Manuscript No.
(will be inserted by the editor)

The Algebraic Structure of the Nonconvex Second-Order Cone

Baha Alzalg and Lilia Benakkouche

Received: 10 August 2023 / Accepted: date

Abstract This paper explores the nonconvex second-order cone as a nonconvex conic extension of the known convex second-order cone in optimization, as well as a higher-dimensional conic extension of the known causality cone in relativity. The nonconvex second-order cone can be used to reformulate nonconvex quadratic programming and nonconvex quadratically constrained quadratic program in conic format. The cone can also arise in real-world applications. We define notions of the algebraic structure of the nonconvex second-order cone, and show that its ambient space is a commutative power-associative magma whose elements always have real eigenvalues; this is remarkable because it is not the case for arbitrary Jordan algebras. We will also find that the magma of this nonconvex cone is rankly independent of its dimension; this is also remarkable because it is not the case for algebras of arbitrary convex cones. Even more remarkably, we prove that the nonconvex second-order cone equals the cone of squares of its magma; this is not the case for all non-Euclidean Jordan algebras. Finally, numerous algebraic properties that already exist in the framework of the convex second-order cone are generalized to the framework of the nonconvex second-order cone.

Keywords Nonconvex bodies · Cones of operators · Finite-dimensional structures · Sums of squares and quadratic forms

Mathematics Subject Classification (2000) 11H16 · 47L07 · 17C55 · 11E25

Baha Alzalg, Corresponding author
The University of Jordan
Amman 11942, Jordan
b.alzalg@ju.edu.jo

Lilia Benakkouche
The University of Jordan
Amman 11942, Jordan
lyl9170462@ju.edu.jo

1 Introduction

This paper studies an attractive family of nonconvex cones algebraically. This family can be viewed as a conic nonconvex extension of the (convex) second-order cone (SOC for short), which is well-known and studied in mathematics and widely used in operations research. This family of nonconvex cones is also a higher-dimensional conic extension of the Einstein-Minkowski causality cone, which is well-known and studied in special and general relativity.

More specifically, we define and establish some notions and concepts associated with this cone, such as its spectral factorization, eigenvalues, eigenvectors, determinant, trace, multiplication operation, positive powers, quadratic representation, and the logarithmic barrier function. We also adopt generalized concepts such as the identity-like element, generalized inverse, and the crane-shaped matrix. We deal with the notion of magma, which generalizes the concept of algebra. A magma consists of a set equipped with a binary operation that must be closed by definition. An algebra is a magma whose binary operation is bilinear. We define a commutative (but rejects being bilinear) binary operation in the underlying vector space of our cone to generate a commutative magma. Although we lose the bilinearity, we will be able to prove that this commutative magma is power-associative. We will also see that the elements of this magma always have real eigenvalues; this is not the case for commutative power-associative algebras or even arbitrary Jordan algebras. We will also see that the magma of this nonconvex cone is rankly independent of its ambient dimension; this is not the case for algebras of arbitrary convex cones or even arbitrary convex symmetric cones. This is a key feature because the time complexity of interior-point algorithms for the conic optimization problems is given in terms of the rank of the underlying cone rather than the ambient dimension.

The magma that we propose is novel enough to characterize the nonconvex SOCs. More specifically, our work finds that the cone of squares of our magma is the nonconvex SOC itself. Amazingly, we are able to provide a rigorous but very simple proof of this central fact in the absence of the bilinearity of the binary operation. This finding is very interesting since it is not found in any non-Euclidean Jordan algebra. By using the introduced notions and tools, we also extend several algebraic properties that already exist in the framework of the convex SOC to the framework of the nonconvex SOC. We will see that the magma of the nonconvex SOC is able to preserve many key properties and features of the algebra of the convex SOC, especially when it comes to differentiation.

The paper is organized as follows. Section 2 formally introduces the nonconvex SOC and motivates the study of this cone. In Section 3, we develop algebraic notions and concepts associated with the cone. In Section 4, we prove some fundamental properties of the cone and its ambient space. In Section 5, we present with proofs some spectral properties and further important characteristics of the quadratic operators associated with the cone. Section 6 is devoted to introducing the logarithmic barrier function associated with the cone, computing its derivatives, and connecting this to a class of optimization problems over nonconvex SOCs. Finally, a conclusion is drawn in Section 7, recapitulating our essential results and leaving some open questions for future research.

2 Motivation

The motivation of this paper stems from its importance in mathematical optimization, but applications in other fields of mathematics and probably in the theory of relativity and modern physics could also be possible.

In optimization, logarithmic barrier interior-point methods for nonconvex programming have recently been the subject of significant studies by the optimization community. For example, the authors in [9,13,16,27,31,32] show their results on standard logarithmic barriers applied to general nonconvex nonlinear programs when the first and second derivatives of the objective and constraint functions are available. There are also successful attempts to extend the concept of self-concordance locally to classes of nonconvex programming [18].

In convex optimization [22], interior-point methods have been extended to general convex programs by using a convex conic reformulation of convex programming. For example, it is known that convex quadratic programs and quadratically constrained convex quadratic programs can be reformulated as optimization problems over the well-known (convex) SOC (also called the Lorentz cone, quadratic cone, or ice-cream cone) [6, 15]

$$\mathcal{E}_+^{n+1} := \left\{ \boldsymbol{x} = \begin{bmatrix} x_1 \\ \bar{\boldsymbol{x}} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n : x_1 \geq \|\bar{\boldsymbol{x}}\| \right\}, \tag{1}$$

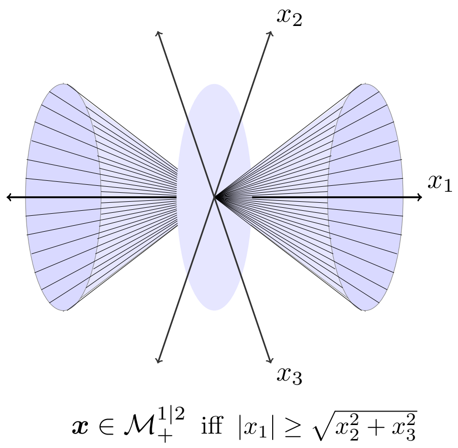
where $\|\cdot\|$ denotes the Euclidean norm.

When passing again to nonconvex extensions, it is then natural to think about a nonconvex conic reformulation of nonconvex programs. For example, it is interesting to study a conic formulation that can be applied to classes of nonconvex programming, such as nonconvex quadratic programs and quadratically constrained nonconvex quadratic programs, and whose corresponding first and second barrier derivatives can be calculated.

We give our definition of the proposed nonconvex cone based on the simple and well-known polarization identity:

$$\langle \boldsymbol{y}, \boldsymbol{z} \rangle = \frac{\|\boldsymbol{y} + \boldsymbol{z}\|^2 - \|\boldsymbol{y} - \boldsymbol{z}\|^2}{4},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. For example, if G be a symmetric matrix, the (generally nonconvex) quadratic constraint $\langle \boldsymbol{y}, G\boldsymbol{y} \rangle \leq 0$ can be reformulated as $\|(I - G)\boldsymbol{y}\| \geq \|(I + G)\boldsymbol{y}\|$.



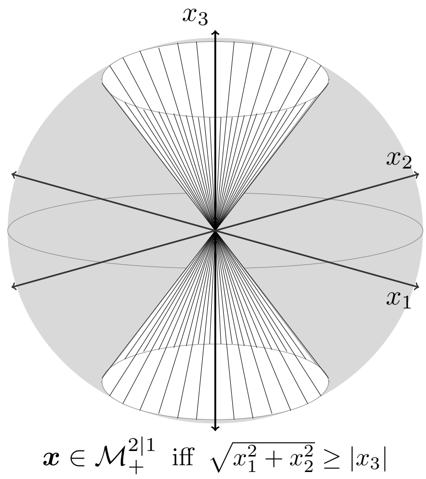


Fig. 1: Graphs of the nonconvex SOCs $\mathcal{M}_+^{1|2}$ and $\mathcal{M}_+^{2|1}$. Clearly, the complement of $\mathcal{M}_+^{1|2}$ in $\mathbb{R}^3$ is the interior of $\mathcal{M}_+^{2|1}$ if x_1 and x_3 are interchanged.

We define the $(m+n)$th-dimensional nonconvex SOC (or the extended causality cone) as

$$\mathcal{M}_+^{m|n} := \left\{ \boldsymbol{x} = \begin{bmatrix} \hat{\boldsymbol{x}} \\ \bar{\boldsymbol{x}} \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : \|\hat{\boldsymbol{x}}\| \geq \|\bar{\boldsymbol{x}}\| \right\}. \tag{2}$$

Figure 1 shows the graphs of $\mathcal{M}_+^{1|2}$ and $\mathcal{M}_+^{2|1}$. Note that $\mathcal{M}_+^{m|n}$ is not contained in any closed half-space. The sets $\operatorname{bd} \mathcal{M}_+^{m|n} := \{\boldsymbol{x} \in \mathcal{M}^{m|n} : \|\hat{\boldsymbol{x}}\| = \|\bar{\boldsymbol{x}}\|\}$ and $\operatorname{int} \mathcal{M}_+^{m|n} := \{\boldsymbol{x} \in \mathcal{M}^{m|n} : \|\hat{\boldsymbol{x}}\| > \|\bar{\boldsymbol{x}}\|\}$ represent the boundary and interior of the nonconvex SOC, respectively.

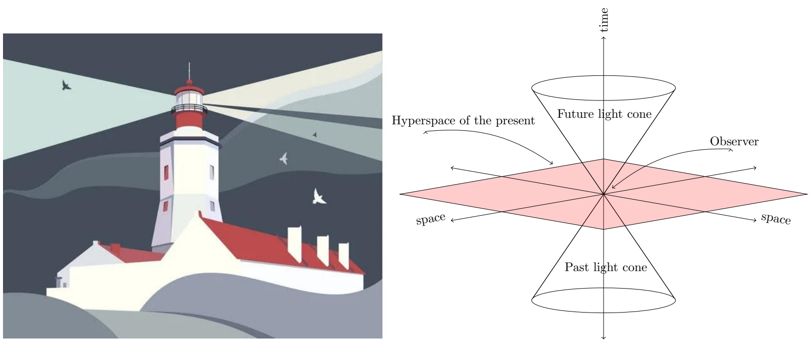
The $(m+n)$th-dimensional nonconvex SOC $\mathcal{M}_+^{m|n}$ includes the Einstein-Minkowski causality cone $\mathcal{M}_+^{1|3}$ as a special case (which occurs when $m = 1$ and $n = 3$). In a broader definition of the causality cone, any $n > 1$ is allowed. In the astrophysics sense, any vector $\boldsymbol{x} \in \operatorname{int} \mathcal{M}_+^{1|n}$ is called a timelike vector, any vector $\boldsymbol{x} \in \operatorname{bd} \mathcal{M}_+^{1|n}$ is called a lightlike vector (also called a null vector), and any vector $\boldsymbol{x} \notin \mathcal{M}_+^{1|n}$ is called a spacelike vector. A vector is called a causal vector if it is not spacelike. Likewise, we call $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ (resp., $\boldsymbol{x} \in \operatorname{int} \mathcal{M}_+^{m|n}, \boldsymbol{x} \in \operatorname{bd} \mathcal{M}_+^{m|n}$, and $\boldsymbol{x} \notin \mathcal{M}_+^{m|n}$) a causal (resp., timelike, lightlike, and spacelike) vector in $\mathcal{M}^{m|n}$.

The nonconvex SOC $\mathcal{M}_+^{1|n}$ consists of all causal vectors forming the causality cone. The sets $\operatorname{bd} \mathcal{M}_+^{1|n}$ and $\operatorname{int} \mathcal{M}_+^{1|n}$ represent the light cone and time cone, respectively. The causality cone is easier to visualize with n reduced from three to two (as we show its boundary in the right-hand side picture in Figure 2), but in reality the spatial dimension of the Einstein-Minkowski causality cone equals three and its time dimension equals one. The light in the causality (resp., light) cone generates an expanding spherical ball (resp., sphere) in $\mathbb{R}^3$ rather than a circular disk (resp., circle) in $\mathbb{R}^2$, and hence the light cone is a 4th-dimensional version of the right-hand side picture in Figure 2. This is exactly analogous to the cone drawn by the light rays from a lighthouse in which the rays generate an expanding two-dimensional circular disk in $\mathbb{R}^2$ as shown in the left-hand side picture in Figure 2.

The $(n+1)$st-dimensional nonconvex SOC $\mathcal{M}_+^{1|n}$ is the union of the $(n+1)$st-dimensional convex SOC $\mathcal{E}_+^{n+1}$ (the future-pointing causality cone) and the $(n+1)$st-dimensional convex SOC $-\mathcal{E}_+^{n+1}$ (the past-pointing causality cone). That is, $\mathcal{M}_+^{1|n} := \{\boldsymbol{x} = (x_1, \bar{\boldsymbol{x}}^\mathsf{T})^\mathsf{T} \in \mathbb{R} \times \mathbb{R}^n : |x_1| \geq \|\bar{\boldsymbol{x}}\|\} = \mathcal{E}_+^{n+1} \cup -\mathcal{E}_+^{n+1}$, where $\mathcal{E}_+^{n+1}$ is defined in (1).

We emphasize that $\mathcal{M}_+^{m|n}$ generalizes both $\mathcal{M}_+^{1|n}$ and $\mathcal{E}_+^{n+1}$, and that the nonconvex SOC $\mathcal{M}_+^{m|n}$ reduces to the convex SOC $\mathcal{E}_+^{n+1}$ when both $m = 1$ and $x_1 \geq 0$, or equivalently, when $\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|$ is nothing but the number one. Imposing the linear constraint $x_1 \geq 0$ on $\mathcal{M}_+^{1|n}$ avoids the construction of a nonconvex double cone, but this is not the case for $\mathcal{M}_+^{m|n}$ when $m \geq 2$. That is, the cone $\mathcal{M}_+^{m\geq 2|n}$ is generally nonconvex even if the linear inequality $\hat{\boldsymbol{x}} \geq \boldsymbol{0}$ is imposed. To visualize this, see, for example the right-hand side picture in Figure 1.

To see the conicity of the body $\mathcal{M}_+^{m|n}$, note that $\theta\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ for every $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ and $\theta \geq 0$, hence $\mathcal{M}_+^{m|n}$ is a cone. To see the nonconvexity of the cone $\mathcal{M}_+^{m|n}$, note that $\boldsymbol{u} = (1,0,1,0)^\mathsf{T}, \boldsymbol{v} = (0,1,1,0)^\mathsf{T} \in \mathcal{M}_+^{2|2}$, but $0.8\boldsymbol{u} + 0.2\boldsymbol{v} = (0.8, 0.2, 1, 0)^\mathsf{T} \notin \mathcal{M}_+^{2|2}$ as $((0.8)^2 + (0.2)^2)^{1/2} = 0.68 \not\geq 1 = (1^2 + 0^2)^{1/2}$. This paper underlines that the cone $\mathcal{M}_+^{m|n}$ is the simplest and most important nontrivial nonpolyhedral nonconvex cone whose analytical geometry and algebraic structure can be understood.



-The light rays from the lighthouse draw the cone $\mathcal{M}_+^{1|2}$. –The light cone, bd $\mathcal{M}_+^{1|3}$, is a 4D version of this picture.

Fig. 2: The causality cone $\mathcal{M}_+^{1|n=3}$ is easier to visualize with n reduced from 3 to 2. The picture of the lighthouse on the left is from *https://www.everypixel.com*.

Table 1: Comparison of some features between the convex and nonconvex SOCs.

Features	Convex SOC $\mathcal{E}_+^{n+1}$	Nonconvex SOC $\mathcal{M}_+^{m\|n}$
Closedness	✓	✓
Solidity	✓	✓
Pointedness	✓	N/A
Self-duality	✓	✗
Homogeneity	✓	OQA

Table 1[1] compares some features between the convex and nonconvex SOCs.

The nonconvex SOCs arise in some real-world applications, such as facility location problems when some existing facilities are more likely to be closer to the new facility (or new facilities) than other existing facilities. Mathematically speaking, if we need to add a new facility, say $\boldsymbol{x} \in \mathbb{R}^n$, among $r+s$ existing facilities, say $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_r, \tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2, \ldots, \tilde{\boldsymbol{a}}_s \in \mathbb{R}^n$, in such a way that the Euclidean distance between $\boldsymbol{x}$ and each of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_r$ is required to be smaller than or equal to the Euclidean distance between $\boldsymbol{x}$ and each of $\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2, \ldots, \tilde{\boldsymbol{a}}_s$. In this case, we need to add to the optimization model a constraint like this: $(\boldsymbol{x}^\mathsf{T} - \tilde{\boldsymbol{a}}_j^\mathsf{T}, \boldsymbol{x}^\mathsf{T} - \boldsymbol{a}_i^\mathsf{T})^\mathsf{T} \in \mathcal{M}_+^{n|n}$, for $i = 1, 2, \ldots, r$, and $j = 1, 2, \ldots, s$.

We end this section by introducing some notations that will be applied in the sequel.

2.1 Notations

Scalars will always be denoted by lower case characters such as x, vectors will always be denoted by lower case boldface characters such as $\boldsymbol{x}$, and matrices will always be denoted by upper case characters such as X. We denote by $\mathbf{0}$ a zero vector of appropriate dimension, and denote by O and I zero and identity matrices of appropriate sizes, respectively. Dimensions and sizes are known from the context unless it is necessary to be given. For instance, I_n denotes the identity matrix of order n. We use "," for adjoining vectors and matrices in a row, and we use ";" for adjoining them in a column. Therefore, for column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we have $(\boldsymbol{x}^\mathsf{T}, \boldsymbol{y}^\mathsf{T})^\mathsf{T} = (\boldsymbol{x}; \boldsymbol{y})$.

Throughout this paper, unless otherwise stated, we let m and n be arbitrary positive integers. For each vector $\boldsymbol{x} \in \mathbb{R}^{m+n}$, we denote by $\hat{\boldsymbol{x}}$ the sub-vector starting from the first entry to the mth entry, and denote by $\bar{\boldsymbol{x}}$ the sub-vector of $\boldsymbol{x}$ starting from the $(m+1)$st entry to the $(m+n)$th entry; therefore, $\boldsymbol{x} = (\hat{\boldsymbol{x}}; \bar{\boldsymbol{x}}) \in \mathbb{R}^m \times \mathbb{R}^n$.

[1] In Table 1, N/A means Not Applicable, H/E means However, and OQA means Open Question Argument.

Table 2: Notations in spaces under consideration.

Space	$\mathcal{E}^{n+1}$	$\mathcal{M}^{m\|n}$
Vector	$\boldsymbol{x} = (x_1; \bar{\boldsymbol{x}})$	$\boldsymbol{x} = (\hat{\boldsymbol{x}}; \bar{\boldsymbol{x}})$
First sub-vector	$x_1 \in \mathbb{R}$	$\hat{\boldsymbol{x}} \in \mathbb{R}^m$
Second sub-vector	$\bar{\boldsymbol{x}} \in \mathbb{R}^n$	$\bar{\boldsymbol{x}} \in \mathbb{R}^n$
Normalized first sub-vector	$\text{sgn}(x_1)$	$\hat{\boldsymbol{x}}/\left\|\hat{\boldsymbol{x}}\right\|$
Normalized second sub-vector	$\bar{\boldsymbol{x}}/\left\|\bar{\boldsymbol{x}}\right\|$	$\bar{\boldsymbol{x}}/\left\|\bar{\boldsymbol{x}}\right\|$

By $\mathcal{M}^{m|n}$, we mean the $(m+n)$th-dimensional real vector space $\mathbb{R}^m \times \mathbb{R}^n$ equipped with a standard inner product. That is,

$$\mathcal{M}^{m|n} := \{\boldsymbol{x} = (\hat{\boldsymbol{x}}; \bar{\boldsymbol{x}}) : \hat{\boldsymbol{x}} \in \mathbb{R}^m, \bar{\boldsymbol{x}} \in \mathbb{R}^n\}.$$

For basic literature review purposes, we will also deal with the following space

$$\mathcal{E}^{n+1} := \{\boldsymbol{x} = (x_1; \bar{\boldsymbol{x}}) : x_1 \in \mathbb{R}, \bar{\boldsymbol{x}} \in \mathbb{R}^n\} = \mathcal{M}^{1|n}.$$

We define the reflection matrix $R^{m|n}$ as

$$R^{m|n} := \begin{bmatrix} I_m & O \\ O & -I_n \end{bmatrix}, \tag{3}$$

to be a generalization of the traditional reflection matrix:

$$R^{1|n} := \begin{bmatrix} 1 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & -I_n \end{bmatrix}. \tag{4}$$

Let $\boldsymbol{x} \in \mathcal{M}^{m|n}$. If $\hat{\boldsymbol{x}} = \mathbf{0}$, the vector $\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|$ is considered to be any vector in $\mathbb{R}^m$ of Euclidean norm one. Similarly, if $\bar{\boldsymbol{x}} = \mathbf{0}$, the vector $\bar{\boldsymbol{x}}/\|\bar{\boldsymbol{x}}\|$ is considered to be any vector in $\mathbb{R}^n$ of Euclidean norm one.

Table 2 shows some notations in spaces under consideration. In this table, $\text{sgn}(\cdot)$ is the sign function that extracts the sign of a real number. If $x \in \mathbb{R}$, we assume that $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$.

3 The Algebraic Structure of the Ambient Space

In this section, we extend the notions, concepts, and results that exist in [1, Section 4] in the framework of the convex SOC to the framework of the nonconvex SOC. We also present several fundamental results to form and completely understand the algebraic structure of the nonconvex SOC.

Before starting this section, it must be noted that the algebraic structure of the convex cone $\mathcal{E}_+^{n+1}$ is strong enough to be extremely rigid because it is linked to solid mathematical abstractions such as Euclidean Jordan algebras. While such linkage is not claimed for the algebraic structure of $\mathcal{M}_+^{m|n}$, which is the reason why we have not reviewed those algebras in this part, we will see that the algebraic structure of the nonconvex cone $\mathcal{M}_+^{m|n}$ is still rigid because it preserves many key properties of that of $\mathcal{E}_+^{n+1}$, especially when it comes to differentiation.

The nonconvex SOC (2) can be redefined as

$$\mathcal{M}_+^{m|n} := \left\{ \boldsymbol{x} \in \mathcal{M}^{m|n} : \boldsymbol{x}^{\mathsf{T}} R^{m|n} \boldsymbol{x} \geq 0 \right\},$$

where $R^{m|n}$ is the reflection matrix defined in (3).

The characteristic polynomial of $\boldsymbol{x} \in \mathcal{M}^{m|n}$ is given by the quadratic equation

$$\begin{aligned} p^{m|n}(\lambda, \boldsymbol{x}) :=& \begin{bmatrix} (\lambda - \|\hat{\boldsymbol{x}}\|) \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ -\bar{\boldsymbol{x}} \end{bmatrix}^{\mathsf{T}} R^{m|n} \begin{bmatrix} (\lambda - \|\hat{\boldsymbol{x}}\|) \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ -\bar{\boldsymbol{x}} \end{bmatrix} \\ =& \lambda^2 - 2\|\hat{\boldsymbol{x}}\|\lambda + \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right). \end{aligned}$$

The two solutions of the equation $p^{m|n}(\lambda, \boldsymbol{x}) = 0$ (see also (9)), which are $\lambda_{1,2}(\boldsymbol{x}) := \|\hat{\boldsymbol{x}}\| \pm \|\bar{\boldsymbol{x}}\|$, are called the eigenvalues of $\boldsymbol{x}$. It is obvious that $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ ($\boldsymbol{x} \in \text{int}\, \mathcal{M}_+^{m|n}$) if and only if $\lambda_{1,2}(\boldsymbol{x}) \geq 0\, (\lambda_{1,2}(\boldsymbol{x}) > 0)$. We call $\boldsymbol{x}$ positive semidefinite (positive definite) if $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ ($\boldsymbol{x} \in \text{int}\, \mathcal{M}_+^{m|n}$).

For $\boldsymbol{x} \in \mathcal{M}^{m|n}$, we call the values

$$\begin{aligned} \text{trace}(\boldsymbol{x}) &:= \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) = 2\|\hat{\boldsymbol{x}}\|, \\ \det(\boldsymbol{x}) &:= \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}) \quad = \|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2, \end{aligned}$$

the trace and determinant of $\boldsymbol{x}$, respectively.

Let $\boldsymbol{x} \in \mathcal{M}^{m|n}$. We denote an identity-like element in the space $\mathcal{M}^{m|n}$ by $\boldsymbol{e}(\boldsymbol{x})$, and define this element as

$$\boldsymbol{e}(\boldsymbol{x}) := \boldsymbol{c}_1(\boldsymbol{x}) + c_1(\boldsymbol{x}) = \frac{1}{2} \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ -\dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix},$$

where $\boldsymbol{c}_{1,2}(\boldsymbol{x})$ are the eigenvectors of $\boldsymbol{x}$. Thus, we have

$$\boldsymbol{e}(\boldsymbol{x}) := \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \boldsymbol{0} \end{bmatrix}.$$

Note that $\boldsymbol{e}(\boldsymbol{x})$ depends on the choice of $\boldsymbol{x} \in \mathcal{M}^{m|n}$, and hence it is not unique in $\mathcal{M}^{m|n}$. We emphasize that the space $\mathcal{M}^{m|n}$ has multiple identity-like elements because it will be seen that $\boldsymbol{e}(\boldsymbol{x})$ is neutral with the powers of $\boldsymbol{x} \in \mathcal{M}^{m|n}$ only. Note also that $\text{trace}(\boldsymbol{e}(\boldsymbol{x})) = 2$ and $\det(\boldsymbol{e}(\boldsymbol{x})) = 1$ (since all the eigenvalues of $\boldsymbol{e}(\boldsymbol{x})$ are equal to one) for any $\boldsymbol{x} \in \mathcal{M}^{m|n}$.

$$\begin{bmatrix} \star & & & & & \star & \star & \star & \star & \star \\ & \star & & & & \star & \star & \star & \star & \star \\ & & \star & & & \star & \star & \star & \star & \star \\ & & & \star & & \star & \star & \star & \star & \star \\ & & & & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & & & & \\ \star & \star & \star & \star & \star & & \star & & & \\ \star & \star & \star & \star & \star & & & \star & & \\ \star & \star & \star & \star & \star & & & & \star & \\ \star & \star & \star & \star & \star & & & & & \star \end{bmatrix}$$

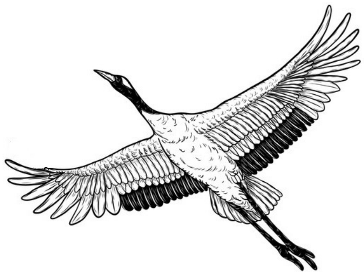


Fig. 3: We have found the matrix in $\mathcal{M}^{m|n}$ that generalizes the arrow-shaped matrix in $\mathcal{E}^{n+1}$ looks like a flying crane. Therefore, we believe that it is appropriate to name this matrix 'the crane-shaped matrix', denoted as $\mathsf{Crn}(\cdot)$. The matrix $\mathsf{Crn}(\star)$ shown on the left is associated with vectors in $\mathcal{M}^{5|5}$. The picture on the right is from *https://www.freepik.com.*

Associated with each vector $\boldsymbol{x} \in \mathcal{M}^{m|n} / \mathrm{bd}\, \mathcal{M}_+^{m|n}$, we define the element

$$\boldsymbol{x}^{\mathrm{g}} := \tfrac{1}{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})} \left(\lambda_2(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_1(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) = \tfrac{1}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2} \begin{bmatrix} \hat{\boldsymbol{x}} \\ -\bar{\boldsymbol{x}} \end{bmatrix} = \frac{R^{m|n}}{\det(\boldsymbol{x})} \boldsymbol{x}. \tag{5}$$

Note that the vector $\boldsymbol{x}^{\mathrm{g}}$ is not defined when $\boldsymbol{x} \in \mathrm{bd}\, \mathcal{M}_+^{m|n}$ (i.e., $\det(\boldsymbol{x}) = 0$), and it acts as a generalized inverse for each $\boldsymbol{x} \in \mathrm{int}\, \mathcal{M}_+^{m|n}$ as it will be seen shortly. In the astrophysics sense, all timelike vectors have generalized inverses and all lightlike vectors are singular.

Associated with each vector $\boldsymbol{x} \in \mathcal{M}^{m|n}$, we define a crane-shaped matrix, $\mathsf{Crn}(\boldsymbol{x})$ (see Figure 3), so that $\mathsf{Crn}(\boldsymbol{x})\, \boldsymbol{x}^{\mathrm{g}} = \boldsymbol{e}(\boldsymbol{x})$. Let $\boldsymbol{x} \in \mathcal{M}^{m|n}$, the crane-shaped matrix is defined as

$$\begin{aligned} \mathsf{Crn}(\boldsymbol{x}) &\triangleq \boldsymbol{x}\, \boldsymbol{e}^{\mathsf{T}}(\boldsymbol{x}) + \boldsymbol{e}(\boldsymbol{x})\, \boldsymbol{x}^{\mathsf{T}} - \frac{\mathrm{trace}(\boldsymbol{x})}{2} \left(2\boldsymbol{e}(\boldsymbol{x})\boldsymbol{e}^{\mathsf{T}}(\boldsymbol{x}) - I_{m+n}\right) \\ &= \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}^{\mathsf{T}} & O \\ \bar{\boldsymbol{x}} \left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^{\mathsf{T}} & O \end{bmatrix} + \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}^{\mathsf{T}} & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ O & O \end{bmatrix} - \|\hat{\boldsymbol{x}}\| \begin{bmatrix} 2\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^{\mathsf{T}} - I_m & O \\ O & -I_n \end{bmatrix}. \end{aligned}$$

As a result, we have

$$\mathsf{Crn}(\boldsymbol{x}) := \begin{bmatrix} \|\hat{\boldsymbol{x}}\|\, I_m & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}} \left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix}.$$

Note that the matrix $\mathsf{Crn}(\boldsymbol{x})$ is symmetric, and that $\mathsf{Crn}(\boldsymbol{e}(\boldsymbol{x})) = I_{m+n}$ and $\mathsf{Crn}(\boldsymbol{x})\, \boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{x}$.

The quadratic representation $\mathsf{P}_{\boldsymbol{x}}$ of each vector $\boldsymbol{x} \in \mathcal{M}^{m|n}$ is defined so that $\mathsf{P}_{\boldsymbol{x}}\boldsymbol{x}^{\mathrm{g}} = \boldsymbol{x}$. This quadratic representation is defined as

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x}} &:= 2\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}} - \det(\boldsymbol{x})R^{m|n} \\
&= 2\begin{bmatrix} \hat{\boldsymbol{x}} \\ \bar{\boldsymbol{x}} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}}^{\mathsf{T}} & \bar{\boldsymbol{x}}^{\mathsf{T}} \end{bmatrix} - \det(\boldsymbol{x})R^{m|n} \\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \end{bmatrix} - \begin{bmatrix} \det(\boldsymbol{x})I_m & O \\ O & -\det(\boldsymbol{x})I_n \end{bmatrix} \\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \det(\boldsymbol{x})I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix} \\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix}.
\end{aligned}$$

Note that the matrix $\mathsf{P}_{\boldsymbol{x}}$ is symmetric. The map $\mathsf{P}_{\cdot} : \mathcal{M}^{m|n} \longrightarrow \mathcal{M}^{m|n}$ is extended to another map $\mathsf{P}_{\cdot,\cdot} : \mathcal{M}^{m|n} \times \mathcal{M}^{m|n} \longrightarrow \mathcal{M}^{m|n}$ that is also symmetric, meets the requirement $\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}} = \mathsf{P}_{\boldsymbol{x}}$, and is defined as

$$\mathsf{P}_{\boldsymbol{x},\boldsymbol{y}} := \frac{1}{2}\left(\mathsf{P}_{\boldsymbol{x}+\boldsymbol{y}} - \mathsf{P}_{\boldsymbol{x}} - \mathsf{P}_{\boldsymbol{y}}\right), \text{ for } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}^{m|n}. \tag{6}$$

Note that

$$\begin{aligned}
&\mathsf{P}_{\boldsymbol{x}+\boldsymbol{y}} \\
&= \begin{bmatrix} 2\left(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}^{\mathsf{T}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}^{\mathsf{T}}\right) + \left(\|\bar{\boldsymbol{x}} + \bar{\boldsymbol{y}}\|^2 - \|\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\|^2\right) I_m & 2\left(\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \hat{\boldsymbol{x}}\bar{\boldsymbol{y}}^{\mathsf{T}} + \hat{\boldsymbol{y}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \hat{\boldsymbol{y}}\bar{\boldsymbol{y}}^{\mathsf{T}}\right) \\ 2\left(\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \bar{\boldsymbol{x}}\hat{\boldsymbol{y}}^{\mathsf{T}} + \bar{\boldsymbol{y}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \bar{\boldsymbol{y}}\hat{\boldsymbol{y}}^{\mathsf{T}}\right) & 2\left(\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \bar{\boldsymbol{x}}\bar{\boldsymbol{y}}^{\mathsf{T}} + \bar{\boldsymbol{y}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \bar{\boldsymbol{y}}\bar{\boldsymbol{y}}^{\mathsf{T}}\right) + \left(\|\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{x}} + \bar{\boldsymbol{y}}\|^2\right) I_n \end{bmatrix}, \\
&\mathsf{P}_{\boldsymbol{x}} = \begin{bmatrix} -\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_n + 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \end{bmatrix}, \\
&\mathsf{P}_{\boldsymbol{y}} = \begin{bmatrix} -\left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) I_m + 2\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}^{\mathsf{T}} & 2\hat{\boldsymbol{y}}\bar{\boldsymbol{y}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{y}}\hat{\boldsymbol{y}}^{\mathsf{T}} & \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) I_n + 2\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}^{\mathsf{T}} \end{bmatrix}.
\end{aligned}$$

From (6), it follows that

$$\mathsf{P}_{\boldsymbol{x},\boldsymbol{y}} := \begin{bmatrix} \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}^{\mathsf{T}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \left(\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}} - \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\right) I_m & \hat{\boldsymbol{x}}\bar{\boldsymbol{y}}^{\mathsf{T}} + \hat{\boldsymbol{y}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}}\hat{\boldsymbol{y}}^{\mathsf{T}} + \bar{\boldsymbol{y}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \bar{\boldsymbol{x}}\bar{\boldsymbol{y}}^{\mathsf{T}} + \bar{\boldsymbol{y}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \left(\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}} - \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\right) I_n \end{bmatrix}.$$

We define the square of $\boldsymbol{x} \in \mathcal{M}^{m|n}$ as

$$\boldsymbol{x}^2 := \mathsf{Crn}(\boldsymbol{x})\,\boldsymbol{x} = \begin{bmatrix} \|\hat{\boldsymbol{x}}\|\, I_m & \dfrac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \dfrac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \bar{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}} + \dfrac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} \\ 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}} \end{bmatrix}.$$

As a result, we have

$$\boldsymbol{x}^2 := \begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{\mathsf{T}}\boldsymbol{x}\,\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}} \end{bmatrix}. \tag{7}$$

It can be seen that $\mathsf{P}_{\boldsymbol{x}}\boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{x}^2$ (see Theorem 5.2). The cone of squares of the set $\mathcal{M}^{m|n}$ is denoted by $\mathcal{S}(\mathcal{M}^{m|n})$ and is defined as

$$\mathcal{S}(\mathcal{M}^{m|n}) := \left\{\boldsymbol{x}^2 : \boldsymbol{x} \in \mathcal{M}^{m|n}\right\}.$$

Similarly, $\mathcal{S}(\mathcal{E}^{n+1}) := \left\{\boldsymbol{x}^2 : \boldsymbol{x} \in \mathcal{E}^{n+1}\right\}$ is the cone of squares of $\mathcal{E}^{n+1}$. The result in the following lemma connects the convex SOC with the algebra $(\mathcal{E}^{n+1}, \circ)$. The proof of this lemma can be found in [1, Section 4].

Lemma 3.1 $\mathcal{S}(\mathcal{E}^{n+1}) = \mathcal{E}^{n+1}_{+}$.

It is now natural and important to ask whether $\mathcal{S}(\mathcal{M}^{m|n}) = \mathcal{M}^{m|n}_{+}$ or not. This is a central question and it is very important to emphasize, because if we can answer in the affirmative, we add a feature to the vector space $\mathcal{M}^{m|n}$ that does not exist in all non-Euclidean Jordan algebras. In fact, it does not seem easy to prove that $\mathcal{M}^{m|n}_{+} = \mathcal{S}(\mathcal{M}^{m|n})$ by generalizing the proof idea for Lemma 3.1. Apparently, to prove the desired result differently, there would be a need to introduce more tools for the proof.

The multiplication in (7) can be extended to a product $\circledcirc : \mathcal{M}^{m|n} \times \mathcal{M}^{m|n} \to \mathcal{M}^{m|n}$ defined as

$$\begin{aligned} \boldsymbol{x} \circledcirc \boldsymbol{y} :=& \frac{1}{2}\left(\mathsf{Crn}(\boldsymbol{x})\boldsymbol{y} + \mathsf{Crn}(\boldsymbol{y})\boldsymbol{x}\right) \\ =& \frac{1}{2}\left(\begin{bmatrix} \|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{y}} + \dfrac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}} \\ \dfrac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}} + \|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{y}} \end{bmatrix} + \begin{bmatrix} \|\hat{\boldsymbol{y}}\|\,\hat{\boldsymbol{x}} + \dfrac{1}{\|\hat{\boldsymbol{y}}\|}\hat{\boldsymbol{y}}\bar{\boldsymbol{y}}^{\mathsf{T}}\bar{\boldsymbol{x}} \\ \dfrac{1}{\|\hat{\boldsymbol{y}}\|}\bar{\boldsymbol{y}}\hat{\boldsymbol{y}}^{\mathsf{T}}\hat{\boldsymbol{x}} + \|\hat{\boldsymbol{y}}\|\,\bar{\boldsymbol{x}} \end{bmatrix}\right), \end{aligned}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}^{m|n}$. As a result, we have

$$\boldsymbol{x} \circledcirc \boldsymbol{y} := \frac{1}{2}\begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|\,\|\hat{\boldsymbol{y}}\| + \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\right)\left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} + \dfrac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|}\right) \\ \left(\|\hat{\boldsymbol{x}}\| + \hat{\boldsymbol{x}}^{\mathsf{T}}\dfrac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|}\right)\bar{\boldsymbol{y}} + \left(\|\hat{\boldsymbol{y}}\| + \hat{\boldsymbol{y}}^{\mathsf{T}}\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)\bar{\boldsymbol{x}} \end{bmatrix}. \tag{8}$$

The map trace$(\boldsymbol{x} \circledcirc \boldsymbol{y})$ is a positive definite, symmetric, but it is not an inner product in the usual sense because it is not bilinear; the formula in (8) involves square roots. Consequently, the product "$\circledcirc$" is not bilinear, and therefore it does not constitute an algebra with $\mathcal{M}^{m|n}$. A magma generalizes the notion of algebra; it consists of a set equipped with a single binary operation that must be closed by definition (so, the bilinearity or any other properties are not imposed). This term was introduced in 1970 by Bourbaki (see the last edition, [10]), who defined a magma as a set having a composition law. Our results confirm that the structure $(\mathcal{M}^{m|n}, \circledcirc)$ is a commutative power-associative magma whose elements always have real eigenvalues; this is not the case for arbitrary commutative power-associative algebras or even arbitrary Jordan algebras.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}^{m|n}$. One can verify that the product "$\circledcirc$" satisfies the following properties

$\boldsymbol{x}^2 = \boldsymbol{x} \circledcirc \boldsymbol{x}$,
$\boldsymbol{x} \circledcirc \boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{e}(\boldsymbol{x}) \circledcirc \boldsymbol{x} = \boldsymbol{x}$ (unitary-like of the submagma generated by $\boldsymbol{x}$),
$\boldsymbol{x} \circledcirc \boldsymbol{x}^{\mathrm{g}} = \boldsymbol{x}^{\mathrm{g}} \circledcirc \boldsymbol{x} = \boldsymbol{e}(\boldsymbol{x})$ provided that $\det(\boldsymbol{x}) > 0$ (generalized invertibility in the submagma generated by$\boldsymbol{x} \in \operatorname{int} \mathcal{M}_+^{m|n}$),
$\boldsymbol{x} \circledcirc (\boldsymbol{x}^{\mathrm{g}} \circledcirc \boldsymbol{x}) = (\boldsymbol{x} \circledcirc \boldsymbol{x}^{\mathrm{g}}) \circledcirc \boldsymbol{x} = \boldsymbol{x}$ provided that $\det(\boldsymbol{x}) > 0$ (generalized invertibility in $\operatorname{int} \mathcal{M}_+^{m|n}$),
$\boldsymbol{x}^{\mathrm{g}} \circledcirc (\boldsymbol{x} \circledcirc \boldsymbol{x}^{\mathrm{g}}) = \boldsymbol{x}^{\mathrm{g}}$, provided that $\det(\boldsymbol{x}) > 0$,
$\boldsymbol{x} \circledcirc \boldsymbol{y} = \boldsymbol{y} \circledcirc \boldsymbol{x}$ (commutativity).

We also define the positive power $\boldsymbol{x}^p$ recursively as $\boldsymbol{x}^p := \boldsymbol{x} \circledcirc \boldsymbol{x}^{p-1}$ for $p \geq 2$.

We can easily verify that every vector $\boldsymbol{x} \in (\mathcal{M}^{m|n}, \circledcirc)$ satisfies the quadratic equation

$$\boldsymbol{x}^2 - 2\,\|\hat{\boldsymbol{x}}\|\,\boldsymbol{x} + \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) \boldsymbol{e}(\boldsymbol{x}) = \mathbf{0}. \tag{9}$$

As a result, $\operatorname{rk}(\mathcal{M}^{m|n}) = 2$ is called the rank of the magma $(\mathcal{M}^{m|n}, \circledcirc)$. It is important to note that the value $\operatorname{rk}(\mathcal{M}^{m|n})$ is independent of any of m and n, i.e., the nonconvex SOC $\mathcal{M}_+^{m|n}$ is rankly independent (or rankly free) of its ambient dimension; this is not the case for algebras of arbitrary convex cones or even arbitrary convex symmetric (i.e., self-dual and homogeneous) cones (see Table 5).

The spectral decomposition (or spectral factorization) of $\boldsymbol{x} \in \mathcal{M}^{m|n}$ is a decomposition of $\boldsymbol{x}$ into eigenvectors (say $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$) together with its eigenvalues so that $\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})$. From (5), we have $\boldsymbol{x} = R^{m|n}(\lambda_2(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_1(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}))$. This can be expanded as

$$\boldsymbol{x} = \underbrace{(\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|)}_{\lambda_1(\boldsymbol{x})} \underbrace{\left(\frac{1}{2}\right) \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \dfrac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix}}_{\boldsymbol{c}_1(\boldsymbol{x})} + \underbrace{(\|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|)}_{\lambda_2(\boldsymbol{x})} \underbrace{\left(\frac{1}{2}\right) \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \dfrac{-\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix}}_{\boldsymbol{c}_2(\boldsymbol{x})}. \tag{10}$$

The decomposition in (10) is defined to be the spectral decomposition of $\boldsymbol{x} \in \mathcal{M}^{m|n}$. We point out that the above decomposition is different from the singular value decomposition in Minkowski space studied in [25]. The pair of eigenvectors $\{\boldsymbol{c}_1(\boldsymbol{x}), \boldsymbol{c}_2(\boldsymbol{x})\}$ in the spectral factorization (10) satisfies the properties

$$\begin{array}{l}\boldsymbol{c}_1(\boldsymbol{x}), \boldsymbol{c}_2(\boldsymbol{x}) \in \operatorname{bd} \mathcal{M}_+^{m|n},\\ \boldsymbol{c}_1(\boldsymbol{x}) \circledcirc \boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{0}, \text{ and } \langle \boldsymbol{c}_1(\boldsymbol{x}), \boldsymbol{c}_2(\boldsymbol{x})\rangle = 0,\\ \boldsymbol{c}_1(\boldsymbol{x}) \circledcirc \boldsymbol{c}_1(\boldsymbol{x}) = \boldsymbol{c}_1(\boldsymbol{x}) \text{ and } \boldsymbol{c}_2(\boldsymbol{x}) \circledcirc \boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{c}_2(\boldsymbol{x}),\\ R^{m|n}\boldsymbol{c}_1(\boldsymbol{x}) = \boldsymbol{c}_2(\boldsymbol{x}), R^{m|n}\boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{c}_1(\boldsymbol{x}), \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) = \widehat{\boldsymbol{c}_2}(\boldsymbol{x}), \text{ and } \overline{\boldsymbol{c}_1}(\boldsymbol{x}) = -\overline{\boldsymbol{c}_2}(\boldsymbol{x}),\\ \lambda_1(\boldsymbol{c}_1(\boldsymbol{x})) = \lambda_1(\boldsymbol{c}_2(\boldsymbol{x})) = 1 \text{ and } \lambda_2(\boldsymbol{c}_1(\boldsymbol{x})) = \lambda_2(\boldsymbol{c}_2(\boldsymbol{x})) = 0,\\ \|\widehat{\boldsymbol{c}_1}(\boldsymbol{x})\| = \|\widehat{\boldsymbol{c}_2}(\boldsymbol{x})\| = \|\overline{\boldsymbol{c}_1}(\boldsymbol{x})\| = \|\overline{\boldsymbol{c}_2}(\boldsymbol{x})\| = \frac{1}{2}.\end{array} \tag{11}$$

All the above identities can be rigorously checked algebraically.

Table 3 summarizes our computational findings in this subsection for the magma of the nonconvex SOC and compares them with those of the algebra of the convex SOC.

4 Fundamental Properties of the Cone

In this section, we present with proofs some fundamental properties of the cone $\mathcal{M}_+^{m|n}$ and its ambient space $(\mathcal{M}^{m|n}, \circledcirc)$.

We start by going back now to our central question that we asked earlier: Does the nonconvex SOC equal the cone of squares of its ambient space? A first, but not deep, look at this question tells us that the mere asking of this question answers it negatively. One of the reasons for that is that a surface answer could be based on a fundamental fact we already know: A set is the cone of squares of a formally real Jordan algebra if and only if it is a symmetric cone [26, Theorem 2]. Being not symmetric implies that the nonconvex SOC is not the cone of squares of its underlying magma. However, this does not apply to $(\mathcal{M}^{m|n}, \circledcirc)$ because it is not even an algebra. Another reason is that a surface answer could be also based on another fundamental fact we already know: The cone of squares is self-dual with respect to the inner product trace$(\boldsymbol{x} \cdot \boldsymbol{y})$. Being not self-dual implies that the nonconvex SOC is not the cone of squares of its underlying magma. This does not also apply to $(\mathcal{M}^{m|n}, \circledcirc)$ because in our case, the map trace$(\boldsymbol{x} \circledcirc \boldsymbol{y})$ is not an inner product. The absence of the bilinearity of the product "$\circledcirc$" does not cause any problematic effect on the proof of the following theorem.

Table 3: Comparing the algebraic notions and concepts associated with the convex SOC $\mathcal{E}_+^{n+1}$ and the nonconvex SOC $\mathcal{M}_+^{m|n}$.

Cone	Convex SOC $\mathcal{E}_+^{n+1}$	Nonconvex SOC $\mathcal{M}_+^{m\|n}$
Cone constraint, dimension, & rank	$x_1 \geq \|\bar{\boldsymbol{x}}\|,\ \dim(\mathcal{E}^{n+1}) = n+1,\ \mathrm{rk}(\mathcal{E}^{n+1}) = 2$	$\|\hat{\boldsymbol{x}}\| \geq \|\bar{\boldsymbol{x}}\|,\ \dim(\mathcal{M}^{m\|n}) = m+n,\ \mathrm{rk}(\mathcal{M}^{m\|n}) = 2$
Binary operation	$\boldsymbol{x} \circ \boldsymbol{y} := \begin{bmatrix} x_1 y_1 + \bar{\boldsymbol{y}}^\mathsf{T} \bar{\boldsymbol{x}} \\ x_1 \bar{\boldsymbol{y}} + y_1 \bar{\boldsymbol{x}} \end{bmatrix}$	$\boldsymbol{x} \circledcirc \boldsymbol{y} := \frac{1}{2} \begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\| \|\hat{\boldsymbol{y}}\| + \bar{\boldsymbol{x}}^\mathsf{T} \bar{\boldsymbol{y}}\right) \left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} + \frac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|} \right) \\ \left(\|\hat{\boldsymbol{x}}\| + \hat{\boldsymbol{x}}^\mathsf{T} \frac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|} \right) \bar{\boldsymbol{y}} + \left(\|\hat{\boldsymbol{y}}\| + \hat{\boldsymbol{y}}^\mathsf{T} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \right) \bar{\boldsymbol{x}} \end{bmatrix}$
Algebraic structure	The Euclidean Jordan algebra $(\mathcal{E}^{n+1}, \circ)$	The magma $(\mathcal{M}^{m\|n}, \circledcirc)$
Reflection matrix	$R^{1\|n} := \begin{bmatrix} 1 & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & -I_n \end{bmatrix}$	$R^{m\|n} := \begin{bmatrix} I_m & O \\ O & -I_n \end{bmatrix}$
Identity(-like) element	$\boldsymbol{e} := \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix}$	$\boldsymbol{e}(\boldsymbol{x}) := \begin{bmatrix} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \boldsymbol{0} \end{bmatrix}$
Eigenvalues	$\lambda_{1,2}(\boldsymbol{x}) := x_1 \pm \|\bar{\boldsymbol{x}}\|$	$\lambda_{1,2}(\boldsymbol{x}) := \|\hat{\boldsymbol{x}}\| \pm \|\bar{\boldsymbol{x}}\|$
Eigenvectors	$\boldsymbol{c}_{1,2}(\boldsymbol{x}) := \frac{1}{2} \begin{bmatrix} 1 \\ \pm \frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix}$	$\boldsymbol{c}_{1,2}(\boldsymbol{x}) := \frac{1}{2} \begin{bmatrix} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \pm \frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix}$
Trace	$\mathrm{trace}(\boldsymbol{x}) := 2x_1$	$\mathrm{trace}(\boldsymbol{x}) := 2\,\|\hat{\boldsymbol{x}}\|$
Determinant	$\det(\boldsymbol{x}) := x_1^2 - \|\bar{\boldsymbol{x}}\|^2$	$\det(\boldsymbol{x}) := \|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2$
Spectral factorization	$\boldsymbol{x} = (x_1 + \|\bar{\boldsymbol{x}}\|) \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ \frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix} + (x_1 - \|\bar{\boldsymbol{x}}\|) \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ \frac{-\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix}$	$\boldsymbol{x} = (\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|) \left(\frac{1}{2}\right) \begin{bmatrix} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \frac{\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix} + (\|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|) \left(\frac{1}{2}\right) \begin{bmatrix} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \frac{-\bar{\boldsymbol{x}}}{\|\bar{\boldsymbol{x}}\|} \end{bmatrix}$
Square	$\boldsymbol{x}^2 := \begin{bmatrix} \boldsymbol{x}^\mathsf{T} \boldsymbol{x} \\ 2 x_1 \bar{\boldsymbol{x}} \end{bmatrix}$	$\boldsymbol{x}^2 := \begin{bmatrix} \boldsymbol{x}^\mathsf{T} \boldsymbol{x} \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ 2\,\|\hat{\boldsymbol{x}}\|\, \bar{\boldsymbol{x}} \end{bmatrix}$
(Generalized) inverse	$\boldsymbol{x}^{-1} := \frac{R^{1\|n}}{\det(\boldsymbol{x})}\, \boldsymbol{x}$	$\boldsymbol{x}^{\mathrm{g}} := \frac{R^{m\|n}}{\det(\boldsymbol{x})}\, \boldsymbol{x}$
Arrow/Crane-shaped matrix	$\mathsf{Arw}(\boldsymbol{x}) := \begin{bmatrix} x_1 & \bar{\boldsymbol{x}}^\mathsf{T} \\ \bar{\boldsymbol{x}} & x_1 I_n \end{bmatrix}$	$\mathsf{Crn}(\boldsymbol{x}) := \begin{bmatrix} \|\hat{\boldsymbol{x}}\|\, I_m & \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \bar{\boldsymbol{x}}^\mathsf{T} \\ \bar{\boldsymbol{x}} \left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \right)^\mathsf{T} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix}$
Quadratic representation matrix	$\mathsf{Q}_{\boldsymbol{x}} := \begin{bmatrix} \boldsymbol{x}^\mathsf{T} \boldsymbol{x} & 2 x_1 \bar{\boldsymbol{x}}^\mathsf{T} \\ 2 x_1 \bar{\boldsymbol{x}} & 2 \bar{\boldsymbol{x}} \bar{\boldsymbol{x}}^\mathsf{T} + \det(\boldsymbol{x}) I_n \end{bmatrix}$	$\mathsf{P}_{\boldsymbol{x}} := \begin{bmatrix} 2 \hat{\boldsymbol{x}} \hat{\boldsymbol{x}}^\mathsf{T} - \det(\boldsymbol{x}) I_m & 2 \hat{\boldsymbol{x}} \bar{\boldsymbol{x}}^\mathsf{T} \\ 2 \bar{\boldsymbol{x}} \hat{\boldsymbol{x}}^\mathsf{T} & 2 \bar{\boldsymbol{x}} \bar{\boldsymbol{x}}^\mathsf{T} + \det(x) I_n \end{bmatrix}$
Quadratic operator (2 vars)	$\mathsf{Q}_{\boldsymbol{x},\boldsymbol{y}} := \begin{bmatrix} \boldsymbol{x}^\mathsf{T} \boldsymbol{y} & x_1 \bar{\boldsymbol{y}}^\mathsf{T} + y_1 \bar{\boldsymbol{x}}^\mathsf{T} \\ x_1 \bar{\boldsymbol{y}} + y_1 \bar{\boldsymbol{x}} & \left(\bar{\boldsymbol{x}} \bar{\boldsymbol{y}}^\mathsf{T} + \bar{\boldsymbol{y}} \bar{\boldsymbol{x}}^\mathsf{T} \right) + \left(x_1 y_1 - \bar{\boldsymbol{x}}^\mathsf{T} \bar{\boldsymbol{y}} \right) I_n \end{bmatrix}$	$\mathsf{P}_{\boldsymbol{x},\boldsymbol{y}} := \begin{bmatrix} \hat{\boldsymbol{x}} \hat{\boldsymbol{y}}^\mathsf{T} + \hat{\boldsymbol{y}} \hat{\boldsymbol{x}}^\mathsf{T} - \left(\hat{\boldsymbol{x}}^\mathsf{T} \hat{\boldsymbol{y}} - \bar{\boldsymbol{x}}^\mathsf{T} \bar{\boldsymbol{y}} \right) I_m & \hat{\boldsymbol{x}} \bar{\boldsymbol{y}}^\mathsf{T} + \hat{\boldsymbol{y}} \bar{\boldsymbol{x}}^\mathsf{T} \\ \bar{\boldsymbol{x}} \hat{\boldsymbol{y}}^\mathsf{T} + \bar{\boldsymbol{y}} \hat{\boldsymbol{x}}^\mathsf{T} & \bar{\boldsymbol{x}} \bar{\boldsymbol{y}}^\mathsf{T} + \bar{\boldsymbol{y}} \bar{\boldsymbol{x}}^\mathsf{T} + \left(\hat{\boldsymbol{x}}^\mathsf{T} \hat{\boldsymbol{y}} - \bar{\boldsymbol{x}}^\mathsf{T} \bar{\boldsymbol{y}} \right) I_n \end{bmatrix}$

Theorem 4.1 $\mathcal{S}(\mathcal{M}^{m|n}) = \mathcal{M}_+^{m|n}$.

Proof Let $\boldsymbol{x} \in \mathcal{S}(\mathcal{M}^{m|n})$, then there exists $\boldsymbol{y} \in \mathcal{M}^{m|n}$ such that

$$\boldsymbol{x} = \boldsymbol{y}^2 = \begin{bmatrix} \left(\|\hat{\boldsymbol{y}}\|^2 + \|\bar{\boldsymbol{y}}\|^2\right) \dfrac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|} \\ 2\,\|\hat{\boldsymbol{y}}\|\,\bar{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \|\boldsymbol{y}\|^2 \dfrac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|} \\ 2\,\|\hat{\boldsymbol{y}}\|\,\bar{\boldsymbol{y}} \end{bmatrix}.$$

It follows that

$$\|\hat{\boldsymbol{x}}\| = \left\| \|\boldsymbol{y}\|^2 \frac{\hat{\boldsymbol{y}}}{\|\hat{\boldsymbol{y}}\|} \right\| = \|\boldsymbol{y}\|^2 = \|\hat{\boldsymbol{y}}\|^2 + \|\bar{\boldsymbol{y}}\|^2 \geq 2\,\|\hat{\boldsymbol{y}}\|\,\|\bar{\boldsymbol{y}}\| = \|(2\,\|\hat{\boldsymbol{y}}\|)\,\bar{\boldsymbol{y}}\| = \|\bar{\boldsymbol{x}}\|,$$

which means that $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$. Thus, $\mathcal{S}(\mathcal{M}^{m|n}) \subseteq \mathcal{M}_+^{m|n}$.

To complete the proof, we need to show that $\mathcal{M}_+^{m|n} \subseteq \mathcal{S}(\mathcal{M}^{m|n})$. Let $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$. We want to prove that there is a vector $\boldsymbol{y} \in \mathcal{M}^{m|n}$ such that $\boldsymbol{x} = \boldsymbol{y}^2$. From the spectral factorization, we have $\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})$. Note that $\lambda_{1,2}(\boldsymbol{x}) \geq 0$ because $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$. Define $\boldsymbol{y} := \lambda_1^{1/2}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{1/2}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})$. Then, by using (7) and (11), we have

$$\begin{aligned}
\boldsymbol{y}^2 &= \left(\sqrt{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \sqrt{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x})\right)^2 \\
&= \left(\sqrt{\lambda_1(\boldsymbol{x})}\begin{bmatrix} \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} + \sqrt{\lambda_2(\boldsymbol{x})}\begin{bmatrix} \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ -\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix}\right)^2 \\
&= \begin{bmatrix} \left(\sqrt{\lambda_1(\boldsymbol{x})} + \sqrt{\lambda_2(\boldsymbol{x})}\right)\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\sqrt{\lambda_1(\boldsymbol{x})} - \sqrt{\lambda_2(\boldsymbol{x})}\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix}^2 \\
&= \begin{bmatrix} \left(\left(\frac{\sqrt{\lambda_1(\boldsymbol{x})} + \sqrt{\lambda_2(\boldsymbol{x})}}{2}\right)^2 + \left(\frac{\sqrt{\lambda_1(\boldsymbol{x})} - \sqrt{\lambda_2(\boldsymbol{x})}}{2}\right)^2\right)\left(\frac{2}{\sqrt{\lambda_1(\boldsymbol{x})} + \sqrt{\lambda_2(\boldsymbol{x})}}\right)\left(\sqrt{\lambda_1(\boldsymbol{x})} + \sqrt{\lambda_2(\boldsymbol{x})}\right)\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ 2\left(\frac{\sqrt{\lambda_1(\boldsymbol{x})} + \sqrt{\lambda_2(\boldsymbol{x})}}{2}\right)\left(\sqrt{\lambda_1(\boldsymbol{x})} - \sqrt{\lambda_2(\boldsymbol{x})}\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\lambda_1(\boldsymbol{x}) + 2\sqrt{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})} + \lambda_2(\boldsymbol{x})}{2} + \frac{\lambda_1(\boldsymbol{x}) - 2\sqrt{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})} + \lambda_2(\boldsymbol{x})}{2}\right)\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \begin{bmatrix} \left(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\right)\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \lambda_1(\boldsymbol{x})\begin{bmatrix} \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} + \lambda_2(\boldsymbol{x})\begin{bmatrix} \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ -\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) \quad = \quad \boldsymbol{x}.
\end{aligned}$$

The result is established. □

The above proof is intuitive but rigorous. In this proof it is important to notice that, if the operation "⊚" is bilinear (which is not the case), then using the orthogonality and idempotency properties of the eigenvectors $\boldsymbol{c}_1(\boldsymbol{x})$ and

$\boldsymbol{c}_2(\boldsymbol{x})$ stated in (11), it would be immediate that

$$\begin{aligned}\boldsymbol{y}^2 =& \left(\sqrt{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \sqrt{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x})\right) \circledcirc \left(\sqrt{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \sqrt{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x})\right) \\ =& \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{x}.\end{aligned}$$

Therefore, unlike the bilinear product "$\circ$" of the algebra of the convex SOC, we have to be cautious in dealing with the non-bilinear product "$\circledcirc$" of the magma of the nonconvex SOC, even if the eigenvectors $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ are both orthogonal and idempotent.

In the proof of Theorem 4.1, we showed that $\boldsymbol{y}^2 = \boldsymbol{x}$. Thus $\boldsymbol{x}^{1/2} = \boldsymbol{y}$, i.e.,

$$\boldsymbol{x}^{1/2} = (\lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}))^{1/2} = \lambda_1^{1/2}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{1/2}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}).$$

Thus, if $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$, then there exists a unique vector in $\mathcal{M}_+^{m|n}$, which we denote by $\boldsymbol{x}^{1/2}$ such that $(\boldsymbol{x}^{1/2})^2 = \boldsymbol{x}$. For any $\boldsymbol{x} \in \mathcal{M}^{m|n}$, we have $\boldsymbol{x}^2 \in \mathcal{M}_+^{m|n}$. Consequently, there is a unique vector $(\boldsymbol{x}^2)^{1/2} \in \mathcal{M}_+^{m|n}$, which is indicated by $|\boldsymbol{x}|$. It is clear that we have $\boldsymbol{x}^2 = |\boldsymbol{x}|^2$. One can also show that

$\lambda_1^{-1}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{-1}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = (1/\det(\boldsymbol{x}))R^{m|n}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{g}}$, provided that $\det(\boldsymbol{x}) \neq 0$ (i.e., $\lambda_{1,2}(\boldsymbol{x}) \neq 0$),
$\lambda_1^2(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = (\|\boldsymbol{x}\|^2 (\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|); 2\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}) = \boldsymbol{x}^2$.

Items of the following corollary will be utilized in the proof of Lemma 4.1.

Corollary 4.1 *Let* $\boldsymbol{x} = (\hat{\boldsymbol{x}}; \bar{\boldsymbol{x}})$ *and* $\boldsymbol{y} = (\hat{\boldsymbol{y}}; \bar{\boldsymbol{y}}) \in (\mathcal{M}^{m|n}, \circledcirc)$ *have spectral values* $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})$ *and* $\lambda_1(\boldsymbol{y}), \lambda_2(\boldsymbol{y})$*, respectively. Then we have*

(i) $|\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})| = 2\|\bar{\boldsymbol{x}}\| = \lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})$.
(ii) $|\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})| = 2\|\hat{\boldsymbol{x}}\| = \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})$.

Proof The properties given in the corollary can be proved as follows.

(i) Note that $|\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})| = |\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\| - \|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|| = 2\|\bar{\boldsymbol{x}}\| = \lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})$, as desired.
(ii) Note that $|\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})| = |\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\| + \|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|| = 2\|\hat{\boldsymbol{x}}\| = \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})$, as desired.

□

Recall that, by its recursive definition, we have $\boldsymbol{x}^p = \boldsymbol{x} \circledcirc \boldsymbol{x}^{p-1}$. The following lemma presents a more general result than the statement written before about $\boldsymbol{x}^2$. Again, it is fortunate that losing the bilinearity of the operation "$\circledcirc$" does not corrupt the induction proof of the following lemma.

Lemma 4.1 *For any nonnegative integer p, we have that*

$$\boldsymbol{x}^p = \lambda_1^p(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}). \tag{12}$$

Proof The equality in (12) trivially holds when $p = 0, 1$. Assume that $p \geq 2$. We write $\boldsymbol{x}^{(p)}$ to mean the right hand side of (12). That is,

$$\boldsymbol{x}^{(p)} := \lambda_1^p(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = \begin{bmatrix} (\lambda_1^p(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x}))\,\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ (\lambda_1^p(\boldsymbol{x}) - \lambda_2^p(\boldsymbol{x}))\,\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix}.$$

In particular,

$$\boldsymbol{x} = \boldsymbol{x}^{(1)} = \begin{bmatrix} (\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\,\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ (\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}))\,\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix}.$$

Notice that, from Corollary 4.1, we have

$$\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) = 2\,\|\hat{\boldsymbol{x}}\| \geq 0, \ \text{ and } \ \lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}) = 2\,\|\bar{\boldsymbol{x}}\| \geq 0. \tag{13}$$

More generally, using the binomial theorem, we also notice that

$$\begin{aligned}\lambda_1^p(\boldsymbol{x}) \pm \lambda_2^p(\boldsymbol{x}) &= (\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|)^p \pm (\|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|)^p \\ &= \sum_{n=0}^{p} \left(\binom{p}{n} \|\hat{\boldsymbol{x}}\|^{p-n} \|\bar{\boldsymbol{x}}\|^n \right) \pm \sum_{n=0}^{p} \left(\binom{p}{n} \|\hat{\boldsymbol{x}}\|^{p-n} (-\|\bar{\boldsymbol{x}}\|)^n \right) \\ &= \sum_{n=0}^{p} \left(\binom{p}{n} \|\hat{\boldsymbol{x}}\|^{p-n} \|\bar{\boldsymbol{x}}\|^n (1 \pm (-1)^n) \right) \geq 0,\end{aligned} \tag{14}$$

for any nonnegative integer p.

Now, we are ready to prove that $\boldsymbol{x}^p = \boldsymbol{x}^{(p)}$ by induction on p. For $p = 2$, it is easy to check that

$$\boldsymbol{x}^{(2)} = (\lambda_1(\boldsymbol{x}))^2\boldsymbol{c}_1(\boldsymbol{x}) + (\lambda_2(\boldsymbol{x}))^2\boldsymbol{c}_2(\boldsymbol{x}) = \begin{bmatrix} \|\boldsymbol{x}\|^2 \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{x} \circledcirc \boldsymbol{x} = \boldsymbol{x}^2.$$

Assume that $\boldsymbol{x}^k = \boldsymbol{x}^{(k)}$ for some $k > 2$. Then, by using (8), (11), (13), and (14), we have

$$\begin{aligned}&\boldsymbol{x}^{k+1} = \boldsymbol{x} \circledcirc \boldsymbol{x}^k \\ &= \boldsymbol{x} \circledcirc \boldsymbol{x}^{(k)} \\ &= (\lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})) \circledcirc \left(\lambda_1^k(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \\ &= \begin{bmatrix} (\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\,\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ (\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}))\,\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \circledcirc \begin{bmatrix} \left(\lambda_1^k(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\right)\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1^k(\boldsymbol{x}) - \lambda_2^k(\boldsymbol{x})\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \left(\frac{1}{4}(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\left(\lambda_1^k(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\right) + (\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}))\left(\lambda_1^k(\boldsymbol{x}) - \lambda_2^k(\boldsymbol{x})\right)\|\overline{\boldsymbol{c}_1}(\boldsymbol{x})\|^2\right)(2\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) + 2\widehat{\boldsymbol{c}_1}(\boldsymbol{x})) \\ \left(\frac{1}{2}(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})) + 2(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\|\widehat{\boldsymbol{c}_1}(\boldsymbol{x})\|^2\right)\left(\lambda_1^k(\boldsymbol{x}) - \lambda_2^k(\boldsymbol{x})\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \quad + \left(\frac{1}{2}\left(\lambda_1^k(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\right) + 2\left(\lambda_1^k(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\right)\|\widehat{\boldsymbol{c}_1}(\boldsymbol{x})\|^2\right)(\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}))\,\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \left((\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\left(\lambda_1^k(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\right) + (\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}))\left(\lambda_1^k(\boldsymbol{x}) - \lambda_2^k(\boldsymbol{x})\right)\right)\widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left((\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}))\left(\lambda_1^k(\boldsymbol{x}) - \lambda_2^k(\boldsymbol{x})\right) + \left(\lambda_1^k(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\right)(\lambda_1(\boldsymbol{x}) - \lambda_2(\boldsymbol{x}))\right)\overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\begin{bmatrix} \left(\lambda_1^{k+1}(\boldsymbol{x}) + \lambda_1(\boldsymbol{x})\lambda_2^k(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\lambda_1^k(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x}) + \lambda_1^{k+1}(\boldsymbol{x}) - \lambda_1(\boldsymbol{x})\lambda_2^k(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})\lambda_1^k(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\right) \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1^{k+1}(\boldsymbol{x}) - \lambda_1(\boldsymbol{x})\lambda_2^k(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\lambda_1^k(\boldsymbol{x}) - \lambda_2^{k+1}(\boldsymbol{x}) + \lambda_1^{k+1}(\boldsymbol{x}) - \lambda_1^k(\boldsymbol{x})\lambda_2(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\lambda_1(\boldsymbol{x}) - \lambda_2^{k+1}(\boldsymbol{x})\right) \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \begin{bmatrix} \left(\lambda_1^{k+1}(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\right) \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1^{k+1}(\boldsymbol{x}) - \lambda_2^{k+1}(\boldsymbol{x})\right) \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \lambda_1^{k+1}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = \boldsymbol{x}^{(k+1)}.
\end{aligned}
$$

Thus, the equality in (12) holds for any integer $p \geq 0$. □

The following theorem is fundamental and its proof relies on Lemma 4.1.

Theorem 4.2 *The magma* $(\mathcal{M}^{m|n}, \circledcirc)$ *is power-associative. That is,* $\boldsymbol{x}^p \circledcirc \boldsymbol{x}^q = \boldsymbol{x}^{p+q}$ *for any positive integers* p *and* q*.*

Proof Using (8), (11), (12), and (14), we have

$$
\begin{aligned}
\boldsymbol{x}^p \circledcirc \boldsymbol{x}^q &= \boldsymbol{x}^{(p)} \circledcirc \boldsymbol{x}^{(q)} \\
&= \left(\lambda_1^p(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \circledcirc \left(\lambda_1^q(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^q(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \\
&= \begin{bmatrix} (\lambda_1^p(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x}))\, \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ (\lambda_1^p(\boldsymbol{x}) - \lambda_2^p(\boldsymbol{x}))\, \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \circledcirc \begin{bmatrix} (\lambda_1^q(\boldsymbol{x}) + \lambda_2^q(\boldsymbol{x}))\, \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ (\lambda_1^q(\boldsymbol{x}) - \lambda_2^q(\boldsymbol{x}))\, \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \left((\lambda_1^p(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x}))\,(\lambda_1^q(\boldsymbol{x}) + \lambda_2^q(\boldsymbol{x})) + (\lambda_1^p(\boldsymbol{x}) - \lambda_2^p(\boldsymbol{x}))\,(\lambda_1^q(\boldsymbol{x}) - \lambda_2^q(\boldsymbol{x}))\right) \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left((\lambda_1^p(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x}))\,(\lambda_1^q(\boldsymbol{x}) - \lambda_2^q(\boldsymbol{x})) + (\lambda_1^q(\boldsymbol{x}) + \lambda_2^q(\boldsymbol{x}))\,(\lambda_1^p(\boldsymbol{x}) - \lambda_2^p(\boldsymbol{x}))\right) \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \left(\lambda_1^{p+q}(\boldsymbol{x}) + \lambda_1^p(\boldsymbol{x})\lambda_2^q(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\lambda_1^q(\boldsymbol{x}) + \lambda_2^{p+q}(\boldsymbol{x}) + \lambda_1^{p+q}(\boldsymbol{x}) - \lambda_1^p(\boldsymbol{x})\lambda_2^q(\boldsymbol{x}) - \lambda_2^p(\boldsymbol{x})\lambda_1^q(\boldsymbol{x}) + \lambda_2^{p+q}(\boldsymbol{x})\right) \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1^{p+q}(\boldsymbol{x}) - \lambda_1^p(\boldsymbol{x})\lambda_2^q(\boldsymbol{x}) + \lambda_2^p(\boldsymbol{x})\lambda_1^q(\boldsymbol{x}) - \lambda_2^{p+q}(\boldsymbol{x}) + \lambda_1^{p+q}(\boldsymbol{x}) - \lambda_1^q(\boldsymbol{x})\lambda_2^p(\boldsymbol{x}) + \lambda_2^q(\boldsymbol{x})\lambda_1^p(\boldsymbol{x}) - \lambda_2^{p+q}(\boldsymbol{x})\right) \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \begin{bmatrix} \left(\lambda_1^{p+q}(\boldsymbol{x}) + \lambda_2^{p+q}(\boldsymbol{x})\right) \widehat{\boldsymbol{c}_1}(\boldsymbol{x}) \\ \left(\lambda_1^{p+q}(\boldsymbol{x}) - \lambda_2^{p+q}(\boldsymbol{x})\right) \overline{\boldsymbol{c}_1}(\boldsymbol{x}) \end{bmatrix} \\
&= \lambda_1^{p+q}(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^{p+q}(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) \\
&= \boldsymbol{x}^{(p+q)} = \boldsymbol{x}^{p+q}.
\end{aligned}
$$

The proof is complete. □

It is known [15] that Jordan identity holds in the algebra $(\mathcal{E}^{n+1}, \circ)$. That is, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}^{n+1}$, we have

$$
(\boldsymbol{x} \circ \boldsymbol{y}) \circ \boldsymbol{x}^2 = \boldsymbol{x} \circ (\boldsymbol{y} \circ \boldsymbol{x}^2) \quad \text{(Jordan identity)}. \tag{15}
$$

Due to the commutativity of the algebra, Jordan identity for $(\mathcal{E}^{n+1}, \circ)$ can be written as

$$
\boldsymbol{x}^2 \circ (\boldsymbol{x} \circ \boldsymbol{y}) = \boldsymbol{x} \circ (\boldsymbol{x}^2 \circ \boldsymbol{y}), \text{ or equivalently, } \mathsf{Arw}(\boldsymbol{x}^2)\mathsf{Arw}(\boldsymbol{x}) = \mathsf{Arw}(\boldsymbol{x})\mathsf{Arw}(\boldsymbol{x}^2),
$$

because $\boldsymbol{x} \circ \boldsymbol{y} = \mathsf{Arw}(\boldsymbol{x})\boldsymbol{y}$. In other words, Jordan identity is satisfied in $(\mathcal{E}^{n+1}, \circ)$ due to a known fact about the arrow-shaped matrix: $\mathsf{Arw}(\boldsymbol{x})$ and $\mathsf{Arw}(\boldsymbol{x}^2)$ commute.

Regardless of the fact that $(\mathcal{M}^{m|n}, \circledcirc)$ is not an algebra, it is interesting to know whether the Jordan identity holds for $(\mathcal{M}^{m|n}, \circledcirc)$. We have seen that the magma $(\mathcal{M}^{m|n}, \circledcirc)$ is also commutative, so the Jordan identity for $(\mathcal{M}^{m|n}, \circledcirc)$ is

$$
(\boldsymbol{x} \circledcirc \boldsymbol{y}) \circledcirc \boldsymbol{x}^2 = \boldsymbol{x} \circledcirc (\boldsymbol{y} \circledcirc \boldsymbol{x}^2), \text{ or equivalently, } \boldsymbol{x}^2 \circledcirc (\boldsymbol{x} \circledcirc \boldsymbol{y}) = \boldsymbol{x} \circledcirc (\boldsymbol{x}^2 \circledcirc \boldsymbol{y}). \tag{16}
$$

The following lemma demonstrates (16) for $(\mathcal{M}^{1|n}, \circledcirc)$.

Lemma 4.2 *The Jordan identity holds for* $(\mathcal{M}^{1|n}, \circledcirc)$.

Proof For $\boldsymbol{x}, \boldsymbol{y} \in (\mathcal{M}^{1|n}, \circledcirc)$, it can be shown that

$$\boldsymbol{x} \circledcirc \boldsymbol{y} = \begin{cases} \boldsymbol{x} \circ \boldsymbol{y}, & \text{if } x_1, y_1 \geq 0; \\ -(\boldsymbol{x} \circ \boldsymbol{y}), & \text{if } x_1, y_1 < 0; \\ \boldsymbol{0}, & \text{if } x_1 y_1 \ < 0. \end{cases}$$

The desired result follows from (15). □

It does not seem easy to prove the statement in Lemma 4.2 for $(\mathcal{M}^{m\geq 2|n}, \circledcirc)$. Theorem 4.3 proves an interesting fact about the crane-shaped matrix: $\mathsf{Crn}(\boldsymbol{x})$ and $\mathsf{Crn}(\boldsymbol{x}^2)$ commute. This generalizes the fact that $\mathsf{Arw}(\boldsymbol{x})$ and $\mathsf{Arw}(\boldsymbol{x}^2)$ commute, which was the evidence for the satisfaction of the Jordan identity for $(\mathcal{E}^{n+1}, \circ)$. However, we should be aware that the result in Theorem 4.3 is not an evidence for the satisfaction of the Jordan identity for $(\mathcal{M}^{m|n}, \circledcirc)$. In other words, the identity in (16) has not yet been ascertained for the magma $(\mathcal{M}^{m\geq 2|n}, \circledcirc)$ (see Table 4) because it is not generally true that $\boldsymbol{x} \circledcirc \boldsymbol{y} = \mathsf{Crn}(\boldsymbol{x})\boldsymbol{y}$ due to losing the bilinearity in "$\circledcirc$" (remember $\boldsymbol{x} \circledcirc \boldsymbol{y} = (\mathsf{Crn}(\boldsymbol{x})\boldsymbol{y} + \mathsf{Crn}(\boldsymbol{y})\boldsymbol{x})/2$). Nevertheless, we believe that the result of the following theorem provides supportive evidence for the powerful of the magma $(\mathcal{M}^{m|n}, \circledcirc)$.

Theorem 4.3 *For any* $\boldsymbol{x} \in (\mathcal{M}^{m|n}, \circledcirc)$*, the matrices* $\mathsf{Crn}(\boldsymbol{x})$ *and* $\mathsf{Crn}(\boldsymbol{x}^2)$ *commute.*

Proof Using a direct calculation, we have

$$\mathsf{Crn}\left(\boldsymbol{x}^2\right) = \begin{bmatrix} \left\| \dfrac{\|\boldsymbol{x}\|^2}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}} \right\| I_m & \dfrac{\dfrac{2\,\|\boldsymbol{x}\|^2\,\|\hat{\boldsymbol{x}}\|}{\|\hat{\boldsymbol{x}}\|}}{\left\| \dfrac{\|\boldsymbol{x}\|^2}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}} \right\|} \hat{\boldsymbol{x}} \bar{\boldsymbol{x}}^\mathsf{T} \\ \dfrac{\dfrac{2\,\|\boldsymbol{x}\|^2\,\|\hat{\boldsymbol{x}}\|}{\|\hat{\boldsymbol{x}}\|}}{\left\| \dfrac{\|\boldsymbol{x}\|^2}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}} \right\|} \bar{\boldsymbol{x}} \hat{\boldsymbol{x}}^\mathsf{T} & \left\| \dfrac{\|\boldsymbol{x}\|^2}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}} \right\| I_n \end{bmatrix} = \begin{bmatrix} \|\boldsymbol{x}\|^2 I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \|\boldsymbol{x}\|^2 I_n \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathsf{Crn}(\boldsymbol{x})\mathsf{Crn}\left(\boldsymbol{x}^2\right) &= \begin{bmatrix} \|\hat{\boldsymbol{x}}\| I_m & \dfrac{1}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ \dfrac{1}{\|\hat{\boldsymbol{x}}\|} \bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \|\hat{\boldsymbol{x}}\| I_n \end{bmatrix} \begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \left(\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \\ &= \begin{bmatrix} (\|\hat{\boldsymbol{x}}\|^3 + \|\hat{\boldsymbol{x}}\|\,\|\bar{\boldsymbol{x}}\|^2) I_m + \dfrac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 3\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \dfrac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|} \hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 3\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} + \dfrac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|} \bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \left(\|\hat{\boldsymbol{x}}\|^3 + \|\hat{\boldsymbol{x}}\|\,\|\bar{\boldsymbol{x}}\|^2\right) I_n + 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix}, \end{aligned}$$

Table 4: A comparison of some properties between $(\mathcal{E}^{n+1}, \circ)$ and $(\mathcal{M}^{m|n}, \circledcirc)$.

Properties	Algebra $(\mathcal{E}^{n+1}, \circ)$	Magma $(\mathcal{M}^{m|n}, \circledcirc)$
Commutativity	✓	✓
Power-associativity	✓	✓
Associativity	✗	✗
Jordan identity	✓	OQA. H/E $\mathsf{Crn}(\boldsymbol{x})$ and $\mathsf{Crn}(\boldsymbol{x}^2)$ commute (see Theorem 4.3)
Unitary	✓	✗ H/E the submagma generated by any element is unitary-like
Invertibility	✓	✗ H/E generalized invertibility holds in int $\mathcal{M}_+^{m|n}$
Reality of eigenvalues	✓	✓
Rankly independence	✓	✓
Cone of squares = SOC	✓	✓

and

$$\mathsf{Crn}\left(\boldsymbol{x}^2\right)\mathsf{Crn}(\boldsymbol{x}) = \begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \left(\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \begin{bmatrix} \|\hat{\boldsymbol{x}}\| I_m & \frac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ \frac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \|\hat{\boldsymbol{x}}\| I_n \end{bmatrix}$$
$$= \begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|^3 + \|\hat{\boldsymbol{x}}\|\,\|\bar{\boldsymbol{x}}\|^2\right) I_m + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 3\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 3\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \left(\|\hat{\boldsymbol{x}}\|^3 + \|\hat{\boldsymbol{x}}\|\,\|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix}.$$

Thus, $\mathsf{Crn}(\boldsymbol{x})\mathsf{Crn}(\boldsymbol{x}^2) = \mathsf{Crn}(\boldsymbol{x}^2)\mathsf{Crn}(\boldsymbol{x})$. The proof is complete. □

We summarize key properties and features of the magma $(\mathcal{M}^{m|n}, \circledcirc)$ in Table 4[2] which also compares them with those of the algebra $(\mathcal{E}^{n+1}, \circ)$.

5 Further Algebraic and Spectral Properties

This section generalizes further important properties of the algebra $(\mathcal{E}^{n+1}, \circ)$ associated with the convex SOC to the magma $(\mathcal{M}^{m|n}, \circledcirc)$ associated with the nonconvex SOC by using the definitions, notions, and results obtained in Section 3. The following lemma is due to [11] and will be used in the proof of Theorem 5.1.

Lemma 5.1 *For any nonzero vector $\boldsymbol{x} \in \mathbb{R}^d$, the matrix $\boldsymbol{x}\boldsymbol{x}^\mathsf{T}$ is positive semidefinite with only one nonzero eigenvalue, namely $\|\boldsymbol{x}\|^2$.*

In Theorem 4.3, we stated one of the fundamental properties of $\mathsf{Crn}(\boldsymbol{x})$ matrix. The following theorem presents two more fundamental properties of $\mathsf{Crn}(\boldsymbol{x})$. This theorem can be viewed as a generalization of items (1) and (2) in [1, Theorem 3].

Theorem 5.1 *Let $\boldsymbol{x} \in (\mathcal{M}^{m|n}, \circledcirc)$. Then*

(i) $\mathsf{Crn}(\boldsymbol{x})$ *and* $\mathsf{P}_{\boldsymbol{x}}$ *commute and thus share a system of eigenvectors.*
(ii) $\lambda_1(\boldsymbol{x}) = \|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|$ *and* $\lambda_2(\boldsymbol{x}) = \|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|$ *are eigenvalues of* $\mathsf{Crn}(\boldsymbol{x})$*. Moreover, if* $\lambda_1(\boldsymbol{x}) \neq \lambda_2(\boldsymbol{x})$ *then each one has multiplicity one; the corresponding eigenvectors are* $\boldsymbol{c}_1(\boldsymbol{x})$ *and* $\boldsymbol{c}_2(\boldsymbol{x})$*. Furthermore,* $\|\hat{\boldsymbol{x}}\|$ *is an eigenvalue of* $\mathsf{Crn}(\boldsymbol{x})$ *and has a multiplicity* $m + n - 2$ *when* $\boldsymbol{x} \neq \boldsymbol{0}$*.*

[2] In Table 4, H/E means However and OQA means Open Question Argument.

Proof We prove the theorem by a direct calculation.

(i) We have that

$$\begin{aligned}
\mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}} &= \begin{bmatrix} \|\hat{\boldsymbol{x}}\| I_m & \frac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \frac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\| I_n \end{bmatrix} \begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix} \\
&= \begin{bmatrix} -\det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_m + 2\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + 4\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 4\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix} \\
&= \begin{bmatrix} -\det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_m + 2\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + 3\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 3\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 4\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix},
\end{aligned}$$

and that

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x}}\mathsf{Crn}(\boldsymbol{x}) &= \begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix} \begin{bmatrix} \|\hat{\boldsymbol{x}}\| I_m & \frac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \frac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\| I_n \end{bmatrix} \\
&= \begin{bmatrix} -\det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_m + 2\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -\frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + 4\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 4\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix} \\
&= \begin{bmatrix} -\det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_m + 2\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\,\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + 3\,\|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 3\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 4\,\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})\,\|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix}.
\end{aligned}$$

Thus, $\mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}} = \mathsf{P}_{\boldsymbol{x}}\mathsf{Crn}(\boldsymbol{x})$. In matrix algebra, two matrices commute if and only if they share a common system of eigenvectors. The result is obtained.

(ii) Note that

$$\det\left(\lambda I_{m+n} - \mathsf{Crn}(\boldsymbol{x})\right) = \begin{vmatrix} (\lambda - \|\hat{\boldsymbol{x}}\|)\, I_m & -\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\bar{\boldsymbol{x}}\left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^{\mathsf{T}} & (\lambda - \|\hat{\boldsymbol{x}}\|)\, I_n \end{vmatrix}.$$

If $\det((\lambda - \|\hat{\boldsymbol{x}}\|)\, I_m) \neq 0$, then the matrix

$$S := (\lambda - \|\hat{\boldsymbol{x}}\|)\, I_n - \left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}}\right)^{\mathsf{T}} ((\lambda - \|\hat{\boldsymbol{x}}\|)\, I_m)^{-1} \left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}}\right)$$

is the Schur complement of $(\lambda - \|\hat{\boldsymbol{x}}\|)I_m$ in $\lambda I_{m+n} - \mathsf{Crn}(\boldsymbol{x})$. We then have

$$\det\left(\lambda I_{m+n} - \mathsf{Crn}(\boldsymbol{x})\right) = \det((\lambda - \|\hat{\boldsymbol{x}}\|)I_m)\det(S) = (\lambda - \|\hat{\boldsymbol{x}}\|)^m \det(S).$$

Using Lemma 5.1, we also have

$$
\begin{aligned}
\det(S) &= \det\left((\lambda - \|\hat{\boldsymbol{x}}\|)\, I_n - \left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \bar{\boldsymbol{x}}^{\mathsf{T}} \right)^{\mathsf{T}} \left(\frac{1}{\lambda - \|\hat{\boldsymbol{x}}\|} I_m \right) \left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \bar{\boldsymbol{x}}^{\mathsf{T}} \right) \right) \\
&= \det\left((\lambda - \|\hat{\boldsymbol{x}}\|)\, I_n - \frac{1}{\lambda - \|\hat{\boldsymbol{x}}\|} \bar{\boldsymbol{x}} \bar{\boldsymbol{x}}^{\mathsf{T}} \right) \\
&= \det\left((\lambda - \|\hat{\boldsymbol{x}}\|)\, I_n \left(I_n - \frac{1}{(\lambda - \|\hat{\boldsymbol{x}}\|)^2} \bar{\boldsymbol{x}} \bar{\boldsymbol{x}}^{\mathsf{T}} \right) \right) \\
&= \det\left((\lambda - \|\hat{\boldsymbol{x}}\|)\, I_n \right) \det\left(I_n - \frac{1}{(\lambda - \|\hat{\boldsymbol{x}}\|)^2} \bar{\boldsymbol{x}} \bar{\boldsymbol{x}}^{\mathsf{T}} \right) \\
&= (\lambda - \|\hat{\boldsymbol{x}}\|)^n \left(\prod_{i=1}^{n} \left(1 - \lambda_i \left(\frac{1}{(\lambda - \|\hat{\boldsymbol{x}}\|)^2} \bar{\boldsymbol{x}} \bar{\boldsymbol{x}}^{\mathsf{T}} \right) \right) \right) \\
&= (\lambda - \|\hat{\boldsymbol{x}}\|)^n \left(1 - \frac{\|\bar{\boldsymbol{x}}\|^2}{(\lambda - \|\hat{\boldsymbol{x}}\|)^2} \right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\det\left(\lambda I_{m+n} - \mathsf{Crn}(\boldsymbol{x}) \right) &= (\lambda - \|\hat{\boldsymbol{x}}\|)^m \left((\lambda - \|\hat{\boldsymbol{x}}\|)^n \left(1 - \frac{\|\bar{\boldsymbol{x}}\|^2}{(\lambda - \|\hat{\boldsymbol{x}}\|)^2} \right) \right) \\
&= (\lambda - \|\hat{\boldsymbol{x}}\|)^{m+n} - \|\bar{\boldsymbol{x}}\|^2 (\lambda - \|\hat{\boldsymbol{x}}\|)^{m+n-2} \\
&= (\lambda - \|\hat{\boldsymbol{x}}\|)^{m+n-2} \left((\lambda - \|\hat{\boldsymbol{x}}\|)^2 - \|\bar{\boldsymbol{x}}\|^2 \right).
\end{aligned}
$$

Consequently, $\|\hat{\boldsymbol{x}}\|$ is an eigenvalue of $\mathsf{Crn}(\boldsymbol{x})$ with multiplicity $m+n-2$. In addition, $(\lambda - \|\hat{\boldsymbol{x}}\|)^2 - \|\bar{\boldsymbol{x}}\|^2 = 0$ implies that $(\lambda - \|\hat{\boldsymbol{x}}\|)^2 = \|\bar{\boldsymbol{x}}\|^2$ or $\lambda - \|\hat{\boldsymbol{x}}\| = \pm \|\bar{\boldsymbol{x}}\|$, hence $\lambda = \|\hat{\boldsymbol{x}}\| \pm \|\bar{\boldsymbol{x}}\|$ are two eigenvalues of $\mathsf{Crn}(\boldsymbol{x})$. Finally, it is clear that $\boldsymbol{c}_1(\boldsymbol{x})$ and $\boldsymbol{c}_2(\boldsymbol{x})$ are the corresponding eigenvectors of $\|\hat{\boldsymbol{x}}\| + \|\bar{\boldsymbol{x}}\|$ and $\|\hat{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|$, respectively. The proof is complete. □

Item (ii) in Theorem 5.1 gives a certificate that the might be termed the "nonconvex second-order cone programming" is a special case of the nonlinear semidefinite programming [12, 17, 24, 28, 29, 33].

The operator $\mathsf{P}.$ is of significant importance because it is being used to express the Hessian of the logarithmic barrier function associated with the cone (see Section 6). The following theorem establishes numerous fundamental properties of $\mathsf{P}.$ operator and generalizes the corresponding items in [1, Theorem 8].

Theorem 5.2 *Let $\boldsymbol{x}, \boldsymbol{y} \in (\mathcal{M}^{m|n}, \circledcirc)$. Assume also that $\boldsymbol{x}^g$ exists (i.e., $\det(\boldsymbol{x}) \neq 0$) wherever it is necessary. Then*

(i) $\mathsf{P}_{\boldsymbol{x}^g} = \mathsf{P}_{\boldsymbol{x}}^{-1}$.

(ii) $\mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}^g} = \mathsf{P}_{\boldsymbol{x}^g}\mathsf{Crn}(\boldsymbol{x}) = \mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}^g}$.

(iii) $\mathsf{P}_{\boldsymbol{x}^g}\boldsymbol{x} = \mathsf{P}_{\boldsymbol{x}}^{-1}\boldsymbol{x} = \boldsymbol{x}^g$.

(iv) $\mathsf{P}_{\alpha\boldsymbol{x}} = \alpha^2 \mathsf{P}_{\boldsymbol{x}}$, *for* $\alpha \in \mathbb{R}$.

(v) $\mathsf{P}_{\boldsymbol{x}}\boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{x}^2$.

(vi) $\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^g}\mathsf{P}_{\boldsymbol{x}} = \mathsf{P}_{\boldsymbol{x}}\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^g} = \mathsf{Crn}(\boldsymbol{x}^2)$.

(vii) $\dfrac{1}{(\det(\boldsymbol{x}))^2} R^{m|n} \mathsf{P}_{\boldsymbol{x}} R^{m|n} = \mathsf{P}_{\boldsymbol{x}^g}$.

(viii) $\det(\mathsf{P}_{\boldsymbol{x}}\boldsymbol{y}) = (\det(\boldsymbol{x}))^2 \det(\boldsymbol{y})$.

Proof Most statements are shown by direct calculations. We prove the theorem item by item.

(i) Let

$$M(\boldsymbol{x}) := \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \det(\boldsymbol{x})I_n \end{bmatrix}.$$

Then

$$\begin{aligned}
&M(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}} \\
&= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \det(\boldsymbol{x})I_n \end{bmatrix}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \det(\boldsymbol{x})I_n \end{bmatrix} \\
&= \begin{bmatrix} I_m - \frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top + \frac{4\|\hat{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top - \frac{4\|\bar{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -\frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4\|\hat{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top - \frac{4\|\bar{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ \frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top - \frac{4\|\hat{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top + \frac{4\|\bar{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -\frac{4\|\hat{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4\|\hat{\boldsymbol{x}}\|^2}{(\det(\boldsymbol{x}))^2}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + I_n \end{bmatrix} \\
&= \begin{bmatrix} I_m - \frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top + \frac{4\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{(\det(\boldsymbol{x}))^2}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -\frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{(\det(\boldsymbol{x}))^2}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ \frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top - \frac{4\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{(\det(\boldsymbol{x}))^2}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -\frac{4\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{(\det(\boldsymbol{x}))^2}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top}{\det(\boldsymbol{x})} + I_n \end{bmatrix} \\
&= \begin{bmatrix} I_m - \frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top + \frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -\frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ \frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top - \frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & -\frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \frac{4}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + I_n \end{bmatrix} \\
&= \begin{bmatrix} I_m & O \\ O & I_n \end{bmatrix} = I_{m+n}.
\end{aligned}$$

Similarly, a simple computation can also show that $\mathsf{P}_{\boldsymbol{x}}M(\boldsymbol{x}) = I_{m+n}$. It follows that $M(\boldsymbol{x}) = \mathsf{P}_{\boldsymbol{x}}^{-1}$.
To prove the result, it is enough to show that $\mathsf{P}_{\boldsymbol{x}^{\rm g}} = M(\boldsymbol{x})$. Recall that

$$\boldsymbol{x}^{\rm g} = \frac{R^{m|n}}{\det(\boldsymbol{x})}\boldsymbol{x} = \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix}\hat{\boldsymbol{x}} \\ -\bar{\boldsymbol{x}}\end{bmatrix}, \text{ hence } \widehat{\boldsymbol{x}^{\rm g}} = \frac{1}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}} \text{ and } \overline{\boldsymbol{x}^{\rm g}} = \frac{-1}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}.$$

Recall also that

$$\lambda_{1,2}(\boldsymbol{x}^{\rm g}) = \frac{1}{\lambda_{1,2}(\boldsymbol{x})},$$

and hence

$$\det(\boldsymbol{x}^{\rm g}) = \lambda_1(\boldsymbol{x}^{\rm g})\lambda_2(\boldsymbol{x}^{\rm g}) = \frac{1}{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})} = \frac{1}{\det(\boldsymbol{x})}.$$

It follows that

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x}^{\rm g}} &= \begin{bmatrix} 2\widehat{\boldsymbol{x}^{\rm g}}\widehat{\boldsymbol{x}^{\rm g}}^\top - \det(\boldsymbol{x}^{\rm g})I_m & 2\widehat{\boldsymbol{x}^{\rm g}}\overline{\boldsymbol{x}^{\rm g}}^\top \\ 2\overline{\boldsymbol{x}^{\rm g}}\widehat{\boldsymbol{x}^{\rm g}}^\top & 2\overline{\boldsymbol{x}^{\rm g}}\,\overline{\boldsymbol{x}^{\rm g}}^\top + \det(\boldsymbol{x}^{\rm g})I_n \end{bmatrix} \\
&= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top - \det(\boldsymbol{x})I_m & -2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top \\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\top + \det(\boldsymbol{x})I_n \end{bmatrix} = M(\boldsymbol{x}).
\end{aligned}$$

(ii) From item (i) in Theorem 5.1, we have $\mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}} = \mathsf{P}_{\boldsymbol{x}}\mathsf{Crn}(\boldsymbol{x})$. Multiplying both sides from right by $\mathsf{P}_{\boldsymbol{x}}^{-1}$ we get $\mathsf{Crn}(\boldsymbol{x}) = \mathsf{P}_{\boldsymbol{x}}\mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}}^{-1}$. Multiplying both sides from left by $\mathsf{P}_{\boldsymbol{x}}^{-1}$ we get $\mathsf{P}_{\boldsymbol{x}}^{-1}\mathsf{Crn}(\boldsymbol{x}) = \mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}}^{-1}$, or using item (i), $\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}\mathsf{Crn}(\boldsymbol{x}) = \mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}$. The last equality in item (ii) is proven by noting that

$$\begin{aligned}
&\mathsf{Crn}(\boldsymbol{x})\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}\\
&= \begin{bmatrix} \|\hat{\boldsymbol{x}}\| I_m & \frac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \frac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\| I_n \end{bmatrix} \frac{1}{(\det(\boldsymbol{x}))^2} \begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix}\\
&= \frac{1}{(\det(\boldsymbol{x}))^2} \begin{bmatrix} -\|\hat{\boldsymbol{x}}\|\det(\boldsymbol{x})I_m + 2\|\hat{\boldsymbol{x}}\|\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \frac{2\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -2\|\hat{\boldsymbol{x}}\|\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \frac{2\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} + 2\|\hat{\boldsymbol{x}}\|\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - 2\|\hat{\boldsymbol{x}}\|\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -2\|\hat{\boldsymbol{x}}\|\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + 2\|\hat{\boldsymbol{x}}\|\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \|\hat{\boldsymbol{x}}\|\det(\boldsymbol{x})I_n \end{bmatrix}\\
&= \frac{1}{(\det(\boldsymbol{x}))^2} \begin{bmatrix} -\|\hat{\boldsymbol{x}}\|\det(\boldsymbol{x})I_m + \frac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -\frac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\|\det(\boldsymbol{x})I_n \end{bmatrix}\\
&= \frac{1}{(\det(\boldsymbol{x}))^2} \begin{bmatrix} -\|\hat{\boldsymbol{x}}\|\det(\boldsymbol{x})I_m + \frac{2\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -\frac{2\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{\det(\boldsymbol{x})}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\|\det(\boldsymbol{x})I_n \end{bmatrix}\\
&= \frac{1}{\det(\boldsymbol{x})} \begin{bmatrix} -\|\hat{\boldsymbol{x}}\| I_m + \frac{2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -\frac{1}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -\frac{1}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\| I_n \end{bmatrix} = \mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}^{\mathrm{g}}}.
\end{aligned}$$

(iii) In light of item (i), it suffices to show that $\mathsf{P}_{\boldsymbol{x}}\boldsymbol{x}^{\mathrm{g}} = \boldsymbol{x}$. A straightforward computation finds that

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x}}\boldsymbol{x}^{\mathrm{g}} &= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \det(\boldsymbol{x})I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix} \frac{1}{\det(\boldsymbol{x})} \begin{bmatrix} \hat{\boldsymbol{x}} \\ -\bar{\boldsymbol{x}} \end{bmatrix}\\
&= \begin{bmatrix} \left(-\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \begin{bmatrix} \frac{1}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} \\ -\frac{1}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} \end{bmatrix}\\
&= \begin{bmatrix} \frac{2\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} - \frac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} - \frac{2\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} \\ \frac{2\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} - \frac{2\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} - \frac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} \end{bmatrix}\\
&= \begin{bmatrix} \frac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} - \frac{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} \\ \frac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} - \frac{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} \end{bmatrix}\\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}} \\ 2\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{x}.
\end{aligned}$$

(iv) If $\alpha \in \mathbb{R}$, then

$$\begin{aligned}
\mathsf{P}_{\alpha \boldsymbol{x}} &= \begin{bmatrix} -\left(\|\alpha\hat{\boldsymbol{x}}\|^2 - \|\alpha\bar{\boldsymbol{x}}\|^2\right) I_m + 2\alpha\hat{\boldsymbol{x}}\alpha\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha\hat{\boldsymbol{x}}\alpha\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\alpha\bar{\boldsymbol{x}}\alpha\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha\bar{\boldsymbol{x}}\alpha\bar{\boldsymbol{x}}^\mathsf{T} + \left(\|\alpha\hat{\boldsymbol{x}}\|^2 - \|\alpha\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \\
&= \begin{bmatrix} -\left(|\alpha|^2\|\hat{\boldsymbol{x}}\|^2 - |\alpha|^2\|\bar{\boldsymbol{x}}\|^2\right) I_m + 2\alpha^2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha^2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\alpha^2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha^2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \left(|\alpha|^2\|\hat{\boldsymbol{x}}\|^2 - |\alpha|^2\|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \\
&= \begin{bmatrix} -\alpha^2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_m + 2\alpha^2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha^2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\alpha^2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha^2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \alpha^2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \\
&= \begin{bmatrix} -\alpha^2\det(\boldsymbol{x}) I_m + 2\alpha^2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha^2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\alpha^2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\alpha^2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \alpha^2\det(\boldsymbol{x}) I_n \end{bmatrix} \;=\; \alpha^2\mathsf{P}_{\boldsymbol{x}}.
\end{aligned}$$

(v) A straightforward computation finds that

$$\begin{aligned}
&\mathsf{P}_{\boldsymbol{x}}\boldsymbol{e}(\boldsymbol{x}) \\
&= \begin{bmatrix} \left(-\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_n \end{bmatrix} \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\dfrac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} + \dfrac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} + \dfrac{2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}\hat{\boldsymbol{x}} \\ \dfrac{2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T}\hat{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \dfrac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} + \dfrac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} \\ 2\|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}} \end{bmatrix} \;=\; \boldsymbol{x}^2.
\end{aligned}$$

(vi) Note that

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}} &= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} -\|\bar{\boldsymbol{x}}\|^2 I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} - \|\hat{\boldsymbol{x}}\|^2 I_m & O \\ O & \left(\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2\right) I_n - 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix} \\
&= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} -\|\boldsymbol{x}\|^2 I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & O \\ O & \|\boldsymbol{x}\|^2 I_n - 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}}\mathsf{P}_{\boldsymbol{x}} \\
&= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} -\|\boldsymbol{x}\|^2 I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & O \\ O & \|\boldsymbol{x}\|^2 I_n - 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix}\begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} - \det(\boldsymbol{x}) I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \det(\boldsymbol{x}) I_n \end{bmatrix} \\
&= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} \left(-2\|\boldsymbol{x}\|^2 + 4\|\hat{\boldsymbol{x}}\|^2 - 2\det(\boldsymbol{x})\right)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2\det(\boldsymbol{x}) I_m & -2\|\boldsymbol{x}\|^2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + 4\|\hat{\boldsymbol{x}}\|^2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\|\boldsymbol{x}\|^2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} - 4\|\bar{\boldsymbol{x}}\|^2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \left(2\|\boldsymbol{x}\|^2 - 4\|\bar{\boldsymbol{x}}\|^2 - 2\det(\boldsymbol{x})\right)\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2\det(\boldsymbol{x}) I_m \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2 I_m - 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \dfrac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\det(\boldsymbol{x})}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ \dfrac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \dfrac{2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)}{\det(\boldsymbol{x})}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2 I_n - 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix} \\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2 I_m - 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2 I_n - 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix} \\
&= \begin{bmatrix} \|\boldsymbol{x}\|^2 I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \|\boldsymbol{x}\|^2 I_n \end{bmatrix} \;=\; \mathsf{Crn}(\boldsymbol{x}^2).
\end{aligned}$$

Similarly, a straightforward computation can show that $\mathsf{P}_{\boldsymbol{x}}\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}} = \mathsf{Crn}(\boldsymbol{x}^2)$.

(vii) With a little computation, we have

$$
\begin{aligned}
&\frac{1}{(\det(\boldsymbol{x}))^2} R^{m|n}\mathsf{P}_{\boldsymbol{x}}R^{m|n}\\
&= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} I_m & O\\ O & -I_n\end{bmatrix}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n\end{bmatrix}\begin{bmatrix} I_m & O\\ O & -I_n\end{bmatrix}\\
&= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -(2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n)\end{bmatrix}\begin{bmatrix} I_m & O\\ O & -I_n\end{bmatrix}\\
&= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n\end{bmatrix}\\
&= \mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}
\end{aligned}
$$

as desired.

(viii) From the definition of $\mathsf{P}_{\cdot}$, we have $\mathsf{P}_{\boldsymbol{x}}\boldsymbol{y} = 2(\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y})\boldsymbol{x} - \det(\boldsymbol{x})R^{m|n}\boldsymbol{y}$. On the other hand, we also have

$$
\begin{aligned}
\mathsf{P}_{\boldsymbol{x}}\boldsymbol{y} &= \begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{y}}\\ \bar{\boldsymbol{y}}\end{bmatrix}\\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}}\hat{\boldsymbol{x}} + 2\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\hat{\boldsymbol{x}} - \det(\boldsymbol{x})\hat{\boldsymbol{y}}\\ 2\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}}\bar{\boldsymbol{x}} + 2\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\bar{\boldsymbol{x}} + \det(\boldsymbol{x})\bar{\boldsymbol{y}}\end{bmatrix}.
\end{aligned}
$$

Letting $\boldsymbol{z} = \mathsf{P}_{\boldsymbol{x}}\boldsymbol{y}$, $\alpha = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{y}$ and $\gamma = \det(\boldsymbol{x})$, we get $\boldsymbol{z} = 2\alpha\boldsymbol{x} - \gamma R^{m|n}\boldsymbol{y}$. Note that

$$
\begin{aligned}
\|\hat{\boldsymbol{z}}\|^2 = \|2\alpha\hat{\boldsymbol{x}} - \gamma\hat{\boldsymbol{y}}\|^2 &= 4\alpha^2\|\hat{\boldsymbol{x}}\|^2 - 4\alpha\gamma\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}} + \gamma^2\|\hat{\boldsymbol{y}}\|^2,\\
\|\bar{\boldsymbol{z}}\|^2 = \|2\alpha\bar{\boldsymbol{x}} + \gamma\bar{\boldsymbol{y}}\|^2 &= 4\alpha^2\|\bar{\boldsymbol{x}}\|^2 + 4\alpha\gamma\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}} + \gamma^2\|\bar{\boldsymbol{y}}\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\det(\mathsf{P}_{\boldsymbol{x}}\boldsymbol{y}) = \det(\boldsymbol{z})\\
&= \|\hat{\boldsymbol{z}}\|^2 - \|\bar{\boldsymbol{z}}\|^2\\
&= 4\alpha^2\|\hat{\boldsymbol{x}}\|^2 - 4\alpha\gamma\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}} + \gamma^2\|\hat{\boldsymbol{y}}\|^2 - \left(4\alpha^2\|\bar{\boldsymbol{x}}\|^2 + 4\alpha\gamma\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}} + \gamma^2\|\bar{\boldsymbol{y}}\|^2\right)\\
&= 4\alpha^2\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) - 4\alpha\gamma\left(\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{y}} + \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\right) + \gamma^2\left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right)\\
&= 4\alpha^2\gamma - 4\alpha^2\gamma + \gamma^2\left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right)\\
&= \gamma^2\left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) \;=\; (\det(\boldsymbol{x}))^2\det(\boldsymbol{y}).
\end{aligned}
$$

This completes the proof. □

We mentioned that the $\mathsf{P}_{\cdot}$ operator is important because it is used to express the Hessian of the cone's barrier. Likewise, the operator $\mathsf{P}_{\cdot,\cdot}$ is also of fundamental importance because it is being used to express the third derivative of the logarithmic barrier function associated with the cone (see Section 6). The following theorem presents some fundamental properties of $\mathsf{P}_{\cdot,\cdot}$ operator.

Theorem 5.3 *Let* $\boldsymbol{x} \in (\mathcal{M}^{m|n}, \circledcirc)$*. Assume also that* $\boldsymbol{x}^g$ *exists (i.e.,* $\det(\boldsymbol{x}) \neq 0$*) wherever it is necessary. Then*

(i) $\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}}\boldsymbol{x}^g = \mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^g}\boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{e}(\boldsymbol{x})$.
(ii) $\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}}\boldsymbol{e}(\boldsymbol{x}) = \mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^g}\boldsymbol{x} = \boldsymbol{x}$.
(iii) $\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^g}\boldsymbol{x}^g = \boldsymbol{x}^g$.
(iv) $\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}}\boldsymbol{x} = \boldsymbol{x}^2$.

Proof Note that, from the definition of $\mathsf{P}_{\cdot,\cdot}$, we have

$$\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}} = \begin{bmatrix} 2\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}^\mathsf{T} - \|\hat{\boldsymbol{x}}\|\, I_m & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^\mathsf{T} \\ \bar{\boldsymbol{x}}\left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^\mathsf{T} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix}, \text{ and}$$

$$\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}} = \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} - \|\boldsymbol{x}\|^2 I_m & O \\ O & -2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \|\boldsymbol{x}\|^2 I_n \end{bmatrix}.$$

We are ready to prove the theorem item by item.

(i) If $\det(\boldsymbol{x}) \neq 0$, we have

$$\begin{aligned}\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}}\boldsymbol{x}^{\mathrm{g}} &= \begin{bmatrix} 2\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}^\mathsf{T} - \|\hat{\boldsymbol{x}}\|\, I_m & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^\mathsf{T} \\ \bar{\boldsymbol{x}}\left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^\mathsf{T} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix} \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} \hat{\boldsymbol{x}} \\ -\bar{\boldsymbol{x}} \end{bmatrix} \\ &= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} \dfrac{2\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} - \|\hat{\boldsymbol{x}}\|\,\hat{\boldsymbol{x}} - \dfrac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} \\ \dfrac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}} - \|\hat{\boldsymbol{x}}\|\,\bar{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{e}(\boldsymbol{x}).\end{aligned}$$

Similarly, a straightforward computation can show that $\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{e}(\boldsymbol{x})$.

(ii) Note that

$$\begin{aligned}\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}}\boldsymbol{e}(\boldsymbol{x}) &= \begin{bmatrix} 2\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}^\mathsf{T} - \|\hat{\boldsymbol{x}}\|\, I_m & \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^\mathsf{T} \\ \bar{\boldsymbol{x}}\left(\dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^\mathsf{T} & \|\hat{\boldsymbol{x}}\|\, I_n \end{bmatrix}\begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \boldsymbol{0} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{2\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2}\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}} \\ \dfrac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|^2}\bar{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}} \\ \bar{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{x}.\end{aligned}$$

Similarly, a straightforward computation can show that $\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{x} = \boldsymbol{x}$ provided that $\det(\boldsymbol{x}) \neq 0$.

(iii) If $\det(\boldsymbol{x}) \neq 0$, we get

$$\begin{aligned}\mathsf{P}_{\boldsymbol{x},\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{x}^{\mathrm{g}} =& \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix}2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \|\boldsymbol{x}\|^2 I_m & O \\ O & -2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \|\boldsymbol{x}\|^2 I_n\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}} \\ -\bar{\boldsymbol{x}}\end{bmatrix} \\ =& \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix}2\|\hat{\boldsymbol{x}}\|^2\hat{\boldsymbol{x}} - \|\boldsymbol{x}\|^2\hat{\boldsymbol{x}} \\ 2\|\bar{\boldsymbol{x}}\|^2\bar{\boldsymbol{x}} - \|\boldsymbol{x}\|^2\bar{\boldsymbol{x}}\end{bmatrix} \\ =& \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix}\hat{\boldsymbol{x}} \\ -\bar{\boldsymbol{x}}\end{bmatrix} = \boldsymbol{x}^{\mathrm{g}}.\end{aligned}$$

(iv) With a little computation, we have

$$\begin{aligned}\mathsf{P}_{\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{x}}\boldsymbol{x} =& \begin{bmatrix}2\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}}^{\mathsf{T}} - \|\hat{\boldsymbol{x}}\| I_m & \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}}\left(\frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|}\right)^{\mathsf{T}} & \|\hat{\boldsymbol{x}}\| I_n\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}} \\ \bar{\boldsymbol{x}}\end{bmatrix} \\ =& \begin{bmatrix}\frac{2\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} - \|\hat{\boldsymbol{x}}\|\hat{\boldsymbol{x}} + \frac{\|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} \\ \frac{\|\hat{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\bar{\boldsymbol{x}} + \|\hat{\boldsymbol{x}}\|\bar{\boldsymbol{x}}\end{bmatrix} \\ =& \begin{bmatrix}\frac{\|\hat{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{x}}\|^2}{\|\hat{\boldsymbol{x}}\|}\hat{\boldsymbol{x}} \\ 2\|\hat{\boldsymbol{x}}\|\bar{\boldsymbol{x}}\end{bmatrix} = \boldsymbol{x}^2\end{aligned}$$

as desired. The proof is complete.

□

In the following theorem, we explore some characterizations of spectral values, determinant, and trace of $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$. The inequalities and identities stated in the following theorem are parallel results analogous to those associated with the cone of positive semidefinite matrices [20]. These results are also applied to the convex cone $\mathcal{E}_+^{n+1}$ (see also [3]).

Theorem 5.4 *Let* $\boldsymbol{x}, \boldsymbol{y} \in (\mathcal{M}^{m|n}, \circledcirc)$, *and* $\alpha, \beta \in \mathbb{R}$. *Then*

(i) $\operatorname{trace}(\alpha\boldsymbol{x}) = |\alpha| \operatorname{trace}(\boldsymbol{x})$.
(ii) $\operatorname{trace}(\boldsymbol{x}+\boldsymbol{y}) \leq \operatorname{trace}(\boldsymbol{x}) + \operatorname{trace}(\boldsymbol{y})$ *with equality holds if and only if* $m = 1$ *and* $x_1, y_1 \geq 0$.
(iii) $\det(\alpha\boldsymbol{x}) = \alpha^2 \det(\boldsymbol{x})$.
(iv) $\det(\boldsymbol{x}+\boldsymbol{y}) \leq ((\operatorname{trace}(\boldsymbol{x}))^2 + (\operatorname{trace}(\boldsymbol{y}))^2)/2$. *Moreover, if* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}_+^{m|n}$, $|\det(\boldsymbol{x}+\boldsymbol{y})| \leq ((\operatorname{trace}(\boldsymbol{x}))^2 + (\operatorname{trace}(\boldsymbol{y}))^2)/2$.
(v) $\det(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) \leq (\alpha^2(\operatorname{trace}(\boldsymbol{x}))^2 + \beta^2(\operatorname{trace}(\boldsymbol{y}))^2)/2$. *Moreover,* $|\det(\alpha\boldsymbol{x} + \beta\boldsymbol{y})| \leq (\alpha^2(\operatorname{trace}(\boldsymbol{x}))^2 + \beta^2(\operatorname{trace}(\boldsymbol{y}))^2)/2$ *provided that* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}_+^{m|n}$ *and* $\alpha, \beta \geq 0$.
(vi) $\det(\boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{x}) \geq (1 + \det(\boldsymbol{x}^{1/2}))^2$, *provided that* $\boldsymbol{x}^{1/2}$ *is defined (i.e.,* $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$*).*
(vii) $\det(\boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{x} + \boldsymbol{e}(\boldsymbol{y}) + \boldsymbol{y}) \leq 2((1 + \|\hat{\boldsymbol{x}}\|)^2 + (1 + \|\hat{\boldsymbol{y}}\|)^2)$.

Proof We prove the theorem item by item. For any $\boldsymbol{x}, \boldsymbol{y} \in (\mathcal{M}^{m|n}, \circledcirc)$, and $\alpha, \beta \in \mathbb{R}$, we have

(i) $\operatorname{trace}(\alpha\boldsymbol{x}) = 2\|\widehat{\alpha\boldsymbol{x}}\| = 2\|\alpha\hat{\boldsymbol{x}}\| = 2|\alpha|\,\|\hat{\boldsymbol{x}}\| = |\alpha|\operatorname{trace}(\boldsymbol{x})$.

(ii) $\operatorname{trace}(\boldsymbol{x}+\boldsymbol{y}) = 2\|\widehat{\boldsymbol{x}+\boldsymbol{y}}\| = 2\|\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\| \le 2(\|\hat{\boldsymbol{x}}\|+\|\hat{\boldsymbol{y}}\|) = \operatorname{trace}(\boldsymbol{x}) + \operatorname{trace}(\boldsymbol{y})$. Clearly, the inequality becomes an identity if and only if $\hat{\boldsymbol{x}} = x_1 \ge 0$ and $\hat{\boldsymbol{y}} = y_1 \ge 0$. In this case, $\operatorname{trace}(\boldsymbol{x}+\boldsymbol{y}) = 2(x_1+y_1) = \operatorname{trace}(\boldsymbol{x}) + \operatorname{trace}(\boldsymbol{y})$.

(iii) $\det(\alpha\boldsymbol{x}) = \|\widehat{\alpha\boldsymbol{x}}\|^2 - \|\overline{\alpha\boldsymbol{x}}\|^2 = \|\alpha\hat{\boldsymbol{x}}\|^2 - \|\alpha\bar{\boldsymbol{x}}\|^2 = \alpha^2(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2) = \alpha^2\det(\boldsymbol{x})$.

(iv) We have that

$$\begin{aligned}
&\det(\boldsymbol{x}+\boldsymbol{y})\\
&= \left\|\widehat{\boldsymbol{x}+\boldsymbol{y}}\right\|^2 - \|\overline{\boldsymbol{x}+\boldsymbol{y}}\|^2\\
&= \|\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}\|^2\\
&= \|\hat{\boldsymbol{x}}\|^2 + 2\langle\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}\rangle + \|\hat{\boldsymbol{y}}\|^2 - \left(\|\bar{\boldsymbol{x}}\|^2 + 2\langle\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}\rangle + \|\bar{\boldsymbol{y}}\|^2\right)\\
&= \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) + \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) + 2\langle\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}\rangle + 2\langle-\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}\rangle\\
&\le \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) + \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) + 2\,\|\hat{\boldsymbol{x}}\|\,\|\hat{\boldsymbol{y}}\| + 2\,\|\bar{\boldsymbol{x}}\|\,\|\bar{\boldsymbol{y}}\|\\
&\le \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) + \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) + \|\hat{\boldsymbol{x}}\|^2 + \|\hat{\boldsymbol{y}}\|^2 + \|\bar{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{y}}\|^2\\
&= 2\,\|\hat{\boldsymbol{x}}\|^2 + 2\,\|\hat{\boldsymbol{y}}\|^2 = \frac{1}{2}\left((\operatorname{trace}(\boldsymbol{x}))^2 + (\operatorname{trace}(\boldsymbol{y}))^2\right).
\end{aligned}$$

In addition, if $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}_+^{m|n}$, we also have that

$$\begin{aligned}
&\det(\boldsymbol{x}+\boldsymbol{y})\\
&= \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) + \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) + 2\langle\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}\rangle + 2\langle-\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}\rangle\\
&\ge \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) + \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) - 2\,\|\hat{\boldsymbol{x}}\|\,\|\hat{\boldsymbol{y}}\| - 2\,\|\bar{\boldsymbol{x}}\|\,\|\bar{\boldsymbol{y}}\|\\
&\ge \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) + \left(\|\hat{\boldsymbol{y}}\|^2 - \|\bar{\boldsymbol{y}}\|^2\right) - \left(\|\hat{\boldsymbol{x}}\|^2 + \|\hat{\boldsymbol{y}}\|^2\right) - \left(\|\bar{\boldsymbol{x}}\|^2 + \|\bar{\boldsymbol{y}}\|^2\right)\\
&= -2\,\|\bar{\boldsymbol{x}}\|^2 - 2\,\|\bar{\boldsymbol{y}}\|^2\\
&\ge -2\,\|\hat{\boldsymbol{x}}\|^2 - 2\,\|\hat{\boldsymbol{y}}\|^2 = -\frac{1}{2}\left((\operatorname{trace}(\boldsymbol{x}))^2 + (\operatorname{trace}(\boldsymbol{y}))^2\right),
\end{aligned}$$

where the last inequality follows from the fact that $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}_+^{m|n}$. Thus,

$$|\det(\boldsymbol{x}+\boldsymbol{y})| \le \frac{1}{2}\left((\operatorname{trace}(\boldsymbol{x}))^2 + (\operatorname{trace}(\boldsymbol{y}))^2\right).$$

(v) The result immediately follows from items (i) and (iv).

(vi) Note that

$$\boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{x} = \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} + \hat{\boldsymbol{x}} \\ \bar{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} (\|\hat{\boldsymbol{x}}\| + 1)\, \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \\ \bar{\boldsymbol{x}} \end{bmatrix}.$$

If $\boldsymbol{x} \in \mathcal{M}_+^{m|n}$ is defined (i.e., $\lambda_{1,2}(\boldsymbol{x}) \geq 0$), then

$$\begin{aligned} \det\left(\boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{x}\right) &= \left\| (\|\hat{\boldsymbol{x}}\| + 1)\, \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} \right\|^2 - \|\bar{\boldsymbol{x}}\|^2 \\ &= (\|\hat{\boldsymbol{x}}\| + 1)^2 - \|\bar{\boldsymbol{x}}\|^2 \\ &= 1 + 2\,\|\hat{\boldsymbol{x}}\| + \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) \\ &= 1 + \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) + \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}) \\ &\geq 1 + 2\sqrt{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})} + \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x}) \\ &= \left(1 + \sqrt{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})}\right)^2 = \left(1 + \det\left(\boldsymbol{x}^{\frac{1}{2}}\right)\right)^2. \end{aligned}$$

(vii) Note that $\operatorname{trace}(\boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{x}) = 2(1 + \|\hat{\boldsymbol{x}}\|)$. The inequality in this item immediately follows from item (iv). The proof of the theorem is now complete.

□

Items (i) and (ii) in Theorem 5.4 proves the convexity of the function trace(·) on any convex subset of the space $\mathcal{M}^{m|n}$.

6 The Barrier Function Associated With the Cone

In optimization, particularly conic programming, self-concordant functions [22] are useful in the analysis of Newton's method. More specifically, self-concordant barriers satisfy certain smoothness conditions, allowing potential efficient interior-point algorithms to solve the underlying optimization problem. Such conditions are given in terms of the Hessian and the third derivatives of the cone's barrier. In this section, we introduce the logarithmic barrier function associated with the nonconvex SOC, compute its derivatives, and introduce a class of optimization problems over nonconvex SOCs.

Following the standard way of defining the logarithmic barriers in conic programming, we define the logarithmic barrier associated with the nonconvex SOC $\ell : \operatorname{int} \mathcal{M}_+^{m|n} \longrightarrow \mathbb{R}$ as

$$\ell(\boldsymbol{x}) := -\ln\det(\boldsymbol{x}) = -\ln\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right). \tag{17}$$

The barrier function proposed in this section is the counterpart of very well-known barriers in the interior-point theory of conic programming; see Table 5.

Table 5: The barriers, dimensions, and ranks of the most well-known cones.

Cone	Constraint	Log Barrier	Ambient dimension	Rank
The nonnegative orthant cone	$\boldsymbol{x} \in \mathbb{R}^n_+$	$-\sum_{i=1}^n \ln x_i$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
The convex SOC	$\boldsymbol{x} \in \mathcal{E}^{n+1}_+$	$-\ln\left(x_1^2 - \\|\bar{\boldsymbol{x}}\\|^2\right)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$
The cone of positive semidefinite matrices	$X \in \mathcal{P}^n_+$	$-\ln\det(X)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n)$
The nonconvex SOC	$\boldsymbol{x} \in \mathcal{M}^{m\|n}_+$	$-\ln\left(\\|\hat{\boldsymbol{x}}\\|^2 - \\|\bar{\boldsymbol{x}}\\|^2\right)$	$\mathcal{O}(n+m)$	$\mathcal{O}(1)$

Items (i) and (ii) in Theorem 6.1 generalize the corresponding ones in [1, Theorem 8 (item 6)] for convex SOC (see also [4]).

Theorem 6.1 *Let $\boldsymbol{x} \in int\,\mathcal{M}^{m|n}_+$ and $\boldsymbol{v} \in \mathcal{M}^{m|n}$. Then*

(i) The gradient $\nabla_{\boldsymbol{x}} \ell(\boldsymbol{x}) = -2\boldsymbol{x}^g$.
(ii) The Hessian $\nabla^2_{\boldsymbol{xx}} \ell(\boldsymbol{x}) = 2\mathsf{P}_{\boldsymbol{x}^g}$.
(iii) The Jacobian $\mathsf{J}_{\boldsymbol{x}} \mathsf{P}_{\boldsymbol{x}} \boldsymbol{v} = 2\mathsf{P}_{\boldsymbol{x},\boldsymbol{v}}$.
(iv) The third derivative $\nabla^3_{\boldsymbol{xxx}} \ell(\boldsymbol{x})[\boldsymbol{v}] = -4\mathsf{P}_{\mathsf{P}_{\boldsymbol{x}^g}\boldsymbol{v},\boldsymbol{x}^g}$.

Proof The logarithmic barrier function is defined as

$$\ell(\boldsymbol{x}) = -\log\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) = -\log\left(\hat{\boldsymbol{x}}^\mathsf{T}\hat{\boldsymbol{x}} - \bar{\boldsymbol{x}}^\mathsf{T}\bar{\boldsymbol{x}}\right).$$

Since $\det(\boldsymbol{x}) = \|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2 > 0$, we have

$$\nabla_{\boldsymbol{x}} \ell(\boldsymbol{x}) = \begin{bmatrix} \dfrac{-2\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2} \\ \dfrac{2\bar{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2} \end{bmatrix} = \frac{-1}{\det(\boldsymbol{x})}\begin{bmatrix} 2\hat{\boldsymbol{x}} \\ -2\bar{\boldsymbol{x}} \end{bmatrix} = \frac{-2}{\det(\boldsymbol{x})} R^{m|n}\boldsymbol{x} = -2\boldsymbol{x}^{\mathrm{g}}.$$

The result in item (i) is obtained.

To prove item (ii), note that the Jacobian of $\boldsymbol{x}^{\mathrm{g}}$ is

$$\begin{aligned}
\mathsf{J}_{\boldsymbol{x}} \boldsymbol{x}^{\mathrm{g}} &= \mathsf{J}_{\boldsymbol{x}} \begin{bmatrix} \dfrac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2} \\ \dfrac{-\bar{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2} I_m - \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & -\dfrac{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2} I_n - \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix} \\
&= \frac{1}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2} \begin{bmatrix} \left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_m - 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & -\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right) I_n - 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \end{bmatrix} \\
&= \frac{1}{(\det(\boldsymbol{x}))^2} \begin{bmatrix} -2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} + \det(\boldsymbol{x}) I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^\mathsf{T} & -\left(2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^\mathsf{T} + \det(\boldsymbol{x}) I_n\right) \end{bmatrix} = -\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}.
\end{aligned}$$

As a result, $\nabla^2_{\boldsymbol{xx}} \ell(\boldsymbol{x}) = -2\mathsf{J}_{\boldsymbol{x}} \boldsymbol{x}^{\mathrm{g}} = 2\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}$. This proves item (ii).

Moving to item (iii), note that

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x}}\boldsymbol{v} &= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} - \det(\boldsymbol{x})I_m & 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{v}} \\ \bar{\boldsymbol{v}} \end{bmatrix} \\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} - \det(\boldsymbol{x})\hat{\boldsymbol{v}} + 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} + 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} + \det(\boldsymbol{x})\bar{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} - \|\hat{\boldsymbol{x}}\|^2\hat{\boldsymbol{v}} + \|\bar{\boldsymbol{x}}\|^2\hat{\boldsymbol{v}} + 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} + 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} + \|\hat{\boldsymbol{x}}\|^2\bar{\boldsymbol{v}} - \|\bar{\boldsymbol{x}}\|^2\bar{\boldsymbol{v}} \end{bmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathsf{J}_{\boldsymbol{x}}\mathsf{P}_{\boldsymbol{x}}\boldsymbol{v} \\
&= \begin{bmatrix} 2\hat{\boldsymbol{x}}\hat{\boldsymbol{v}}^{\mathsf{T}} + 2\hat{\boldsymbol{v}}\hat{\boldsymbol{x}}^{\mathsf{T}} - 2\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}}I_m + 2\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}}I_m & 2\hat{\boldsymbol{v}}\bar{\boldsymbol{x}}^{\mathsf{T}} + 2\hat{\boldsymbol{x}}\bar{\boldsymbol{v}}^{\mathsf{T}} \\ 2\bar{\boldsymbol{x}}\hat{\boldsymbol{v}}^{\mathsf{T}} + 2\bar{\boldsymbol{v}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}}I_n + 2\bar{\boldsymbol{x}}\bar{\boldsymbol{v}}^{\mathsf{T}} + 2\bar{\boldsymbol{v}}\bar{\boldsymbol{x}}^{\mathsf{T}} - 2\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}}I_n \end{bmatrix} \\
&= 2\begin{bmatrix} \hat{\boldsymbol{x}}\hat{\boldsymbol{v}}^{\mathsf{T}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{x}}^{\mathsf{T}} + \left(\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} - \hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}}\right)I_m & \hat{\boldsymbol{v}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \hat{\boldsymbol{x}}\bar{\boldsymbol{v}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}}\hat{\boldsymbol{v}}^{\mathsf{T}} + \bar{\boldsymbol{v}}\hat{\boldsymbol{x}}^{\mathsf{T}} & \bar{\boldsymbol{x}}\bar{\boldsymbol{v}}^{\mathsf{T}} + \bar{\boldsymbol{v}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \left(\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} - \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}}\right)I_n \end{bmatrix} = 2\mathsf{P}_{\boldsymbol{x},\boldsymbol{v}}.
\end{aligned}$$

The result in item (iii) is obtained.

Now, we compute the third derivative of $\ell(\boldsymbol{x})$. Note that

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{v} &= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -\det(\boldsymbol{x})I_m + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & -2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} \\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}} & 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}} + \det(\boldsymbol{x})I_n \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{v}} \\ \bar{\boldsymbol{v}} \end{bmatrix} \\
&= \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -\det(\boldsymbol{x})\hat{\boldsymbol{v}} + 2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} - 2\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} \\ -2\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} + 2\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} + \det(\boldsymbol{x})\bar{\boldsymbol{v}} \end{bmatrix} \\
&= \begin{bmatrix} -\dfrac{1}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\hat{\boldsymbol{v}} + \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} - \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\hat{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} \\ -\dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\bar{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{v}} + \dfrac{2}{\left(\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2\right)^2}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{v}} + \dfrac{1}{\|\hat{\boldsymbol{x}}\|^2 - \|\bar{\boldsymbol{x}}\|^2}\bar{\boldsymbol{v}} \end{bmatrix}.
\end{aligned}$$

Now, by taking some lengthy but not complicated calculations, one can show that $\mathsf{J}_{\boldsymbol{x}}\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{v} = -2\mathsf{P}_{\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{v},\boldsymbol{x}^{\mathrm{g}}}$. Thus, $\nabla^3_{\boldsymbol{xxx}}\ell(\boldsymbol{x}) = 2\mathsf{J}_{\boldsymbol{x}}\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}[\boldsymbol{v}] = -4\mathsf{P}_{\mathsf{P}_{\boldsymbol{x}^{\mathrm{g}}}\boldsymbol{v},\boldsymbol{x}^{\mathrm{g}}}$ as desired. The proof is complete. □

The new vectors, matrices, traces, determinants, and functions introduced in this paper can also be introduced in a block setting. After the primary advances that have been made in the preceding sections, what might be termed "nonconvex second-order cone programming" can now be defined to exclusively deal with nonconvex optimization problems in which a linear objective function is minimized over the intersection of an affine linear manifold with the Cartesian product of nonconvex SOCs. See Figure 4 which shows a plane intersecting the novconvex SOC $\mathcal{M}_{+}^{1|2}$.

The construction of the nonconvex second-order cone programming as a new class of optimization problems is broad enough to include challenging problems such as nonconvex quadratic programs, quadratically constrained nonconvex quadratic programs, and reverse convex programs as special cases. On the other side, the results in Theorem 5.1 give a certificate that the nonconvex second-order cone programming is a special case of the nonlinear semidefinite programming [12,17,24,28,29,33]. In Figure 5 we show a Venn diagram of seven different classes of optimization problems. However, similar to (and not less importantly than) the convex second-order cone programming [1, 2, 5, 7] (which is a special case of the linear semidefinite programming [30]), we believe that the nonconvex second-order cone programming warrants its own study and requires special-purpose algorithmic methodologies.

For $i = 1, 2, \ldots, r$, let n_i, m_i and l be positive integers, $\boldsymbol{b} \in \mathbb{R}^l$, $\boldsymbol{c}_i, \boldsymbol{x}_i \in \mathcal{M}^{m_i|n_i}$, and $A_i \in \mathbb{R}^{l\times(m_i+n_i)}$. A typical nonconvex second-order cone programming problem looks like this:

$$\begin{array}{ll} \min & \boldsymbol{c}_1^{\mathsf{T}}\boldsymbol{x}_1 + \boldsymbol{c}_2^{\mathsf{T}}\boldsymbol{x}_2 + \cdots + \boldsymbol{c}_r^{\mathsf{T}}\boldsymbol{x}_r \\ \text{s.t.} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 + \cdots + A_r\boldsymbol{x}_r = \boldsymbol{b}, \\ & \boldsymbol{x}_i \in \mathcal{M}_+^{m_i|n_i},\ i = 1, 2, \ldots, r. \end{array} \tag{18}$$

Let $\mu > 0$ be a barrier parameter. Using the logarithmic barrier function $\ell(\cdot)$ defined in (17), the conicity constraints of Problem (18) are then replaced with a logarithmic barrier term in the objective function, resulting in the barrier problem:

$$\begin{array}{ll} \min & \displaystyle\sum_{i=1}^{r} \boldsymbol{c}_i^{\mathsf{T}}\boldsymbol{x}_i + \mu \sum_{i=1}^{r} \ell(\boldsymbol{x}_i) \\ \text{s.t.} & \displaystyle\sum_{i=1}^{r} A_i\boldsymbol{x}_i - \boldsymbol{b} = \boldsymbol{0}. \end{array} \tag{19}$$

The Lagrangian for Problem (19) is

$$L(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{r} \boldsymbol{c}_i^{\mathsf{T}}\boldsymbol{x}_i + \mu \sum_{i=1}^{r} \ell(\boldsymbol{x}_i) - \boldsymbol{y}^{\mathsf{T}} \left(\sum_{i=1}^{r} A_i\boldsymbol{x}_i - \boldsymbol{b} \right). \tag{20}$$

The first-order optimality conditions for a minimum as well as a weak version of the second-order sufficient optimality conditions are determined by computing the derivatives of $L(\cdot, \cdot)$, which in turn require the derivatives of the barrier function $\ell(\cdot)$ that were calculated in Theorem 6.1.

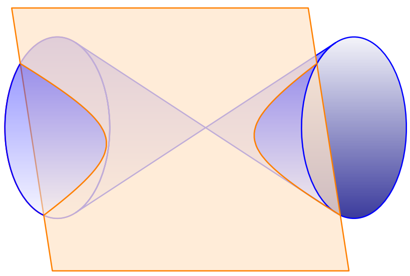


Fig. 4: The nonconvex second-order cone programming manifests in the intersection of a nonconvex SOC with an affine linear manifold.

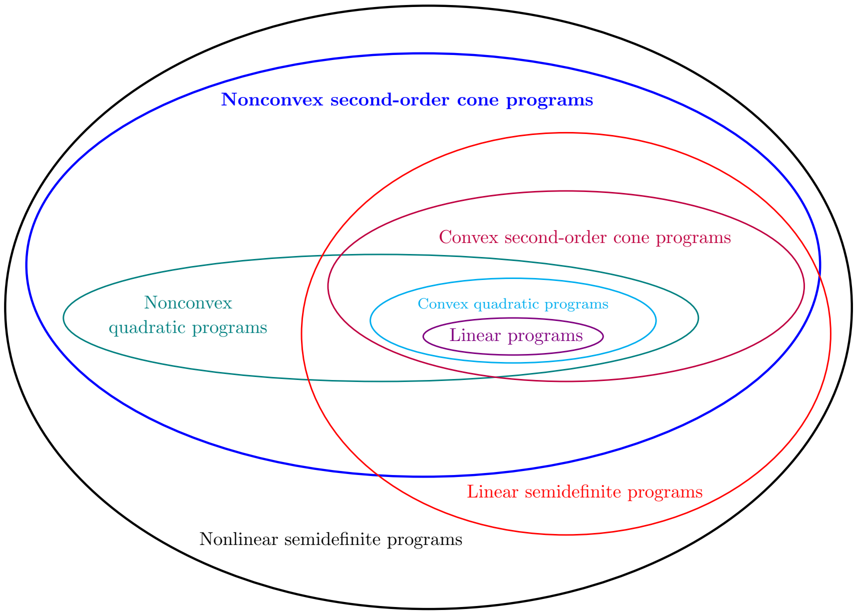


Fig. 5: Graphical relationships among different classes of optimization problems.

7 Conclusions and Open Questions

A mathematical foundation of what we termed the "nonconvex second-order cone" (nonconvex SOC) has been established in this paper. The foundation of this nonconvex cone enjoys many advantages and algebraic properties incomparable with those of other nonconvex cones or even arbitrary convex cones. Among such advantages and properties, we distinguish four of them:

A1. All the elements in the underlying magma of the cone have real eigenvalues. This is not the case for arbitrary commutative power-associative algebras or even arbitrary Jordan algebras.
A2. The rank of the underlying magma of the cone is independent of its dimension. This is not the case for algebras of arbitrary convex cones or even arbitrary symmetric cones.
A3. The nonconvex SOC equals the cone of squares of its underlying magma.
A4. The derivatives of the logarithmic barrier function associated with the cone have explicit expressions. This is not the case for the barriers of arbitrary convex nonsymmetric cones.

This foundation, however, has some disadvantages or limitations. We highlight three of them:

D1. The underlying magma of the cone with the binary operation "$\circledcirc$" fails to be an algebra. However, our results have shown that our structure is a commutative power-associative magma and the submagma generated by any element is an associative algebra.

D2. The underlying magma of the cone disregards being unitary. However, the submagma generated by any element is unitary-like, and more importantly, we were able to find identity-like elements in the magma that made the generalized invertibility holds for the interior of the cone.

D3. The map trace$(\boldsymbol{x} \circledcirc \boldsymbol{y})$ rejects being an inner product in the vector space of the nonconvex SOC. In contrast, the map trace$(\boldsymbol{x} \circ \boldsymbol{y})$ is indeed an inner product in the vector space of the convex SOC due to the linearity of the function trace$(\cdot)$ in the space. However, it is noted that trace$(\boldsymbol{x} \circledcirc \boldsymbol{y})$ is a positive definite, symmetric.

Despite the above limitations, generalizing old concepts such as the determinant, spectral factorization, square power vector, and the quadratic representation, plus adopting new concepts such as the identity-like element, generalized inverse, and the crane-shaped matrix, were sufficient for our subsequent purposes in the paper. More specifically, we succeeded in generalizing many fundamental algebraic properties that already exist in the framework of the algebra of the convex SOC to the framework of the magma of the nonconvex SOC, and we also succeeded in writing explicit expressions for the gradient, Hessian, and the third derivative of the logarithmic barrier function associated with the cone.

Our paper puts forth the following open questions:

OQ1. Is the nonconvex SOC homogeneous?
This question is factual and important to emphasize because homogeneous, but not necessarily convex, cones have proven useful in applications (cf. [14,19,21]). So, an important open question in our research line is whether it is possible to prove the homogeneity of the nonconvex SOC.

OQ2. Is the Jordan identity satisfied in the underlying magma of the cone?
This question is also factual as well as normative because the Jordan identity is a special case of the associative law, which is a property of binary operations in magmas, not only algebras. Like the algebra of the convex SOC, the magma of the nonconvex SOC loses the associativity property. Theorem 4.3 evidences that $\mathsf{Crn}(\boldsymbol{x})$ and $\mathsf{Crn}(\boldsymbol{x}^2)$ commute. This is great because it generalizes the fact that $\mathsf{Arw}(\boldsymbol{x})$ and $\mathsf{Arw}(\boldsymbol{x}^2)$ commute, which demonstrates the satisfaction of Jordan identity for the algebra of the convex SOC. We indicated in Section 4 that the result in Theorem 4.3 does not demonstrate the satisfaction of the Jordan identity for the magma of the nonconvex SOC. In other words, we can say that Theorem 4.3 (as well as Lemma 4.2) would highly support, but does not replace, the Jordan identity for our magma. So, an interesting but challenging open question in this research line is whether it is possible to prove the Jordan identity in the underlying magma of our cone.

Because it does not seem easy to answer OQ1 and OQ2 in the affirmative or in the negative, we leave them for future research. From an optimization point of view, it is now possible to extend locally the concept of self-concordance to the nonconvex second-order cone programming problem (18) (see, for example, [18, 23]). The entire analysis of future research work would also be based on this important framework. Over and above, in a companion paper [8], the vector-valued functions associated with the nonconvex SOC are defined analogously to those for convex SOC (see [11]) and also used in solutions methods for nonconvex second-order cone programs and nonconvex second-order cone complementarity problems.

We believe that the development of this paper reveals an exploration of relations among algebra, analysis, and optimization. It is also our belief that this paper has high academic value and will significantly influence researchers in optimization and other disciplines.

References

1. Alizadeh, F., Goldfarb, D.: Second-order cone programming. Math. Program. 95, 3–51 (2003)
2. Alzalg, B.: Decomposition-based interior point methods for stochastic quadratic second-order cone programming. Appl. Math. Comput. 249, 1-18 (2014)
3. Alzalg, B.: The Jordan algebraic structure of the circular cone. Oper. Matrices, 11, 1–21 (2017)
4. Alzalg, B.: Primal interior-point decomposition algorithms for two-stage stochastic extended second-order cone programming. Optim. 67, 2291-2323 (2018)
5. Alzalg, B., Alioui, H.: Applications of stochastic mixed-integer second-order cone optimization. IEEE Access. 10, 3522-3547 (2022)
6. Alzalg, B., Pirhaji, M.: Elliptic cone optimization and primal–dual path-following algorithms. Optim. 66, 2245-2274 (2017)
7. Alzalg, B.M.: Stochastic second-order cone programming: Applications models. Appl. Math. Model. 36, 5122-5134 (2012)
8. Benakkouche, L., Alzalg, B.: Functions associated with the nonconvex second-order cone. Submitted for publication (2023)
9. Benson, H.Y., Shanno, D.F.: Interior-point methods for nonconvex nonlinear programming: regularization and warmstarts. Comput. Optim. Appl. 40, 143–189 (2008)
10. Bourbaki, N.: Algèbre: Chapitres 1 à 3, Springer Science & Business Media (2007)
11. Chen, J-Sh.: The convex and monotone functions associated with second-order cone. Optim. 55, 363–385 (2006)
12. Correa, R.: A global algorithm for nonlinear semidefinite programming. SIAM J. Optim. 1, 303–318 (2004)
13. Curtis, F.E., Schenk, O., Wächter, A.: An interior-point algorithm for large-scale nonlinear optimization with inexact step computations. author=Curtis, Frank E and Schenk, Olaf and Wächter, Andreas. SIAM J. Sci. Comput. 32, 3447–3475 (2010)
14. Faraut, J., Gindikin, S.: Pseudo-Hermitian symmetric spaces of tube type. In Memory of Jozef D'atri (ed.): Topics in geometry, pp. 123–154. Springer (1996)
15. Faraut, J., Korányi, A.: Analysis on symmetric cones, The Clarendon Press Oxford University Press, New York (1994)
16. Forsgren, A., Gill, Ph.E.: Primal-dual interior methods for nonconvex nonlinear programming. SIAM J. Optim. 8, 1132–1152 (1998)
17. Freund, R.W., Jarre, F., Vogelbusch, Ch.H.: Nonlinear semidefinite programming: sensitivity, convergence, and an application in passive reduced-order modeling. Math. Program. 109, 581–611 (2007)

18. Garcés, R., Gómez, W., Jarre, F.: A self-concordance property for nonconvex semidefinite programming. Math. Oper. Res. 74, 77–92, (2011)
19. Gindikin, S.G.: Analysis inhomogeneous domains. Russ. Math. Surv. 19, 1–89 (1964)
20. Horn, R.A, Johnson, Ch.R.: Matrix analysis, Cambridge University Press (1985)
21. Koecher, M.: The Minnesota notes on Jordan algebras and their applications, Springer Science & Business Media (1999)
22. Nesterov, Y., Nemirovskii, A.: Interior-point polynomial algorithms in convex programming, Society for Industrial and Applied Mathematics (1994)
23. Neuenhofen, M.: Weakly polynomial efficient minimization of a non-convex quadratic function with logarithmic barriers in a trust-region. arXiv preprint arXiv:1806.06936 (2018)
24. Qi, H.: Local duality of nonlinear semidefinite programming. Math. Oper. Res. 34, 124–141 (2009)
25. Renardy. M.: Singular value decomposition in Minkowski space. Linear Algebra Appl. 236, 53-58 (1996)
26. Schmieta, S.H and Alizadeh, F.: Extension of primal-dual interior point algorithms to symmetric cones. Math. Program. 96, 409–438 (2003)
27. Shanno, D.F, Vanderbei, R.J: Interior-point methods for nonconvex nonlinear programming: orderings and higher-order methods. Math. Program. 87, 303–316 (2000)
28. Sun, D.: The strong second-order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications. Math. Oper. Res. 31, 761–776 (2006)
29. Sun, W., Li, Ch., Sampaio, R. JB.: On duality theory for non-convex semidefinite programming. 186, 331–343 (2011)
30. Todd, M. J.: Semidefinite optimization, Acta Numerica, Cambridge University Press (2001)
31. Tuy, H.: Nonconvex quadratic programming. In Memory of Jozef D'atri (ed.): Convex Analysis and Global Optimization, pp. 337–390. Springer (2016)
32. Vanderbei, R.J, Shanno, D.F.: An interior-point algorithm for nonconvex nonlinear programming. Comput. Optim. Appl. 13, 231–252 (1999)
33. Zhang, L., Li, Y., Wu, J.: Nonlinear rescaling Lagrangians for nonconvex semidefinite programming. Optim. 63, 899–920 (2014)