Abstract  We call an optimization problem an optimization problem with controllable uncertainty if a) it contains uncertain input data and b) prior to deciding the optimization variables, the optimizer can for a certain cost reduce the scenario set of this uncertain data. In particular, we are interested in situations where each uncertain parameter is a priori known to lie in a closed, bounded interval and the optimizer has a fractional choice to shrink each interval. We study this setting for optimization problems with uncertain non-negative linear cost functions and non-negative decision variables. Moreover, we assume the scenario set to be restricted by some polyhedron, e.g., we consider so-called budgeted uncertainty. In particular, we show that the resulting three-level problem can be solved as a single mixed integer program for binary queries, if the underlying optimization problem is a linear program.
1 Introduction

1.1 Motivation

We propose and study a model for optimization under uncertainty in which one can pay to reduce the uncertainty. We call an optimization problem an \textit{optimization problem with controllable uncertainty} if a) it contains uncertain input data and b) prior to deciding the optimization variables, the optimizer can for a certain cost reduce the scenario set of this uncertain data.

Such problems have been studied before. In particular, in Explorable Uncertainty input parameters are initially only known to lie within certain intervals. The optimizer can choose to buy exact information for individual parameters. In these models the uncertainty for each parameter is either kept or fully erased.

We study models of controllable uncertainty where the optimizer for each of the initially given (closed and bounded) intervals has a continuous choice how much to shrink it. This may eventually but not necessarily reduce an interval to a single value. Models in which one can continuously shrink the intervals of uncertain data are a natural extension to models in which one can either buy full information or keep the interval as initially given.

We do not assume distributional knowledge. Instead we consider a worst case approach as done in Robust Optimization such that the resulting problem has multiple levels.

In this paper we consider minimization problems with uncertain linear cost functions and non-negative decision variables.

Besides the continuous choice, the second important aspect of the models studied in this paper are restricted scenario sets, i.e., the possible scenarios for the malign adversary to choose from, are a subset of the Cartesian product of the (reduced) intervals. Such restricted scenario sets are widely used in robust optimization. For the present work we use the so-called budgeted uncertainty set [BS03]. Following the rational underlying this uncertainty set, it is over-conservative to protect against a worst case where all uncertain parameters are realized as the maximal value in their interval. Instead one assumes that the sum over all parameters of all relative deviations is limited by a budget parameter $\Gamma$.

In this setting, we consider two significantly different concepts for shrinking an interval. When fully shrunk, the interval reduces to a single point. We call this point the hedging point, as it is the value we get when fully averting uncertainty. We explore two alternative models for the hedging points. In one model we are given known hedging points as part of the initial input. This does not mean that we know in advance which value the adversary in the second level will choose. But, we know in advance to which value we can force the adversary, if we buy full information.

In the other model the hedging points are \textit{not} known in the initial input.

For the case of known hedging points and binary queries we show that the three level problem can be reformulated as a (single level) mixed integer program, if the underlying optimization problem is a linear program.

1.2 Related work

Optimization with controllable uncertainty is closely related to at least robust optimization, bilevel optimization and explorable uncertainty. In robust and stochastic optimization, the uncertainty
realizes after the decision has been taken. In contrast, the uncertainty is first handled before the underlying problem is solved in the model of explorable uncertainty. Combining, in decision-dependent information discovery first some parts of the uncertainty are handled before taking decisions after which the remaining uncertainty realizes. The described approaches result in optimization problems with a multilevel structure. Problems with two levels are regarded in bilevel optimization that is thus also closely related to optimization with controllable uncertainty.

In the following, we give a brief account of robust optimization and explorable uncertainty as well as bilevel optimization including interdiction games and its extension of fortification games. In particular for robust and bilevel optimization, this is far from exhaustive.

**Robust optimization** In robust optimization, optimization problems with uncertain costs are regarded where the scenario is chosen adversarially after the decision of the optimization problem has been fixed, see e.g., the standard textbook on robust optimization [BEN09]. A widely used scenario set is the so-called *budgeted uncertainty set* $U$ introduced in [BS03, BS04] that is restricted in two ways. For each decision variable the uncertain costs are restricted to an interval from the nominal value $c$ to the nominal value plus the maximal increase $d$. In each scenario the actual increase normalized by the maximal increase and summed over all rows is limited by a budget traditionally denoted by $\Gamma$. This budgeted uncertainty set $U$ is a polytope whose number of vertices grows exponentially with $\Gamma$. Robust counterparts of discrete optimization problems with polynomial runtime are still solvable in polynomial time [BS03].

**Stochastic optimization** Stochastic optimization is another approach where the uncertainty realizes after the decisions of the optimization problem are fixed, e.g., [BL11]. In stochastic optimization the realization of the uncertainty happens randomly instead of adversarially as in robust optimization.

**Explorable uncertainty** In the model of explorable uncertainty, the underlying problem is only solved once this can be done exactly after precise values have been revealed elementwise at some cost. Thus, the order of handling uncertainty and solving the underlying problem is the other way around in contrast to both robust and stochastic optimization. In explorable uncertainty, the data uncertainty can be reduced elementwise to a single value at some investment or effort. We refer to such revealing of precise data points as *query*. The goal is to make as few queries as possible to solve the underlying problem exactly. This model of explorable uncertainty goes back to the seminal work [Kah91]. There exists a wide range of possible application areas. Many different studies investigate the concept of explorable uncertainty on basic combinatorial problems like Minimum and Selection [GSS11] as well as classical discrete problems like Shortest Path [FMO+03], Minimum Spanning Tree [EHK+08, FMM17], knapsack [GGI+15], and matroids [Mei18, MS19]. The paper [GSS11] extended the binary query selection revealing an exact value to the concept of returning a refined interval. The literature distinguishes online and offline approaches for query selection. The online query selection is an adaptive model where queries are selected sequentially and for each decision one can use the outcome of all previous value determinations. The offline query model requests a non-adaptive selection simultaneously choosing and revealing as many queries as required to ensure the existence of an exact solution of the underlying problem.
Decision-Dependent information discovery  Decision-dependent revelation of uncertain parameters has mainly been regarded in stochastic optimization, see [VGY22] and references therein. Recently, this idea has been combined with robust optimization instead resulting in the problem of decision-dependent information discovery (DDID) [VGY22, PGDT22, OP23]. In DDID the optimizer has a 0-1-choice in the first step to reveal some exact values. The chosen values then realize in a worst-case manner. Afterwards, the nominal problem is solved in a robust approach for the remaining uncertainties. Thus, DDID can be formulated as a four-level min-max-min-max problem. For general polyhedral uncertainty sets, both exact algorithms and approximations have been proposed. Note that uncertain values are only either revealed completely or kept uncertain until later in DDID.

Bilevel optimization  In bilevel optimization, two optimization problems are nested. Two players or levels control disjoint sets of variables, optimizing their own objectives with constraints that both can depend on the other’s decisions. Foundations on bilevel programming are explained in the textbook [Dem02] and further advances in bilevel optimization can be found in [DZ21] which includes an extensive bibliography in the last chapter. For mixed integer bilevel programs, see also the survey [KLLS21] and references therein.

Connections between robust and bilevel optimization, in particular possible reformulations of problems in the one setting in the other one and vice versa, are investigated in [GKST23].

Two types of uncertainty that have been regarded for bilevel problems are data uncertainty and decision uncertainty, see [BLS23a] and references therein. In data uncertainty, there is an uncertainty about the follower’s data that is either realized just before the follower takes decisions after the leader has chosen (wait-and-see follower) or after the follower’s decision is fixed (here-and-now follower). In decision uncertainty, one or both level’s hedge against the other level’s decision that for example might not be optimal but only near-optimal due to limited resources. In contrast to controllable uncertainty, neither of the two levels can influence the uncertainty in these approaches.

A special type of bilevel optimization problems in which the two players share the same objective function though optimize in opposite directions are interdiction games. In interdiction games, the upper-level problem interdicts some lower-level elements such that the lower level is as much as possible prevented in pursuing its goal, see [SS20] and references therein. Interdiction games with a monotone \( \Gamma \)-robust follower have been regarded in [BLS23b, BLS23c].

An extension of interdiction games are fortification games where a third level is added. In fortification games, some items can be defended before the opponent destroys some of the remaining items. Binary fortification games can be solved with a decomposition approach [BCSW06]. A generalized solution method is to use a branch-and-cut algorithm with fortification cuts [LLM+23].

Structure of the paper  The remaining paper is structured as follows: In Section 2 we explain the general concept of optimization under controllable uncertainty and how one can modify the scenario set for a certain cost by reducing the uncertain intervals around the so-called hedging points. In Section 3 we assume that the hedging points are known and part of the input data. For binary queries, we show how one could reformulate the problem to a single level one, if the underlying optimization problem is given by a linear program. One could otherwise assume that
the hedging points are not initially known and model them as a variable that is chosen in a worst-case fashion. We consider this four-level problem in Section 4 where we investigate a robust optimization approach and present an equivalent formulation that only has two levels.

2 Controllable uncertainty for uncertain cost coefficients

We study controllable uncertainty for optimization problems with uncertain cost function and non-negative decision variables \( y \in \mathbb{R}^n_0 \). In the following, we introduce a controllable uncertainty set for polyhedral uncertainty sets in which the intervals of the uncertainty can be reduced to some cost. We assume that the cost function \( f \) of the underlying optimization problem is parameterized by non-negative uncertain coefficients \( \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n) \). For all findings in this paper, the cost function is assumed to be linear, i.e., given by \( f(y) = \tilde{c}^\top y \).

In our model of uncertainty, we assume that each uncertain cost coefficient lies within a bounded, closed interval, i.e., \( \tilde{c}_e \in [c_e, c_e + d_e] \) for all \( e \in [n] := \{1, \ldots, n\} \). It is convenient to associate each realization of an uncertain parameter \( \tilde{c}_e \in [c_e, c_e + d_e] \) with its normalized value \( u_e \in [0, 1] \), namely \( u_e := (\tilde{c}_e - c_e)/d_e \). Hence, for the underlying cost function, we have \( f(y) = f(u, y) = (c + d \cdot u)^\top y \).

We assume that the scenario set is given as some polyhedron \( \mathcal{U} \subseteq [0, 1]^n \). Thus, the set of initially possible scenarios is

\[
\{\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n) = (c_1 + u_1 d_1, \ldots, c_n + u_n d_n) \in \mathbb{R}^n : u \in \mathcal{U}\}.
\]

Before choosing a solutions for the optimization problem, each interval can be narrowed at a certain cost. For each uncertain parameter \( \tilde{c}_e \), the continuous variable \( x_e \) expresses how much the interval for \( u_e \) (and consequently for \( \tilde{c}_e \)) is narrowed. We call \( x_e \) a query. The set of possible queries is given by \( X \subseteq \mathbb{R}^n_0 \). For example, the choice of \( X \) can only allow a query for element \( e_1 \) if also element \( e_2 \) is queried. Costs for all queries \( x \) are denoted by \( q(x) \) with a monotone function \( q : \mathbb{R}^n_0 \rightarrow \mathbb{R}_\geq 0 \). We assume that there is no query cost if no queries are made, i.e., \( q(0) = 0 \). Note that in general we do not assume that the costs for queries are elementwise, i.e., \( q(x) = \sum_{e \in [n]} q_e(x_e) \) does not generally hold for functions \( q_e : \mathbb{R}_\geq 0 \rightarrow \mathbb{R}_\geq 0 \). Thus, in general depending on the choice of allowed queries \( X \) and query costs \( q \), queries are not independent of each other.

The influence of a query \( x_e \) on the lower and upper boundary of interval \( [c_e, c_e + d_e] \) are modeled via upper and lower query functions \( \phi^L_e : \mathbb{R}_\geq 0 \rightarrow [0, 1] \) and \( \phi^U_e : \mathbb{R}_\geq 0 \rightarrow [0, 1] \) that shift the boundaries of the corresponding \([0, 1]\) interval for \( u_e \). We require each query function \( \phi^L_e, \phi^U_e \) to be monotone and to fulfill \( \phi^L_e(0) = \phi^U_e(0) = 0 \).

**Assumption 2.1.** Whenever the query functions \( \phi^L_e \) and \( \phi^U_e \) are equal, we refer to them with \( \phi_e : \mathbb{R}_\geq 0 \rightarrow [0, 1] \) for the sake of convenience. Also we will drop the index and only write \( \phi \) if \( \phi = \phi_e \) for all \( e \in [n] \). Similarly, we will also drop the index and only write \( \phi^L, \phi^U \) if \( \phi^L = \phi^L_e \) and \( \phi^U = \phi^U_e \) for all \( e \in [n] \) respectively.

To define the influence of \( \phi^L_e(x_e) \) and \( \phi^U_e(x_e) \) on the \( e \)-th interval \([c_e, c_e + d_e]\), we consider the corresponding interval \([0, 1]\) for \( u_e \). The maximal effect of a query is to reduce the \([0, 1]\) interval...
Assumption 2.4.

Definition 2.3

In the following, we will assume that $\mathcal{U}$ is the budgeted uncertainty set

$$
\mathcal{U} = \left\{ u \in [0,1]^n \left| \sum_{e \in [n]} u_e \leq \Gamma \right. \right\}
$$

that has been introduced in [BS03] and is widely used in robust optimization.
For the model to be well-defined, the set of scenarios must not be empty after any feasible query. We devise a technical condition in Observation 2.5 how to ensure this requirement of adversarial feasibility.

**Observation 2.5** (Adversarial feasibility). The condition $\mathcal{U}(x, b) \neq \emptyset$ has to be fulfilled for any query $x \in \mathbb{R}^n_{\geq 0}$ and hedging point $b \in \mathcal{B}$ for the model of controllable uncertainty to be well-defined.

If $\phi^e(x) = \phi^u(x) = 1$ for some query $x \in X$, then the controllable uncertainty set is reduced to the singleton $b$, i.e., $\mathcal{U}(x, b) = \{b\}$.

Using different objectives for the two decisions of making the queries and solving the underlying problem, we obtain a similar setting as bilevel problems with uncertainty in the follower’s data, see [BLS23a] and references therein. Often, one distinguishes whether the uncertainty realizes between the leader’s and the follower’s decision (wait-and-see follower) or after the follower’s decision (here-and-now follower). In controllable uncertainty, it is in general not possible to distinguish beforehand which parts of the underlying problem’s data uncertainties realize before and which after the problem is solved as this can be dependent on the queries made. Regarding some coordinates, a here-and-now follower deciding before the uncertainty realizes might be turned into a wait-and-see follower if remaining uncertainty is completely removed.

In the following, we regard a single optimizer making the queries and solving the underlying problem. Note that this is similar to the setting in fortification games. The overall objective function $F(x, u, y)$ encompasses two distinct components, the query costs $q(x)$, dependent on the queries $x$, and the objective function of the underlying problem $f(y) = f(u, y)$ that depends on both the realization of the uncertainty $u$ and the decision variables for the underlying optimization problem $y$. This combination of costs captures the dual nature of our approach, where resources are allocated both to uncertainty mitigation and the core optimization problem,

$$F(x, u, y) = q(x) + f(u, y).$$  \hfill (Overall-Objective)

**Observation 2.6.** There are the following relations between the query cost $q$ and the query function $\phi$, if $X = \mathbb{R}^n_{\geq 0}$:

(a) If $q$ is strongly monotone, it is equivalent to use either $q$, $\phi_e^l$ and $\phi_u^u$ or $\text{Id}$, $q^{-1}\phi_e^l$ and $q^{-1}\phi_u^u$ as query cost and query function respectively.

(b) If $\phi = \phi_e^l = \phi_u^u$ is strongly monotone, it is equivalent to use either $q$ and $\phi$ or $\phi^{-1}q$ and $\text{Id}$ as query cost and query function respectively.

**Proof.** If there is a bijection $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $g(0) = 0$, we can get another instance of our problem by applying this coordinates transformation and replacing $x$ with $\tilde{x} = g(x)$ in all occurrences. If $q$ or respectively $\phi$ is strictly monotone, the inverse $q^{-1}$ or $\phi^{-1}$ exist. If $g = q^{-1}$, then $\phi_e^l(\tilde{x}) = \phi_e^l(q^{-1}(x))$, $\phi_u^u(\tilde{x}) = \phi_u^u(q^{-1}(x))$ and $q(\tilde{x}) = x$. The same argument follows, if $g = \phi^{-1}$. \hfill \qed

**Example 2.7** (Concept of controllable uncertainty). Consider the digraph given in Figure 2 under controllable uncertainty for uncertain edge costs. Further, assume the query function $\phi_e(x_e) = \min\{x_e, 1\}$. The uncertainty interval of edge $e_1$ is $[1, 3]$ which can be symmetrically narrowed
3 Optimization with known hedging points

We regard the problem of determining an optimal query \( x \) such that the underlying problem is minimized for the worst outcome in the uncertainty \( u \) with known hedging points (KHP) \( b \), i.e., assume that the values of \( b \) are parameters to the problem

\[
\min_{x \in X} \max_{u \in U(x, b)} \min_{y \in Y} F(x, u, y). \quad \text{(KHP)}
\]
We use this min-max form although an optimal query \( x \) might not exist. For example, consider \( \phi \) being the asymptotic query function suggested in Example 2.2 in combination with zero query costs, i.e., \( q = 0 \). If no restrictions are imposed on the possible queries, i.e., \( X = \mathbb{R}_{\geq 0} \), for every query \( x \) there is a larger query \( x' \) resulting in a smaller objective value.

Note that for (KHP), we assume that the hedging points \( b \) are known, i.e., parameters to the problem. To simplify notation, as \( b \) is assumed to be fixed, we will write in this section the uncertainty set only in dependence of the query, i.e., \( U(x, b) \). In case of (KHP), we can ensure the adversarial feasibility by the assumption \( \sum_{e \in [n]} \phi_e^I(x_e) b_e \leq \Gamma \) which is substantiated by Observation 3.4.

If the query \( x \) is fixed, the problem becomes a continuous interdiction problem where interdiction only affects the objective. In the remaining bilevel problem, the upper level choosing \( u \) from intervals has a budget of \( \Gamma \) to maximize the minimal outcome of the lower level. For underlying convex problems, this can be solved via a dualization approach, see e.g., [SS20].

Note that (KHP) relates to fortification. In (KHP), a query \( x \) in (KHP) can be used to reduce the maximal size of the weight modification by the uncertainty \( u \) that realizes in a worst-case manner. Thus, making a query in (KHP) resembles fortifying elements before they can be interdicted in fortification games. However, in (KHP), in order to reduce the adversary weight modification for an element \( e \), another element \( e' \) might be queried. We refer to this phenomenon as budget deflection and describe it in Section 3.3.

Similar to bounds for fortification games, simple upper bounds on the objective of (KHP) are obtained by fixing some decisions while a lower bound arises from omitting the query costs and assuming that no uncertainty is left. Regarding any feasible query \( x \) and realization of the uncertainty \( u \) provides an upper bound on the objective. However, in general the remaining bilevel problem may not be easily solved. Thus, in order to be able to efficiently evaluate bounds, further decisions like an uncertainty \( u \) or a feasible lower-level problem’s solution \( y \) need to be fixed. Alternatively, using a query \( \pi \) for which the uncertainty set reduces to the singleton of the hedging point \( b \), only the lower-level problem remains, providing an upper bound for which only the underlying problem needs to be solved.

**Observation 3.1.** Let \( \pi \) denote a query such that no uncertainty is left for this query, i.e., \( \phi(\pi) = 1 \). For an optimal solution \((x^*, u^*, y^*)\), the following lower and upper bounds hold:

\[
\min_{y \in Y} f(b, y) \leq F(x^*, u^*, y^*) \leq q(\pi) + \min_{y \in Y} f(b, y).
\]

### 3.1 Single-level reformulation with underlying LP

A problem under controllable uncertainty (KHP) can be reformulated as a single-level non-linear problem based on strong duality if the underlying problem can be described with a linear program (LP), e.g., a totally dual integral (TDI) system. We consider monotone query costs \( q(x) \) and underlying problem costs \( f(u, y) = c^T y + d^T (y \cdot u) \). Upper bounds for the new introduced variables \( \beta \) and \( \theta \) arise directly from the single-level reformulation.

In the following, we require the underlying problem to be given as an LP. Note that this includes in particular discrete problems described by a totally dual integral (TDI) system. An inequality system \( Ax \leq b \) with \( A \in \mathbb{Q}^{m \times n} \), \( b \in \mathbb{Q}^m \), and free \( x \in \mathbb{R}^n \) is called **totally dual integral (TDI)**
if, for every integral vector \( c \in \mathbb{Z}^n \), there exists an integral optimal solution of the (dual) LP
\[
\min_y \left\{ \mathbf{b}^\top y \mid A^\top y = c, y \geq 0 \right\},
\]
provided the LP value is bounded. As a consequence, if \( Ax \leq b \) is TDI and \( b \) is integral, then the polyhedron \( \{ x \mid Ax \leq b \} \) is integral, i.e., all its vertices are integral. For several optimization problems including Shortest Path, Minimum Spanning Tree, Maximum Flow, and Minimum Cut TDI systems are known. For a more comprehensive overview on TDI, we refer to [KV18].

**Assumption 3.2.** We assume that the underlying problem is bounded and given in the form
\[
\min_y \left\{ c^\top y \mid A^\top y = a, y \geq 0 \right\}.
\]

**Theorem 3.3 (Single-level NLP).** An optimization problem under controllable uncertainty
\[
\min_{x \in X} \max_{u \in \mathcal{U}(x)} \min_{y \in Y} q(x) + c^\top y + (d \cdot u)^\top y
\]
with uncertainty set \( \mathcal{U}(x) \) as given in Definition 2.3 and underlying problem given by an LP as in Assumption 3.2 can be rewritten as a single-level non-linear problem (NLP)
\[
\begin{align*}
\min_{x,y,\beta,\theta} & \quad \Gamma \theta + q(x) + \sum_{e \in [n]} \beta_e + c_ey_e + \phi^l_e(x_e)b_e (d_ey_e - \beta_e - \theta) + \phi^u_e(x_e)\beta_e(b_e - 1) \\
\text{s.t.} & \quad Ay = a \\
& \quad d_ey_e - \beta_e \leq \theta \quad \forall e \in [n] \\
& \quad x, y, \beta, \theta \geq 0.
\end{align*}
\]

**Proof.** We start to consider the underlying problem in the lower level only and successively add the next outer level by using strong duality. Thus, for the moment, assume \( x \) and \( u \) to be fixed. We receive
\[
\begin{align*}
\min_y & \quad (c + d \cdot u)^\top y \\
\text{s.t.} & \quad Ay = a \\
& \quad y \geq 0
\end{align*}
\]
with corresponding dual
\[
\max_z \quad z^\top a \\
\text{s.t.} \quad z^\top A \leq c + d \cdot u.
\]
In the next step, we combine the two inner levels and assume only \( x \) to be fixed, i.e., regard
\[
\begin{align*}
\max_{u \in \mathcal{U}(x)} \min_{y \in Y} & \quad c^\top y + d^\top (y \cdot u).
\end{align*}
\]
Replacing the underlying problem on the lower level by its dual and accounting \( \mathcal{U}(x) \), we obtain
the following LP in $z$ and $u$:

$$\begin{align*}
\max_{z,u} & \quad z^\top a \\
\text{s.t.} & \quad zA^\top - d \cdot u \leq c \\
& \quad u_e \geq \phi^u_e(x_e) b_e \quad \forall e \in [n] \\
& \quad u_e \leq 1 - \phi^u_e(x_e)(1 - b_e) \quad \forall e \in [n] \\
& \quad u^\top 1 \leq \Gamma.
\end{align*}$$

Assigning the dual variables $y$, $\alpha$, $\beta$, and $\theta$ respectively, we generate the dual problem

$$\begin{align*}
\min_{y,\alpha,\beta,\theta} & \quad \Gamma \theta \ + \sum_{e \in E} c_e y_e + \phi^f_e(x_e) b_e \alpha_e + (1 - \phi^u_e(x_e)(1 - b_e)) \beta_e \\
\text{s.t.} & \quad Ay = a \\
& \quad -d_e y_e + \alpha_e + \beta_e + \theta = 0 \quad \forall e \in [n] \\
& \quad \alpha \leq 0 \\
& \quad y, \beta \geq 0 \\
& \quad \theta \geq 0.
\end{align*}$$

Finally, we combine all three levels and substitute $\alpha_e = d_e y_e - \beta_e - \theta$ to derive the single-level NLP equivalent

$$\begin{align*}
\min_{x,y,\beta,\theta} & \quad \Gamma \theta + q(x) + \sum_{e \in [n]} \beta_e + c_e y_e + \phi^f_e(x_e) b_e (d_e y_e - \beta_e - \theta) - \phi^u_e(x_e) \beta_e (1 - b_e) \\
\text{s.t.} & \quad Ay = a \\
& \quad d_e y_e - \beta_e \leq \theta \quad \forall e \in [n] \\
& \quad x, y, \beta, \theta \geq 0.
\end{align*}$$

Regarding the derived single-level non-linear problem, we make the following observations and deduce several bounds.

**Observation 3.4.** If the accumulated lower bounds of the controllable uncertainty intervals with known hedging points $b$ do not exceed the full budget $\Gamma$, i.e.,

$$\Gamma - \sum_{e \in [n]} \phi^u_e(x_e) b_e \geq 0,$$

the objective is bounded for all $\theta$.

**Proof.** It follows immediately from a transformation of the objective function

$$F(x, u, y) = \Gamma \theta + q(x) + \sum_{e \in [n]} \beta_e + c_e y_e + \phi^f_e(x_e) b_e (d_e y_e - \beta_e - \theta) - \phi^u_e(x_e) \beta_e (1 - b_e)$$

$$= \left( \Gamma - \sum_{e \in [n]} \phi^u_e(x_e) b_e \right) \theta + q(x) + \sum_{e \in [n]} c_e y_e + \phi^f_e(x_e) b_e d_e y_e$$

$$+ \beta_e \left( 1 + \phi^u_e(x_e)(b_e - 1) - \phi^f_e(x_e) b_e \right).$$

$\square$
Note that for $\phi_e \equiv 0$ the problem is bounded for all dual variables $\theta$. This implies that the adversarial feasibility from Observation 2.5 always holds.

The following Lemma derives bounds for the optimal dual variables $\beta^*$ and $\theta^*$ as well as the objective value $F(x^*, u^*, y^*)$. Parts of these conclusions will come handy later for further problem simplification and linearization.

**Lemma 3.5.** For an optimal solution $(x^*, u^*, y^*, \beta^*, \theta^*)$ we obtain the following upper bounds on $\beta, \theta$ and the objective:

$$\theta^* \leq \max_e d_e =: D$$

and

$$F(x^*, u^*, y^*) \leq \Gamma D + \sum_{e \in [n]} d_e + \min_{y \in \{0,1\}^n} \left\{ c^\top y \mid Ay = a \right\}.$$

**Proof.** Consider the constraints $d_e y_e - \beta_e \leq \theta$ for all $e \in [n]$. Since $\beta_e$ and $\theta$ are to be minimized with non-negative factors, we have $\theta^* \leq \max_e d_e =: D$, and $\beta_e^* \leq d_e$.

Not querying any edge ($x = 0$ so $\phi(x) = 0$), then $q(x) = 0$ and thus

$$F(0, u, y) = \Gamma \theta + \sum_{e \in [n]} \beta_e + y_e c_e.$$  

Using the bounds on $\beta, \theta$, we get the upper bound

$$F(x^*, u^*, y^*) \leq \Gamma D + \sum_{e \in [n]} d_e + \min_{y \in \{0,1\}^n} \left\{ c^\top y \mid Ay = a \right\}.$$

\[\square\]

### 3.2 Optimality conditions for elementwise query costs

In this subsection, we assume elementwise query costs. This assumption allows us to derive several optimality conditions.

**Assumption 3.6.** We assume that the query costs are elementwise defined, i.e.,

$$q(x) = \sum_{e \in [n]} q_e(x_e)$$

for functions $q_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Note that the assumption of no query costs for no queries, i.e., $q(0) = 0$, holds elementwise for elementwise query costs, i.e., $q_e(0) = 0$. First, we give relations on the optimal query selection depending on the size of the uncertainty interval $d$, and the query costs $q$.

**Observation 3.7.** There are the following elementwise relations between an optimal query selection $x^*$, the size of the uncertainty interval $d$, and elementwise query costs $q_e(x_e)$:

(a) If $d_e \leq q_e$, there is at least one optimal solution that does not query the according element, i.e., $x_e^* = 0$.

(b) If an element $e$ is queried in an optimal solution, i.e., $x_e^* > 0$, then $d_e \geq q_e$. 


(c) If an element \( e \) is part of the lower-level problem’s solution, i.e., \( y_e^* = 1 \) and there is an active query for this element, i.e., \( x_e^* > 0 \), it holds that
\[
q_e(x_e^*) \leq \phi_e^u(x_e^*)d_e(1 - b_e).
\]

Proof. The first two statements follow directly from the objective function. For the third statement, let \( y_e^* = 1 \) and \( x_e^* > 0 \). The according costs for this element \( e \) are \( c_e + d_e \cdot u_e + q_e(x_e) \) where \( c_e, d_e \) are constants and \( u_e \in [0, 1] \). Increasing \( x_e \) increases the query costs \( q_e(x_e) \) while it decreases \( d_e \cdot u_e \) as the upper bound on \( u_e \) in the controllable uncertainty set is limited by \( 1 - \phi_e^u(x_e)(1 - b_e) \).

The upper bound of \( u_e \) is also the worst case that can happen in the uncertainty set. The minimal gain has to be at least as large as the query costs as otherwise a contradiction to the optimality arises, that is \( q_e(x_e^*) \leq \phi_e^u(x_e^*)d_e(1 - b_e) \).

\[\square\]

Observation 3.8. There is no active query for an element in an optimal solution, i.e., \( x_e^* = 0 \), if
\[q_e(x_e) > b_eD + d_e.\]

Proof. Consider the objective function derived in Theorem 3.3
\[F_{\text{NLP}}(x, y, \beta, \theta) = \Gamma \theta + \sum_{e \in [n]} q_e(x_e) + \beta_e + c_e y_e + \phi_e^f(x_e) b_e (d_e y_e - \beta_e - \theta) + \phi_e^u(x_e) \beta_e (b_e - 1).\]

Applying Lemma 3.5 and \( y_e, \phi_e^f(x_e), \phi_e^u(x_e) \in [0, 1] \), we receive for the query-dependent terms
\[
q_e(x_e) + \phi_e^f(x_e) b_e (d_e y_e - \beta_e - \theta) + \phi_e^u(x_e) \beta_e (b_e - 1) \\
\geq q_e(x_e) - b_e d_e - b_e D - d_e(1 - b_e) \\
= q_e(x_e) - b_e D - d_e
\]

Hence, there is no incentive to query element \( e \) if \( q_e(x_e) > b_e D + d_e \). \[\square\]

Depending on whether an element \( e \) is used in a lower-level optimal solution, \( y_e^* \in \{0, 1\} \), the following conditions on the query costs hold.

Observation 3.9. Let \((x^*, y^*, \beta^*, \theta^*)\) be an optimal solution.
- If \( y_e^* = 0 \), then \( \beta_e^* = 0 \) and \( q_e(x_e^*) \leq b_e D \).
- If \( y_e^* = 1 \) and \( \phi = \phi_e^f = \phi_e^u \), then
\[q_e(x_e^*) \leq b_e (D - d_e) + d_e.\]

Proof. If \( y_e^* = 0 \), the constraint \( d_e y_e - \beta_e \leq \theta \) simplifies to \( 0 \leq \beta_e + \theta \) which is always fulfilled as \( \beta, \theta \geq 0 \) are required. Due to Observation 2.5 we know that \( \phi_e^f(x_e) b_e \leq 1 - \phi_e^u(x_e)(1 - b_e) \).

Considering the objective reformulated as in Observation 3.4, variable \( \beta_e \) is included with the non-negative factor \( (1 - \phi_e^u(x_e)(1 - b_e) - \phi_e^f(x_e) b_e) \). As there are no further constraints on \( \beta_e \), we can set \( \beta_e = 0 \). Then by Lemma 3.5 and \( \phi_e^f(x_e) \leq 1 \)
\[
q_e(x_e^*) \leq \phi_e^u(x_e^*) \beta_e^*(1 - b_e) - \phi_e^f(x_e^*) b_e (d_e y_e^* - \beta_e^* - \theta^*) \\
= \phi_e^f(x_e^*) b_e \theta^* \leq b_e D
\]
If \( y^*_e = 1 \), the constraint \( d_e y_e - \beta_e \leq \theta \) becomes \( d_e - \beta_e - \theta \leq 0 \). By Lemma 3.5, it follows that
\[
ge_e(x_e^*) \leq \phi_e^u(x_e^*)\beta_e^*(1 - b_e) + \phi_e^l(x_e^*)b_e(\beta_e^* + \theta - d_e y_e^*) \\
\leq \beta_e^* b_e + \beta_e^* b_e + b_e \theta - b_e d_e \\
\leq d_e + b_e D - b_e d_e = b_e (D - d_e) + d_e.
\]

\[\square\]

### 3.3 Budget deflection

A query in a problem setting with controllable uncertainty usually serves the purposes of gaining information and/or protecting against very unwelcome realizations of the uncertainty at a certain cost.

However, in problem (KHP) a query may be made with the sole purpose of decreasing the uncertainty budget. It forces an uncertainty investment of at least \( b_e \) (for \( x_e = 1 \)) at a relatively cheap price while it may be clear that the queried element \( e \) will never be part of a lower-level solution. We call this phenomenon **budget deflection** (or just deflection).

**Example 3.10** (Budget deflection). Let a digraph with three nodes and two edges be given as shown in Figure 3. Consider the task to find a shortest \( s\)-\( t \)-path under controllable uncertainty for uncertain edge costs. Let the adversary have a budget of \( \Gamma = 1 \).

Then, it is possible to exhaust the adversary’s budget by querying edge \( e_1 \) which has an hedging point at the upper bound of the uncertainty interval. At the same time, the player takes \( e_2 \) as the only available \( s\)-\( t \)-path that now has a certain length of 2, i.e., the lower bound of the uncertainty interval. The adversary is forced to spend its entire budget on \( e_1 \) and has no chance to put some extra weight on \( e_2 \) any more.

It depends on the application and the actual multi-level structure whether budget deflection makes sense and should be allowed. It can be seen as a natural consequence of the concept of budgeted uncertainty, which creates a dependency between uncertain quantities that would otherwise be assumed to be independent.

In case of (KHP) and the adversary representing uncertainty in the worst-case scenario rather than an actual adversarial agent, budget deflection seems unwanted. It is indeed up to us to exclude it by setting the lower query function to zero:

![Figure 3: Example for budget deflection](image-url)
Theorem 3.11. Let $X = \mathbb{R}$ and assume that the query costs are elementwise. Let $x^*$ be an optimal solution of (KHP) with query costs $q_e(x^*_e) > 0$ for some $e \in [n]$ where $\phi^*_e = 0$. Then there exists $u \in U(x^*)$ such that there is an optimal solution $y^*$ to the third-level problem with $y^*_e > 0$.

Proof. For an optimal query selection $x^*$, we regard $e \in [n]$ with $q(x^*_e) > 0$. Assume that for all $u^* \in U(x^*)$ we have $y^*_e = 0$ for all optimal solutions $y^*$ to the third level.

Then, we can set $u^*_e = 0$ without loss of generality: First, note that setting $u^*_e = 0$ is feasible due to $\phi^*_e = 0$ as the other constraints in $U$ are upper bounds on $u$. As $Y$ is independent of $u$, it suffices to regard the objective for optimality. For the only summands $d_e u_e y_e$ including $u_e$ in the objective, we already have $d_e u_e y_e = 0$ as we are regarding $e \in [n]$ with $y^*_e = 0$. Thus, when choosing $u$ there is no advantage in setting $u_e > 0$ such that we can set $u^*_e = 0$ without loss of generality.

Due to $u^*_e = 0$ and $X = \mathbb{R}$, we can set $x_e = 0$ which reduces the objective value as $q_e(x^*_e) > 0$ without changing latter decisions for choosing $u, y$ which contradicts the optimality of $x^*$.

In some sense setting $\phi^*_e = 0$ does nothing else to the instance but disabling budget deflection. At first glance one might assume that increasing the lower uncertainty bound by an amount specified by $\phi^*_e$ is an additional price to pay; after all it reduces positive outcomes. However, in robust optimization we are interested in dealing with the worst-possible outcome. In fact, $\phi^*_e \neq 0$ can only benefit the minimization player, who can exploit it for budget deflection:

Lemma 3.12. Let $F^*$ be the optimal objective value of an instance of (KHP) and $F^*_0$ be the optimal objective value of the same instance but with $\phi^e = 0$. Assume that for the modified instance with $\phi^e = 0$, the optimum is attained in some $x^*_0 \in X$. Then, $F^* \leq F^*_0$.

Proof. Let $\psi(x, u) := \min_{y \in Y} F(x, u, y)$, i.e., let $\psi(x, u)$ denote the optimal value function for the underlying problem. Note that the optimal query $x^*_0$ for the modified instance is feasible for the original problem, though it might not be optimal. Furthermore, let $U_0(x)$ denote the uncertainty set where $\phi^e = 0$. Due to $\phi^e \geq 0$ for all $e \in [n]$ in the original instance, we have $U(x) \subseteq U_0(x)$ for every $x$. We obtain

$$F^* = \min_{x \in X} \max_{u \in \hat{U}(x)} \psi(x, u) \leq \max_{u \in U(x^*_0)} \psi(x^*_0, u) \leq \max_{u \in U_0(x^*_0)} \psi(x^*_0, u) = \psi(x^*_0, u^*_0) = F^*_0.$$

More generally, the optimal objective value might increase if the query functions are replaced by smaller ones. Furthermore, the assumption that there is an optimal query for the modified version can be dropped by regarding according converging sequences. Formally, we get:

Theorem 3.13. Let $F^*$ be the optimal objective value of an instance of (KHP) and $\tilde{F}_F$ be the optimal objective value of the same instance but with $\tilde{\phi}^e$ instead of $\phi^e$ and $\tilde{\phi}^u$ instead of $\phi^u$. Let $\tilde{\phi}^e \leq \phi^e$ and $\tilde{\phi}^u \leq \phi^u$ for all $e \in [n]$. Then, $F^* \leq \tilde{F}_F$. 

3.4 Single-level MIP

For linear query costs $q$ and binary query function $\phi$ being $\phi(x) = 0$ for $x \in [0, 1)$ and $\phi(x) = 1$ for $x \geq 1$, the above derived single-level problem in Theorem 3.3 can be formulated as a single-level mixed integer problem by applying McCormick envelopes [McC76]. The resulting constraints define the convex and the concave envelopes of a bilinear function on a rectangular domain and can be used as an exact, linear reformulation of a bilinear term if at least one of the two factors is binary. The case of binary query functions is restrictive compared to our original approach, but it is precisely this case that is being studied in the current literature on decision-dependent information discovery [VGY22, PGDT22, OP23].

**Theorem 3.14.** Let the query costs be linear $q(x) = q^\top x$ and the query function $\phi$ be given as a binary function by $\phi(x) = 0$ for $x \in [0, 1)$ and $\phi(x) = 1$ for $x \geq 1$. Then an optimal query $x^*$ exists and the optimization problem under controllable uncertainty for a problem given as an LP can be formulated as the following mixed integer linear problem with $O(n)$ constraints.

$$\min_{x,y,z,\theta} \Gamma \theta + \sum_{e \in [n]} q_e x_e + c_e y_e + \beta_e + z_e$$

s.t.  
$$Ay = a$$
$$d_e y_e - \beta_e \leq \theta \quad \forall e \in [n]$$
$$z_e \geq -(b_e D + d_e) x_e \quad \forall e \in [n]$$
$$z_e \geq b_e d_e (x_e + y_e - 1) - b_e \theta - \beta_e \quad \forall e \in [n]$$
$$\beta \geq 0$$
$$\theta \geq 0$$
$$y_e \in [0, 1] \quad \forall e \in [n]$$
$$x_e \in \{0, 1\} \quad \forall e \in [n].$$

**Proof.** Instead of using the binary query function $\phi$, we can equivalently require queries $x$ to be binary, simplifying notation. This also results in only finitely many feasible solutions for a query $x$, which means that the optimum $x^*$ exists. Regard the single-level problem with the assumptions on $q$ and $\phi$ plugged in:

$$\min_{x,y,z,\theta} \Gamma \theta + \sum_{e \in [n]} q_e x_e + c_e y_e + \beta_e + x_e (b_e d_e y_e - b_e \theta - \beta_e)$$

s.t.  
$$Ay = a$$
$$d_e y_e - \beta_e \leq \theta \quad \forall e \in [n]$$
$$0 \leq \beta$$
$$0 \leq \theta$$
$$y_e \in [0, 1] \quad \forall e \in [n]$$
$$x_e \in \{0, 1\} \quad \forall e \in [n].$$
The only non-linear part of the objective is the bilinear term \( x_e (b_e d_e y_e - b_e \theta - \beta_e) \). We introduce a new variable \( z_e = x_e (b_e d_e y_e - b_e \theta - \beta_e) \) and bound it by constraints which can be derived from McCormick envelopes. For this, note that \( x_e \in \{0,1\} \) and \( b_e d_e y_e - b_e \theta - \beta_e \in [-b_e D - d_e, b_e d_e] \) using the bounds on \( \beta \) and \( \theta \) obtained in Lemma 3.5. The lower bounds suffice as \( z \) is minimized. Since we assume that \( x_e \) is binary, this reformulation is exact. 

Note that the application of McCormick envelopes does not require linearity of the functions \( q \) and \( \phi \). The final program class only depends on the structure and properties of \( q \) and \( \phi \).

4 Robustness against uncertain hedging points

Unlike the previous chapter, where hedging points \( b \) were treated as fixed values, we now consider these points as variables that can change. Their selection is done as a worst-case scenario for our objective function. This perspective provides a contrast to our earlier approach, where hedging points were assumed to be fixed and based on specific instances. Our optimization problem is structured in four levels, providing a comprehensive perspective:

1. Choosing queries: In the initial phase, we tackle the task of determining the queries denoted by \( x \) for the uncertain intervals. These queries determine the size of the uncertainty intervals.

2. Selecting hedging points: In contrast to our previous approach, we introduce the concept of adaptable hedging points \( b \in B \). These points are chosen strategically to magnify the worst possible effect on our objective function within the given budget \( \Gamma \). The hedging points determine the center point of the uncertainty intervals but have to fulfill adversarial feasibility as mentioned in Observation 2.5.

3. The third phase introduces a robust discrete optimization problem. Here, we choose decision variables \( y \in Y \) that give the cheapest worst-case cost for all scenarios left after the first two levels. We assume \( Y \subseteq \{0,1\}^n \).

4. Realization of uncertainties: In the final step the previously hidden uncertain cost vectors \( u \) are unveiled within the budget \( \Gamma \), determining the worst-case cost of our objective function. The feasible scenarios for this step are determined by the decisions made in the first two levels.

The interplay between query choices, adaptable hedging points, robust discrete optimization, and worst-case cost vectors leads to the following four-level optimization problem

\[
\min_{x \in X} \max_{b \in B} \min_{y \in Y} \max_{u \in \mathcal{U}(x,b)} F(x, u, y) \tag{UHP}
\]

where \( Y \subseteq \{0,1\}^n \) is a feasible set for a discrete optimization problem. Note that we assumed in Section 2 that the overall objective function \( F(x, u, y) \) is the sum of costs \( q(x) \) for the queries \( x \) and the in \( y \) linear objective of the underlying problem \( f(u, y) = (c + d \cdot u)^T y \). Furthermore, we assumed to consider controllable uncertainty for budgeted uncertainty, i.e., have

\[
\mathcal{U}(x,b) = \left\{ u \in \mathbb{R}^n \mid \phi_e^f(x_e) b_e \leq u_e \leq 1 - \phi_e^u(x_e)(1 - b_e), \sum_{e\in[n]} u_e \leq \Gamma \right\}.
\]
In general, queries can also be made only partially in (UHP). However, using appropriate query functions $\phi^\ell_e$, $\phi^u_e$, one can also model to either completely query uncertain parameters or not as it is done in the recently introduced concept of decision-dependent information discovery, e.g., [PGDT22, OP23]. Note that we allow that making queries adds to the overall objective.

One could argue that exhausting the budget is an essential part of querying, since a reduction to the budget is the only drawback for the adversary when they reply with $b_e = 1$. As introduced in Section 3.3, budget deflection occurs for independently made queries, if a query is made for an element that is never used in the third level. However, the issue of budget deflection for independent queries does not occur in the setting of (UHP). Note that in contrast to the corresponding Theorem 3.11 for (KHP), we do not need any additional assumption on $\phi^\ell$ when regarding (UHP) in the following theorem.

**Theorem 4.1.** For an instance of (UHP) with $X = \mathbb{R}_{\geq 0}$ and elementwise query costs, let $x^*$ be an optimal solution with $q_e(x^*_e) > 0$ for some $e \in [n]$. Then there exists an optimal $b^*_e$ such that there is an optimal solution $y^*$ with $y^*_e > 0$.

**Proof.** Due to the assumption of elementwise query costs, the overall objective function has the elementwise structure $F(x, u, y) = \sum_{e \in [n]} q_e(x_e) + c_e y_e + d_e u_e y_e$. For a proof by contradiction, assume that for a fixed optimal query $x^*$ with $q_e(x^*_e) > 0$ we have $y^*_e = 0$ for all optimal $b^*_e$. Then, without loss of generality $u^*_e = \phi^\ell_e(x^*_e) b^*_e$: There is no other lower bound on $u_e$ and $u_e$ appears in the objective only in the summand $u_e b_e y_e$ that is already zero. As a consequence, we can have $b^*_e = 0$ without loss of generality by a similar argument. Now setting $x_e = 0$ reduces the objective value as $q_e(x^*_e) > 0$ without changing the latter decisions for $b, y, u$ which contradicts the optimality of $x^*$.

Note that there is a corresponding result for the related problem of decision-dependent information discovery [OP23, Remark 1].

### 4.1 Reformulation of the inner robust problem

For fixed $x$ and $b$ and by dropping the constant term in the objective, this reduces to the following problem:

$$\begin{align*}
\min_y \max_u & \sum_{e \in [n]} c_e y_e + d_e u_e y_e \\
\text{s.t.} & \quad u_e \geq \phi^\ell_e(x_e) b_e & \forall e \in [n] \\
& \quad u_e \leq 1 - \phi^u_e(x_e) (1 - b_e) & \forall e \in [n] \\
& \quad \sum_{e \in [n]} u_e \leq \Gamma \\
& \quad y \in Y.
\end{align*}$$

(Dual-Robust)

Dualizing the inner problem with dual variables $\alpha_e$, $\beta_e$, and $\theta$ yields after recombining with the
minimization of $y$:

$$\min_{\alpha, \beta, \theta, y} \Gamma \theta + \sum_{e \in [n]} c_e y_e - \phi_e^e(x_e) b_e \alpha_e + (1 - \phi_e^u(x_e) (1 - b_e)) \beta_e$$

s.t. $-\alpha_e + \theta + \beta_e - d_e y_e = 0 \quad \forall e \in [n]$

$$\begin{align*}
\alpha, \beta, \theta & \geq 0 \\
y & \in Y.
\end{align*}$$

Substituting $\alpha_e = \theta + \beta_e - d_e y_e \geq 0$ gives

$$\begin{align*}
\min_{\beta, \theta, y} & \left( \Gamma - \sum_{e \in [n]} \phi_e^e(x_e) b_e \right) \theta \\
& + \sum_{e \in [n]} \left( c_e + \phi_e^e(x_e) b_e d_e \right) y_e + \left( 1 - \phi_e^u(x_e) + (\phi_e^u(x_e) - \phi_e^e(x_e)) b_e \right) \beta_e \\
\text{s.t.} & \quad \theta + \beta_e - d_e y_e \geq 0 \quad \forall e \in [n] \\
& \quad \beta, \theta \geq 0 \\
& \quad y \in Y.
\end{align*}$$

From this, we can make the following observation.

**Observation 4.2.** If the accumulated lower bounds of the controllable uncertainty intervals with fixed hedging points $b$ do not exceed the full budget $\Gamma$, i.e.,

$$\Gamma - \sum_{e \in [n]} \phi_e^e(x_e) b_e \geq 0,$$

the objective is bounded for all $\theta$.

**Proof.** This follows immediately from the objective function in (1). \hfill \Box

Following this observation, to always have a bounded objective, we therefore define the set of all feasible hedging points in this case as $\mathcal{B} := \mathcal{B}(x) = \{ b \in [0,1]^n \mid \sum_{e \in [n]} \phi_e^e(x_e) b_e \leq \Gamma \}$. Note that for $\phi_e^e \equiv 0$ the problem is bounded for all dual variables $\theta$. This implies that the adversarial feasibility from Observation 2.5 always holds.

In the following, we will assume $\phi_e^e = \phi_e^u = \phi_e^e$. Note that this leads to a simplification in the last part of the objective in (1).

In the work of Bertsimas and Sim in [BS03], the authors establish that the robust counterpart of a discrete optimization problem can be effectively optimized by solving a set of $n + 1$ nominal optimization problems, each featuring a different cost vector. With a modified budget of

$$\tilde{\Gamma} = \tilde{\Gamma}(x,b) = \Gamma - \sum_{e \in [n]} \phi_e(x_e) b_e$$

and a modified cost vector

$$\tilde{c}_e = \tilde{c}_e(x_e, b_e) = c_e + \phi_e(x_e) b_e d_e,$$

we get an integer problem that mirrors the integer problem outlined in the proof of their paper, only differing in the coefficient $1 - \phi(x_e)$ of the dual variables in the objective function. For $b \in \mathcal{B}$
Theorem 4.3. This problem is well-defined, since $\Gamma \geq 0$ and $\tilde{c}_e \geq 0$. Furthermore, we assume without loss of generality that the entries of the vector $d$ are ordered in a non-increasing way, which results in $d_1 \geq d_2 \geq \cdots \geq d_n \geq d_{n+1} = 0$, including the additional, artificial value $d_{n+1} = 0$. Therefore, by using the insights of their proof, we can conclude, that the Problem (Inner-Robust) can be solved, by solving the following $n+1$ nominal problems:

$$
\min_{\ell \in [n+1]} G^\ell(x, b) = \min_{\ell \in [n+1]} \tilde{\Gamma} d_\ell + \min_{y \in Y} \sum_{e \in [n]} \tilde{c}_e y_e + \sum_{j=1}^\ell (d_j - d_\ell)(1 - \phi_e(x_j)) y_j.
$$

Note that generally $G^\ell(x, b)$ is not (piecewise, affine) linear in $b$. There are several results in the literature that aim to reduce the number of subproblems that have to be solved, but this will not be the focus of this work. For further information, see [LK14, HRS18, BGK23].

4.2 Solving idea for fixed queries

In the following, we assume that the queries $x$ are fixed. After combining (2) with the maximization step of finding the worst-case hedging points $b \in B = \{ b \in [0, 1]^n \mid \sum_{e \in [n]} \phi_e(x_b) b_e \leq \Gamma \}$, we get:

$$
\max_{b \in B} \min_{\ell \in [n+1]} \tilde{\Gamma}(x, b)d_\ell + \min_{y \in Y} \sum_{e \in [n]} \tilde{c}_e(x, b) y_e + \sum_{j=1}^{\ell-1} (d_j - d_\ell)(1 - \phi_j(x_j)) y_j.
$$

This problem can be regarded as a bilevel mutli-follower problem, providing a possibility to parallelize computations. The upper level maximizes the objective value $R$ and variables $b$ while $n+1$ followers solve the underlying problem with different objectives:

$$
\max_{R \in \mathbb{R}, b \in B} R
\quad \text{s.t.} \quad R - \Gamma(x, b) \leq \min_{y \in Y} \sum_{e \in [n]} \tilde{c}_e(x, b_e) y_e + \sum_{j=1}^{\ell-1} (d_j - d_\ell)(1 - \phi_j(x_j)) \quad \forall \ell \in [n+1].
$$

As the underlying problem has a finite feasible set, we can further reformulate our problem:

$$
\max_{R \in \mathbb{R}, b \in B} R \quad \text{s.t.} \quad R \leq \left( \Gamma - \sum_{e \in [n]} \phi_e(x_b) b_e \right) d_\ell + \sum_{e \in [n]} (c_e + \phi_e(x) b_e d_e) y_e + \sum_{j=1}^{\ell-1} (d_j - d_\ell)(1 - \phi_j(x_j)) \quad \forall \ell \in [n+1], \ y \in Y.
$$

Theorem 4.3. For a fixed query $x$, if the nominal problem of finding the optimal $y^* \in Y$ can be solved in polynomial time, then the robust problem (3) can be solved in polynomial time.

Proof. Using our reformulation, for a given $x$, $\tilde{b}$ and $\ell$, one can find the cheapest $y \in Y$ that minimizes the right-hand side of the constraint. Combined with the linear $n+1$ possible offsets $(\Gamma - \sum_{e \in [n]} \phi_e(x_b) b_e) d_\ell$ that are independent of $y$, one could find a constraint that violates feasibility or conclude that there is none. Therefore the separation problem consists of solving $n+1$ nominal problems with different objectives. Based on the ellipsoid method, the problem thus can be solved in polynomial time, see e.g., [KV18, Section 4.6].
4.2 Solving idea for fixed queries

To efficiently solve this problem with a potential exponential number of constraints, we replace $Y$ by a subset $\hat{Y} \subseteq Y$ and proceed as follows

1. Calculate an initial set $\hat{Y} \subseteq Y$, e.g., by solving the nominal ($x = 1$) or robust optimization problem with $x = 0$, where $\hat{b}$ becomes redundant.

2. Solve (3) with the set $\hat{Y}$ and get optimal $\hat{b}, \hat{R}$. This is an upper bound for (3).

3. Solve the $n + 1$ nominal problems (2) with parameters $x, \hat{b}$. This results in $n + 1$ solutions $\hat{y}^\ell$ with costs $G^\ell(x, \hat{b})$ and $G(x, \hat{b}) = \min_{\ell \in [n+1]} G^\ell(x, \hat{b})$ as an upper bound for (3).

4. If $\hat{R} = G(x, \hat{b})$, then the optimal solution for (3) is found, else add new elements to $\hat{Y} \leftarrow \hat{Y} \cup \{\hat{y}^\ell, \forall \ell \in [n+1]\}$ and go to step 2.

**Corollary 4.4.** If the nominal problem of finding the optimal $y^* \in Y$ can be solved in polynomial time, then each iteration of the steps 1-4 can be done in polynomial time.

**Proof.** Step 4 is done in constant time. If there is a polynomial time algorithm for $Y$, then the optimization in steps 1 and 3 can be solved efficiently and by Theorem 4.3, step 2 can be done in polynomial time.

More details on the complexity and theory of separation problems for specific optimization problems can be found in [KV18]. This Lemma does not necessarily mean, that the whole problem for fixed $x$ can be solved in polynomial time by this specific algorithm, because one may need an exponential number of iterations to get the optimal solution for step 4. We also have to note that up to this point, we only considered the case of fixed queries $x$.

In summary, we were able to reduce the four-level problem (UHP) to a two-level one, where the inner problem (3) with fixed queries $x$ can be solved in polynomial time if there is a polynomial time algorithm to find the optimal $y^* \in Y$ for the nominal problem without uncertainties. Alternatively, we presented an iterative solution algorithm for the inner problem.

5 Conclusion and outlook

In this paper, we introduced the concept of controllable uncertainty. It models robust optimization problems in which one can shrink intervals for uncertain parameters at a certain cost. Exact hedging points may or may not be known, so in the latter case one influences when some of the uncertainty reveals relative to the optimization decision rather than securing an acceptable cost for a solution element. The concept is highly flexible and can be applied to a large variety of robust optimization problems, differing in the number, type and order of levels, the structure of the query function and the parameters that are subject to uncertainty. In particular, one may distinguish between whether hedging points are given as part of the input or chosen by an adversary. For each of these two cases, we consider an example problem setting with three and four levels, respectively. In both cases, we were able to simplify the problems to manageable, though still difficult, problem classes. Thereby, we illustrate that optimization models using the concept of controllable uncertainty can still be accessible to methods seeking global optimality, despite an initially daunting number of levels.
Future research may aim to identify and classify further classes of optimization problems under controllable uncertainty for which levels can be reduced significantly, possibly also discovering polynomially solvable ones. Moreover, the concept could be used in economic applications for quantifying the value of benefits achieved by queries. Depending on the setting, leader queries can serve multiple purposes, namely hedging against highly unwelcome realizations of the uncertainty, quite similar to fortification games, but also gaining access to uncertain information earlier – or, in the interpretation of a two-player game, forcing the adversary’s hand. A third and sometimes unwanted purpose is just to exhaust the uncertainty budget. We discussed some modeling subtleties in conjunction with this as a first step to quantitatively differentiate between these purposes. Finally, specialized algorithmic techniques could lead to significant speedups for solving problems under controllable uncertainty.

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