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# Robust Optimization Under Controllable Uncertainty

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**Abstract** Applications for optimization with uncertain data in practice often feature a possibility to reduce the uncertainty at a given *query cost*, e.g., by conducting measurements, surveys, or paying a third party in advance to limit the deviations. To model this type of applications we introduce the concept of *optimization problems under controllable uncertainty (OCU)*. For an OCU, we assume the uncertain cost parameters to lie in bounded, closed intervals. The optimizer can shrink each of these intervals around a certain value (*hedging point*), possibly reducing it to a single point. Depending on whether the hedging points are known in advance or not, different types of OCU arise. Moreover, the models may differ with respect to when the narrowing down, the underlying optimization, and the revelation of true data take place.

We study two different problem settings – one with known and one with unknown hedging points – in more detail, in which we handle the remaining uncertainty by the paradigm of robust optimization. For both settings, we draw connections to the existing literature, provide bounds on the optimal objective value, and give a single-level non-linear reformulation. Furthermore, we state assumptions under which the three- respectively four-level problem can be solved as a single-level mixed-integer linear program. We also show that in robust OCU an optimizer might query a parameter solely to reduce the uncertainty for other parameters (*budget deflection*). We give necessary conditions for this phenomenon.

**Keywords:** Multi-Level Optimization, Controllable Uncertainty, Mixed-Integer Programming, Robust Optimization, Budgeted Uncertainty Set, Single-Level Reformulation

# 1 Introduction

## 1.1 Motivation

Optimization problems in applications often come with uncertainty in the data input. We propose and study a new concept for optimization under uncertainty in which one can pay to reduce some of the uncertainty before solving the underlying optimization problem. We call this new concept *optimization under controllable uncertainty (OCU)*. In OCU, uncertain parameters are initially only known to lie within bounded, closed intervals. The optimizer can continuously shrink each of the initially given intervals for the uncertain parameters at a *query cost*. This may eventually but not necessarily reduce an interval to a single point. We call this point the *hedging point*, as it is the value we get when fully averting uncertainty. However, shrinking of the interval to a singleton might not be possible. Further, we distinguish in our concept OCU whether hedging points are known or unknown. If hedging points are known as part of the initial input, this does not mean that the values to which the uncertainties realize are given in advance. These values are only fixed if we buy full information.

OCU has several applications, both with known and unknown hedging points. Management of currency risks [FWR12] or protection against damage in networks like electric power [BCSW06], supply chain [CS07], or transportation networks [JLSY15, FHEED22] involve investments to reduce uncertainty in the underlying problem. The hedging points are known in these applications and model the known costs of a decision that is made now instead of postponed to the future. If this decision is postponed, a risk is taken by betting on better costs in the future. In contrast, revenues are only revealed after some investment is made in research and development portfolio optimization [SCJB10], production planning [JWW98], pharmaceutical clinical trial planning

[CM10], or offshore gas-field development [GG04].

In *robust optimization under controllable uncertainty (ROCU)*, we use the worst-case approach of *robust optimization* to deal with the remaining uncertainty in OCU. The possible scenarios for the malign adversary to choose from are restricted to a subset of the Cartesian product of intervals. In the present work, we use the so-called *budgeted uncertainty* set that is widely used in robust optimization [BS03]. The rationale underlying this uncertainty set is that it would be over-conservative to protect against a worst case where all uncertain parameters are realized as the maximal value in their interval. Instead, we assume that the sum over all parameters of all relative deviations is limited by a budget parameter.

In ROCU, for elementwise query costs, an optimizer might query a parameter solely to control the uncertainty for other parameters. We call this phenomenon *budget deflection*. We give an example where budget deflection occurs in a problem setting with known hedging point and provide a necessary condition for it. For another problem setting with unknown hedging point, we show that budget deflection is not possible.

The concept of OCU can also be combined with other approaches to deal with uncertainty, e.g., stochastic optimization. In stochastic optimization, the uncertainty realizes according to a random distribution, e.g., see [BL11] for an introduction. Using the paradigm of robust optimization, we do not need to assume distributional knowledge.

OCU is closely related to existing concepts that allow to reduce uncertainty in data input before solving an optimization problem. Various settings in which the uncertainty set is influenced by the choices of the optimizer have been studied under the more general notion *decision-dependent uncertainty* [**?**, NS18, **?**]. For example, the structure of the underlying probability distribution or uncertainty set is modified or uncertain parameters are removed. In *explorable uncertainty*, the optimizer can buy exact information for individual uncertain parameters [Kah91]. Uncertainty for each parameter is either fully kept or completely erased until the actual optimization problem can be solved exactly. Similarly, in *decision-dependent information discovery (DDID)*, either the exact value is revealed or the full uncertainty for a parameter remains [OPR24]. Costs for exact information are not part of the overall objective. Instead, the optimizer has a fixed budget for revealing some individual uncertain parameters. The possibility to continuously shrink the intervals of uncertain data in OCU extends models in which one can either buy full information or keep the uncertainty as initially given. Furthermore, the query cost for additional information in OCU is an extension of a fixed budget for the reduction of uncertainty. In applications, it might be difficult or unrealistic to completely erase uncertainty in underlying cost and to fix a budget for investments.

In OCU, the optimizer solving the underlying problem and the optimizer choosing the queries is the same entity. In contrast, one could use different objectives for the two decisions of making the queries and solving the underlying problem. Then, we obtain a similar setting as bilevel problems with uncertainty in the follower's data, see [BLS23a] and references therein. Often, one distinguishes whether the uncertainty realizes between the leader's and the follower's decision *(wait-and-see follower)* or after the follower's decision *(here-and-now follower)*. In controllable uncertainty, it is not possible to distinguish beforehand which parts of the underlying problem's data uncertainties realize before and which after the problem is solved as this can be dependent on the queries made. A here-and-now follower who decides before the uncertainty realizes might be turned into a wait-and-see follower if the remaining uncertainty is completely removed.

**Structure of the paper** Our paper is structured as follows: In the remaining of this section we describe related work. Then, in Section 2, we explain the general concept of optimization with controllable uncertainty for uncertain cost. We describe how one can modify the scenario set at a query cost by reducing the uncertain intervals around the so-called hedging points. In Section 3, we assume that the hedging points are known in advance and part of the input data. For binary queries, we show how one can reformulate the problem to a single-level one, if the underlying optimization problem is given by a linear program (LP). In Section 4, we assume that the hedging points are not initially known and model them as a variable that is chosen in a worst-case fashion. We model this setting in a four-level problem. Further, we investigate a robust optimization approach, and present an equivalent nonlinear single-level formulation.

## 1.2 Related work

Optimization under controllable uncertainty is closely related to other concepts like bilevel optimization, robust optimization, explorable uncertainty, decision-dependent uncertainty sets and decision-dependent information discovery. In the following, we give a brief overview of the aforementioned concepts. Please note that the literature review is far from exhaustive. Recent surveys for further reading are given where available.

**Bilevel optimization** In a bilevel optimization problem, two optimization problems are nested in a hierarchical order. Two players usually called *leader* and *follower* control disjoint sets of variables, who optimize their own objectives with constraints that both can depend on the other's decisions. Foundations on bilevel programming are explained in the textbook by [Dem02] and further advances in bilevel optimization can be found in [DZ21] which includes an extensive bibliography in the last chapter. For mixed-integer bilevel programs, see also the survey [KLLS21] and references therein.

Connections between robust and bilevel optimization, in particular possible reformulations of problems in one setting to the other one and vice versa, are discussed in [GKST23].

Two types of uncertainty that have been considered for bilevel problems are data uncertainty and decision uncertainty, see [BLS23a] and references therein. In data uncertainty, there is an uncertainty about the follower's data that is either realized after the leader's but before the follower's decisions (wait-and-see follower) or after the follower's decision is fixed (here-and-now follower). In decision uncertainty, one or both levels hedge against the other level's decision that for example might not be optimal but only near-optimal due to limited resources. In contrast to controllable uncertainty, neither of the two players can influence the uncertainty in these approaches.

A special type of bilevel optimization problems are min-max problems, i.e., problems in which the two players share the same objective function though optimize in opposite directions. A prominent example are interdiction games. In interdiction games, the upper-level problem interdicts some lower-level elements such that the follower is inhibited as much as possible in pursuing their goal [SS20]. Interdiction games with a monotone $\Gamma$-robust follower have been considered in [BLS23b, BLS23c].

An extension of interdiction games are fortification games where a third level is added. In fortification games, some items can be defended before the opponent interdicts some of the remaining items. Binary fortification games can be solved with a decomposition approach [BCSW06]. A generalized solution method is to use a branch-and-cut algorithm with fortification cuts [LLM$^+$23].

**Robust optimization** In robust optimization, optimization problems with uncertain cost are considered where the scenario is chosen adversarially after the decision of the optimization problem has been fixed, e.g., [BEN09].

A widely used scenario set is the so-called *budgeted uncertainty set* introduced in [BS03, BS04] that is restricted in two ways. For each decision variable the uncertain cost is restricted to an interval. Furthermore, there is a budget $\Gamma$ for the sum of actual increases normalized by the interval sizes. This budgeted uncertainty set is a polytope whose number of vertices grows exponentially with $\Gamma$. Robust counterparts of discrete optimization problems with polynomial runtime are still solvable in polynomial time [BS03].

**Explorable uncertainty** In explorable uncertainty, the uncertainty is resolved by revealing precise data values at some cost until an optimal solution of the underlying problem (i.e, a solution that is optimal irrespective of the remaining uncertainty) can be determined. The seminal work was introduced in [Kah91]. Revealing precise data values at some investment or effort is referred to as a *query*. The goal is to minimize query cost while the underlying problem can still be solved exactly.

Studies investigate the concept of explorable uncertainty on basic combinatorial problems like Selection [GSS11] as well as classical discrete problems like Shortest Path [FMO+03], Minimum Spanning Tree [EHK+08, FMM17], knapsack [GGI+15], and matroids [Mei18, MS19]. The binary query selection revealing an exact value is extended to returning a refined uncertainty interval in [GSS11]. We refer to the survey [EH15] for a good research overview on explorable uncertainty.

The query selection is realized in an online or offline approach. The online query selection is an adaptive model where queries are selected sequentially and for each decision one can use the outcome of all previous value determinations [BHKR05, FMM17]. The offline query model requests a non-adaptive selection simultaneously choosing and revealing as many queries as required to ensure the existence of an exact solution of the underlying optimization problem [MS19, FMO+03].

**Decision-dependent uncertainty sets** In optimization with decision-dependent uncertainty sets, the uncertainty set depends on the decisions taken by the optimizer. This captures situations where decisions influence the magnitude or structure of uncertainty. Robust linear problems with binary decisions modifying polyhedral uncertainty sets are NP-complete [NS18], and several mixed-integer linear reformulations have been proposed [NS18, ?]. Extensions include reformulations that shift the decision dependence from the uncertainty set into the problem constraints [?], as well as column-and-constraint generation methods for two-stage settings with linear decision dependence [?], where bilinear terms can be linearized when the decisions are binary. Related approaches are also referred to as *endogenous uncertainty* [LG18, BG22], where decisions influence whether, when, or how uncertainty affects the model. For example, removing a parameter from the uncertainty set can represent cases in which a decision renders this parameter irrelevant, alters the timing of its realization, or changes its distribution.

**Decision-dependent information discovery** Decision-dependent revelation of uncertain parameters has mainly been considered in stochastic optimization, see [VGY22] and references therein. Recently, this idea has been combined with robust optimization instead resulting in the problem of decision-dependent information discovery (DDID) [VGY22, PGDT22, OPR24]. In DDID, the optimizer has a binary choice in the first step to reveal some exact values, i.e., uncertain values are only either revealed completely or kept uncertain. The chosen values realize in a worst-case manner. Afterwards, the nominal problem is solved in a robust approach for the remaining uncertainties. Thus, DDID can be formulated as a four-level min-max-min-max problem. For general polyhedral uncertainty sets, both exact algorithms and approximations have been proposed. Furthermore, due to a budget instead of a cost for made queries, it can only be beneficial to make additional queries and to exhaust the query budget. As a natural consequence, they assume that it is not possible to make all queries, e.g., [OPR24, Assumption 1].

# 2 Controllable uncertainty for uncertain cost

In general, the concept of controllable uncertainty can be used for both uncertain cost and uncertain feasible regions. In this paper we solely discuss the case of uncertain cost. For this case we now formalize the concept of controllable uncertainty and point variations in modeling with this concept. In Section 2.3 we summarize additional assumptions to which we restrict the analysis in the rest of this paper. Finally, in Section 2.4 we discuss a peculiar effect of controllable uncertainty with uncertain cost, namely, budget deflection.

## 2.1 Formalization of the concept

In the following, we introduce the concept of controllable uncertainty. First, we state the optimization problems for which we consider controllable uncertainty. Then, we describe the used model of uncertainty. Afterwards, we introduce queries and explain how they reduce uncertainty. In particular, this includes the definition of the controllable uncertainty set. Finally, we describe the overall objective function of the resulting problem. For a summary of the notation, see Table 1.

Controllable uncertainty is a possibility to model how one can deal with uncertainty in some *underlying (optimization) problem*

$$\min_{y \in Y} f(y).$$

We assume that the underlying problem has non-negative decision variables $y$ chosen from a feasible set $Y \subseteq \mathbb{R}^n_{\geq 0}$. We refer to the indices of vectors like $y$ as *elements* and denote them with $e$. We assume that the objective function $f$ of the underlying problem is parameterized by non-negative uncertain coefficients $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)$. These uncertain coefficients $\tilde{c}$ lie within bounded, closed intervals, i.e.,

$$\tilde{c}_e \in [c_e, c_e + d_e] \text{ for all } e \in [n] := \{1, \ldots, n\} \text{ with } d \in \mathbb{R}^n_{\geq 0}.$$

Each realization of an uncertain parameter $\tilde{c}_e$ is associated with the corresponding normalized value $u_e \in [0, 1]$ such that $\tilde{c}_e = c_e + u_e d_e$.

A possible choice of $u_e$ as well as the whole vector $u$ is a *realization of the uncertainty* or *scenario*. We assume that the uncertainty set is given as some polyhedron $\mathcal{U} \subseteq [0, 1]^n$. To explicitly denote the dependency on the scenario, we also write $f(u, y)$ for the objective of the underlying problem.

The interval for the choice of $u_e$ is narrowed, from $[0, 1]$ to at most a single point $b_e \in [0, 1]$. This point $b_e$ is the *e-th hedging point*. The set of possible hedging points $b$ is denoted by $\mathcal{B} \subseteq [0, 1]^n$.

The continuous variable $x_e$ expresses how much the size of the interval for $u_e$ is narrowed. We call $x_e$ as well as the vector $x$ as a whole a *query*. The set of possible queries is given by $X \subseteq \mathbb{R}^n_{\geq 0}$. We assume that $\mathbf{0} \in X$ and will refer to $x = \mathbf{0}$ as "making no query". Mostly, we think of $X$ as

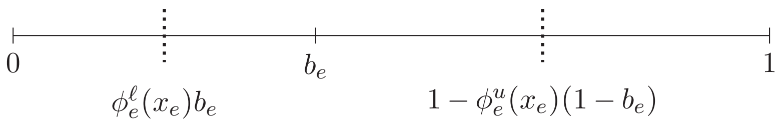


Figure 1: Reduced interval for the choice of $u_e$

$\mathbb{R}^n_{\geq 0}$. However, it can include constraints on the query selection. For example, a query for element $e$ might only be allowed if also element $e'$ is queried to at least the same amount, i.e., $x_e \leq x_{e'}$.

The *lower* and *upper query outcome* $\phi_e^\ell(x_e)$ respectively $\phi_e^u(x_e)$ shift the lower respectively upper boundary of the interval for the realization of the uncertainty $u_e$. We require the functions $\phi_e^\ell \colon \mathbb{R}_{\geq 0} \to [0,1]$ and $\phi_e^u \colon \mathbb{R}_{\geq 0} \to [0,1]$ to be monotone and to fulfill $\phi_e^\ell(0) = \phi_e^u(0) = 0$. The lower bound for the realization of the uncertainty $u_e$ raises within the interval $[0, b_e]$ by the fraction of the lower query outcome $\phi_e^\ell(x_e)$. Similarly, the upper query outcome $\phi_e^u(x_e)$ is the fraction by which the upper bound on the realization of the uncertainty $u_e$ is reduced within the interval $[b_e, 1]$. For a visualization, see Figure 1. More precisely, the query $x_e$ narrows the interval $[0,1]$ for $u_e$ to the interval

$$\left[b_e\phi_e^\ell(x_e), 1-(1-b_e)\phi_e^u(x_e)\right].$$

If $\phi_e^\ell$ and $\phi_e^u$ are strictly less than one for all queries $x_e$, a reduction of the interval for the choice of $u_e$ to a singleton is not possible, see the example of asymptotic behavior below.

**Example 2.1** (Query functions)**.** *The query outcome function $\phi_e$ can be*

*(a) $\phi_e(x_e) = \min\{x_e, 1\}$ for proportional outcome on $[0,1]$ and constant else or*

*(b) $\phi_e(x_e) = \frac{x_e}{x_e+1}$ for an asymptotic behavior or*

*(c) $\phi_e(x_e) = 0$ for $x_e < 1$ and $\phi_e(x_e) = 1$ for $x_e \geq 1$ for a binary query outcome. The same outcome is obtained for $\phi_e$ being the identity and restricting $X$ such that $x_e \in \{0,1\}$.*

For the sake of convenience, we use the following conventions for the query outcome functions. Whenever $\phi_e^\ell$ and $\phi_e^u$ are equal, we refer to them with one function $\phi_e \colon \mathbb{R}_{\geq 0} \to [0,1]$. Also, we drop the index and only write $\phi^\ell, \phi^u$ or $\phi$ if the respective query outcome functions $\phi_e^\ell, \phi_e^u, \phi_e$ are equal for all elements $e \in [n]$.

Next, we define the *controllable uncertainty set* $\mathcal{U}(x,b)$ to capture the reduction of the uncertainty set $\mathcal{U}$. We call elements of $\mathcal{U}(x,b)$ the *remaining uncertainty. Adversarial feasibility* is the requirement that the controllable uncertainty set $\mathcal{U}(x,b)$ is non-empty.

**Definition 2.2** (Controllable uncertainty set)**.** *For a polyhedral uncertainty set $\mathcal{U}$, the* controllable uncertainty set *with respect to query $x$ and hedging point $b$ is the set*

$$\mathcal{U}(x,b) = \left\{ u \in \mathcal{U} \;\middle|\; \phi_e^\ell(x_e) b_e \leq u_e \leq 1 - \phi_e^u(x_e)(1-b_e) \;\forall e \in [n] \right\}.$$

**Observation 2.3.** *We have:*

*(a) If $\phi_e^\ell(x_e) = \phi_e^u(x_e) = 0$ for all $e \in [n]$, then $\mathcal{U}(x,b) = \mathcal{U}$.*

*(b) $\mathcal{U}(\mathbf{0},b) = \mathcal{U}$.*

*(c) Let $\overline{x} \in X$ be a query with $\phi_e^\ell(\overline{x}_e) = \phi_e^u(\overline{x}_e) = 1$ for all $e \in [n]$. Then $\mathcal{U}(\overline{x},b) = \{b\} \cap \mathcal{U}$.*

*Proof.* The statements follow from the definition of the controllable uncertainty set $\mathcal{U}(x,b)$ and the assumption that $\phi_e^\ell(0) = \phi_e^u(0) = 0$. □

If we make a query to narrow an uncertainty interval in our model, a *query cost* $q(x)$ is generated. We assume that the query cost function $q\colon X \to \mathbb{R}_{\geq 0}$ is monotone and that there is no query cost if no query is made, i.e., $q(\mathbf{0}) = 0$.

**Observation 2.4.** *There are the following relations between the query cost $q$ and the query outcome function $\phi$, if $X = \mathbb{R}^n_{\geq 0}$:*

*(a) If $q$ is strictly monotone, it is equivalent to use either $q$, $\phi_e^\ell$ and $\phi_e^u$ or the identity* Id, *$q^{-1}\phi_e^\ell$ and $q^{-1}\phi_e^u$ as query cost and query function respectively.*

*(b) If $\phi = \phi_e^\ell = \phi_e^u$ is strictly monotone, it is equivalent to use either $q$ and $\phi$ or $\phi^{-1}q$ and* Id *as query cost and query function respectively.*

*Proof.* If there is a bijection $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $g(0) = 0$, we can get another instance of our problem if we apply $g$ and replace $x$ with $\tilde{x} = g(x)$ in all occurrences. If $q$ or respectively $\phi$ is strictly monotone, the inverse $q^{-1}$ or $\phi^{-1}$ exist. If $g = q^{-1}$, then $\phi_e^\ell(\tilde{x}) = \phi_e^\ell(q^{-1}(x))$, $\phi_e^u(\tilde{x}) = \phi_e^u(q^{-1}(x))$ and $q(\tilde{x}) = x$. The same argument applies if $g = \phi^{-1}$. □

In optimization under controllable uncertainty, a single optimizer allocates resources to both uncertainty mitigation and the core optimization problem. The overall objective function $F(x,u,y)$ is the sum of the query cost $q(x)$ and the objective function $f(u,y)$ of the underlying problem, i.e.,

$$F(x,u,y) = q(x) + f(u,y).$$

| | |
|---|---|
| $\min_{y \in Y} f(y)$ | underlying problem |
| $\tilde{c}_e = c_e + u_e d_e$ | uncertain coefficients in $f(y)$ |
| $u \in \mathcal{U} \subseteq [0,1]^n$ | realization of the uncertainty |
| $x \in X \subseteq \mathbb{R}^n_{\geq 0}$ | query |
| $q\colon X \to \mathbb{R}_{\geq 0}$ | query cost |
| $\phi^\ell_e, \phi^u_e\colon \mathbb{R}_{\geq 0} \to [0,1]$ | lower, upper query outcome |
| $b \in \mathcal{B} \subseteq [0,1]^n$ | hedging point |
| $\mathcal{U}(x,b) \subseteq \mathcal{U}$ | controllable uncertainty set |
| $F\colon X \times \mathcal{U} \times Y \to \mathbb{R}$ | overall objective function |

Table 1: Summary of notation

## 2.2 Variations in modeling with controllable uncertainty

In the following, we outline possibilities to use controllable uncertainty. One can vary when parts of the uncertainty is reduced around the hedging points, how and when the remaining uncertainty is dealt with or whether all queries are made at once or successively.

In the following, we assume that all queries are chosen in a single, first step. This is the analogue to offline queries in explorable uncertainty. Successive queries like online queries in explorable uncertainty are not considered here.

For the remaining uncertainty, we use a robust approach, i.e., we consider a worst-case scenario. Thus, we obtain min-max settings. Other approaches for dealing with the remaining uncertainty like stochastic optimization are possible though will not be considered here.

In the two possible problem settings with known hedging points, the uncertainty either realizes before or after the underlying problem is solved. If the underlying problem is solved before the remaining uncertainty realizes, the decisions for both the queries and the underlying problem can be made in the same step. Hence, we obtain a robust optimization problem with a decision-dependent uncertainty set. This setting is not further considered. We consider the other setting in which the remaining uncertainty realizes before the underlying problem is solved in Section 3.

Afterwards, in Section 4, we consider a setting with uncertain hedging points. We deal with parts of the uncertainty in the hedging points before and with the remaining uncertainty after solving the underlying problem.

Note that we focus in Sections 3 and 4 on two settings for controllable uncertainty that are closely related to decision-dependent/endogenous uncertainty and decision-dependent-information discovery respectively. Work on these topics typically places fewer assumptions on the uncertainty set but assumes that queries are binary. In contrast, in general, we do not make any assumptions on whether queries are continuous or discrete. Moreover, both decision-dependent uncertainty and decision-dependent information discovery have fixed problem stages, whereas our concept of controllable uncertainty allows for different settings. We point out similar results in the literature as well as differences to related models when considering the two settings for controllable uncertainty.

## 2.3 Additional assumptions

For all findings in the remaining of this paper, we add the following two assumptions.

**Assumption 2.5.** *We assume that the underlying problem has a linear objective function, i.e.,*

$$f(y) = f(u, y) = \tilde{c}^\top y = (c + d \cdot u)^\top y.$$

For example, the underlying problem can be a linear problem (LP) like the diet problem or a discrete problem like shortest path, min cut or TSP. In later sections, we derive some results that only hold for binary problems or problems that can be formulated as an LP.

As uncertainty set, we will only consider the budgeted uncertainty set. This uncertainty set has been introduced in [BS03] and is widely used in robust optimization.

**Assumption 2.6.** *We assume that $\mathcal{U}$ is the budgeted uncertainty set*

$$\mathcal{U} = \left\{ u \in [0,1]^n \;\middle|\; \sum_{e \in [n]} u_e \leq \Gamma \right\}.$$

In order to have adversarial feasibility, i.e., $\mathcal{U}(x, b) \neq \emptyset$, we derive a necessary condition for the choice of a hedging point.

**Observation 2.7** (Necessary condition for adversarial feasibility)**.** *It is*
*$\mathcal{U}(x, b) = \emptyset$ if*

$$\Gamma - \sum_{e \in [n]} \phi_e^\ell(x_e) b_e < 0.$$

*Proof.* The statement follows if we combine the lower bounds $\phi_e^\ell(x_e) b_e \leq u_e$ for all $e \in [n]$ from Definition 2.2 and the budget constraint $\sum_{e \in [n]} u_e \leq \Gamma$ from Assumption 2.6. □

Thus, we assume in the following that for all $x \in X$ we have

$$\mathcal{B} \subseteq \left\{ b \in [0,1]^n \;\middle|\; \Gamma - \sum_{e \in [n]} \phi_e^\ell(x_e) b_e \geq 0 \right\}.$$

## 2.4 Budget deflection for elementwise query cost

In the following, we define the phenomenon of *budget deflection* for *elementwise query cost*.

**Definition 2.8** (Elementwise query cost)**.** *The query cost $q(x)$ is elementwise if there are functions*
$q_e\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that*

$$q(x) = \sum_{e\in[n]} q_e(x_e).$$

In controllable uncertainty, a query serves the purpose to gain information or to protect against very unwelcome realizations of the uncertainty at a certain cost. If the query cost is elementwise, one could assume that the query of an element $e$ directly corresponds to controlling the uncertainty $u_e$. However, in an optimal solution, an element $e$ might be queried though never used in any optimal solution of the underlying problem to control the uncertainty of cost for another element $e'$. The query for element $e$ decreases the uncertainty budget at a relatively cheap price. Due to the reduced uncertainty budget, the cost for element $e'$ is decreased. We call this phenomenon *budget deflection* and give a more formal definition.

**Definition 2.9** (Budget deflection)**.** *Let $x^* \in X = \mathbb{R}^n_{\geq 0}$ be an optimal query for an optimization problem with controllable uncertainty and elementwise query cost. Furthermore, let $Y^*(x^*)$ denote the set of all solutions $y^*$ of the underlying problem that are optimal for at least one realization $u \in \mathcal{U}(x^*, b)$ of the uncertainty when $x^*$ has been fixed.*

*We say that an instance allows for* budget deflection*, if*

$$\exists e \in [n] : q_e(x^*_e) > 0,\ \forall y^* \in Y^*(x^*) :\ y^*_e = 0.$$

*In other words, if there is an element $e \in [n]$ with positive query cost, i.e., $q_e(x^*_e) > 0$, that is not used in any optimal solution $y^* \in Y^*(x^*)$ of the underlying problem.*

We will show that budget deflection can occur in the setting with known hedging points described in Section 3. Depending on the application, the phenomenon of budget deflection can be either considered necessary for correct modeling or unrealistic. The former may apply if the uncertainty models an adversarial agent with a budget, whose actions can be directly influenced by investment. However, if the uncertainty is caused by random events, budget deflection can be seen as a modeling artifact that results from the model used for $\mathcal{U}$; more precisely, by the interdependence of the impact of uncertainty on different elements when using budgeted uncertainty (as well as when using general polyhedral or ellipsoidal uncertainty sets). After we provide a small numeric example, we will give a necessary condition for budget deflection and show the impact of a model adaption that prevents it, see Section 3.3. In contrast, in Section 4.3, we will show that there is no budget deflection in the setting with unknown hedging points considered in Section 4.

# 3 Optimization with known hedging points

We consider the problem with known hedging points (KHP) to determine an optimal query $x$ such that the underlying problem is minimized for the worst-case outcome of the uncertainty $u$. The values of the hedging point $b$ are input parameters of the problem

$$\inf_{x \in X} \max_{u \in \mathcal{U}(x,b)} \min_{y \in Y} F(x,u,y). \tag{KHP}$$

An optimal query $x^*$ and the optimal objective value $F^*$ might not exist. For example, consider $\phi$ to be the asymptotic query function suggested in Example 2.1 in combination with zero query cost, i.e., $q = 0$. If there are no restrictions on the possible queries, i.e., $X = \mathbb{R}_{\geq 0}$, for every query $x$ there exists another query $x'$ that results in a smaller objective value.

Since the hedging points $b$ are input parameters of KHP, in the following we write the uncertainty set only in dependence of the query $x$, i.e., $\mathcal{U}(x) = \mathcal{U}(x, b)$.

First, we derive bounds on the optimal objective value of KHP in Section 3.1. Afterwards, in Section 3.2, we provide an equivalent single-level reformulation for KHP if the underlying problem is given as a linear program (LP). In Section 3.3, we describe budget deflection, which is a phenomenon that in order to reduce the adversary weight modification for an element $e$, a different element $e'$ is queried. Finally, after we add additional assumptions on the query cost and query outcome functions, we show how KHP can be formulated as single-level mixed-integer program.

**Comparison of KHP with similar concepts** In the following, we compare KHP with interdiction and fortification problems. For a fixed query $x$, KHP becomes a continuous interdiction problem where interdiction only affects the objective. The uncertainty $u$ is chosen from intervals within a budget of $\Gamma$ to maximize the minimal outcome of the underlying problem.

Next, we argue how the choice of a query $x$ in KHP resembles the uppermost level of fortification games. In fortification, the uppermost level decides which elements to protect such that they cannot be interdicted. A fortification can be used on the uppermost level to prevent interdiction that realizes in a worst-case manner. In KHP, a query $x$ indicates the reduction of the interval sizes for the weight modification caused by the uncertainty $u$. For binary query outcome functions, see Example 2.1, KHP is a fortification game with binary fortification and continuous interdiction.

Finally, we observe that KHP can also be seen as a special case of optimization with decision-dependent uncertainty sets. In this broader class of models, the decision maker influences the structure of the uncertainty set itself. Related work in this area, such as [NS18, **?**, **?**], typically considers binary decisions affecting polyhedral uncertainty sets via linear constraints. In contrast, our formulation allows for continuous query decisions and nonlinear, query-dependent bounds on the uncertainty. The bilinear interaction between the query $x$ and the adversarial uncertainty $u$ distinguishes our model from those in which uncertainty dependence is linear. Still, the structural similarity allows us to adapt techniques such as single-level reformulations (e.g., as in Proposition 13 of [?]) for our own results, such as Theorems 3.4 and 3.11.

## 3.1 Bounds for KHP

In the following, we show upper and lower bounds on the optimal objective value of KHP. We fix some particular query and then solve the underlying optimization problem. In general, even when a query is fixed, the resulting bilevel problem cannot be easily solved.

For the first upper bound, we consider that no query is made, i.e., $x = \mathbf{0}$. Recall that if no query is made, there is no query cost. We relax the budget constraint for the uncertainty and use that the upper bounds for the uncertainty are at most one. Then, only the underlying problem with objective $c + d$ remains. Thus, we obtain the following upper bound.

**Observation 3.1.** *The optimal objective value $F^*$ of KHP has the upper bound*

$$F^* \leq \min_{y \in Y} (c + d)^\top y.$$

*Proof.* We use query $\mathbf{0} \in X$, the definition of $F(x, u, y)$, the assumptions that $q(\mathbf{0}) = 0$, $\mathcal{U}(x) \subseteq [0,1]^n$ and $c, d, y$ are non-negative, see Section 2, to obtain

$$\begin{aligned}
F^* &= \inf_{x \in X} \max_{u \in \mathcal{U}(x)} \min_{y \in Y} F(x, u, y) \\
&\leq \max_{u \in \mathcal{U}(\mathbf{0})} \min_{y \in Y} F(\mathbf{0}, u, y) = \max_{u \in \mathcal{U}(\mathbf{0})} \min_{y \in Y} (c + d \cdot u)^\top y \\
&\leq \max_{u \in [0,1]^n} \min_{y \in Y} (c + d \cdot u)^\top y \\
&= \min_{y \in Y} (c + d \cdot \mathbf{1})^\top y = \min_{y \in Y} (c + d)^\top y.
\end{aligned}$$

□

For the second upper bound and a lower bound on the optimal objective value of KHP, let $\overline{x}$ denote a query for which the uncertainty set reduces to a singleton. The existence of such a query depends on the allowed queries $X$ and the query outcome functions $\phi_e^\ell$ and $\phi_e^u$. If such a query $\overline{x}$ exists and we plug this in, only the underlying problem remains. Depending on whether we consider the query cost for $\overline{x}$ or not, we obtain an upper respectively lower bound.

**Observation 3.2.** *Let $\overline{x} \in X$ be a query such that no uncertainty is left for it, i.e., $\phi_e^\ell(\overline{x}_e) = \phi_e^u(\overline{x}_e) = 1$ for all $e \in [n]$. Then, for the optimal objective value $F^*$ of KHP, the following lower* *and upper bounds hold:*

$$\min_{y\in Y} f(b,y) \le F^* \le q(\overline{x}) + \min_{y\in Y} f(b,y).$$

*Proof.* We choose $\overline{x} \in X$ and use $\mathcal{U}(\overline{x}) = \{b\}$, see Observation 2.3, to obtain the upper bound:

$$\begin{aligned} F^* &= \inf_{x\in X} \max_{u\in\mathcal{U}(x)} \min_{y\in Y} q(x) + f(u,y) \\ &\le q(\overline{x}) + \max_{u\in\mathcal{U}(\overline{x})} \min_{y\in Y} f(u,y) = q(\overline{x}) + \min_{y\in Y} f(b,y). \end{aligned}$$

For the lower bound, we use that query cost $q(x)$ is non-negative and that $\mathcal{U}(\overline{x}) = \{b\}$. If there is no query cost, $\overline{x}$ is optimal for the outer minimization. We have

$$\begin{aligned} F^* &= \inf_{x\in X} \max_{u\in\mathcal{U}(x)} \min_{y\in Y} f(u,y) + q(x) \\ &\ge \inf_{x\in X} \max_{u\in\mathcal{U}(x)} \min_{y\in Y} f(u,y) = \max_{u\in\mathcal{U}(\overline{x})} \min_{y\in Y} f(u,y) = \min_{y\in Y} f(b,y). \end{aligned}$$

□

## 3.2 Single-level reformulation for KHP

The main result of this section is that KHP can be reformulated as an equivalent single-level non-linear problem (NLP) if the underlying problem is a LP. We call two optimization problems *equivalent*, if they depend on the same parameters and always have the same optimal objective value. In the single-level reformulation, variables $u$ for the realization of the uncertainty are replaced by dual variables for the constraints on the realization of the uncertainty within $\mathcal{U}(x)$. Afterwards, we state conditions on the values of variables in the single-level reformulation that hold for optimal solutions.

**Assumption 3.3.** *For the remaining of Section 3, we assume that $Y \subseteq \mathbb{R}^n_{\ge 0}$ is a nonempty compact, convex feasible set.*

Furthermore, recall that the objective of the underlying problem is linear by Assumption 2.5. Our setting therefore also includes discrete problems given by a totally dual integral (TDI) system, since integrality constraints can be dropped for any linear objective function in this case. TDI systems are known for several optimization problems like Shortest Path, Minimum Spanning Tree, Maximum Flow, and Minimum Cut, see e.g., [KV18].

**Theorem 3.4** (Single-level NLP)**.** *Assume that an optimal query $x^*$ exists. An optimization problem under controllable uncertainty with known hedging points (KHP)*

$$\min_{x\in X} \max_{u\in\mathcal{U}(x)} \min_{y\in Y} q(x) + c^\top y + (d\cdot u)^\top y \tag{1}$$

*is equivalent to the following single-level non-linear problem (NLP)*

$$
\begin{aligned}
\min_{\substack{x\in X,\\ y\in Y,\\ \beta,\theta}} \quad & \Gamma\theta + q(x) \\
& + \sum_{e\in[n]} \beta_e + c_e y_e - \phi_e^\ell(x_e) b_e\left(\theta + \beta_e - d_e y_e\right) - \phi_e^u(x_e)\beta_e(1-b_e) \\
\text{s.t.} \quad & \theta + \beta_e - d_e y_e \geq 0 \qquad \forall e\in[n] \\
& \beta,\theta \geq 0.
\end{aligned} \tag{2}
$$

*Proof.* First, we argue that we can interchange the innermost minimization and maximization step. The objective function $F$ is linear in both variables $u$ and $y$ if all respective other variables are fixed. Furthermore, the feasible sets for $u$ and $y$ are polytopes by assumption and independent of $y, u$ respectively. Thus, we can apply von Neumann's minimax theorem, see [vN28] or in english e.g., [Sim09, Theorem 2], to obtain

$$
\min_{x\in X}\max_{u\in\mathcal{U}(x)}\min_{y\in Y} F(x,u,y) = \min_{x\in X,\, y\in Y}\max_{u\in\mathcal{U}(x)} F(x,u,y). \tag{3}
$$

If the underlying problem is an LP, we can also obtain this equality using strong duality twice.

Next, we use strong duality to replace the inner maximization problem by a minimization problem. The inner maximization problem is the LP

$$
\begin{aligned}
\max_{u} \quad & (d\cdot y)^\top u \\
\text{s.t.} \quad & u_e \geq \phi_e^\ell(x_e) b_e \qquad \forall e\in[n] \\
& u_e \leq 1 - \phi_e^u(x_e)(1-b_e) \qquad \forall e\in[n] \\
& u^\top \mathbf{1} \leq \Gamma
\end{aligned} \tag{4}
$$

with dual problem

$$
\begin{aligned}
\min_{\alpha,\beta,\theta} \quad & \Gamma\theta + \sum_{e\in[n]} \phi_e^\ell(x_e) b_e \alpha_e + (1-\phi_e^u(x_e)(1-b_e))\beta_e \\
\text{s.t.} \quad & \alpha_e + \beta_e + \theta = d_e y_e \qquad \forall e\in[n] \\
& \alpha \leq 0 \\
& \beta,\theta \geq 0.
\end{aligned} \tag{5}
$$

We substitute $\alpha_e = d_e y_e - \beta_e - \theta \leq 0$ which simplifies Problem (5) to

$$\begin{aligned}
\min_{\beta,\theta} \quad & \Gamma\theta + \sum_{e\in[n]} \phi_e^\ell(x_e) b_e (d_e y_e - \beta_e - \theta) + (1 - \phi_e^u(x_e)(1-b_e))\beta_e && (6)\\
\text{s.t.} \quad & \theta + \beta_e - d_e y_e \geq 0 \qquad \forall e \in [n] \\
& \beta, \theta \geq 0.
\end{aligned}$$

Due to strong duality, Problems (4), (5) and (6) have the same optimal objective value. Thus, we can replace the inner maximization problem in (3) by Problem (6). We obtain the single-level reformulation (2) that is equivalent to KHP. □

Note that the second part of our proof, once equivalence to the 2-stage min-max problem is known, has also been established in [NS18]. Furthermore, a similar result to our Theorem 3.4 has been independently given in [**?**, Proposition 13].

Remember that equivalence of (1) and (2) is defined with respect to the optimal objective value. It follows from the proof of Theorem 3.4 that optimal queries for (2) are also optimal for (1), and vice versa. However, the values of the variables $y$ in optimal solutions of both problem formulations don't have to agree. In fact, if the underlying problem is given by a TDI system, the values of the variables $y$ are not necessarily integral in an optimal solution of (2).

Next, we consider elements $e$ that are not used in the underlying problem's optimal solution.

**Lemma 3.5.** *In an optimal solution $(x^*, y^*, \beta^*, \theta^*)$ of Problem (2), if $y_e^* = 0$ then*

$$\beta_e^* \left(1 - \phi_e^u(x_e^*)(1-b_e) - \phi_e^\ell(x_e^*) b_e\right) = 0.$$

*Proof.* If $y_e^* = 0$, the constraint $d_e y_e \leq \theta + \beta_e$ simplifies to $0 \leq \beta_e + \theta$. This is always fulfilled as $\beta$ and $\theta$ are required to be nonnegative. There are no further constraints on $\beta$.

For the objective of Problem (2), we have

$$\begin{aligned}
& \Gamma\theta + q(x) + \sum_{e\in[n]} \beta_e + c_e y_e + \phi_e^\ell(x_e) b_e \left(d_e y_e - \beta_e - \theta\right) - \phi_e^u(x_e)\beta_e(1-b_e) \\
=& \left(\Gamma - \sum_{e\in[n]} \phi_e^\ell(x_e) b_e\right)\theta + q(x) \\
& + \sum_{e\in[n]} c_e y_e + \phi_e^\ell(x_e) b_e d_e y_e + \beta_e \left(1 - \phi_e^u(x_e)(1-b_e) - \phi_e^\ell(x_e) b_e\right).
\end{aligned}$$

Variable $\beta_e$ is included with the factor $\left(1 - \phi_e^u(x_e)(1-b_e) - \phi_e^\ell(x_e) b_e\right)$. This non-negative factor is the size of the remaining interval the adversary chooses $u_e$ from, see Section 2. Thus, in an optimal solution, at least one of the two factors $\beta_e^*$ and $1 - \phi_e^u(x_e^*)(1-b_e) - \phi_e^\ell(x_e^*) b_e$ is zero. □

In the following lemma, we provide upper bounds for the variables $\beta$ and $\theta$ in an optimal solution of Problem (2) if the underlying problem's decisions $y$ are bounded. For example, let the underlying problem be binary.

**Lemma 3.6.** *Let $Y \subseteq \{y \in \mathbb{R}_{\geq 0} \mid |y_e \leq M_e\}$. If $(x^*, y^*, \beta^*, \theta^*)$ is an optimal solution for the single-level reformulation (2), then*

$$\theta^* \leq \max_{e \in [n]} d_e M_e =: D \quad \text{and} \quad \beta_e^* \leq d_e M_e \; \forall e \in [n].$$

*Proof.* Consider the constraints $d_e y_e - \beta_e \leq \theta$ for all $e \in [n]$. Since $\beta_e$ and $\theta$ are required to be nonnegative and are included with nonnegative factors in the objective that is minimized and since $y_e \leq M_e$, we have $\theta^* \leq \max_e d_e M_e =: D$ and $\beta_e^* \leq d_e M_e$. □

## 3.3 Budget deflection for KHP

In the following, we consider *budget deflection* for KHP. Recall that budget deflection is the phenomenon that one coordinate is queried to control the uncertainty for other coordinates, see Section 2.4.

In this section, we first provide examples that budget deflection can occur and describe an example setting where this is crucial for modeling. Further, we give sufficient conditions under which it cannot occur and show in Theorem 3.9 that under these conditions the optimal objective value does not decrease.

**Example 3.7** (Budget deflection)**.** *Let the underlying problem be the LP*

$$\min_{y \in Y} (3 + 4u_1)y_1 + (1 + 4u_2)y_2 \text{ with } Y = \left\{y \in [0,1]^2 \mid y_1 + y_2 = 1\right\}.$$

*Let the hedging points be both in the middle of the intervals, i.e., $b_1 = b_2 = 0.5$ and let the budget for the uncertainty be $\Gamma = 1$. The controllable uncertainty set is then given by*

$$\mathcal{U}(x) = \left\{u \in [0,1]^2 \mid u_1 + u_2 \leq 1, 0.5x_i \leq u_i \leq 1 - 0.5x_i, i \in \{1,2\}\right\}.$$

*Let the query cost be given as linear function $q(x) = 0.5x_1 + x_2$. In total, we consider the following example for KHP:*

$$\min_{x \in [0,1]^2} \max_{u \in \mathcal{U}(x)} \min_{y \in Y} 0.5x_1 + x_2 + (3 + 4u_1)y_1 + (1 + 4u_2)y_2.$$

*In the optimal solution the query is $x^* = (1,0)^\top$ while the underlying solution is $y^* = (0,1)^\top$. Fully querying the first coordinate, i.e., $x_1^* = 1$ results in fixing the uncertainty of the first coordi-*

*nate to the hedging point $u_1^* = b_1 = 0.5$. Then, the problem simplifies to*

$$\max_{u_2 \in [0,0.5]} \min_{y \in Y} 0.5 + 5y_1 + (1 + 4u_2)y_2 = \min_{y \in Y} 0.5 + 5y_1 + 3y_2 = 3.5.$$

It depends on the application whether budget deflection makes sense or not. If the adversary in KHP represents uncertainty in the worst-case scenario rather than an actual adversarial agent, budget deflection seems unwanted. We can prevent budget deflection if we set the lower query function to zero:

**Theorem 3.8.** *Let $X = \mathbb{R}^n_{\geq 0}$ and assume that the query cost is elementwise. Let $x^*$ be an optimal solution of KHP with query cost $q_e(x_e^*) > 0$ for some $e \in [n]$ where $\phi_e^\ell = 0$. Then there exists $u \in \mathcal{U}(x^*)$ such that there is an optimal solution $y^*$ to the underlying problem with $y_e^* > 0$.*

*Proof.* Let $x^*$ be an optimal query. Fix some $e \in [n]$ with positive query cost, i.e., $q(x_e^*) > 0$, which implies that $x_e^* > 0$. Assume for contradiction that for all $u^* \in \mathcal{U}(x^*)$ we have $y_e^* = 0$ for all optimal choices $y^* \in Y$.

First, we argue that without loss of generality, $u_e^* = 0$: Due to $\phi_e^\ell = 0$, the remaining constraints in $\mathcal{U}$ are upper bounds on $u$ such that $u_e^* = 0$ is feasible. The set $Y$ does not depend on $u$. Thus, it suffices to consider the objective for optimality. The value $u_e$ only occurs in the summand $d_e u_e y_e$. Due to $y_e^* = 0$ by assumption, we have $d_e u_e y_e = 0$ regardless of the value for $u_e$. Thus, we can set $u_e^* = 0$ without loss of generality.

Let $u_e^* = 0$ and let $x'$ be an alternative query that is equal to $x^*$ in all elements except $e$ where $x'_e = 0$. The query $x'$ is feasible because $X = \mathbb{R}^n_{\geq 0}$. Due to $q_e(x_e^*) > 0$, we obtain $F(x', u^*, y^*) < F(x^*, u^*, y^*)$ which contradicts the optimality of $x^*$. $\square$

If we set $\phi_e^\ell = 0$, we disable budget deflection. As a corollary of the following theorem, we obtain that the optimal objective value does not decrease if in an instance of KHP budget deflection is disabled while we keep the remaining fixed. More generally, the optimal objective value might increase if query outcome functions are replaced by smaller ones. Formally, we have:

**Theorem 3.9.** *Let $\tilde{\phi}_e^\ell \leq \phi_e^\ell$ and $\tilde{\phi}_e^u \leq \phi_e^u$ for all $e \in [n]$. Assume that the optimal objective value $F^*$ of an instance of KHP with query outcome functions $\phi_e^\ell$ and $\phi_e^u$ exists. Further assume that the optimal objective value $\tilde{F}^*$ of the same instance except $\tilde{\phi}_e^\ell$ instead of $\phi_e^\ell$ and $\tilde{\phi}_e^u$ instead of $\phi_e^u$ exists. Then, $F^* \leq \tilde{F}^*$.*

*Proof.* Let $\psi(x, u) := \min_{y \in Y} F(x, u, y)$. Let $\mathcal{U}(x)$ denote the uncertainty set with $\phi_e^u$ and $\phi_e^\ell$ and let $\tilde{\mathcal{U}}(x)$ denote the uncertainty set with $\tilde{\phi}_e^u$ and $\tilde{\phi}_e^\ell$. Then, for all queries $x$ we have $\mathcal{U}(x) \subseteq \tilde{\mathcal{U}}(x)$. Let $\tilde{x}^*$ be an optimal query for the instance with query outcome functions $\tilde{\phi}_e^u$ and $\tilde{\phi}_e^\ell$. The query

$\tilde{x}^*$ is feasible for the instance with $\phi_e^u$ and $\phi_e^\ell$. Combining, we obtain

$$F^* = \inf_{x\in X} \max_{u\in\mathcal{U}(x)} \psi(x,u) \le \max_{u\in\mathcal{U}(\tilde{x}^*)} \psi(\tilde{x}^*,u) \le \max_{u\in\tilde{\mathcal{U}}(\tilde{x}^*)} \psi(\tilde{x}^*,u) = \tilde{F}^*.$$

□

**Corollary 3.10.** *Let $F^*$ be the optimal objective value of an instance of KHP. Let $F_0^*$ be the optimal objective value of the same instance except that $\phi^\ell \equiv 0$. Then, $F^* \le F_0^*$.*

## 3.4 Single-level MIP for binary problems

KHP can be formulated as a single-level mixed-integer problem for linear query cost $q$, binary queries and binary underlying optimization problem. We apply McCormick envelopes [McC76] on the single-level problem derived in Theorem 3.4.

Binary query functions are restrictive compared to our original approach. However, precisely this case is studied in decision-dependent information discovery, e.g., [OPR24, PGDT22, VGY22] and is an assumption often made for optimization problems with decision-dependent uncertainty sets, e.g., [NS18, **?**].

Several works in the literature discuss single-level reformulations for robust optimization problems with decision-dependent uncertainty, notably in settings where the decision variables are binary. For instance, [NS18] and [**?**] provide reformulations involving bilinear terms that arise when uncertainty appears in constraints, and these require general bounds over all elements of the uncertainty set. Similarly, [**?**] mentions in Remark 5 that when the first-stage decisions are binary, the resulting bilinear program can be linearized into a mixed-integer linear program. The following theorem builds on these ideas by providing an explicit MILP reformulation for the case of known hedging points under binary queries.

In contrast to the more general formulations in [NS18, **?**, **?**], we make additional structural assumptions – most notably, a budgeted uncertainty set and a binary lower-level problem – allowing us to derive a more specific and practically tighter reformulation. In particular, our formulation uses componentwise bounds for the bilinear terms, yielding variable-specific big-M constants $M_e$, which can improve the numerical tightness of the MILP.

**Theorem 3.11.** *Let the query cost be linear, i.e., $q(x) = q^\top x$ and let the query outcome be binary, see Example 2.1. Furthermore, let the convex hull of the feasible set of the underlying problem be bounded and given by a linear number of linear inequalities.*

$$\operatorname{conv}(Y) = \left\{ y \ge 0 \;\middle|\; A^\top y = a \right\} \subseteq \{ y \ge 0 \mid y_e \le M_e \}.$$

*Then, an optimal query exists and KHP can be formulated as an equivalent mixed-integer linear problem with $\mathcal{O}(n)$ variables and constraints.*

*Proof.* As there are finitely many feasible solutions for a query $x$, an optimal query $x^*$ exists.

By Theorem 3.4, in our setting KHP is equivalent to the single-level problem

$$\begin{aligned}
\min_{x,y,\beta,\theta} \quad & \Gamma\theta + \sum_{e\in[n]} q_e x_e + c_e y_e + \beta_e + x_e\left(b_e d_e y_e - b_e\theta - \beta_e\right) \\
\text{s.t.} \quad & \theta + \beta_e - d_e y_e \geq 0 \qquad \forall e \in [n] \\
& Ay = a \\
& y, \beta, \theta \geq 0 \\
& x \in \{0,1\}^n .
\end{aligned}$$

The only non-linear part are the bilinear summands $x_e\left(b_e d_e y_e - b_e\theta - \beta_e\right)$ in the objective. In the following, we obtain an exact reformulation for these bilinear terms by the McCormick envelopes, since $x$ is binary. First, we deduce upper and lower bounds for the latter factor: Recall that $b$ and $d$ are non-negative, see Section 2. Furthermore with $y_e \leq M_e$, in an optimal solution we have $\theta \leq D$ and $\beta_e \leq d_e M_e$, see Lemma 3.6. Together with $y, \beta$ and $\theta$ being non-negative, in an optimal solution we have

$$b_e d_e y_e - b_e\theta - \beta_e \in [-b_e D - d_e M_e, b_e d_e M_e].$$

Next, we introduce new variables $z_e$ for the bilinear summands. The lower bound

$$z_e \geq \max\left\{-(b_e D + d_e M_e)x_e, b_e d_e M_e(x_e + y_e - 1) - b_e\theta - \beta_e\right\}$$

suffices as we minimize. In total, we obtain the following mixed-integer linear problem:

$$\begin{aligned}
\min_{x,y,z,\beta,\theta} \quad & \Gamma\theta + \sum_{e\in[n]} q_e x_e + c_e y_e + \beta_e + z_e \\
\text{s.t.} \quad & z_e - b_e d_e M_e x_e - b_e d_e M_e y_e + b_e\theta + \beta_e \geq -b_e d_e M_e \qquad \forall e \in [n] \\
& z_e + (b_e D + d_e M_e)x_e \geq 0 \qquad \forall e \in [n] \\
& \theta + \beta_e - d_e y_e \geq 0 \qquad \forall e \in [n] \\
& Ay = a \\
& y, \beta, \theta \geq 0 \\
& x \in \{0,1\}^n .
\end{aligned}$$

This problem has both $\mathcal{O}(n)$ variables and constraints. $\square$

# 4 Robustness against uncertain hedging points

We now consider optimization under controllable uncertainty with unknown hedging points (UHP). After we introduce the problem, we compare it to the problem of the previous section and a similar problem from the literature. Then, we provide some bounds on the optimal objective value. Afterwards, we develop an equivalent single-level reformulation of the four-level problem. Finally, we consider whether budget deflection can occur.

In UHP, the hedging points $b$ are chosen adversarially after queries $x$ are made and before the underlying problem's decisions $y$ are fixed. The remaining uncertainty $u$ realizes in a worst-case manner afterwards. In total, we have the following four-level optimization problem

$$\inf_{x\in X} \max_{b\in\mathcal{B}} \min_{y\in Y} \max_{u\in\mathcal{U}(x,b)} F(x,u,y). \tag{UHP}$$

**Comparison of UHP with KHP and DDID** There are two main differences between UHP, considered in this section, and KHP from the previous section. First, in UHP, hedging points $b$ are variables that can change. In contrast, hedging points are fixed values in KHP. Furthermore, in KHP, the uncertainty realizes in-between making queries and solving the underlying problem. However, in UHP, the uncertainty realizes after the decision of the underlying problem is fixed.

UHP is a generalization of decision-dependent information discovery (DDID), e.g., [PGDT22, OPR24]. In DDID, there are only binary queries and the objective function does not depend on the query. Binary queries can be modeled in UHP with appropriate query outcome functions $\phi_e^\ell$, $\phi_e^u$, see Example 2.1. Furthermore, if a query cost is set to zero in UHP, the objective function does not depend on the query anymore. The remaining settings in UHP and DDID are the same.

## 4.1 Bounds for UHP

In the following, we show upper and lower bounds on the optimal objective value of UHP. They are based on the boundaries of the intervals $[c_e, c_e + d_e]$ for the uncertain cost coefficients in the underlying problem's objective.

**Observation 4.1.** *For the optimal objective value $F^*$ of UHP holds*

$$\min_{y\in Y} c^\top y \le F^* \le \min_{y\in Y} \max_{u\in\mathcal{U}} f(u,y) \le \min_{y\in Y} (c+d)^\top y.$$

*Proof.* Recall that $F(x,u,y) = q(x) + (c + d\cdot u)^\top y$, that $q(x), c, d$ are non-negative, $\mathcal{U}, \mathcal{B} \subseteq [0,1]^n$, $\mathbf{0} \in X$ and $\mathcal{U}(\mathbf{0}, b) = \mathcal{U}$, see Section 2. Furthermore, the controllable uncertainty set always contains the hedging point $b$.

If there is no query cost in UHP, an optimal query reduces the upper bounds in the controllable uncertainty set $\mathcal{U}(x,b)$ to a maximal amount. Thus, we have

$$\begin{aligned} F^* = \inf_{x\in X} \max_{b\in\mathcal{B}} \min_{y\in Y} \max_{u\in\mathcal{U}(x,b)} f(u,y) + q(x) &\geq \inf_{x\in X} \max_{b\in\mathcal{B}} \min_{y\in Y} \max_{u\in\mathcal{U}(x,b)} f(u,y) \\ &\geq \max_{b\in\mathcal{B}\cap\mathcal{U}} \min_{y\in Y} f(b,y) \\ &\geq \min_{y\in Y} f(\mathbf{0},y) = \min_{y\in Y} c^\top y. \end{aligned}$$

For the upper bounds, fix $x = \mathbf{0}$ to obtain

$$\begin{aligned} F^* &\leq \max_{b\in\mathcal{B}} \min_{y\in Y} \max_{u\in\mathcal{U}(\mathbf{0},b)} F(\mathbf{0},u,y) = \max_{b\in\mathcal{B}} \min_{y\in Y} \max_{u\in\mathcal{U}} f(u,y) = \min_{y\in Y} \max_{u\in\mathcal{U}} f(u,y) \\ &\leq \min_{y\in Y} \max_{u\in[0,1]^n} (c + d\cdot u)^\top y = \min_{y\in Y} (c+d)^\top y. \end{aligned}$$

□

The robust problem for the underlying problem provides a tighter upper bound. Recall that the uncertainty set $\mathcal{U}$ is the budgeted uncertainty set, see Assumption 2.6. For binary underlying problems, the robust problem can thus be solved by $n+1$ underlying problems [BS03].

## 4.2 Single-level reformulation for UHP

The main result of this section is an equivalent single-level reformulation for UHP. First, we define some terms to simplify notation and consider the bilevel problem for fixed query and fixed hedging points. Then, we reformulate the inner robust problem as $n+1$ nominal problems for binary underlying problems. Afterwards in Theorem 4.7, for binary underlying problems that can be solved as LP, we obtain a single-level NLP that is equivalent to UHP.

**Definition 4.2.** *Let*

$$\begin{aligned} \tilde{\Gamma}(x,b) &:= \Gamma - \sum_{e\in[n]} \phi_e^\ell(x_e) b_e, \\ \tilde{c}_e(x_e,b_e) &:= c_e + \phi_e^\ell(x_e) d_e b_e, \\ h_e(x_e,b_e) &:= 1 - \phi_e^u(x_e) + \left(\phi_e^u(x_e) - \phi_e^\ell(x_e)\right) b_e, \\ h_{n+1}(x_{n+1},b_{n+1}) &:= 0, \\ G_e(x_e,k) &:= c_e + \mathbb{1}_{e<k}(d_e - d_k)\left(1 - \phi_e^u(x_e)\right), \text{ and} \\ g_e(x_e,k) &:= \mathbb{1}_{e<k}(d_e - d_k)\left(\phi_e^u(x_e) - \phi_e^\ell(x_e)\right) + \phi_e^\ell(x_e) d_e \end{aligned}$$

*where $\mathbb{1}_{e<k}$ denotes the indicator whether $e<k$.*

By the assumptions on $b, c, d, \Gamma$ and $\phi_e^\ell(x_e)$, see Section 2, the modified budget $\tilde{\Gamma}(x,b)$ and the modified cost $\tilde{c}_e(x_e,b_e)$ is non-negative.

**Lemma 4.3.** *Let $x \in X$ be a fixed query and fix a hedging point $b \in \mathcal{B}$. Then, the bilevel problem*

$$\min_{y \in Y} \max_{u \in \mathcal{U}(x,b)} f(u,y) \tag{7}$$

*is equivalent to the following LP:*

$$\begin{aligned}
\min_{\beta,\theta,y} \quad & \tilde{\Gamma}(x,b)\theta + \sum_{e\in[n]} \tilde{c}_e(x_e,b_e)y_e + h_e(x_e,b_e)\beta_e && \qquad (8)\\
\text{s.t.} \quad & \theta + \beta_e - d_e y_e \;\geq\; 0 && \forall e \in [n]\\
& \beta,\theta \;\geq\; 0 &&\\
& y \;\in\; Y. &&
\end{aligned}$$

*Proof.* In the following, we explicitly formulate the constraints of $\mathcal{U}(x,b)$, see Definition 2.2 and Assumption 2.6, and the objective function $f(u,y)$, see Assumption 2.5. The lower level of Problem (7) is the LP

$$\begin{aligned}
\max_{u} \quad & \sum_{e\in[n]} c_e y_e + d_e u_e y_e && \qquad (9)\\
\text{s.t.} \quad & u_e \;\geq\; \phi_e^\ell(x_e) b_e && \forall e \in [n]\\
& u_e \;\leq\; 1 - \phi_e^u(x_e)(1-b_e) && \forall e \in [n]\\
& \sum_{e\in[n]} u_e \;\leq\; \Gamma. &&
\end{aligned}$$

The dual problem of Problem (9) is:

$$\begin{aligned}
\min_{\alpha,\beta,\theta} \quad & \Gamma\theta + \sum_{e\in[n]} c_e y_e - \phi_e^\ell(x_e) b_e \alpha_e + \left(1 - \phi_e^u(x_e)(1-b_e)\right)\beta_e && \qquad (10)\\
\text{s.t.} \quad & -\alpha_e + \theta + \beta_e - d_e y_e \;=\; 0 && \forall e \in [n]\\
& \alpha,\beta,\theta \;\geq\; 0. &&
\end{aligned}$$

Due to strong duality, Problems (9) and (10) have the same optimal objective value. Thus, we can replace Problem (9) by Problem (10). Furthermore, we substitute $\alpha_e = \theta + \beta_e - d_e y_e \geq 0$, use Definition 4.2 and combine with the minimization of $y$ in Problem (7) to obtain Problem (8). $\square$

The robust counterpart of a binary optimization problem can be effectively optimized via $n+1$

appropriate nominal optimization problems [BS03, Theorem 3]. These nominal optimization problems only differ in the cost vector. For Problem (7), we obtain the following adaption.

**Assumption 4.4.** *For the remaining, we assume that the elements are ordered such that entries of the vector $d$ are non-increasing for increasing index and add $d_{n+1} = 0$, i.e.,*

$$d_1 \geq d_2 \geq \cdots \geq d_n \geq d_{n+1} = 0.$$

**Theorem 4.5** (Adaption of [BS03])**.** *Let the underlying optimization problem be binary, i.e., $Y \subseteq \{0,1\}^n$. Then, Problem (7) is equivalent to*

$$\min_{k \in [n+1]} \tilde{\Gamma}(x,b) d_k + \min_{y \in Y} \sum_{e \in [n]} \tilde{c}_e(x_e, b_e) y_e + \sum_{j \in [k]} (d_j - d_k) h_j(x_j, b_j) y_j. \tag{11}$$

*Proof.* The single-level reformulation (8) only differs in the coefficient $h_e(x_e, b_e)$ of the dual variables $\beta_e$ in the objective function to the problem considered in [BS03, Theorem 3]. The statement follows from the proof given in [BS03]. □

There are several results that reduce the number of subproblems that have to be solved, e.g., [LK14, HRS18, BGK23]. We will not further consider these results.

If the binary underlying problem is furthermore given as an LP, we provide an equivalent single-level LP for the three innermost levels of UHP in the following lemma.

**Lemma 4.6.** *Let $x$ be a fixed query, assume that the underlying problem be binary, i.e., $Y \subseteq \{0,1\}^n$, fulfilling*

$$\operatorname{conv}(Y) = \left\{ y \;\middle|\; A^\top y = a, y \geq 0 \right\} \text{ with } A \in \mathbb{R}^{m \times n}, a \in \mathbb{R}^m.$$

*Let $\mathcal{B}$ be given by a polynomial-sized set of linear inequalities. Then, the remaining problem, i.e.,*

$$\max_{b \in \mathcal{B}} \min_{y \in Y} \max_{u \in \mathcal{U}(x,b)} f(u,y) \tag{12}$$

*is equivalent to the following LP:*

$$\begin{aligned}
&\max_{R,b,z} \; R && && (13)\\
&\text{s.t.} \quad -g_e(x_e,k) b_e + \sum_{i \in [m]} A_{i,e} z_i^{(k)} \leq G_e(x_e,k) && \forall k \in [n+1], e \in [n] \\
&\qquad R + \sum_{e \in [n]} \phi_e^\ell(x_e) b_e - \sum_{i \in [m]} a_i z_i^{(k)} \leq \Gamma d_k && \forall k \in [n+1] \\
&\qquad b \in \mathcal{B}.
\end{aligned}$$

*Proof.* Based on Theorem 4.5, Problem (12) is equivalent to the problem

$$\begin{aligned} \max_{R\in\mathbb{R},\ b\in\mathcal{B}} \quad & R \\ \text{s.t.} \quad & R - \tilde{\Gamma}(x,b)d_k \quad \leq \quad \psi(x,b,k) \qquad \forall k \in [n+1] \end{aligned}$$

with

$$\psi(x,b,k) := \min_{y\in Y} \sum_{e\in[n]} \tilde{c}_e(x_e,b_e)y_e + \sum_{j\in[k]} (d_j - d_k)h_j(x_j,b_j)y_j. \tag{14}$$

The dual of the minimization problem in (14) is given for every $k \in [n+1]$ by

$$\begin{aligned} \max_{z^{(k)}\in\mathbb{R}^m} \quad & \sum_{i\in[m]} a_i z_i^{(k)} \\ \text{s.t.} \quad & \sum_{i\in[m]} A_{i,e} z_i^{(k)} \quad \leq \quad \tilde{c}_e(x_e,b_e) + c_e + \mathbb{1}_{e<k}(d_e - d_k)\ h_e(x_e,b_e) \qquad \forall e \in [n]. \end{aligned}$$

Based on strong duality, we can combine this maximization problem with the evaluation of variables $R$ and $b$ to obtain the equivalent LP in Problem (13). □

Finally, we obtain a single-level non-linear problem that is equivalent to UHP.

**Theorem 4.7.** *Let an optimal query $x^*$ exist and the underlying problem be binary and solvable as an LP. Furthermore, let the set of hedging points $\mathcal{B}$ be given by a polynomial number of constraints that are linear in $b_e$. Then, for UHP there exists an equivalent single-level NLP.*

*Proof.* By Lemma 4.6, the inner three levels of UHP are equivalent to the LP (13). Due to strong duality, we obtain an equivalent single-level problem to UHP, when we combine the dual of Problem (13) with the minimization over queries $x \in X$. □

**Corollary 4.8.** *Let an optimal query $x^*$ exist. For some $A \in \mathbb{R}^{m\times n}, a \in \mathbb{R}^m$, let*

$$\operatorname{conv}(Y) = \left\{ y \;\middle|\; A^\top y = a, y \geq 0 \right\} \text{ and } \mathcal{B} = \left\{ b \in [0,1]^n \;\middle|\; \Gamma - \sum_e \phi_e^\ell(x_e)b_e \geq 0 \right\}.$$

*Then, UHP is equivalent to the following single-level bilinear problem*

$$\begin{aligned}
\min_{\substack{x,\,y,\,\theta,\\ \tilde{\theta},\,\beta}}\ & q(x) + \Gamma\left(\theta + \sum_{k\in[n+1]} \tilde{\theta}_k d_k\right) + \sum_{e\in[n]} \beta_e + \sum_{k\in[n+1]} G_e(x_e,k) y_e^{(k)} \\
s.t.\ & \beta_e + d_k \phi_e^\ell(x_e)\theta + \sum_{k\in[n+1]} \phi_e^\ell(x_e)\tilde{\theta}_k - g_e(x_e,k) y_e^{(k)} \geq 0 \quad \forall e\in[n] \\
& A^\top y^{(k)} - \tilde{\theta}_k a = 0 \quad \forall k \in [n+1] \\
& \sum_{k\in[n+1]} \tilde{\theta}_k = 1 \\
& \theta,\, \tilde{\theta}_k,\, \beta_e,\, y_e^{(k)} \geq 0 \quad \forall k\in[n+1],\ e\in[n] \\
& x \in X.
\end{aligned}$$

For binary queries given by a polynomially sized MIP, UHP can be reformulated as a MIP based on linearizing bilinear terms, similarly as done for KHP in Theorem 3.11. Note that under comparable assumptions a similar result to Theorem 4.7 can be obtained in the setting of DDID, see [OPR24, Section 4].

## 4.3 Budget deflection for UHP

In the following, we show that budget deflection does not occur in the setting of UHP. Recall that budget deflection is the phenomenon that a query is made for an element to control the uncertainty of other elements, see Section 2.4. In contrast to Theorem 3.8 for KHP, we do not need any additional assumption on $\phi^\ell$ when we consider UHP in the following Theorem.

**Theorem 4.9.** *Let $X = \mathbb{R}_{\geq 0}$ and let $q$ denote elementwise query cost. For an instance of UHP, let $x^*$ be an optimal solution with $q_e(x_e^*) > 0$ for some $e \in [n]$. Then there exists an optimal $b_e^*$ such that there is an optimal solution $y^*$ with $y_e^* > 0$.*

*Proof.* Due to elementwise query cost, the overall objective function has the elementwise structure

$$F(x,u,y) = \sum_{e\in[n]} q_e(x_e) + c_e y_e + d_e u_e y_e.$$

We do a proof by contradiction. Let $x^*$ be a fixed optimal query with $q_e(x_e^*) > 0$ and $y_e^* = 0$ for some $e \in [n]$ for all optimal $b_e^*$. Then, the only summand including $u_e$ in the objective, $u_e b_e y_e$ is already zero. Thus, without loss of generality $u_e^* = \phi_e^\ell(x_e^*) b_e^*$ as there is no other lower bound on $u_e$. As a consequence, we can have $b_e^* = 0$ without loss of generality by a similar argument. We now set $x_e = 0$ to reduce the objective value as $q_e(x_e^*) > 0$. This does not change the latter decisions for $b, y, u$ and contradicts to the optimality of $x^*$. □

**Corollary 4.10.** *In UHP, there is no budget deflection.*

# 5 Conclusion and outlook

We introduced the concept of controllable uncertainty. OCU models optimization problems with uncertainty in which one can shrink intervals for uncertain parameters at a certain cost. The concept is highly flexible. In particular can be applied to a large variety of robust optimization problems that differ in the number, type and order of levels, the structure of the query function and the parameters that are subject to uncertainty. We distinguish whether hedging points are given as part of the input or chosen by an adversary a posteriori. For the first case of known hedging points, we consider the setting where the uncertainty realizes before the underlying optimization problem is solved. In the latter case of unknown hedging points, we consider the setting where the point in time when uncertainty reveals is influenced by the queries. For both cases, we consider an example problem setting with three and four levels, respectively. In both cases, we were able to simplify the problems to manageable, though still difficult, problem classes. Thereby, we illustrate that optimization models that use the concept of controllable uncertainty can still be accessible to methods which seek global optimality, despite an initially daunting number of levels.

Future research may aim to identify and classify further classes of optimization problems under controllable uncertainty for which levels can be reduced significantly. Possibly this could also discover polynomially solvable ones. Moreover, specialized algorithmic techniques could lead to significant speed-ups to solve problems under controllable uncertainty. An advantage of the OCU approach in particular in contrast to DDID is that OCU could be used in economic applications for quantifying the value of benefits achieved by queries. Depending on the setting, leader queries can serve multiple purposes, namely hedging against highly unwelcome realizations of the uncertainty, quite similar to what is done in fortification games, but also gain access to uncertain information earlier – or, in the interpretation of a two-player game, force the adversary's hand. An important aspect of ROCU models is budget deflection. A third and sometimes unwanted purpose is just to exhaust the uncertainty budget. We discuss modeling subtleties in conjunction with this as a first step to quantitatively differentiate between these purposes. We give fundamental results to study this and similar intriguing phenomena arising for ROCU models.

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