(0)\Delta\, \boldsymbol{x} + o(\|\Delta \boldsymbol{x}\|)\,.
\end{aligned}$$

If one (or both) of $\Delta\, \hat{\boldsymbol{x}}$ and $\Delta\, \bar{\boldsymbol{x}}$ is (are) equal to zero, we apply the same reasoning. We only have to be careful about the vectors $\Delta\, \hat{\boldsymbol{x}} / \|\Delta\, \hat{\boldsymbol{x}}\|$ and $\Delta\, \bar{\boldsymbol{x}} / \|\Delta\, \bar{\boldsymbol{x}}\|$. They have to be taken in the same way as in Cases (ii) and (iii). The proof is complete. □

The formula of the differential of vector-valued function may be used as a springboard for exploring the differentiability in broader contexts, such as Bouligand or Clarke Jacobians [6].

5. Concluding remarks

The vector-valued functions associated with convex second-order cones were used in solutions methods for convex second-order-cone programs and convex second-order-cone complementarity problems, and, similarly, the matrix-valued functions were used in solutions methods for semidefinite programs and semidefinite complementarity problems. Likewise, the vector-valued functions associated with nonconvex second-order cones are expected to be used in solution methods for nonconvex second-order-cone programs and nonconvex second-order-cone complementarity problems. In this paper, we have defined and studied the spectral and vector-valued functions associated with the nonconvex SOC analog to those associated with the convex SOC. We have presented several important properties and key characteristics of these nonconvex SOC-related functions.

Continuing in this research line, our future work aims to show whether the vector-valued functions introduced in this paper inherit from the corresponding analytical properties of the real-valued functions such as continuity, Lipschitz continuity, monotonicity and semismoothness. Another direction to be taken is to develop a heuristic approach based on the alternating direction method of multipliers for solving optimization problems over the nonconvex SOC, for which the nonconvex SOC-related functions studied in this paper can play a key role; they can be used to express the Euclidean projection onto the cone. This highlights the substance of our future research papers, which is based on this paper and the methodological part in [7].

Declarations

Competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during this study.

References

[1] B. Alzalg, L. Benakkouche. The algebraic structure of the nonconvex second-order cone. Submitted for publication. Available online at: https://optimization-online.org/?p=24090, 2023.

[2] D. Sun, J. Sun. Löwner's operator and spectral functions in Euclidean Jordan algebras. Math. Oper. Res. 33(2):421-445, 2008.

[3] J.-S. Chen, X. Chen, P. Tseng. Analysis of nonsmooth vector-valued functions associated with second-order cones. Math. Program. 101(1):95-117, 2004.

[4] J.-S. Chen. The convex and monotone functions associated with second-order cone. Optimization. 55(4):363-385, 2006.

[5] M. Fukushima, Z.-Q. Luo, P. Tseng. Smoothing functions for second-order-cone complementarity problems. SIAM J. Optim. 12(2):436-460, 2002.

[6] D. Bartl, M. Fabian, J. Kolář. Clarke Jacobians, Bouligand Jacobians, and compact connected sets of matrices. J. Math. Anal. Appl. 516(1):126491, 2022.

[7] L. Benakkouche. On optimization problems over nonconvex cones. PhD thesis, The University of Jordan, 2023.

Department of Mathematics, The University of Jordan, Amman, Jordan