# The pseudo-Boolean polytope and polynomial-size extended formulations for binary polynomial optimization

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**Abstract**

With the goal of obtaining strong relaxations for binary polynomial optimization problems, we introduce the pseudo-Boolean polytope defined as the set of binary points $z \in \{0,1\}^{V \cup S}$ satisfying a collection of equations of the form $z_s = \prod_{v \in s} \sigma_s(z_v)$, for all $s \in S$, where $\sigma_s(z_v) \in \{z_v, 1 - z_v\}$, and where $S$ is a multiset of subsets of $V$. By representing the pseudo-Boolean polytope via a signed hypergraph, we obtain sufficient conditions under which this polytope has a polynomial-size extended formulation. Our new framework unifies and extends all prior results on the existence of polynomial-size extended formulations for the convex hull of the feasible region of binary polynomial optimization problems of degree at least three.

*Key words:* Binary polynomial optimization; Pseudo-Boolean optimization; Pseudo-Boolean polytope; Signed hypergraph; Polynomial-size extended formulation

## 1 Introduction

We consider the problem of maximizing a multivariate polynomial function over the set of binary points, henceforth referred to as *binary polynomial optimization*. This problem class has numerous applications across science and engineering [7], and is NP-hard in general. Based on the encoding of the polynomial objective function, we obtain two popular optimization problems, which we refer to as "binary multilinear optimization" and "pseudo-Boolean optimization." As binary quadratic optimization has been extensively studied in the literature (see for example [2, 5, 32]), in this paper, we focus on binary polynomial optimization problems of degree at least three.

### 1.1 Binary multilinear optimization

A *hypergraph* $G$ is a pair $(V, E)$, where $V$ is a finite set of nodes and $E$ is a set of *edges*, which are subsets of $V$ of cardinality at least two. Following the approach introduced in [15], with any hypergraph $G = (V, E)$, and cost vector $c \in \mathbb{R}^{V \cup E}$, we associate the following *binary multilinear optimization problem*:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{e \in E} c_e \prod_{v \in e} z_v \\ \text{s.t.} \quad & z_v \in \{0,1\} \qquad \forall v \in V. \end{aligned} \tag{BMO}$$

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It can be checked that any binary polynomial optimization problem has a unique representation of the form Problem BMO. Next, we linearize the objective function of Problem BMO by introducing a new variable $z_e$ for each product term $\prod_{v\in e} z_v$ to obtain an equivalent reformulation of this problem in a lifted space:

$$\begin{aligned} \max \quad & \sum_{v\in V} c_v z_v + \sum_{e\in E} c_e z_e \\ \text{s.t.} \quad & z_e = \prod_{v\in e} z_v \qquad \forall e \in E \\ & z_v \in \{0,1\} \qquad \forall v \in V. \end{aligned} \tag{L-BMO}$$

With the objective of understanding the facial structure of the convex hull of the feasible region of Problem L-BMO, Del Pia and Khajavirad [15] introduced the *multilinear set*, defined as

$$\mathcal{S}(G) := \Big\{ z \in \{0,1\}^{V+E} : z_e = \prod_{v\in e} z_v, \ \forall e \in E \Big\},$$

and its convex hull, which is called the *multilinear polytope* and is denoted by MP($G$).

In [16,19], the authors show that the complexity of the facial structure of MP($G$) is closely related to the "acyclicity degree" of $G$. The most well-known types of acyclic hypergraphs, in increasing order of generality, are Berge-acyclic, $\gamma$-acyclic, $\beta$-acyclic, and $\alpha$-acyclic hypergraphs [3,8,23,24]. We next briefly review the existing results on the facial structure of the multilinear polytope of acyclic hypergraphs. Recall that the *rank* of a hypergraph $G$, denoted by $r$, is the maximum cardinality of any edge in $E$. In [9,16], the authors prove that MP($G$) coincides with its standard linearization if and only if $G$ is Berge-acyclic. This in turn implies that if $G$ is Berge-acyclic, then MP($G$) is defined by $|V| + (r+2)|E|$ inequalities in the original space. In [16], the authors prove that MP($G$) coincides with its flower relaxation if and only if $G$ is $\gamma$-acyclic. This result implies that if $G$ is $\gamma$-acyclic, then MP($G$) has a polynomial-size extended formulation with at most $|V| + 2|E|$ variables and at most $|V| + (r+2)|E|$ inequalities. Subsequently, in [19], the authors present a polynomial-size extended formulation for the multilinear polytope of $\beta$-acyclic hypergraphs with at most $(r-1)|V|+|E|$ variables and at most $(3r-4)|V| + 4|E|$ inequalities.

On the contrary, in [14], the authors prove that Problem BMO is strongly NP-hard over $\alpha$-acyclic hypergraphs. This result implies that, unless P = NP, one cannot construct, in polynomial time, a polynomial-size extended formulation for the multilinear polytope of $\alpha$-acyclic hypergraphs. However, as we detail next, by making further assumptions on the rank of $\alpha$-acyclic hypergraphs, one can construct a polynomial-size extended formulation for the multilinear polytope. In [4,31,33], the authors give extended formulations for the convex hull of the feasible set of (possibly constrained) binary multilinear optimization problems. The size of these extended formulations is parameterized in terms of the "tree-width" of their so-called intersection graphs. For the unconstrained case, as detailed in [18], their result can be equivalently stated as follows: If $G$ is an $\alpha$-acylic hypergraph of rank $r$, with $r = O(\log \text{poly}(|V|, |E|))$, then MP($G$) has a polynomial-size extended formulation, where by poly($|V|, |E|$), we imply a polynomial function in $|V|, |E|$. Henceforth, for brevity, whenever for a hypergraph $G = (V, E)$ we have $r = O(\log \text{poly}(|V|, |E|))$, we say that $G$ has *log-poly* rank.

For further results regarding polyhedral relaxations of multilinear sets, see [12,13,18,20–22,28,30].

### 1.2 Pseudo-Boolean optimization

We define a *signed hypergraph* $H$ as a pair $(V, S)$, where $V$ is a finite set of nodes and $S$ is a set of *signed edges*. A *signed edge* $s \in S$ is a pair $(e, \eta_s)$, where $e$ is a subset of $V$ of cardinality at least two, and $\eta_s$ is a map that assigns to each $v \in e$ a *sign* $\eta_s(v) \in \{-1, +1\}$. The *underlying edge* of a signed edge $s = (e, \eta_s)$ is $e$. Two signed edges $s = (e, \eta_s)$, $s' = (e', \eta_{s'}) \in S$ are said to be *parallel* if $e = e'$, and they are said to be *identical* if $e = e'$ and $\eta_s = \eta_{s'}$. Throughout this paper, we consider signed hypergraphs with no identical signed edges. However, our signed hypergraphs often contain parallel signed edges.

With any signed hypergraph $H = (V, S)$, and cost vector $c \in \mathbb{R}^{V \cup S}$, we associate the following *pseudo-Boolean optimization problem:*

$$\begin{aligned} \max \quad & \sum_{s \in S} c_s \prod_{v \in s} \sigma_s(z_v) \\ \text{s.t.} \quad & z \in \{0,1\}^V, \end{aligned} \tag{PBO}$$

where

$$\sigma_s(z_v) := \begin{cases} z_v & \text{if } \eta_s(v) = +1 \\ 1 - z_v & \text{if } \eta_s(v) = -1. \end{cases}$$

A variety of important applications such as maximum satisfiability problems [25], and graphical models [27] can naturally be formulated as pseudo-Boolean optimization problems. Problem PBO has been extensively studied in the literature; see [7] for a detailed survey of main results. Some of the main topics considered in these works are quadratic pseudo-Boolean optimization problems [5, 26], quadratization of general pseudo-Boolean optimization problems [6], and special problem classes such as super-modular pseudo-Boolean optimization problems [11].

As before, we linearize the objective function of Problem PBO by introducing one new variable $z_s$, for each signed edge $s \in S$, to obtain an equivalent reformulation of Problem PBO in a lifted space:

$$\begin{aligned} \max \quad & \sum_{s \in S} c_s z_s \\ \text{s.t.} \quad & z_s = \prod_{v \in s} \sigma_s(z_v) \qquad \forall s \in S \\ & z \in \{0,1\}^{V \cup S}. \end{aligned} \tag{L-PBO}$$

In this paper, we introduce the *pseudo-Boolean set* of the signed hypergraph $H = (V, S)$, as the feasible region of Problem L-PBO:

$$\text{PBS}(H) := \Big\{ z \in \{0,1\}^{V \cup S} : z_s = \prod_{v \in s} \sigma_s(z_v), \ \forall s \in S \Big\},$$

and we refer to its convex hull as the *pseudo-Boolean polytope* and denote it by $\text{PBP}(H)$. With the objective of constructing strong linear programming (LP) relaxations for Pseudo-Boolean optimization problems, in this paper, we study the facial structure of the pseudo-Boolean polytope.

### 1.3 Binary multilinear optimization versus Pseudo-Boolean optimization

Problem BMO and the multilinear polytope $\text{MP}(G)$ are special cases of Problem PBO and the pseudo-Boolean polytope $\text{PBP}(H)$, obtained by letting $\eta_s(v) = +1$ for every $s \in S$ and $v \in s$. Vice versa, by expanding the objective function, Problem PBO over a signed hypergraph $H = (V, S)$ can be formulated as a binary multilinear optimization problem over the multilinear hypergraph $\text{mh}(H)$, which we define next. Formally, the *multilinear hypergraph* of $H$ is the hypergraph $\text{mh}(H) = (V, E)$, where $E$ is constructed as follows: For each $s \in S$, and every $t \subseteq s$ with $\eta_s(v) = -1$ for all $v \in t$, the set $E$ contains $\{v \in s : \eta_s(v) = +1\} \cup t$, if it has cardinality at least two. However, reformulating Problem PBO as a binary multilinear optimization problem may lead to an exponential increase in the number of monomials obtained, and hence in the number of edges in the multilinear hypergraph. To see this, consider for example the signed hypergraph $H = (V, S)$, where $S$ contains two signed edges $s = (V, \eta_s)$ and $t = (V, \eta_t)$ such that $\eta_s(v) = -\eta_t(v)$ for every $v \in V$. It is then simple to check that the number of edges in $\mathrm{mh}(H)$ is at least $\sum_{i=2}^{m} \binom{m}{i} = 2^m - m - 1$, where $m := \lceil |V|/2 \rceil$.

It is important to note that unlike the multilinear polytope, the pseudo-Boolean polytope may not be full-dimensional. For example, if $H = (V, S)$ contains three signed edges $s_1 = (e, \eta_{s_1})$, $s_2 = (e, \eta_{s_2})$, $s_3 = (e', \eta_{s_3})$, where $e' = e \setminus \{\bar{v}\}$ for some $\bar{v} \in V$, $\eta_{s_1}(v) = \eta_{s_2}(v) = \eta_{s_3}(v)$ for all $v \in e \setminus \{\bar{v}\}$, and $\eta_{s_1}(\bar{v}) = -\eta_{s_2}(\bar{v})$, then $\mathrm{PBP}(H)$ is not full-dimensional as we have $z_{s_1} + z_{s_2} = z_{s_3}$.

While the pseudo-Boolean polytope is significantly more complex than the multilinear polytope, as we demonstrate in this paper, by studying the facial structure of $\mathrm{PBP}(H)$, one can obtain polynomial-size extended formulations for many instances for which such a formulation is not known for the corresponding multilinear polytope; i.e., $\mathrm{MP}(G)$ with $G = \mathrm{mh}(H)$.

### 1.4 Our contributions

In this paper, we demonstrate how the hypergraph framework pioneered in [15] for binary multilinear optimization is relevant and applicable to pseudo-Boolean optimization as well. Namely, using signed hypergraphs to represent pseudo-Boolean sets, we present sufficient conditions under which the pseudo-Boolean polytope admits a polynomial-size extended formulation. Our results unify and extend all prior results on polynomial-size representability of the multilinear polytope [4, 16, 18, 19, 31, 33].

We introduce a new technique, which we refer to as the "recursive inflate-and-decompose" framework to construct polynomial-size extended formulations for the pseudo-Boolean polytope. Our proposed framework relies on a recursive application of three key ingredients, each of which is of independent interest:

1. A sufficient condition for decomposability of pseudo-Boolean polytopes (see Theorem 1). This is the first result on decomposability of pseudo-Boolean polytopes and serves as a significant generalization of Theorem 4 in [19] (Section 2).

2. A polynomial-size extended formulation for the pseudo-Boolean polytope of pointed signed hypergraphs (see Theorem 2). The pseudo-Boolean polytope of pointed signed hypergraphs is the building block of our extended formulations, which appears as a result of applying our decomposition technique of Part 1 (Section 3).

3. An operation, which we refer to as "inflation of signed edges" (see Theorem 3) that we use to transform a large family of signed hypergraphs to those for which our results of Parts 1 and 2 are applicable (Section 4).

As we detailed in Section 1.1, at the time of this writing, the most general sufficient conditions under which one can obtain a polynomial-size extended formulation for the multilinear polytope $\mathrm{MP}(G)$ are:

(i) $G$ is a $\beta$-acyclic hypergraph [19],

(ii) $G$ is an $\alpha$-acyclic hypergraph with log-poly rank [31].

It is important to remark that neither of the above sufficient conditions implies the other one. Furthermore, the two results in [19] and [31] have been proven using entirely different techniques. In Section 5, we show that our recursive inflate-and-decompose framework implies as special cases both sufficient conditions (i) and (ii) above and extends to many more cases of interest. Below, we summarize these results.

Consider a signed hypergraph $H = (V, S)$. We define the *underlying hypergraph* of $H$ as the hypergraph obtained from $H$ by ignoring the signs and dropping parallel edges. In Section 5.1 we prove that, if the underlying hypergraph of $H$ is $\beta$-acyclic, then PBP($H$) has a polynomial-size extended formulation (see Theorem 5). This is a significant generalization of case (i) above, as the multilinear hypergraph of a signed hypergraph $H$ whose underlying hypergraph is $\beta$-acyclic may contain many $\beta$-cycles, in general. In Section 5.2 we prove that if the underlying hypergraph of $H$ is $\alpha$-acyclic and has log-poly rank, then PBP($H$) has a polynomial-size extended formulation (see Theorem 6). This result essentially coincides with case (ii) above. In Section 5.3, we introduce the notion of "gap" for hypergraphs, which roughly speaking, indicates if it is possible to inflate signed edges in an efficient manner. We then show that for certain signed hypergraphs, if the gap is not too large, by combining results of Theorems 3 and 5, one can obtain polynomial-size extended formulations for the pseudo-Boolean polytope (see Propositions 3 and 4). Finally, in Section 5.4 we outline some generalizations and directions of future research.

It is important to remark that the proofs of all our results regarding the existence of polynomial-size extended formulations are constructive and the proposed extended formulations can be constructed in polynomially many operations in $|V|, |S|$.

We would like to conclude this section by further emphasizing on the power of these extended formulations: Not only they serve as polynomial-size LP formulations for special classes of Problem BMO and Problem PBO, they can also be used to construct strong LP *relaxations* for general mixed-integer nonlinear optimization problems whose factorable reformulations contain pseudo-Boolean sets (see for example [20] and [29]).

## 1.5 Hypergraph notation and preliminaries

In the following, we present all hypergraph terminology and notation that we will use throughout this paper.

**Hypergraphs.** Let $G = (V, E)$ be a hypergraph. We define the hypergraph obtained from $G$ by *removing* a node $v \in V$ as the hypergraph $G - v$ with set of nodes $V \setminus \{v\}$ and set of edges $\{e - v : e \in E,\ |e - v| \geq 1\}$.

A node $v \in V$ is a *$\beta$-leaf* of $G$ if the set of the edges of $G$ containing $v$ is totally ordered with respect to inclusion. A *sequence of $\beta$-leaves* of length $t$ for some $1 \leq t \leq |V|$ of $G$ is an ordering $v_1, \ldots, v_t$ of $t$ distinct nodes of $G$, such that $v_1$ is a $\beta$-leaf of $G$, $v_2$ is a $\beta$-leaf of $G - v_1$, and so on, until $v_t$ is a $\beta$-leaf of $G - v_1 - \cdots - v_{t-1}$. The hypergraph $G$ is said to be *$\beta$-acyclic* if it has a sequence of $\beta$-leaves of length $|V|$. An equivalent definition of $\beta$-acyclic hypergraphs can be obtained using the concept $\beta$-cycle. A *$\beta$-cycle* of length $q$ for some $q \geq 3$ in $G$ is a sequence $v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_1$ such that $v_1, v_2, \ldots, v_q$ are distinct nodes, $e_1, e_2, \ldots, e_q$ are distinct edges, and $v_i$ belongs to $e_{i-1}, e_i$ and no other $e_j$ for all $i = 1, \ldots, q$, where $e_0 = e_q$. A hypergraph is $\beta$-acyclic if and only if its does not contain any $\beta$-cycles [23].

A node $v \in V$ is an *$\alpha$-leaf* of $G$ if the set of edges of $G$ containing $v$ has a maximal element for inclusion. A *sequence of $\alpha$-leaves* of length $t$ for some $1 \leq t \leq |V|$ of $G$ is an ordering $v_1, \ldots, v_t$ of $t$ distinct nodes of $G$, such that $v_1$ is an $\alpha$-leaf of $G$, $v_2$ is an $\alpha$-leaf of $G - v_1$, and so on, until $v_t$ is an $\alpha$-leaf of $G - v_1 - \cdots - v_{t-1}$. The hypergraph $G$ is said to be *$\alpha$-acyclic* if it has a sequence of $\alpha$-leaves of length $|V|$.

We say that $G$ is *connected* if for every $u, w \in V$, there exists a sequence $u, e_1, v_2, e_2, \cdots, v_q, e_q, w$ such that $v_2, v_3, \cdots, v_q \in V$, $e_1, e_2, \cdots, e_q \in E$, and $v_i$ belongs to $e_{i-1}, e_i$, for all $i = 2, \ldots, q$. The *connected components* of $G$ are its maximal connected partial hypergraphs. A hypergraph $G' = (V', E')$ is a *partial hypergraph* of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

**Signed hypergraphs.** Let $H = (V, S)$ be a signed hypergraph. We start by defining some useful operations on signed edges. Let $s = (e, \eta_s) \in S$. With a slight abuse of notation, we use set theoretical notation on $s$, with the understanding that $s$ should be replaced with $e$. For example, we denote by $|s|$ the number $|e|$, we say that $s$ is nonempty if $e$ is nonempty, we write $v \in s$ meaning $v \in e$, and for $U \subseteq V$, we write $s \subseteq U$ (resp., $s \supseteq U$, $s = U$) meaning $e \subseteq U$ (resp., $e \supseteq U$, $e = U$). Similarly, we might write $s \subseteq s'$, for $s = (e, \eta_s), s' = (e', \eta_{s'}) \in S$, instead of $e \subseteq e'$. If $v \in s$, we denote by $s - v$ the signed edge $s' = (e', \eta_{s'})$, where $e' := e \setminus \{v\}$, and $\eta_{s'}$ is the restriction of $\eta_s$ that assigns to each $v \in e'$ the sign $\eta_{s'}(v) = \eta_s(v)$. If $v \notin s$, we denote by $s + v^+$ the signed edge $s' = (e', \eta_{s'})$, where $e' := e \cup \{v\}$, and $\eta_{s'}$ is the extension of $\eta_s$ that assigns to each $u \in e'$ the sign $\eta_{s'}(u) = \eta_s(u)$ and assigns to $v$ the sign $\eta_{s'}(v) = +1$. Similarly, $s + v^-$ is defined as $s + v^+$ but with $\eta_{s'}(v) = -1$.

We define the signed hypergraph obtained from $H$ by *removing* a node $v \in V$ as the signed hypergraph $H - v$ with set of nodes $V \setminus \{v\}$ and set of signed edges $\{s - v : s \in S,\ |s - v| \geq 1\}$.

## 2 Decomposability of pseudo-Boolean sets

In this section, we present a sufficient condition for decomposability of the pseudo-Boolean polytope that we will use to obtain our extended formulations. Our decomposition result is the first known sufficient condition for decomposability of the pseudo-Boolean polytope.

Existing decomposability results for the multilinear polytope are Theorem 1 in [17], Theorem 5 in [16], Theorem 1 in [18], Theorem 4 in [13], and Theorem 4 in [19]. Our decomposition result serves as a significant generalization of Theorem 4 in [19].

Consider a signed hypergraph $H = (V, S)$, let $V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$, let $S_1 \subseteq \{s \in S : s \subseteq V_1\}$, $S_2 \subseteq \{s \in S : s \subseteq V_2\}$ such that $S = S_1 \cup S_2$. Let $H_1 := (V_1, S_1)$ and $H_2 := (V_2, S_2)$. We say that the pseudo-Boolean polytope $\mathrm{PBP}(H)$ is *decomposable into the pseudo-Boolean polytopes $PBP(H_1)$ and $PBP(H_2)$*, if the system comprised of a description of $\mathrm{PBP}(H_1)$ and a description of $\mathrm{PBP}(H_2)$, is a description of $\mathrm{PBP}(H)$.

Throughout this paper, for an integer $k$, we define $[k] := \{1, \cdots, k\}$. We are now ready present our decomposition result.

**Theorem 1.** *Let $H = (V, S)$ be a signed hypergraph, let $v$ be a $\beta$-leaf of the underlying hypergraph of $H$, let $s_1 \subseteq s_2 \subseteq \cdots \subseteq s_k$ be the signed edges of $H$ containing $v$, and let $S_v := \{s_1, \ldots, s_k\}$. For each $i = 1, \ldots, k$, let $p_i := s_i - v$. Define $P_v := \{p \in \{p_1, \ldots, p_k\} : |p| \geq 2\}$. Assume that $S$ contains the signed edges in $P_v$. Let $H_1 := (V_1, S_v \cup P_v)$, where $V_1$ is the underlying edge of $s_k$, and let $H_2 := H - v$. Then $PBP(H)$ is decomposable into $PBP(H_1)$ and $PBP(H_2)$.*

*Proof.* We assume $k \geq 1$, as otherwise the result is obvious. We now explain how we write, in the rest of the proof, a vector $z \in \mathbb{R}^{V \cup S}$ by partitioning its components in a number of subvectors. The vector $z_\cap$ contains the components of $z$ corresponding to nodes and signed edges that are both in $H_1$ and in $H_2$, i.e., nodes in $V_1 \setminus \{v\}$ and signed edges in $P_v$. The vector $z_1$ contains the components of $z$ corresponding to nodes and signed edges in $H_1$ but not in $H_2$, i.e., node $v$ and signed edges in $S_v$. Finally, the vector $z_2$ contains the components of $z$ corresponding to nodes and signed edges in $H_2$ but not in $H_1$. Using these definitions, we can now write, up to reordering variables, $z = (z_1, z_\cap, z_2)$. Similarly, we can write a vector $z$ in the space defined by $H_1$ as $(z_1, z_\cap)$, and a vector $z$ in the space defined by $H_2$ as $z = (z_\cap, z_2)$.

We now proceed with the proof of the theorem. We need to show that the system comprised of a description of $\mathrm{PBP}(H_1)$ and a description of $\mathrm{PBP}(H_2)$ results in a description of $\mathrm{PBP}(H)$. It is simple to check that all inequalities valid for $\mathrm{PBP}(H_1)$ and for $\mathrm{PBP}(H_2)$ are also valid for $\mathrm{PBP}(H)$, since all signed edges of $H_1$ and of $H_2$ are also signed edges of $H$. Thus, it suffices to show that a vector that satisfies all inequalities in $\mathrm{PBP}(H_1)$ and $\mathrm{PBP}(H_2)$ is in $\mathrm{PBP}(H)$. Let $\tilde{z}$ such that $(\tilde{z}_1, \tilde{z}_\cap) \in \mathrm{PBP}(H_1)$ and $(\tilde{z}_\cap, \tilde{z}_2) \in \mathrm{PBP}(H_2)$. We show $\tilde{z} \in \mathrm{PBP}(H)$. To do so, we will write $\tilde{z}$ explicitly as a convex combinations of vectors in $\mathrm{PBS}(H)$. In the next claim, we show how a vector in $\mathrm{PBS}(H_1)$ and a vector in $\mathrm{PBS}(H_2)$ can be combined to obtain a vector in $\mathrm{PBS}(H)$.

Note that, for some $i = 1, \dots, k$, we might have $|p_i| = 1$, and so $p_i \notin P_v$, meaning that it is not a signed edge of $H$. In these cases, we have that $p_i$ contains a single node, say $u$, and we call $p_i$ a *signed loop*. Consistently with our notation for signed edges, we will write $z_{p_i}$ and $\sigma_{p_i}(z_u)$ to denote $z_u$, if $\eta_{p_i}(u) = +1$, or to denote $1 - z_u$, if $\eta_{p_i}(u) = -1$.

**Claim 1.** *Let $(z_1, z_\cap) \in PBS(H_1)$ and $(z'_\cap, z'_2) \in PBS(H_2)$ such that $z_{p_i} = z'_{p_i}$ for every $i \in [k]$. Then, $(z_1, z'_\cap, z'_2) \in PBS(H)$.*

*Proof of claim.* It suffices to show that $(z_1, z'_\cap) \in \text{PBS}(H_1)$. The signed edges of $H_1$ are $S_v \cup P_v$. Consider first a signed edge $p_i \in P_v$. In $(z_1, z'_\cap)$, the component of the signed edge $p_i$ is in $z'_\cap$, and all nodes in $p_i$ have components in $z'_\cap$, thus we need to show the equality $z'_{p_i} = \prod_{u \in p_i} \sigma_{p_i}(z'_u)$, which follows directly from $(z'_\cap, z'_2) \in \text{PBS}(H_2)$. Consider now a signed edge $s_i \in S_v$. In $(z_1, z'_\cap)$, the component of the signed edge $s_i$ is in $z_1$, and all the nodes in $s_i$ have components in $z'_\cap$, except for node $v$ that has component in $z_1$. Thus, we need to show the equality

$$z_{s_i} = \sigma_{s_i}(z_v) \prod_{u \in s_i \setminus \{v\}} \sigma_{s_i}(z'_u).$$

Since $s_i$ is a signed edge of $H_1$ and $(z_1, z_\cap) \in \text{PBS}(H_1)$, we know

$$z_{s_i} = \sigma_{s_i}(z_v) \prod_{u \in s_i \setminus \{v\}} \sigma_{s_i}(z_u).$$

We then obtain

$$\prod_{u \in s_i \setminus \{v\}} \sigma_{s_i}(z_u) = \prod_{u \in p_i} \sigma_{p_i}(z_u) = z_{p_i} = z'_{p_i} = \prod_{u \in p_i} \sigma_{p_i}(z'_u) = \prod_{u \in s_i \setminus \{v\}} \sigma_{s_i}(z'_u).$$

Here, the first equality holds by definition of $p_i$ for $i \in [k]$; the second equality holds since $p_i$ is a signed edge of $H_1$ or a signed loop containing a node of $H_1$; in the third equality we use the assumption $z_{p_i} = z'_{p_i}$; the fourth equality holds because $p_i$ is a signed edge of $H_2$ or a signed loop containing a node of $H_2$, and $(z'_\cap, z'_2) \in \text{PBS}(H_2)$; the last equality holds by definition of $p_i$ for $i \in [k]$. $\diamond$

In the remainder of the proof, we show how to write explicitly $\tilde{z}$ as a convex combination of the vectors in $\text{PBS}(H)$ obtained in Claim 1. By assumption, the vector $(\tilde{z}_1, \tilde{z}_\cap)$ is in $\operatorname{conv} \text{PBS}(H_1)$. Thus, it can be written as a convex combination of points in $\text{PBS}(H_1)$; i.e., there exists $\mu \geq 0$ such that

$$\begin{aligned} \sum_{(z_1, z_\cap) \in \text{PBS}(H_1)} \mu_{(z_1, z_\cap)} &= 1 \\ \sum_{(z_1, z_\cap) \in \text{PBS}(H_1)} \mu_{(z_1, z_\cap)} (z_1, z_\cap) &= (\tilde{z}_1, \tilde{z}_\cap). \end{aligned} \tag{1}$$

Similarly, the vector $(\tilde{z}_\cap, \tilde{z}_2)$ is in $\operatorname{conv} \text{PBS}(H_2)$ and it can be written as a convex combination of points in $\text{PBS}(H_2)$; i.e., there exists $\nu \geq 0$ such that

$$\begin{aligned} \sum_{(z_\cap, z_2) \in \text{PBS}(H_2)} \nu_{(z_\cap, z_2)} &= 1 \\ \sum_{(z_\cap, z_2) \in \text{PBS}(H_2)} \nu_{(z_\cap, z_2)} (z_\cap, z_2) &= (\tilde{z}_\cap, \tilde{z}_2). \end{aligned} \tag{2}$$

In the remainder of the proof, given $z_{p_1}, \dots, z_{p_k} \in \{0,1\}$, we will consider the number $\max\{j \in [k] : z_{p_j} = 1\} \in \{0, \dots, k\}$, with the understanding that this number equals 0 when $z_{p_1} = \cdots = z_{p_k} = 0$. In the next technical claim, we study the sums of the multipliers $\mu$ and $\nu$ corresponding to binary vectors with a fixed $\max\{j \in [k] : z_{p_j} = 1\}$. To do so, we define $\mathrm{Next}(0)$ as the set of indices $t \in \{1, \dots, k\}$ such that there is no $r \in \{1, \dots, t-1\}$ with $\pi(p_r) = \pi(p_t) \cap p_r$. For $i = 1, \dots, k$, we denote by $\mathrm{Next}(i)$ the set of indices $t \in \{i+1, \dots, k\}$ such that $\pi(p_i) = \pi(p_t) \cap p_i$ and such that there is no $s \in \{i+1, \dots, t-1\}$ with $\pi(p_s) = \pi(p_t) \cap p_s$. Note that $\mathrm{Next}(k) = \emptyset$. We are now ready to present our technical claim.

**Claim 2.** *For $i \in \{0, \dots, k\}$, we have*

$$\sum_{\substack{(z_1,z_\cap) \in PBS(H_1) \\ \max\{j \in [k] : z_{p_j} = 1\} = i}} \mu_{(z_1,z_\cap)} = \sum_{\substack{(z_\cap,z_2) \in PBS(H_2) \\ \max\{j \in [k] : z_{p_j} = 1\} = i}} \nu_{(z_\cap,z_2)} = \begin{cases} 1 - \sum_{t \in Next(0)} \tilde{z}_{p_t} & \text{if } i = 0 \\ \tilde{z}_{p_i} - \sum_{t \in Next(i)} \tilde{z}_{p_t} & \text{if } i \in \{1, \dots, k\}. \end{cases}$$

*Proof of claim.* By considering the component of (1) corresponding to $p_i$, for $i \in [k]$, we obtain

$$\sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_i} = 1}} \mu_{(z_1,z_\cap)} = \tilde{z}_{p_i}.$$

We first consider the case $i = k$. We have

$$\sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ \max\{j \in [k] : z_{p_j} = 1\} = k}} \mu_{(z_1,z_\cap)} = \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_k} = 1}} \mu_{(z_1,z_\cap)} = \tilde{z}_{p_k} = \tilde{z}_{p_k} - \sum_{t \in \mathrm{Next}(k)} \tilde{z}_{p_t}.$$

Next, we consider the case $i \in \{1, \dots, k-1\}$. We have

$$\begin{aligned} \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ \max\{j \in [k] : z_{p_j} = 1\} = i}} \mu_{(z_1,z_\cap)} &= \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_i} = 1 \\ z_{p_t} = 0 \ \forall t \in \{i+1, \dots, k\}}} \mu_{(z_1,z_\cap)} \\ &= \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_i} = 1 \\ z_{p_t} = 0 \ \forall t \in \mathrm{Next}(i)}} \mu_{(z_1,z_\cap)} \\ &= \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_i} = 1}} \mu_{(z_1,z_\cap)} - \sum_{t \in \mathrm{Next}(i)} \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_t} = 1}} \mu_{(z_1,z_\cap)} \\ &= \tilde{z}_{p_i} - \sum_{t \in \mathrm{Next}(i)} \tilde{z}_{p_t}. \end{aligned}$$

Lastly, we consider the case $i = 0$. We have

$$\begin{aligned} \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ \max\{j \in [k] : z_{p_j} = 1\} = 0}} \mu_{(z_1,z_\cap)} &= \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_t} = 0 \ \forall t \in \{1, \dots, k\}}} \mu_{(z_1,z_\cap)} \\ &= \sum_{\substack{(z_1,z_\cap) \in \mathrm{PBS}(H_1) \\ z_{p_t} = 0 \ \forall t \in \mathrm{Next}(0)}} \mu_{(z_1,z_\cap)} \end{aligned}$$

$$= \sum_{(z_1,z_\cap)\in \mathrm{PBS}(H_1)} \mu_{(z_1,z_\cap)} - \sum_{t\in \mathrm{Next}(0)} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ z_{p_t}=1}} \mu_{(z_1,z_\cap)}$$

$$= 1 - \sum_{t\in \mathrm{Next}(0)} \tilde{z}_{p_t}.$$

The statement for $\nu$ follows symmetrically, starting with (2) rather than (1). ⋄

For ease of notation, we define, for $i \in \{0,\dots,k\}$, the quantity $m(i)$ to be the sum obtained in Claim 2, i.e.,

$$m(i) := \begin{cases} 1 - \sum_{t\in \mathrm{Next}(0)} \tilde{z}_{p_t} & \text{if } i = 0 \\ \tilde{z}_{p_i} - \sum_{t\in \mathrm{Next}(i)} \tilde{z}_{p_t} & \text{if } i \in \{1,\dots,k\}. \end{cases}$$

We are now ready to define the multipliers $\lambda$ that we will use to write explicitly $\tilde{z}$ as a convex combination of the vectors in $\mathrm{PBS}(H)$ obtained in Claim 1. For every $(z_1, z_\cap) \in \mathrm{PBS}(H_1)$ and $(z'_\cap, z'_2) \in \mathrm{PBS}(H_2)$ such that $z_{p_i} = z'_{p_i}$ for every $i \in [k]$, we define

$$\lambda_{(z_1,z'_\cap,z'_2)} := \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)},$$

where $i := \max\{j \in [k] : z_{p_j} = 1\} = \max\{j \in [k] : z'_{p_j} = 1\}$. In the next claim, we show that the multipliers $\lambda$ are nonnegative and sum to one.

**Claim 3.** *We have $\lambda \geq 0$ and*

$$\sum_{\substack{(z_1,z_\cap)\in PBS(H_1)\\ (z'_\cap,z'_2)\in PBS(H_2)\\ z_{p_i}=z'_{p_i}\ \forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)} = 1$$

*Proof of claim.* It follows from Claim 2 that $m(i) \geq 0$ for all $i \in \{0,\dots,k\}$. Thus, using the definition of $\lambda$, we obtain $\lambda \geq 0$. Using Claim 2, we obtain

$$\sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ (z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ z_{p_i}=z'_{p_i}\ \forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)} = \sum_{i=0}^{k} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \lambda_{(z_1,z'_\cap,z'_2)}$$

$$= \sum_{i=0}^{k} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)}$$

$$= \sum_{i=0}^{k} \frac{1}{m(i)} \left( \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \mu_{(z_1,z_\cap)} \right) \left( \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \nu_{(z'_\cap,z'_2)} \right)$$

$$= \sum_{i=0}^{k} \frac{(m(i))^2}{m(i)} = \sum_{i=0}^{k} m(i) = 1.$$

The last equality $\sum_{i=0}^{k} m(i) = 1$ can be seen using the definition of $m(i)$, because each index in $\{1,\dots,k\}$ is exactly in one set among $\mathrm{Next}(0),\dots,\mathrm{Next}(k)$. ⋄

Our last claim, which concludes the proof of the theorem, shows that the multipliers $\lambda$ yield $\tilde{z}$ as a convex combination of the vectors in PBS($H$) obtained in Claim 1.

**Claim 4.** *We have*

$$(\tilde{z}_1, \tilde{z}_\cap, \tilde{z}_2) = \sum_{\substack{(z_1,z_\cap)\in PBS(H_1)\\ (z'_\cap,z'_2)\in PBS(H_2)\\ z_{p_i}=z'_{p_i}\ \ \forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)}(z_1, z'_\cap, z'_2), \tag{3}$$

*Proof of claim.* Using the definition of $\lambda$, we rewrite (3) in the form

$$\begin{aligned}
(\tilde{z}_1, \tilde{z}_\cap, \tilde{z}_2) &= \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ (z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ z_{p_i}=z'_{p_i}\ \ \forall i\in[k]}} \lambda_{(z_1,z'_\cap,z'_2)}(z_1, z'_\cap, z'_2)\\
&= \sum_{i=0}^{k} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \lambda_{(z_1,z'_\cap,z'_2)}(z_1, z'_\cap, z'_2)\\
&= \sum_{i=0}^{k} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)}(z_1, z'_\cap, z'_2).
\end{aligned}$$

We now verify the obtained equality, first for components $\tilde{z}_1$, and then for components $\tilde{z}_\cap, \tilde{z}_2$. We start with components $\tilde{z}_1$. Using Claim 2, we obtain

$$\begin{aligned}
\tilde{z}_1 &= \sum_{i=0}^{k} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)} z_1\\
&= \sum_{i=0}^{k} \frac{1}{m(i)} \left( \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \mu_{(z_1,z_\cap)} z_1 \right) \left( \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \nu_{(z'_\cap,z'_2)} \right)\\
&= \sum_{i=0}^{k} \frac{m(i)}{m(i)} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \mu_{(z_1,z_\cap)} z_1\\
&= \sum_{(z_1,z_\cap)\in \mathrm{PBS}(H_1)} \mu_{(z_1,z_\cap)} z_1,
\end{aligned}$$

and the resulting equation is implied by (1).

Next, we consider components $\tilde{z}_\cap, \tilde{z}_2$. Using Claim 2, we obtain

$$(\tilde{z}_\cap, \tilde{z}_2) = \sum_{i=0}^{k} \sum_{\substack{(z_1,z_\cap)\in \mathrm{PBS}(H_1)\\ \max\{j\in[k]:z_{p_j}=1\}=i}} \sum_{\substack{(z'_\cap,z'_2)\in \mathrm{PBS}(H_2)\\ \max\{j\in[k]:z'_{p_j}=1\}=i}} \frac{\mu_{(z_1,z_\cap)}\nu_{(z'_\cap,z'_2)}}{m(i)} (z'_\cap, z'_2)$$

$$
= \sum_{i=0}^{k} \frac{1}{m(i)} \left( \sum_{\substack{(z_1, z_\cap) \in \mathrm{PBS}(H_1) \\ \max\{j \in [k]: z_{p_j}=1\}=i}} \mu_{(z_1, z_\cap)} \right) \left( \sum_{\substack{(z'_\cap, z'_2) \in \mathrm{PBS}(H_2) \\ \max\{j \in [k]: z'_{p_j}=1\}=i}} \nu_{(z'_\cap, z'_2)}(z'_\cap, z'_2) \right)
$$

$$
= \sum_{i=0}^{k} \frac{m(i)}{m(i)} \sum_{\substack{(z'_\cap, z'_2) \in \mathrm{PBS}(H_2) \\ \max\{j \in [k]: z'_{p_j}=1\}=i}} \nu_{(z'_\cap, z'_2)}(z'_\cap, z'_2)
$$

$$
= \sum_{(z'_\cap, z'_2) \in \mathrm{PBS}(H_2)} \nu_{(z'_\cap, z'_2)}(z'_\cap, z'_2),
$$

and the resulting equation is (2). $\diamond$

$\square$

The overall structure of the proof of Theorem 1 is similar to that of Theorem 4 in [19]. The difference lies in the key construction of the proof, that is, the multipliers introduced to construct the convex combination. Due to the presence of the signed edges, these new multipliers are significantly more involved than the ones presented in [19]. In the special case where all signs are positive, the new multipliers simplify to the ones in the proof of Theorem 4 in [19].

## 3 Pointed signed hypergraphs

The signed hypergraph $H_1$ defined in the statement of Theorem 1 plays a key role in our convex hull characterizations. In this section, we prove that the pseudo-Boolean polytope $\mathrm{PBP}(H_1)$ has a polynomial-size extended formulation. Together with the decomposition technique of Theorem 1, this result enables us to obtain polynomial-size extended formulations for the pseudo-Boolean polytope of a large family of signed hypergraphs.

Consider a signed hypergraph $H = (V, S)$ and let $v \in V$ be a $\beta$-leaf of the underlying hypergraph of $H$. Denote by $S_v$ the set of all signed edges in $S$ containing $v$. Define $P_v := \{s - v : s \in S_v, |s| \geq 3\}$. We say that $H$ is *pointed* at $v$ (or is a *pointed signed hypergraph*), if $V$ coincides with the underlying edge of the signed edge of maximum cardinality in $S_v$ and $S = S_v \cup P_v$. From this definition it follows that the signed hypergraph $H_1$ in the statement of Theorem 1 is pointed at $v$.

In order to characterize the pseudo-Boolean polytope of a signed hypergraph $H$ pointed at $v$, we first characterize the pseudo-Boolean polytope of a simpler class of signed hypergraphs corresponding to faces of $\mathrm{PBP}(H)$ defined by $z_v = 0$ or $z_v = 1$. An extended formulation for $\mathrm{PBP}(H)$ can then be obtained using disjunctive programming.

### 3.1 The pseudo-Boolean polytope of nested signed hypergraphs

In this section, we characterize the pseudo-Boolean polytope of nested signed hypergraphs in the original space. This result will then enable us to obtain a polynomial-size extended formulation for the pseudo-Boolean polytope of pointed signed hypergraphs.

Let $H = (V, S)$ be a signed hypergraph with $V = \{v_1, \cdots, v_n\}$. Denote by $E$ the set of underlying edges of $S$ (note that several signed edges may collapse to the same edge). Define $e_k := \{v_1, \cdots v_{k+1}\}$ for all $k \in [n-1]$ and $\bar{E} := \{e_1, \cdots, e_{n-1}\}$. We say that $H$ is a *nested signed hypergraph*, if it satisfies the following conditions:

(N1) $E = \bar{E}$,

(N2) for any signed edge $s \in S$ with underlying edge $e_k$ for some $k \in [n-1]$, the following two signed edges are also present in $S$: $\ell(s)$, obtained from $s$ by flipping the sign of $v$, and $p(s) := s - v_{k+1}$.

**Remark 1.** *Nested signed hypergraphs have two important properties that we will use to obtain our extended formulations:*

1. *As we prove in Proposition 1, the pseudo-Boolean polytope of a nested signed hypergraph $H = (V, S)$ in the original space is defined by $\frac{|S|}{2}$ equalities, and $|S| + 2|V| - 1$ inequalities; i.e., $2(|S| + |V| - 1)$ inequalities.*

2. *Let $H = (V, S)$ be a signed hypergraph such that for any $s, s' \in S$ we either have $s \subseteq s'$ or $s' \subseteq s$. It then follows that by adding at most $2|S|(|V| - 2) + |S|$ signed edges to $H$ we obtain a nested signed hypergraph. To see this, note that to satisfy property (N2) of nested signed hypergraphs, for each edge $s \in S$ we need to add at most $2(|s| - 2) + 1$ signed edges to $H$. Together with case 1 above, this in turn implies that PBP($H$) has a polynomial-size extended formulation with at most $2|S|(|V| - 1) + |V|$ variables and at most $4|S|(|V| - 1) + 2|V|$ inequalities.*

We now proceed with characterizing the pseudo-Boolean of nested signed hypergraphs. To this end, we first introduce some notation. We denote by $\mathcal{E}_k$, $k \in [n-1]$ the set consisting of signed edges in $S$ whose underlying edges are $e_k$. For each $k \in [n-1] \setminus \{1\}$, we define

$$\mathcal{E}_k^+ := \Big\{s \in \mathcal{E}_k : \eta_s(v_{k+1}) = +1\Big\}, \qquad \text{and} \qquad \mathcal{E}_k^- := \Big\{s \in \mathcal{E}_k : \eta_s(v_{k+1}) = -1\Big\}.$$

For any $s \in S$, we define

$$N(s) := \{s' \in S : s = p(s')\},$$

where as before for any $s' \in S$ with underlying edge $e_k$, we define $p(s') = s' - v_{k+1}$. Notice that by property (N2) of nested signed hypergraphs, $|N(s)|$ equals zero or two.

The following proposition characterizes the pseudo-Boolean polytope of nested signed hypergraphs in the original space. While our proof relies on a standard disjunctive programming technique [1] followed by a projection step using Fourier-Motzkin elimination, the novelty of the proof lies in the manner Fourier-Motzkin elimination is implemented. Namely, it is well-understood that a generic application of Fourier-Motzkin elimination leads to a rapid increase in the number of inequalities defining the polyhedron. In our proof, the auxiliary variables are projected out in a specific order so that the projection does not contain redundant inequalities.

**Proposition 1.** *Let $H = (V, S)$ with $n := |V|$ be a nested signed hypergraph. Suppose that $\mathcal{E}_1 = \{q_1, q_2, q_3, q_4\}$ with $\eta_{q_1}(v_1) = \eta_{q_1}(v_2) = \eta_{q_2}(v_1) = \eta_{q_3}(v_2) = +1$ and $\eta_{q_2}(v_2) = \eta_{q_3}(v_1) = \eta_{q_4}(v_1) = \eta_{q_4}(v_2) = -1$. Then the pseudo-Boolean polytope PBP($H$) is given by:*

$$z_s + z_{\ell(s)} = z_{p(s)}, \quad \forall s \in \mathcal{E}_k^+, k \in [n-1] \setminus \{1\} \tag{4}$$

$$z_s \geq 0, \quad \forall s \in S \tag{5}$$

$$\sum_{s \in \mathcal{E}_k^+} z_s \leq z_{v_{k+1}}, \quad \forall k \in [n-1] \setminus \{1\} \tag{6}$$

$$\sum_{s \in \mathcal{E}_k^-} z_s \leq 1 - z_{v_{k+1}}, \quad \forall k \in [n-1] \setminus \{1\} \tag{7}$$

$$z_{q_1} + z_{q_2} = z_{v_1},\ z_{q_1} + z_{q_3} = z_{v_2},\ z_{q_3} + z_{q_4} = 1 - z_{v_1}. \tag{8}$$

*Proof.* The proof is by induction on the number of nodes $n$ of $H$. The base case is $n = 2$; in this case we have $S = \mathcal{E}_1 = \{q_1, q_2, q_3, q_4\}$. It is simple to check that PBP($H$) is given by:

$$z_{q_1} + z_{q_2} = z_{v_1},\ z_{q_1} + z_{q_3} = z_{v_2},\ z_{q_3} + z_{q_4} = 1 - z_{v_1}$$

$$z_{q_1}, z_{q_2}, z_{q_3}, z_{q_4} \geq 0.$$

Henceforth, let $n \geq 3$. Denote by $H_0$ (resp. $H_1$) the signed hypergraph corresponding to the face of $\mathrm{PBP}(H)$ with $z_{v_n} = 0$ (resp. $z_{v_n} = 1$). We then have:

$$\mathrm{PBP}(H) = \mathrm{conv}(\mathrm{PBP}(H_0) \cup \mathrm{PBP}(H_1)).$$

Denote by $\bar{z}$ the vector consisting of $z_v$ for all $v \in V \setminus \{v_n\}$ and $z_s$ for all $s \in S \setminus \mathcal{E}_{n-1}$. Since both $H_0$ and $H_1$ have one fewer node than $H$ and are nested signed hypergraphs, a description of $\mathrm{PBP}(H_0)$ and $\mathrm{PBP}(H_1)$ follows from the induction hypothesis:

$$\mathrm{PBP}(H_0) = \Big\{z \in \mathbb{R}^{V \cup S} : z_{v_n} = 0,\ z_s = 0,\ \forall s \in \mathcal{E}^+_{n-1}, z_s = z_{p(s)},\ \forall e \in \mathcal{E}^-_{n-1},\ \bar{z} \in \mathcal{Q}\Big\}$$
$$\mathrm{PBP}(H_1) = \Big\{z \in \mathbb{R}^{V \cup S} : z_{v_n} = 1,\ z_s = z_{p(s)},\ \forall e \in \mathcal{E}^+_{n-1}, z_s = 0,\ \forall s \in \mathcal{E}^-_{n-1},\ \bar{z} \in \mathcal{Q}\Big\},$$

where the polytope $\mathcal{Q}$ is defined by the following linear constraints:

$$\begin{aligned}
& z_s + z_{\ell(s)} = z_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-2] \setminus \{1\} \\
& z_s \geq 0, \quad \forall s \in S \setminus \mathcal{E}_{n-1} \\
& \sum_{s \in \mathcal{E}^+_k} z_s \leq z_{v_{k+1}}, \quad \forall k \in [n-2] \setminus \{1\} \\
& \sum_{s \in \mathcal{E}^-_k} z_s \leq 1 - z_{v_{k+1}}, \quad \forall k \in [n-2] \setminus \{1\} \\
& z_{q_1} + z_{q_2} = z_{v_1},\ z_{q_1} + z_{q_3} = z_{v_2},\ z_{q_3} + z_{q_4} = 1 - z_{v_1}.
\end{aligned}$$

Using Balas' formulation for the union of polytopes [1], it follows that $\mathrm{PBP}(H)$ is the projection onto the space of the $z$ variables of the polyhedron defined by the following system (9)–(11):

$$\begin{aligned}
& \lambda_0 + \lambda_1 = 1,\ \lambda_0 \geq 0\ , \lambda_1 \geq 0 \\
& z_v = z^0_v + z^1_v, \quad \forall v \in V \\
& z_s = z^0_s + z^1_s, \quad \forall s \in S
\end{aligned} \tag{9}$$

$$\begin{aligned}
& z^0_{v_n} = 0 \\
& z^0_s = 0, \quad \forall s \in \mathcal{E}^+_{n-1} \\
& z^0_s = z^0_{p(s)}, \quad \forall s \in \mathcal{E}^-_{n-1} \\
& z^0_s + z^0_{\ell(s)} = z^0_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-2] \setminus \{1\} \\
& z^0_s \geq 0, \quad \forall s \in S \setminus \mathcal{E}_{n-1} \\
& \sum_{s \in \mathcal{E}^+_k} z^0_s \leq z^0_{v_{k+1}}, \quad \forall k \in [n-2] \setminus \{1\} \\
& \sum_{s \in \mathcal{E}^-_k} z^0_s \leq \lambda_0 - z^0_{v_{k+1}}, \quad \forall k \in [n-2] \setminus \{1\} \\
& z^0_{q_1} + z^0_{q_2} = z^0_{v_1},\ z^0_{q_1} + z^0_{q_3} = z^0_{v_2},\ z^0_{q_3} + z^0_{q_4} = \lambda_0 - z^0_{v_1}
\end{aligned} \tag{10}$$

$$
\begin{aligned}
& z^1_{v_n} = \lambda_1 \\
& z^1_s = z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-1} \\
& z^1_s = 0, \quad \forall s \in \mathcal{E}^-_{n-1} \\
& z^1_s + z^1_{\ell(s)} = z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-2] \setminus \{1\} \\
& z^1_s \geq 0, \quad \forall s \in S \setminus \mathcal{E}_{n-1} \\
& \sum_{s \in \mathcal{E}^+_k} z^1_s \leq z^1_{v_{k+1}}, \quad \forall k \in [n-2] \setminus \{1\} \\
& \sum_{s \in \mathcal{E}^-_k} z^1_s \leq \lambda_1 - z^1_{v_{k+1}}, \quad \forall k \in [n-2] \setminus \{1\} \\
& z^1_{q_1} + z^1_{q_2} = z^1_{v_1}, \; z^1_{q_1} + z^1_{q_3} = z^1_{v_2}, \; z^1_{q_3} + z^1_{q_4} = \lambda_1 - z^1_{v_1}.
\end{aligned}
\tag{11}
$$

In the remainder of this proof, we project out $z^0, z^1, \lambda_0, \lambda_1$ from system (9)–(11) and obtain a description of PBP$(H)$ in the original space. From $z_{v_n} = z^0_{v_n} + z^1_{v_n}$, $z^0_{v_n} = 0$, $z^1_{v_n} = \lambda_1$, and $\lambda_0 + \lambda_1 = 1$, it follows that

$$
\lambda_0 = 1 - z_{v_n}, \quad \lambda_1 = z_{v_n}. \tag{12}
$$

By $z_s = z^0_s + z^1_s$ for all $s \in S$, $z^0_s = 0$, and $z^1_s = z^1_{p(s)}$ for all $s \in \mathcal{E}^+_{n-1}$, we get

$$
z^1_s = z^1_{p(s)} = z_s, \quad z^0_{p(s)} = z_{p(s)} - z_s, \quad \forall s \in \mathcal{E}^+_{n-1}. \tag{13}
$$

Similarly, by $z^1_{\ell(s)} = 0$ and $z^0_{\ell(s)} = z^0_{p(s)}$ for all $s \in \mathcal{E}^+_{n-1}$, we get

$$
z^0_{\ell(s)} = z^0_{p(s)} = z_{\ell(s)}, \quad z^1_{p(s)} = z_{p(s)} - z_{\ell(s)}, \quad \forall s \in \mathcal{E}^+_{n-1}. \tag{14}
$$

Using (9) to project out $z^0_v$, $v \in V$, $z^0_s$, $s \in S$, using (12) to project out $\lambda_0, \lambda_1$, and using (13) and (14) to project out $z^0_s, z^1_s$, $s \in \mathcal{E}_{n-1}$ with $N(s) \neq \emptyset$, we deduce that system (9)-(11) simplifies to:

$$
z_{s+v^-_n} + z_{\ell(s)+v^-_n} = z_{p(s)} - z^1_{p(s)}, \; \forall s \in \mathcal{E}^+_{n-2} : s + v^-_n, \ell(s) + v^-_n \in \mathcal{E}^-_{n-1} \tag{15}
$$

$$
z_{s+v^-_n} + z_{\ell(s)} - z^1_{\ell(s)} = z_{p(s)} - z^1_{p(s)}, \; \forall s \in \mathcal{E}_{n-2} : s + v^-_n \in \mathcal{E}^-_{n-1}, N(\ell(s)) = \emptyset \tag{16}
$$

$$
z_s - z^1_s + z_{\ell(s)} - z^1_{\ell(s)} = z_{p(s)} - z^1_{p(s)}, \; \forall s \in \mathcal{E}^+_k : k \in [n-3] \setminus \{1\}, \text{ or } k = n-2, N(s) = N(\ell(s)) = \emptyset \tag{17}
$$

$$
z_s - z^1_s \geq 0, \quad \forall s \in \mathcal{E}_k : k \in [n-3] \text{ or } k = n-2, N(s) = \emptyset \tag{18}
$$

$$
\sum_{\substack{s \in \mathcal{E}^+_{n-2}: \\ s+v^-_n \in \mathcal{E}^-_{n-1}}} z_{s+v^-_n} + \sum_{\substack{s \in \mathcal{E}^+_{n-2}: \\ N(s)=\emptyset}} (z_s - z^1_s) \leq z_{v_{n-1}} - z^1_{v_{n-1}}, \tag{19}
$$

$$
\sum_{\substack{s \in \mathcal{E}^-_{n-2}: \\ s+v^-_n \in \mathcal{E}^-_{n-1}}} z_{s+v^-_n} + \sum_{\substack{s \in \mathcal{E}^-_{n-2}: \\ N(s)=\emptyset}} (z_s - z^1_s) \leq 1 - z_{v_n} - z_{v_{n-1}} + z^1_{v_{n-1}}, \tag{20}
$$

$$
\sum_{s \in \mathcal{E}^+_k} (z_s - z^1_s) \leq z_{v_{k+1}} - z^1_{v_{k+1}}, \quad \forall k \in [n-3] \setminus \{1\} \tag{21}
$$

$$
\sum_{s \in \mathcal{E}^-_k} (z_s - z^1_s) \leq 1 - z_{v_n} - z_{v_{k+1}} + z^1_{v_{k+1}}, \quad \forall k \in [n-3] \setminus \{1\} \tag{22}
$$

$$z_{q_1} - z^1_{q_1} + z_{q_2} - z^1_{q_2} = z_{v_1} - z^1_{v_1},\ z_{q_1} - z^1_{q_1} + z_{q_3} - z^1_{q_3} = z_{v_2} - z^1_{v_2},$$
$$z_{q_3} - z^1_{q_3} + z_{q_4} - z^1_{q_4} = 1 - z_{v_n} - z_{v_1} + z^1_{v_1}, \tag{23}$$

and

$$z_{s+v_n^+} + z_{\ell(s)+v_n^+} = z^1_{p(s)},\ \forall s \in \mathcal{E}^+_{n-2} : s + v_n^+, \ell(s) + v_n^+ \in \mathcal{E}^+_{n-1} \tag{24}$$
$$z_{s+v_n^+} + z^1_{\ell(s)} = z^1_{p(s)},\ \forall s \in \mathcal{E}_{n-2} : s + v_n^+ \in \mathcal{E}^+_{n-1}, N(\ell(s)) = \emptyset \tag{25}$$
$$z^1_s + z^1_{\ell(s)} = z^1_{p(s)},\ \forall s \in \mathcal{E}^+_k : k \in [n-3] \setminus \{1\}, \text{or } k = 2, N(s) = N(\ell(s)) = \emptyset \tag{26}$$
$$z^1_s \geq 0, \quad \forall s \in \mathcal{E}_k : k \in [n-3],\ \text{or } k = n-2, N(s) = \emptyset \tag{27}$$
$$\sum_{\substack{s\in\mathcal{E}^+_{n-2}:\\ s+v_n^+\in\mathcal{E}^+_{n-1}}} z_{s+v_n^+} + \sum_{\substack{s\in\mathcal{E}^+_{n-2}:\\ N(s)=\emptyset}} z^1_s \leq z^1_{v_{n-1}} \tag{28}$$
$$\sum_{\substack{s\in\mathcal{E}^-_{n-2}:\\ s+v_n^+\in\mathcal{E}^+_{n-1}}} z_{s+v_n^+} + \sum_{\substack{s\in\mathcal{E}^-_{n-2}:\\ N(s)=\emptyset}} z^1_s \leq z_{v_n} - z^1_{v_{n-1}} \tag{29}$$
$$\sum_{s\in\mathcal{E}^+_k} z^1_s \leq z^1_{v_{k+1}}, \quad \forall k \in [n-3] \setminus \{1\} \tag{30}$$
$$\sum_{s\in\mathcal{E}^-_k} z^1_s \leq z_{v_n} - z^1_{v_{k+1}}, \quad \forall k \in [n-3] \setminus \{1\} \tag{31}$$
$$z^1_{q_1} + z^1_{q_2} = z^1_{v_1},\ z^1_{q_1} + z^1_{q_3} = z^1_{v_2},\ z^1_{q_3} + z^1_{q_4} = z_{v_n} - z^1_{v_1}, \tag{32}$$

together with

$$z_s + z_{\ell(s)} = z_{p(s)},\ \forall s \in \mathcal{E}^+_{n-1} \tag{33}$$
$$z_s \geq 0,\ \forall s \in \mathcal{E}_{n-1}. \tag{34}$$

Note that equalities (33) are present among equalities (4) and inequalities (34) are present among inequalities (5).

In the remainder of the proof, we project out variables $z^1_v$, $v \in V \setminus \{v_n\}$, $z^1_s$, $s \in S \setminus \{s \in \mathcal{E}_{n-1} : N(s) \neq \emptyset\}$ in a specific order:

**Projecting out $z^1_{v_k}$, $k \in \{1, \ldots, n-1\}$:**

- The variable $z^1_{v_1}$ appears only in equalities (23) and (32); i.e., the following equations:

$$z^1_{q_1} + z^1_{q_2} = z^1_{v_1}$$
$$z_{q_1} - z^1_{q_1} + z_{q_2} - z^1_{q_2} = z_{v_1} - z^1_{v_1}$$
$$z^1_{q_3} + z^1_{q_4} = z_{v_n} - z^1_{v_1}$$
$$z_{q_3} - z^1_{q_3} + z_{q_4} - z^1_{q_4} = 1 - z_{v_n} - z_{v_1} + z^1_{v_1}$$

  Hence projecting out $z^1_{v_1}$ we obtain:

$$z_{q_1} + z_{q_2} = z_{v_1} \tag{35}$$
$$z_{q_3} + z_{q_4} = 1 - z_{v_1} \tag{36}$$
$$z^1_{q_1} + z^1_{q_2} + z^1_{q_3} + z^1_{q_4} = z_{v_n}. \tag{37}$$

Equalities (35) and (36) are present among equalities (8). Moreover, the above equations imply:

$$z_{q_1} - z^1_{q_1} + z_{q_2} - z^1_{q_2} + z_{q_3} - z^1_{q_3} + z_{q_4} - z^1_{q_4} = 1 - z_{v_n} \tag{38}$$

We will use equality (38) to simplify our derivations.

- The variable $z^1_{v_2}$ appears only in equalities (23) and (32); i.e., the following equations:

$$z^1_{q_1} + z^1_{q_3} = z^1_{v_2},$$
$$z_{q_1} - z^1_{q_1} + z_{q_3} - z^1_{q_3} = z_{v_2} - z^1_{v_2}$$

  Hence projecting out this variable we obtain:

$$z_{q_1} + z_{q_3} = z_{v_2}, \tag{39}$$

  which is present among equalities (8).

- The variable $z^1_{v_{n-1}}$ appears only in inequalities (19), (20), (28) and (29). Projecting out $z^1_{v_{n-1}}$ from (19) and (20) gives:

$$\sum_{s \in \mathcal{E}^-_{n-1}} z_s + \sum_{s \in \mathcal{E}_{n-2}: N(s)=\emptyset} (z_s - z^1_s) \leq 1 - z_{v_n}, \tag{40}$$

  while projecting out $z^1_{v_{n-1}}$ from (28) and (29) gives:

$$\sum_{s \in \mathcal{E}^+_{n-1}} z_s + \sum_{s \in \mathcal{E}_{n-2}: N(s)=\emptyset} z^1_s \leq z_{v_n}. \tag{41}$$

  As we argue shortly, inequality (40) (resp. (41)) is implied by equaly (38) (resp. (37)) and inequality (18) (resp. (27)). Projecting out $z^1_{v_{n-1}}$ from (19) and (28) gives:

$$\sum_{s \in \mathcal{E}^+_{n-2}} z_s \leq z_{v_{n-1}}, \tag{42}$$

  which coincides with inequalities (6) for $k = n-2$. Lastly, projecting out $z^1_{v_{n-1}}$ from (20) and (29) we obtain:

$$\sum_{s \in \mathcal{E}^-_{n-2}} z_s \leq 1 - z_{v_{n-1}}, \tag{43}$$

  which coincides with inequalities (7) for $k = n-2$.

- Variables $z^1_{v_{k+1}}$, $k \in \{2, \cdots, n-3\}$ are only present in inequalities (21), (22), (30), and (31). Projecting out $z^1_{v_{k+1}}$, $k \in \{2, \cdots, n-3\}$ from (21) and (22) gives:

$$\sum_{s \in \mathcal{E}_k} (z_s - z^1_s) \leq 1 - z_{v_n}, \quad \forall k \in [n-3] \setminus \{1\},$$

  which by equalities (17) can be equivalently written as:

$$\sum_{s \in \mathcal{E}^+_k} (z_{p(s)} - z^1_{p(s)}) \leq 1 - z_{v_n}. \quad \forall k \in [n-3] \setminus \{1\}. \tag{44}$$

By (15)–(17), for each $k \in [n-1] \setminus \{1\}$, we have $\sum_{s \in \mathcal{E}_{k+1}} (z_s - z_s^1) \leq \sum_{s \in \mathcal{E}_k} (z_s - z_s^1)$, implying that inequalities (40) and (44) are implied by the following inequality:

$$\sum_{s \in \mathcal{E}_2^+} (z_{p(s)} - z_{p(s)}^1) \leq 1 - z_{v_n},$$

which in turn is implied by inequalities (18) and equality (38). Similarly, projecting out $z_{v_{k+1}}^1$, $k \in \{2, \cdots, n-3\}$ from (30) and (31) gives:

$$\sum_{s \in \mathcal{E}_k} z_s^1 \leq z_{v_n}, \quad \forall k \in [n-3] \setminus \{1\},$$

which by equalities (26) can be equivalently written as:

$$\sum_{s \in \mathcal{E}_k^+} z_{p(s)}^1 \leq z_{v_n}, \quad \forall k \in [n-3] \setminus \{1\}. \tag{45}$$

Since for each $k \in [n-1] \setminus \{1\}$, by (24)–(26), we have $\sum_{s \in \mathcal{E}_{k+1}} z_s^1 \leq \sum_{s \in \mathcal{E}_k} z_s^1$, inequalities (41) and (45) are implied by:

$$\sum_{s \in \mathcal{E}_2^+} z_{p(s)}^1 \leq z_{v_n},$$

which in turn is implied by inequalities (27) and equality (37). Projecting out $z_{v_{k+1}}^1$, $k \in \{2, \cdots, n-3\}$ from (21) and (30) we obtain:

$$\sum_{s \in \mathcal{E}_k^+} z_s \leq z_{v_{k+1}}, \quad \forall k \in [n-3] \setminus \{1\}, \tag{46}$$

which are present among inequalities (6). Similarly, projecting out $z_{v_{k+1}}^1$, $k \in \{2, \cdots, n-3\}$ from (22) and (31), we obtain

$$\sum_{s \in \mathcal{E}_k^-} z_s \leq 1 - z_{v_{k+1}}, \quad \forall k \in [n-3] \setminus \{1\}, \tag{47}$$

which are present among inequalities (7).

Hence, we have shown that projecting out variables $z_{v_k}^1$, $k \in \{1, \ldots, n-1\}$ from (19)-(23) and (28)–(32), we obtain inequalities (6)–(7) when $k \neq n-1$, equalities (8), and equalities (37) (38).

**Projecting out $z_s^1$, $s \in \mathcal{E}_{n-2}$ with $N(e) = \emptyset$:** Consider $z_{\bar{s}}^1$ for some $\bar{s} \in \mathcal{E}_{n-2}^+$ with $N(\bar{s}) = \emptyset$. This variable is only present in (16)–(18) and (25)–(27). Two cases arise:

- If $N(\ell(\bar{s})) \neq \emptyset$, then $z_{\bar{s}}^1$ is only present in (16), (18), (25), (27); i.e., the following system:

$$\begin{aligned}
&z_{\bar{s}} - z_{\bar{s}}^1 + z_{\ell(\bar{s})+v_n^-} = z_{p(\bar{s})} - z_{p(\bar{s})}^1 \\
&z_{\bar{s}} - z_{\bar{s}}^1 \geq 0 \\
&z_{\bar{s}}^1 + z_{\ell(\bar{s})+v_n^+} = z_{p(\bar{s})}^1 \\
&z_{\bar{s}}^1 \geq 0.
\end{aligned}$$

  Projecting out $z_{\bar{s}}^1$ from the above system yields:

$$z_{\bar{s}} + z_{\ell(\bar{s})} = z_{p(\bar{s})}$$

$$
\begin{aligned}
&z_{\bar{s}} \geq 0\\
&z_{\ell(\bar{s})+v_n^-} \leq z_{p(\bar{s})} - z^1_{p(\bar{s})}\\
&z_{\ell(\bar{s})+v_n^+} \leq z^1_{p(\bar{s})}
\end{aligned}
$$

where to obtain the first equality we made use of equalities (33).

- If $N(\ell(\bar{s})) = \emptyset$, then $z^1_{\bar{s}}$ and $z^1_{\ell(\bar{s})}$ are present only in (17)–(18), and (26)–(27); i.e., the following system:

$$
\begin{aligned}
&z_{\bar{s}} - z^1_{\bar{s}} + z_{\ell(\bar{s})} - z^1_{\ell(\bar{s})} = z_{p(\bar{s})} - z^1_{p(\bar{s})},\\
&z_{\bar{s}} - z^1_{\bar{s}} \geq 0\\
&z_{\ell(\bar{s})} - z^1_{\ell(\bar{s})} \geq 0\\
&z^1_{\bar{s}} + z^1_{\ell(\bar{s})} = z^1_{p(\bar{s})},\\
&z^1_{\bar{s}} \geq 0\\
&z^1_{\ell(\bar{s})} \geq 0.
\end{aligned}
$$

Projecting out $z^1_{\bar{s}}$ and $z^1_{\ell(\bar{s})}$ from the above system yields:

$$
\begin{aligned}
&z_{\bar{s}} + z_{\ell(\bar{s})} = z_{p(\bar{s})}\\
&z_{\bar{s}} \geq 0\\
&z_{\ell(\bar{s})} \geq 0\\
&z^1_{p(\bar{s})} \geq 0\\
&z_{p(\bar{s})} - z^1_{p(\bar{s})} \geq 0.
\end{aligned}
$$

By a recursive application of the two steps detailed above to project out all $z^1_s$, $s \in \mathcal{E}_{n-2}$ with $N(s) = \emptyset$ from (16)–(18) and (25)–(27), we obtain:

$$
\begin{aligned}
&z_{s+v_n^+} \leq z^1_{p(s)}, \quad \forall s \in \mathcal{E}_{n-2} : s + v_n^+ \in \mathcal{E}^+_{n-1},\ N(\ell(s)) = \emptyset,\\
&z_{s+v_n^-} \leq z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}_{n-2} : s + v_n^- \in \mathcal{E}^-_{n-1},\ N(\ell(s)) = \emptyset,\\
&z^1_{p(s)} \geq 0,\ z_{p(s)} - z^1_{p(s)} \geq 0, \quad \forall s \in \mathcal{E}_{n-2} : N(s) = N(\ell(s)) = \emptyset
\end{aligned}
$$

together with

$$
\begin{aligned}
&z_s + z_{\ell(s)} = z_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-2}\\
&z_s \geq 0, \quad \forall s \in \mathcal{E}_{n-2}.
\end{aligned}
\tag{48}
$$

Hence, to complete the projection, it suffices to project out variables $z^1_s$, $e \in \mathcal{E}_k$, $k \in [n-3]$ from the following system:

$$
\begin{aligned}
&z_{s+v_n^-} + z_{\ell(s)+v_n^-} = z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-2} : s + v_n^-, \ell(s) + v_n^- \in \mathcal{E}^-_{n-1} \\
&z_{s+v_n^+} + z_{\ell(s)+v_n^+} = z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-2} : s + v_n^+, \ell(s) + v_n^+ \in \mathcal{E}^+_{n-1} \\
&z_{s+v_n^-} \leq z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}_{n-2} : s + v_n^- \in \mathcal{E}^-_{n-1},\ N(\ell(s)) = \emptyset \\
&z_{s+v_n^+} \leq z^1_{p(s)}, \quad \forall s \in \mathcal{E}_{n-2} : s + v_n^+ \in \mathcal{E}^+_{n-1},\ N(\ell(s)) = \emptyset \\
&z_s - z^1_s + z_{\ell(s)} - z^1_{\ell(s)} = z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-3] \setminus \{1\} \\
&z^1_s + z^1_{\ell(s)} = z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-3] \setminus \{1\} \\
&z_s - z^1_s \geq 0, \quad \forall s \in \mathcal{E}_k : k \in [n-3] \\
&z^1_s \geq 0, \quad \forall s \in \mathcal{E}_k : k \in [n-3] \\
&z_{q_1} - z^1_{q_1} + z_{q_2} - z^1_{q_2} + z_{q_3} - z^1_{q_3} + z_{q_4} - z^1_{q_4} = 1 - z_{v_n} \\
&z^1_{q_1} + z^1_{q_2} + z^1_{q_3} + z^1_{q_4} = z_{v_n}.
\end{aligned} \tag{49}
$$

For each $s \in \mathcal{E}^+_{n-3}$ define

$$N_2^+(p(s)) := \{s' \in \mathcal{E}^+_{n-1} : p(p(s')) = s\}$$

and

$$N_2^-(p(s)) := \{s' \in \mathcal{E}^-_{n-1} : p(p(s')) = s\}.$$

Note that $0 \leq |N_2^+(p(s))| \leq 4$ and $0 \leq |N_2^-(p(s))| \leq 4$ for all $s \in \mathcal{E}^+_{n-3}$. Projecting out $z^1_s$, $s \in \mathcal{E}_{n-3}$ from system (49) yields:

$$
\begin{aligned}
&z_s \geq 0, \quad \forall s \in \mathcal{E}_{n-3} \\
&z_s + z_{\ell(s)} = z_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-3},
\end{aligned}
$$

together with

$$
\begin{aligned}
&\sum_{s' \in N_2^-(p(s))} z_{s'} = z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-3} : |N_2^-(p(s))| = 4 \\
&\sum_{s' \in N_2^+(p(s))} z_{s'} = z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-3} : |N_2^+(p(s))| = 4 \\
&\sum_{s' \in N_2^-(p(s))} z_{s'} \leq z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-3} : |N_2^-(s)| < 4 \\
&\sum_{s' \in N_2^+(p(e))} z_{s'} \leq z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_{n-3} : |N_2^+(s)| < 4 \\
&z_s - z^1_s + z_{\ell(s)} - z^1_{\ell(s)} = z_{p(s)} - z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-4] \setminus \{1\} \\
&z^1_s + z^1_{\ell(s)} = z^1_{p(s)}, \quad \forall s \in \mathcal{E}^+_k, k \in [n-4] \setminus \{1\} \\
&z_s - z^1_s \geq 0,\ \forall k \in [n-4] \\
&z^1_s \geq 0,\ \forall k \in [n-4] \\
&z^1_{q_1} + z^1_{q_2} + z^1_{q_3} + z^1_{q_4} = z_{v_n} \\
&z_{q_1} - z^1_{q_1} + z_{q_2} - z^1_{q_2} + z_{q_3} - z^1_{q_3} + z_{q_4} - z^1_{q_4} = 1 - z_{v_n}
\end{aligned}
$$

By a recursive application of the above argument $n-5$ times to project out variables $z^1_s$, $s \in \mathcal{E}_k$, $k \in [n-4] \setminus \{1\}$ from the above system we obtain:

$$
\begin{aligned}
&\sum_{s \in \mathcal{E}^-_{n-1} : s \supset q_i} z_s \leq z_{q_i} - z^1_{q_i}, \quad \forall i \in \{1, \cdots, 4\} \\
&\sum_{s \in \mathcal{E}^+_{n-1} : s \supset q_i} z_s \leq z^1_{q_i}, \quad \forall i \in \{1, \cdots, 4\} \\
&z_{q_i} - z^1_{q_i} \geq 0, \quad \forall i \in \{1, \cdots, 4\} \\
&z^1_{q_i} \geq 0, \quad i \in \{1, \cdots, 4\} \\
&z^1_{q_1} + z^1_{q_2} + z^1_{q_3} + z^1_{q_4} = z_{v_n} \\
&z_{q_1} - z^1_{q_1} + z_{q_2} - z^1_{q_2} + z_{q_3} - z^1_{q_3} + z_{q_4} - z^1_{q_4} = 1 - z_{v_n},
\end{aligned} \tag{50}
$$

together with

$$
\begin{aligned}
&z_s \geq 0, \quad \forall s \in \mathcal{E}_k, k \in [n-3] \setminus \{1\} \\
&z_s + z_{\ell(s)} = z_{p(s)}, \quad \forall e \in \mathcal{E}^+_k, k \in [n-3] \setminus \{1\}.
\end{aligned} \tag{51}
$$

Hence it remains to project out $z^1_{q_i}$, $i \in \{1, \cdots, 4\}$ from system (50). Since $\cup_{i \in \{1, \cdots 4\}} \{s \in \mathcal{E}^-_{n-1} : s \supset q_i\} = \mathcal{E}^-_{n-1}$ and $\cup_{i \in \{1, \cdots 4\}} \{s \in \mathcal{E}^+_{n-1} : s \supset q_i\} = \mathcal{E}^+_{n-1}$, it follows that the projection of system (50) onto the space of $z$ is given by:

$$
\begin{aligned}
&\sum_{s \in \mathcal{E}^+_{n-1}} z_s \leq z_{v_n} \\
&\sum_{s \in \mathcal{E}^-_{n-1}} z_s \leq 1 - z_{v_n} \\
&z_{q_i} \geq 0, \quad \forall i \in \{1, \cdots, 4\} \\
&z_{q_1} + z_{q_2} + z_{q_3} + z_{q_4} = 1,
\end{aligned} \tag{52}
$$

where the last equality is implied by equalities (8). From (33)–(36), (39), (42), (43),(46)-(48), (51), and (52), it follows that the pseudo-Boolean polytope PBP$(H)$ is defined by system (4)–(8) and this completes the proof. □

**Remark 2.** *In Proposition 1, the assumption on the structure of $\mathcal{E}_1$ is not restrictive. Recall that by property (N2) of a nested signed hypergraph we must either have $|\mathcal{E}_1| = 2$ or $|\mathcal{E}_1| = 4$. Moreover, if $|\mathcal{E}_1| = 2$, then we must either have $\mathcal{E}_1 = \{q_1, q_3\}$ or $\mathcal{E}_1 = \{q_2, q_4\}$. Without loss of generality, suppose that $\mathcal{E}_1 = \{q_1, q_3\}$. Then to obtain a description of PBP$(H)$ in the original space, it suffices to project out $z_{q_2}, z_{q_4}$ from system* (4)–(8)*. To do so, notice that $z_{q_2}, z_{q_4}$ appear only in the following constraints of system* (4)–(8)*:*

$$
\begin{aligned}
&z_{q_2} \geq 0, \; z_{q_4} \geq 0 \\
&z_{q_1} + z_{q_2} = z_{v_1}, \; z_{q_3} + z_{q_4} = 1 - z_{v_1}.
\end{aligned}
$$

*Projecting out $z_{q_2}, z_{q_4}$, we obtain:*

$$
z_{q_1} \leq z_{v_1}, \; z_{q_3} \leq 1 - z_{v_1},
$$

*which together with the remaining equalities and inequalities of system* (4)–(8) *gives a description of PBP$(H)$ in the original space.*

It is important to note that in spite of its simple structure, the constraint matrix of the pseudo-Boolean polytope of a nested signed hypergraph is *not* totally unimodular. The following example demonstrates this fact. In this example, for ease of notation, we write a signed edge in a compact form by listing its nodes with their signs as superscripts. For example, the signed edge $s = (e, \eta_s)$, where $e = \{v_1, v_2, v_3\}$ and $\eta_s(v_1) = +1$, $\eta_s(v_2) = -1$, $\eta_s(v_3) = +1$, will be written compactly as $s = \{v_1^+, v_2^-, v_3^+\}$.

**Example 1.** *Consider the nested signed hypergraph* $H = (V, S)$*, with* $V = \{v_1, v_2, v_3, v_4\}$*,* $\mathcal{E}_1 = \{s_1, s_2, s_3, s_4\}$*,* $s_1 = \{v_1^+, v_2^+\}$*,* $s_2 = \{v_1^-, v_2^+\}$*,* $s_3 = \{v_1^+, v_2^-\}$*,* $s_4 = \{v_1^-, v_2^-\}$*,* $\mathcal{E}_2 = \{s_5, s_6, s_7, s_8\}$*,* $s_5 = \{v_1^+, v_2^+, v_3^+\}$*,* $s_6 = \{v_1^+, v_2^+, v_3^-\}$ $s_7 = \{v_1^-, v_2^+, v_3^-\}$*,* $s_8 = \{v_1^-, v_2^+, v_3^+\}$*,* $\mathcal{E}_3 = \{s_9, s_{10}, s_{11}, s_{12}\}$*,* $s_9 = \{v_1^+, v_2^+, v_3^-, v_4^+\}$*,* $s_{10} = \{v_1^+, v_2^+, v_3^-, v_4^-\}$*,* $s_{11} = \{v_1^-, v_2^+, v_3^+, v_4^+\}$*,* $s_{12} = \{v_1^-, v_2^+, v_3^+, v_4^-\}$*. Then by Proposition 1, the following are present in the description of PBP*$(H)$*:*

$$z_{s_5} + z_{s_6} = z_{s_1}$$
$$z_{s_9} + z_{s_{10}} = z_{s_6}$$
$$z_{s_{11}} + z_{s_{12}} = z_{s_8}$$
$$z_{s_5} + z_{s_8} \leq z_{v_3}$$
$$z_{s_9} + z_{s_{11}} \leq z_{v_4}$$

*Consider the submatrix of the above equalities and inequalities corresponding to* $z_{s_5}, z_{s_6}, z_{s_8}, z_{s_9}, z_{s_{11}}$*. It can be checked that the determinant of this submatrix equals 2, implying the constraint matrix of PBP*$(H)$ *is not totally unimodular. In addition, each row and each column of this submatrix has two non-zero entries, implying it is not a balanced matrix either [10].*

### 3.2 The pseudo-Boolean polytope of pointed signed hypergraphs

Thanks to the compact description for the pseudo-Boolean polytope of nested signed hypergraphs given by Proposition 1, we now provide a polynomial-size extended formulation for the pseudo-Boolean polytope of pointed signed hypergraphs. Recall that in a signed hypergraph $H = (V, S)$ pointed at $\bar{v}$ for some $\bar{v} \in V$, we have $S = S_{\bar{v}} \cup P_{\bar{v}}$; the set $S_{\bar{v}}$ consists of all signed edges containing $\bar{v}$, and the set $P_{\bar{v}}$ consists of all signed edges not containing $\bar{v}$ and has the form $P_{\bar{v}} = \{s - \bar{v} : s \in S_{\bar{v}}, |s| \geq 3\}$.

**Remark 3.** *Consider a pointed signed hypergraph* $H = (V, S)$*. There are two special cases for which a polynomial-size description of PBP*$(H)$ *follows from previously known results:*

*(i) Suppose that for each* $v \in V$ *we have* $\eta_s(v) = \eta_{s'}(v)$ *for all signed edges* $s, s'$ *in* $H$ *containing* $v$*. Then (possibly after a one-to-one linear transformation) the explicit description of PBP*$(H)$ *in the original space consisting of at most* $5|V| + 2$ *inequalities is given in Theorem 5 of [19].*

*(ii) Suppose that all signed edges in* $S_{\bar{v}}$ *are parallel. It then follows that for any* $s, s' \in S$ *we either have* $s \subseteq s'$ *or* $s' \subseteq s$*. Then by Part 2 of Remark 1 and the definition of* $P_{\bar{v}}, S_{\bar{v}}$*, the pseudo-Boolean polytope PBP*$(H)$ *has a polynomial-size extended formulation with at most* $|S|(|V| - 1) + |V|$ *variables and at most* $2(|S|(|V| - 1) + |V|)$ *inequalities.*

The following theorem gives a polynomial-size extended formulation for the pseudo-Boolean polytope of general pointed signed hypergraphs.

**Theorem 2.** *Let* $H = (V, S)$ *be a pointed signed hypergraph. Then the pseudo-Boolean polytope PBP*$(H)$ *has a polynomial-size extended formulation with at most* $2|V|(|S| + 1)$ *variables and at most* $4(|S|(|V| - 2) + |V|)$ *inequalities. Moreover, all coefficients and right-hand side constants in the system defining PBP*$(H)$ *are* $0, \pm 1$*.*

*Proof.* Suppose that $H = (V, S)$ is pointed at $\bar{v}$ for some $\bar{v} \in V$, implying that $S = S_{\bar{v}} \cup P_{\bar{v}}$. Define the signed hypergraph $L = (V \setminus \{\bar{v}\}, P_{\bar{v}})$. By definition of $P_{\bar{v}}$, for any two edges $s, s' \in P_{\bar{v}}$ we either have $s \subseteq s'$ or $s' \subseteq s$. Hence after the addition of at most $2|P_{\bar{v}}|(|V|-3) + |P_{\bar{v}}|$ edges to $L$ we can construct a nested signed hypergraph denoted by $L'$. Denote by $P'_{\bar{v}}$ the set of signed edges of $L'$. Note that $|P'_{\bar{v}}| = 2|P_{\bar{v}}|(|V|-2) \leq |S|(|V|-2)$. Define $H' := (V, S_{\bar{v}} \cup P'_{\bar{v}})$. Since a description of $\text{PBP}(H')$ serves as an extended formulation for $\text{PBP}(H)$, to complete the proof, it suffices to show that $\text{PBP}(H')$ has a polynomial-size extended formulation. Denote by $H^0$ (resp. $H^1$) the signed hypergraph corresponding to the face of $\text{PBP}(H')$ with $z_{\bar{v}} = 0$ (resp. $z_{\bar{v}} = 1$). We then have:

$$\text{PBP}(H') = \text{conv}\Big(\text{PBP}(H^0) \cup \text{PBP}(H^1)\Big).$$

Denote by $\bar{z}$ the vector consisting of $z_v$, $v \in V \setminus \{\bar{v}\}$ and $z_s$, $s \in P'_{\bar{v}}$. It then follows that:

$$\begin{aligned}
\text{PBP}(H^0) = \Big\{ & z \in \mathbb{R}^{V \cup S_{\bar{v}} \cup P'_{\bar{v}}} : z_{\bar{v}} = 0,\ z_s = 0, \forall s \in S_{\bar{v}} \text{ with } \eta_s(\bar{v}) = +1, z_s = z_{p(s)}, \forall s \in S_{\bar{v}} \text{ with} \\
& \eta_s(\bar{v}) = -1,\ \bar{z} \in \text{PBP}(L'_{\bar{v}}) \Big\} \\
\text{PBP}(H^1) = \Big\{ & z \in \mathbb{R}^{V \cup S_{\bar{v}} \cup P'_{\bar{v}}} : z_{\bar{v}} = 1,\ z_s = z_{p(s)}, \forall s \in S_{\bar{v}} \text{ with } \eta_s(\bar{v}) = +1, z_s = 0, \forall s \in S_{\bar{v}} \text{ with} \\
& \eta_s(\bar{v}) = -1,\ \bar{z} \in \text{PBP}(L'_{\bar{v}}) \Big\}.
\end{aligned}$$

By Propositon 1, the polytope $\text{PBP}(L'_{\bar{v}})$ is given by system (4)-(8) and this description has at most $2(|S|(|V|-2) + |V|)$ inequalities. For notational simplicity, let us write $\text{PBP}(L'_{\bar{v}})$ compactly as

$$\text{PBP}(L'_{\bar{v}}) = \{\bar{z} : A\bar{z} \leq b, C\bar{z} = d\}.$$

Then, using Balas' formulation for union of polytopes [1], we obtain a polynomial-size extended formulation for $\text{PBP}(H')$:

$$\begin{aligned}
\text{PBP}(H') = \Big\{ & z \in \mathbb{R}^{V \cup S_{\bar{v}} \cup P'_{\bar{v}}} :\ \exists (z^0, z^1, z, \lambda) \text{ s.t. } z = z^0 + z^1,\ \lambda_0 + \lambda_1 = 1,\ z^0_{\bar{v}} = 0, \\
& z^0_s = 0,\ \forall s \in S_{\bar{v}} \text{ with } \eta_s(\bar{v}) = +1,\ z^0_s = z^0_{p(s)},\ \forall s \in S_{\bar{v}} \text{ with } \eta_s(\bar{v}) = -1, \\
& z^1_{\bar{v}} = \lambda_1, z^1_s = z^1_{p(s)}, \forall s \in S_{\bar{v}} \text{ with } \eta_s(\bar{v}) = +1,\ z^1_s = 0,\ \forall s \in S_{\bar{v}} \text{ with } \eta_s(\bar{v}) = -1, \\
& Az^0 \leq b\lambda_0,\ Cz^0 = d\lambda_0, Az^1 \leq b\lambda_1,\ Cz^1 = d\lambda_1,\ \lambda_0, \lambda_1 \geq 0 \Big\}.
\end{aligned}$$

The size of the above extended formulation can further reduced by projecting out variables $\lambda_0, \lambda_1, z^0$ using the equalities $z_{\bar{v}} = \lambda_1$, $\lambda_0 + \lambda_1 = 1$, and $z = z^0 + z^1$. Hence, we obtain an extended formulation for $\text{PBP}(H')$ with at most

$$\begin{aligned}
2(|V| + |S_{\bar{v}}| + |P'_{\bar{v}}|) = 2(|V| + |S_{\bar{v}}| + 2|P_{\bar{v}}|(|V|-2)) = 2(|V| + |S| + |P_{\bar{v}}|(2|V|-5)) \\
\leq 2(|V| + |S|(|V| - \frac{3}{2})) \leq 2|V|(|S|+1), \qquad (53)
\end{aligned}$$

variables and at most

$$2(2(|S|(|V|-2) + |V|)) \leq 4(|S|(|V|-2) + |V|), \qquad (54)$$

inequalities. The second equality in (53) follows from $|S| = |S_{\bar{v}}| + |P_{\bar{v}}|$ and the inequality in (53) follows from $|P_{\bar{v}}| \leq \frac{|S|}{2}$.

By Proposition 1, all coefficients and right-hand side constants in the system defining $\text{PBP}(L'_{\bar{v}})$ are $0, \pm 1$. This together with the fact that $\lambda_0 = 1 - z_{\bar{v}}$ and $\lambda_1 = z_{\bar{v}}$ implies that the same statement holds for the extended formulation of $\text{PBP}(H')$ and as a result for the extended formulation of $\text{PBP}(H)$ as well. □

# 4 Inflation of signed edges

Our decomposition result stated in Theorem 1 relies on the key assumption that the underlying hypergraph of $H$ has at least one $\beta$-leaf. We argue that in many cases of interest, this restrictive assumption can be removed. To this end, in the following, we introduce an operation on signed hypergraphs, which we refer to as the *inflation of signed edges*. Starting from a signed hypergraph $H$ whose underlying hypergraph does not contain any $\beta$-leaves, by inflating certain edges, we obtain a new signed hypergraph $H'$ whose underlying hypergraph has a sequence of $\beta$-leaves. By relating the extended formulations of PBP($H$) and PBP($H'$), we are able to obtain polynomial-size extended formulations for the pseudo-Boolean polytope of various classes of signed hypergraphs whose underlying hypergraphs *contain* $\beta$-cycles.

Let $H = (V, S)$ be a signed hypergraph, let $s \in S$, and let $e \subseteq V$ such that $s \subset e$. Denote by $I(s, e)$ the set of all possible signed edges $s'$ parallel to $e$ such that $\eta_s(v) = \eta_{s'}(v)$ for every $v \in s$. We say that $H' = (V, S')$ is obtained from $H$ by *inflating* $s$ to $e$, if $S' = S \cup I(s, e) \setminus \{s\}$. We also say that $H'$ is obtained from $H$ via an *inflation operation*. The next theorem indicates that if an extended formulation for PBP($H'$) is available, one can obtain an extended formulation for PBP($H$) as well.

**Theorem 3.** *Let $H = (V, S)$ be a signed hypergraph, let $s \in S$, and let $e \subseteq V$ such that $s \subset e$. Let $H' = (V, S')$ be obtained from $H$ by inflating $s$ to $e$. Then an extended formulation of PBP($H$) can be obtained by juxtaposing an extended formulation of PBP($H'$) and the equality constraint*

$$z_s = \sum_{s' \in I(s,e)} z_{s'}. \tag{55}$$

*Moreover, if PBP($H'$) has a polynomial-size extended formulation and $|e| - |s| = O(\log \mathrm{poly}(|V|, |S|))$, then PBP($H$) has a polynomial-size extended formulation as well.*

*Proof.* Given a signed hypergraph $H$ and a set $C$, we denote by $\mathrm{proj}_H(C)$ the orthogonal projection of the set $C$ onto the subspace of variables corresponding to nodes and signed edges of $H$. Let $A(z, y) \leq b$ be an extended formulation of PBP($H'$). Let $H'' = (V, S'')$, where $S'' := S \cup I(s, e) = S' \cup \{s\}$. It suffices to prove that $A(z, y) \leq b$ together with (55) is an extended formulation of PBP($H''$). That is, we need to show that

$$Q := \mathrm{proj}_{H''}\{(z, y) : A(z, y) \leq b,\ (55)\} = \mathrm{PBP}(H'').$$

Since $S \subseteq S''$, it follows that an extended formulation of PBP($H''$) is also an extended formulation of PBP($H$). For every binary $z$, we have $z \in \mathrm{PBP}(H'')$ if and only if

$$z_s = \prod_{v \in s} \eta_s(z_v) = \prod_{v \in s} \eta_s(z_v) \prod_{v \in e \setminus s} (z_v + (1 - z_v)) = \sum_{s' \in I(s,e)} z_{s'}. \tag{56}$$

Thus, it suffices to show that the vertices of $Q$ are binary. We have

$$\begin{aligned} Q &= \mathrm{proj}_{H''}\{(z, y) : A(z, y) \leq b\} \cap \{z : (55)\} \\ &= \mathrm{proj}_{H'}\{(z, y) : A(z, y) \leq b\} \cap \{z : (55)\} \\ &= \mathrm{PBP}(H') \cap \{z : (55)\}, \end{aligned} \tag{57}$$

where the first equality holds since all variables that appear with nonzero coefficients in (55) correspond to signed edges in $H''$, the second equality holds because $\{s\} = S'' \setminus S'$ and so the variable $z_s$ does not appear in the system $A(z, y) \leq b$, and the third equality holds since $A(z, y) \leq b$ is an the extended formulation of PBP($H'$).

Let $\bar{z}$ be a vertex of $Q$. The variable $\bar{z}_s$ is only present in (55), thus we have $\bar{z}_s = \sum_{s' \in I(s,e)} \bar{z}_{s'}$. The other constraints of the system defining $Q$ that are active at $\bar{z}$ are all from the projection of the system $A(z, y) \leq b$, thus they determine a vertex of $\text{PBP}(H')$, which is binary. This implies that all components of $\bar{z}$, except for $\bar{z}_s$, are binary. It then follows from (56) that $\bar{z}_s$ is binary as well.

Finally, since $|I(s, e)| \leq 2^{|e|-|s|}$, we conclude that, $\text{PBP}(H)$ has a polynomial-size extended formulation, if $\text{PBP}(H')$ has a polynomial-size extended formulation and $|e| - |s| = O(\log \text{poly}(|V|, |S|))$. □

The inflation operation is of independent interest as it enables us to obtain polynomial-size extended formulations for the pseudo-Boolean polytope of certain signed hypergraphs. The following proposition is an illustration of this fact:

**Proposition 2.** *Consider a signed hypergraph $H = (V, S)$. Suppose that each $s \in S$ contains at least $|V| - k$ nodes. Then PBP$(H)$ has an extended formulation with $O(2^k|V||S|)$ variables and inequalities. In particular, if $k = O(\log \text{poly}(|V|, |S|))$, then PBP$(H)$ has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP$(H)$ are* $0, \pm 1$.

*Proof.* Consider the signed hypergraph $H' = (V, S')$ obtained from $H$ by inflating every $s \in S$ with $|s| < |V|$ to $V$. We then have $|S'| \leq 2^k|S|$. By Part 2 of Remark 1, the polytope $\text{PBP}(H')$ has a polynomial-size extended formulation with at most $2^{k+1}|S|(|V| - 1) + |V|$ variables and at most $2^{k+2}|S|(|V| - 1) + 2|V|$ inequalities. Therefore, from Lemma 3 we deduce that an extended formulation for $\text{PBP}(H)$ is obtained by adding at most $|S|$ equality constraints (containing $|S|$ additional variables) to the extended formulation of $\text{PBP}(H')$. Since all coefficients and right-hand side constants in the system defining $\text{PBP}(H')$ as well as in equalities (55) are $0, \pm 1$, the statement holds for the system defining $\text{PBP}(H)$. □

# 5 The recursive inflate-and-decompose framework

In order to construct extended formulations for the pseudo-Boolean polytope, by combining our decomposition scheme of Theorem 1, our convex hull characterization of Theorem 2, and the inflation operation of Theorem 3, we introduce a new framework, which we refer to as the *recursive inflate-and-decompose* framework. We show that this framework enables us to obtain polynomial-size extended formulations for the pseudo-Boolean polytope of large families of signed hypergraphs.

As we detail in this section, our new framework unifies and extends all existing results on polynomial-size extended formulations for the convex hull of the feasible region of Problem (L-BMO) [4, 16, 18, 19, 31, 33]. In the following, we present our recursive inflate-and-decompose framework.

**The Recursive inflate-and-decompose (RID) framework**

**Input.** A signed hypergraph $H = (V, S)$.
**Output.** An extended formulation for $\text{PBP}(H)$.
**Step 0.** Set $H^{(0)} := H$, $i := 0$.
**Step 1.** If we can obtain $\bar{H}^{(i)}$ from $H^{(i)}$ via a number of inflation operations, such that a suitable extended formulation for $\text{PBP}(\bar{H}^{(i)})$ is available, then by Theorem 3 we are done. Otherwise, proceed to Step 2.
**Step 2.** Choose a node $\bar{v}$ of $H^{(i)}$. If $\bar{v}$ is a $\beta$-leaf of the underlying hypergraph of $H^{(i)}$, then set $\bar{H}^{(i)} := H^{(i)}$ and proceed to Step 3. Otherwise, construct $\bar{H}^{(i)}$ from $H^{(i)}$ via inflation operations, such that $v$ is a $\beta$-leaf of the underlying hypergraph of $\bar{H}^{(i)}$. By Theorem 3, it suffices to find an extended formulation for $\text{PBP}(\bar{H}^{(i)})$.
**Step 3.** Using Theorem 1, decompose $\text{PBP}(\bar{H}^{(i)})$ into $\text{PBP}(\bar{H}^{(i)}_1)$ and $\text{PBP}(\bar{H}^{(i)}_2)$, where $\bar{H}^{(i)}_1$ denotes the signed hypergraph containing node $\bar{v}$. Since an extended formulation for $\text{PBP}(\bar{H}^{(i)}_1)$ is given by

Theorem 2, it suffices to find an extended formulation for $\text{PBP}(\bar{H}_2^{(i)})$. Set $H^{(i+1)} := \bar{H}_2^{(i)}$, increment $i$ by one, and go to Step 1.

Clearly, Steps 1 and 2 of the RID framework can be performed in many different ways. That is, we have not specified how node $\bar{v}$ in Step 2 should be selected or how the inflation operations of Step 1 and Step 2 should be performed. A simple way to obtain a $\beta$-leaf in Step 2 is to inflate each signed edge containing $\bar{v}$ to the union of all signed edges containing $\bar{v}$. While the RID framework is fairly general, we are naturally interested in specifying conditions under which this framework results in a polynomial-size extended formulation for the pseudo-Boolean polytope. To this end, first note that at each iteration, one node of the signed hypergraph $H^{(i)}$ is removed. Hence, the number of iterations of the RID framework is upper bounded by the number of nodes of $H$. It then follows that RID provides a polynomial-size extended formulation for $\text{PBP}(H)$, if following conditions are satisfied:

(I) In Step 1, the algorithm should terminate, only if a polynomial-size extended formulation for $\text{PBP}(\bar{H}^{(i)})$ is available.

(II) The total number of new edges introduced as a result of inflation operations in Steps 1 and 2 should be upper bounded by a polynomial in $|V|, |S|$.

In the remainder of this section, we consider various types of signed hypergraphs for which one can customize the RID framework so that conditions (I)–(II) above are satisfied and hence polynomial-size extended formulations for the corresponding pseudo-Boolean polytope can be constructed.

Our results are stated in terms of easily verifiable conditions on the structure of the underlying hypergraphs, namely, $\beta$-acyclic hypergraphs, $\alpha$-acyclic hypergraphs with log-poly rank, and certain classes of hypergraphs with log-poly "gaps."

### 5.1 $\beta$-acyclic hypergraphs

Consider a signed hypergraph $H = (V, S)$. In this section, we show that if the underlying hypergraph of $H$ has a sequence of $\beta$-leaves, then $\text{PBP}(H)$ can be described in terms of pseudo-Boolean polytopes of simpler signed hypergraphs. In particular, if the underlying hypergraph of $H$ is $\beta$-acyclic, then $\text{PBP}(H)$ has an extended formulation of size polynomial in $|V|, |S|$. These theorems serve as significant generalizations of the main results in [19].

**Theorem 4.** *Let $H = (V, S)$ be a signed hypergraph of rank $r$, and let $v_1, \cdots, v_t$ for some $t \geq 1$ be a sequence of $\beta$-leaves of the underlying hypergraph of $H$. Then an extended formulation for PBP($H$) is given by a description of PBP($H - v_1 - \cdots - v_t$) together with a system of at most $4rt(2|S|+1)$ inequalities consisting of at most $2rt(2|S|+1)$ variables. Moreover, all coefficients and right-hand side constants in the latter are $0, \pm 1$.*

*Proof.* Denote by $S_{v_1}$ the set of all signed edges of $H$ containing $v_1$. Since $v_1$ is a $\beta$-leaf of the underlying hypergraph of $H$, the set $S_{v_1}$ is totally ordered. Define the signed hypergraph $H'_1 := (V, S'_1)$ with $S'_1 = S \cup P_{v_1}$, where $P_{v_1} := \{s - v_1 : s \in S_{v_1}, |s| \geq 3\}$. Clearly, an extended formulation for $\text{PBP}(H'_1)$ serves as an extended formulation for $\text{PBP}(H'_1)$ as well. Now define the pointed signed hypergraph $H_{v_1} := (V_1, S_{v_1} \cup P_{v_1})$, where $V_1$ denotes the underlying edge of a signed edge of maximum cardinality in $S_{v_1}$. We then have $H'_1 = H_{v_1} \cup (H - v_1)$, where we used the fact that $H'_1 - v_1 = H - v_1$. Hence by Theorem 1, the pseudo-Boolean polytope $\text{PBP}(H'_1)$ is decomposable into pseudo-Boolean polytopes $\text{PBP}(H_{v_1})$ and $\text{PBP}(H - v_1)$.

Next consider the signed hypergraph $H - v_1$. By definition $v_2$ is a $\beta$-leaf of the underlying hypergraph of $H - v_1$. Denote by $S_{v_2}$ the set of signed edges $H - v_1$ containing $v_2$. Define $S'_2 := (S \setminus S_{v_1}) \cup P_{v_2}$, where $P_{v_2} := \{s - v_2 : s \in S_{v_2}, |s| \geq 3\}$, and define the signed hypergraph $H'_2 := (V \setminus \{v_1\}, S'_2)$. Again, an extended formulation for $\text{PBP}(H'_2)$ serves as an extended formulation for $\text{PBP}(H - v_1)$ as well.

Define the pointed signed hypergraph $H_{v_2} := (V_2, S_{v_2} \cup P_{v_2})$, where $V_2$ denotes the underlying edge of a signed edge of maximum cardinality in $S_{v_2}$. Then by Theorem 1, PBP$(H'_2)$ is decomposable into PBP$(H_{v_2})$ and PBP$(H - v_1 - v_2)$.

By a recursive application of the above argument after $t$ times, we conclude that an extended formulation for PBP$(H)$ is given by a description of PBP$(H - v_1 - \cdots - v_t)$ together with a system of inequalities defining PBP$(H_{v_i})$ for all $i \in \{1, \cdots, t\}$, where $H_{v_i}$ is a signed hypergraph pointed at $v_i$ defined as $H_{v_i} := (V_i, S_{v_i} \cup P_{v_i})$, where $S_{v_i}$ denotes the set of signed edge of $H - v_1 - \cdots - v_{i-1}$ (we define $H - v_1 - v_0 = H$) containing $v_i$, $P_{v_i} := \{s - v_i : s \in S_{v_i}, |s| \geq 3\}$, and $V_i$ denotes the underlying edge of a signed edge of maximal cardinality in $S_{v_i}$.

By Theorem 2 the polytopes PBP$(H_{v_i})$, $i \in \{1, \cdots, t\}$ have polynomial-size extended formulations each of which consists of at most $2|V_i|(|S_{v_i}| + |P_{v_i}| + 1) \leq 2r(2|S| + 1)$ variables and at most $4(|S_{v_i}| + |P_{v_i}|)(|V_i| - 2) + 4|V_i| \leq 8(r-2)|S| + 4r \leq 4r(2|S| + 1)$ inequalities, where the inequalities follow from $|V_i| \leq r$ and $|P_{v_i}| \leq |S_{v_i}| \leq |S|$. □

The proof of Theorem 4 follows from the RID framework. To see this, first note that the node $\bar{v}$ chosen in Step 2 of the $i$th iteration of the RID framework is the $\beta$-leaf $v_i$ for all $i \in [t]$. Moreover, at iteration $t+1$, the algorithm terminates at Step 1 with a description of PBP$(H - v_1 - \cdots - v_t)$. In this case, no inflation operation at any step of RID is needed.

**Theorem 5.** *Let $H = (V, S)$ be a signed hypergraph of rank $r$ whose underlying hypergraph is $\beta$-acyclic. Then the pseudo-Boolean polytope PBP$(H)$ has a polynomial-size extended formulation with at most $2r(2|S| + 1)|V|$ variables, and at most $4r(2|S| + 1)|V|$ inequalities. Moreover, all coefficients and right-hand side constants in the system defining PBP$(H)$ are $0, \pm 1$.*

*Proof.* Since the underlying hypergraph of $H$ is $\beta$-acyclic, it has a sequence of $\beta$-leaves of length $|V|$. The proof then follows directly from Theorem 4. □

**Remark 4.** *Let $G = (V, E)$ be a $\beta$-acyclic hypergraph of rank $r$. Theorem 1 in [19] gives an extended formulation for the multilinear polytope of $G$ with at most $(r-1)|V|$ variables and at most $(3r - 4)|V| + 4|E|$ inequalities. Theorem 5 is a significant generalization of this result as it only requires the $\beta$-acyclicity of the underlying hypergraph of $H$. Indeed the multilinear hypergraph* mh$(H)$ *may contain many $\beta$-cycles. Nonetheless, this generalization has a cost: while the size of the extended formulation for MP$(G)$ is quadratic in $|V|, |E|$, the size of the extended formulation for PBP$(H)$ is cubic in $|V|, |S|$.*

## 5.2 $\alpha$-acyclic hypergraphs with log-poly ranks

As we mentioned before, $\alpha$-acyclic hypergraphs are the most general type of acyclic hypergraphs. In [14], the authors prove that Problem BMO is strongly NP-hard over $\alpha$-acyclic hypergraphs. This result implies that, unless P = NP, one cannot construct, in polynomial time, a polynomial-size extended formulation for the multilinear polytope of $\alpha$-acyclic hypergraphs. In [4, 31, 33], the authors give extended formulations for the convex hull of the feasible set of (possibly constrained) binary multilinear optimization problems. For the unconstrained case, their result can be equivalently stated as follows (see Theorem 5 in [18]): If $G = (V, E)$ is an $\alpha$-acylic hypergraph of rank $r$ such that $r = O(\log \text{poly}(|V|, |E|))$, then MP$(G)$ has a polynomial-size extended formulation. In the following we show that this result follows from Theorems 1–3; that is, it is a special case of the RID framework.

**Theorem 6.** *Let $H = (V, S)$ be a signed hypergraph of rank $r$ whose underlying hypergraph is $\alpha$-acyclic. Then PBP$(H)$ has an extended formulation with at most $(\frac{2}{3}3^r + 2(r-1)(2^r + 1))|V|$ variables and inequalities. In particular, if $k = O(\log \text{poly}(|V|, |S|))$, then PBP$(H)$ has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP$(H)$ are $0, \pm 1$.*

*Proof.* By definition of $\alpha$-acyclic, the underlying hypergraph of $H$ has a sequence of $\alpha$-leaves of length $n := |V|$. Let us denote this sequence by $v_1, v_2, \ldots, v_n$. Since $v_1$ is an $\alpha$-leaf of the underlying hypergraph of $H$, there exists a maximal (for inclusion) signed edge $\bar{s}$ of $H$ containing $v_1$. Let $H'$ be obtained from $H$ by first inflating each signed edge $s'$ containing $v_1$ such that $s' \subset \bar{s}$ to $\bar{s}$, and then by adding all signed edges $s - v_1$ for all signed edges containing $v_1$. Denote by $N_{v_1}$ the number of signed edges $s \in S \setminus \{\bar{s}\}$ containing $v_1$ such that $s \subset \bar{s}$. By Theorem 3, it suffices to find an extended formulation of PBP($H'$). An extended formulation for PBP($H$) is then obtained by juxtaposing the extended formulation for PBP($H'$) and $N_{v_1}$ equalities containing at most $N_{v_1}$ additional variables. Since $H$ is a rank-$r$ signed hypergraph we have:

$$N_{v_1} \leq N_{\max} = 2\Big(\sum_{k=1}^{r-2} \binom{r-1}{k} 2^{r-1-k}\Big) = 2(3^{r-1} - 2^{r-1} - 1).$$

Moreover, by construction, $v_1$ is a $\beta$-leaf of of the underlying hypergraph of $H'$, implying we can apply the decomposition result of Theorem 1. Namely, let $V_1$ be the underlying edge of $\bar{s}$, let $S_{v_1}$ be the set of signed edges of $H'$ parallel to $\bar{s}$, and let $P_{v_1} := \{s - v_1 : |s - v_1| \geq 2\}$. Define the signed hypergraphs $H_{v_1} := (V_1, S_{v_1} \cup P_{v_1})$ and $H'_1 = H' - v_1$. Then, by Theorem 1, PBP($H'$) is decomposable into PBP($H_{v_1}$) and PBP($H'_1$). An extended formulation for the pseudo-Boolean polytope of $H_{v_1}$ can then be obtained from Part 2 of Remark 1.

Next, we show that we can apply the above construction recursively to $H' - v_1$ with sequence of $\alpha$-leaves $v_2, \ldots, v_n$. Note that the underlying hypergraph of $H' - v_1$ is obtained from the underlying hypergraph of $H - v_1$ by possibly removing some edges contained in $\bar{s} - v_1$. First, note that the rank of $H' - v_1$ is at most $r$, as removing nodes and edges from a hypergraph cannot increase its rank. Next, we show that $v_2, \ldots, v_n$ is a sequence of $\alpha$-leaves of the underlying hypergraph of $H' - v_1$. For $k \in \{1, \ldots, n\}$, we show that $v_k$ is an $\alpha$-leaf of the underlying hypergraph of $H'_k := H' - v_1 - v_2 - \cdots - v_{k-1}$. Since $v_k$ is an $\alpha$-leaf of the underlying hypergraph of $H_k := H - v_1 - v_2 - \cdots - v_{k-1}$, let $\hat{s}$ be a signed edge of $H_k$ containing $v_k$ that is maximal for inclusion. Let $\tilde{s}$ be a signed edge of $H$ such that $\hat{s} = \tilde{s} - v_1 - v_2 - \cdots - v_{k-1}$. If $\tilde{s}$ is also a signed edge of $H'$ we are done. Otherwise, by definition of $H'$, we have $\tilde{s} \subseteq \bar{s}$. But then $\bar{s}' := \bar{s} - v_1 - v_2 - \cdots - v_{k-1}$ is a signed edge of $H'_k$ and we have $\hat{s} \subseteq \bar{s}'$. Therefore, $\bar{s}'$ is a signed edge of $H'_k$ containing $v_k$ that is maximal for inclusion. Hence, we can apply the above construction recursively. The proof follows by induction on $n$.

Define the signed hypergraph $H_{v_k} = (V_k, S_{v_k} \cup P_{v_k})$, $k \in \{1, \cdots, n\}$, where $S_{v_k}$ is a set of parallel signed edges containing $v_k$, and $P_{v_k} := \{s - v_k : |s - v_k| \geq 2\}$. Moreover, $V_k$ is the underlying edge of a signed edge in $S_{v_k}$. By the above argument an extended formulation for PBP($H$) is obtained by juxtaposing extended formulations of PBP($H_{v_k}$) for all $k \in \{1, \cdots, n\}$ and at most $N_{\max}|V|$ equalities containing at most $N_{\max}|V|$ additional variables.

Finally, consider the pointed signed hypergraph $H_{v_k}$ for some $k \in \{1, \cdots, n\}$. Since the rank of $H$ is $r$, we have $|S_{v_k}| \leq 2^r$. Moreover, since by construction all signed edges in $S_{v_k}$ are parallel, by Part (ii) of Remark 3 we conclude that PBP($H_{v_k}$) has an extended formulation with at most $(r-1)2^r + r$ variables and at most $2((r-1)2^r + r)$ inequalities. Therefore, the pseudo-Boolean polytope PBP($H$) has an extended formulation with at most

$$((r-1)2^r + r)|V| + 2(3^{r-1} - 2^{r-1} - 1)|V| = (\frac{2}{3}3^r + (r-2)(2^r + 1))|V|,$$

variables, and at most

$$(2(r-1)2^r + 2r)|V| + 2(3^{r-1} - 2^{r-1} - 1)|V| \leq (\frac{2}{3}3^r + 2(r-1)(2^r + 1))|V|$$

inequalities. □

The proof of Theorem 6 follows from the RID framework. To see this, first note that the node $\bar{v}$ chosen in Step 2 of the $i$th iteration of the RID framework is the $\alpha$-leaf $v_i$ for all $i \in [n]$. Since by definition of $\alpha$-leaves each $v_i$ is contained in a maximal edge $\bar{s}$, the inflation operation of Step 2 is defined as follows: inflate all edges containing $v_i$ to $\bar{s}$. Since $H$ is a rank-$r$ hypergraph, from the proof Theorem 6 it follows that the total number of edges added due to inflation operations are upper bounded by $2^r|V|$, which is a polynomial in $|V|, |S|$, if we have $r = O(\log \text{poly}(|V|, |S|))$.

### 5.3 Hypergraphs with log-poly gaps

Consider a hypergraph $G = (V, E)$; we define the *gap* of $G$ as

$$\text{gap}(G) := \max \Big\{ | \cup_{f \in E} f| - |e| : e \in E \Big\}.$$

For a subset $F \subseteq E$, the gap of $F$, denoted by $\text{gap}(F)$, is defined as the gap of the hypergraph $(V, F)$. For a signed hypergraph $H = (V, S)$, the gap of $H$, denoted by $\text{gap}(H)$, is defined as the gap of the underlying hypergraph of $H$. Furthermore, for $S' \subseteq S$, the gap of $S'$, denoted by $\text{gap}(S')$ is defined as the gap of the signed hypergraph $(V, S')$. It then follows that the gap of a rank-$r$ hypergraph is upper bounded by $r - 2$. However, the gap of a hypergraph can be significantly smaller than its rank, in general. Consider the signed hypergraph $H$ defined in the statement of Proposition 2. It can be checked that while the rank of $H$ is at least $|V| - k$, the gap of $H$ is upper bounded by $k$.

In this section, we provide polynomial-size extended formulations for the pseudo-Boolean polytope of certain classes of signed hypergraphs with log-poly gaps, where again by log-poly gap we imply that the gap is upper bounded by $\log(\text{poly}(|V|, |S|))$. These results follow from Step 1 of the RID framework: given a signed hypergraph $H = (V, S)$, via a number of inflation operations, we obtain the hypergraph $H' = (V, S')$ such that

(i) the underlying hypergraph of $H'$ is $\beta$-acyclic, hence, by Theorem 5, the polytope $\text{PBP}(H')$ has a polynomial-size extended formulation in $|V|, |S'|$,

(ii) $|S'| \leq |S| + \text{poly}(|V|, |S|)$, where this inequality is satisfied because of the log-poly gaps.

Let $H = (V, S)$ be a signed hypergraph. We say that $s \in S$ is a *maximal signed edge*, if there is no $s' \in S$ with $s' \supset s$. Denote by $\bar{S}$ the set of maximal signed edges of $H$. The next proposition implies that for a signed hypergraph $H$ with log-poly gaps, if the underlying hypergraph of $(V, \bar{S})$ is $\beta$-acyclic, then $\text{PBP}(H)$ has a polynomial-size extended formulation.

**Proposition 3.** *Consider a signed hypergraph $H = (V, S)$ of rank $r$. For each $s \in S$, among all maximal signed edges of $H$ containing $s$, denote by $f_s$ one with minimum cardinality. Let $k$ be such that*

$$\text{gap}(\{s, f_s\}) \leq k, \quad \forall s \in S. \tag{58}$$

*Denote by $\bar{S}$ the set of maximal signed edges of $H$. If the underlying hypergraph of $(V, \bar{S})$ is $\beta$-acyclic, then PBP$(H)$ has an extended formulation with $O(r 2^k |V||S|)$ variables and inequalities. In particular, if $k = O(\log \text{poly}(|V|, |S|))$, then PBP$(H)$ has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP$(H)$ are $0, \pm 1$.*

*Proof.* Denote by $H' = (V, S')$ the signed hypergraph obtained from $H$ by inflating every non-maximal signed edge $s \in S$ to $f_s$, where $f_s$ is defined in the statement. From (58) it follows that $|S'| \leq 2^k(|S| - |\bar{S}|) + |\bar{S}| \leq 2^k |S|$. Notice that the underlying hypergraph of $H'$ coincides with the underlying hypergraph of $(V, \bar{S})$, which by assumption is $\beta$-acyclic. Hence by Theorem 5, $\text{PBP}(H')$ has an extended formulation with at most $2r|V|(2^{k+1}|S|+1)$ variables, and at most $4r|V|(2^{k+1}|S|+1)$

inequalities. Therefore, from Lemma 3 we deduce that an extended formulation for PBP($H$) is obtained by adding at most $|S| - |\bar{S}|$ equalities consisting of at most $|S| - |\bar{S}|$ additional variables to PBP($H'$), and this completes the proof. □

Let $G = (V, E)$ be a hypergraph and let $C = v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_1$ for some $q \geq 3$ be a $\beta$-cycle of $G$. Then the *support hypergraph* of $C$ is the hypergraph $G[C] = (V[C], E[C])$, where $V[C] = \cup_{i=1}^{q} e_i$ and $E[C] = \{e_1, \ldots, e_q\}$. The next proposition essentially indicates that if edge sets of the support hypergraphs of $\beta$-cycles of the underlying hypergraph of $H$ have log-poly gaps, then one can *remove* these $\beta$-cycles by inflating the edges of the cycle, and obtain a polynomial-size extended formulation for PBP($H$).

**Proposition 4.** *Consider a signed hypergraph $H = (V, S)$ of rank $r$, and denote by $G = (V, E)$ its underlying hypergraph. Let $\mathcal{C}$ denote the set of all $\beta$-cycles in $G$, and for each $C \in \mathcal{C}$, let $G[C] = (V[C], E[C])$ be the support hypergraph of $C$. Let $\tilde{G}$ be the hypergraph $(\cup_{C\in\mathcal{C}} V[C], \cup_{C\in\mathcal{C}} E[C])$, let $(V_1, E_1), (V_2, E_2), \ldots, (V_\omega, E_\omega)$ be the connected components of $\tilde{G}$, and let $k$ be such that*

$$\text{gap}(E_i) \leq k, \quad \forall i \in [\omega]. \tag{59}$$

*Then, PBP($H$) has an extended formulation with $O(r2^k|V||S|)$ variables and inequalities. In particular, if $k = O(\log \text{poly}(|V|, |S|))$, then PBP($H$) has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP($H$) are $0, \pm 1$.*

*Proof.* Each $\beta$-cycle in $G$ is contained in precisely one of the connected components of $\tilde{G}$. Let $H'$ be obtained from $H$ by inflating, for each $j \in \{1, 2, \ldots, \omega\}$, every signed edge $s$ contained in $V_j$, to $V_j$. Let $G'$ be the underlying hypergraph of $H'$. We show that $G'$ contains no $\beta$-cycle. Assume for a contradiction that $G'$ contains a $\beta$-cycle $C'$. Consider first the case where $C'$ does not contain any set $V_j$ as an edge. Then $C'$ is a $\beta$-cycle in $G$ not contained in any connected component of $\tilde{G}$, which gives us a contradiction. Next, consider the case where $C'$ contains at least one set $V_j$ as an edge. We now show how we can modify $C'$ to obtain a $\beta$-cycle $C$ in $G$, not contained in any connected component of $\tilde{G}$, which gives us again a contradiction. Assume that $C'$ contains the edge $V_j$, and let $u, v$ be the nodes in $C'$ before and after $V_j$. We then define $C$ by replacing, in $C'$, the edge $V_j$ with a minimal sequence $f_1, w_2, \ldots, w_t, f_t$, where $f_j$ are edges in $E_j$ and $w_j$ are nodes in $V_j$, such that $u \in f_1$ and $v \in f_t$. Note that this sequence exists because $(V_j, E_j)$ is connected. Clearly, nodes $w_2, \ldots, w_t$ and edges $f_1, \ldots, f_t$ did not belong to $C'$. It is simple to check that applying recursively the above construction to all sets among $V_1, \ldots, V_\omega$ contained in $C'$ yields a $\beta$-cycle in $C$ not contained in any connected component of $\tilde{G}$, which gives us a contradiction. We have therefore shown that $G'$ is $\beta$-acyclic.

By Lemma 3 and assumption (59), an extended formulation for PBP($H$) can be obtained by juxtaposing an extended formulation for PBP($H'$) together with at most $\cup_{C\in\mathcal{C}}|E[C]| \leq |S|$ equalities, containing at most $\cup_{C\in\mathcal{C}}|E[C]| \leq |S|$ additional variables. Now consider the signed hypergraph $H' = (V, S')$. We then have $|S'| \leq |S| - \sum_{i\in[\omega]} |E_i| + \sum_{i\in[\omega]} 2^k|E_i| \leq 2^k|S|$. Since the underlying hypergraph of $H'$ is $\beta$-acyclic, by Corollary 5, PBP($H'$) has an extended formulation with at most $2r|V|(2^{k+1}|S|+1)$ variables, and at most $4r|V|(2^{k+1}|S| + 1)$ inequalities. □

**Remark 5.** *Under more restrictive assumptions, the removal of $\beta$-cycles can be performed in a more efficient manner than the technique in the proof of Proposition 4. Two $\beta$-cycles are* equivalent *if one can be obtained from the other via a circular permutation and/or reversing the order of the edges. Now consider the case where all beta-cycles in each $(V_i, E_i)$, $i \in [\omega]$ are equivalent. Then for each $i \in [\omega]$, it suffices to inflate one edge of maximum cardinality in $E_i$, denoted by $f_i$, to $V_i$. That is, in this case, one can replace assumption* (59) *by the weaker assumption*

$$\text{gap}(\{f_i, V_i\}) \leq k, \quad \forall C \in \mathcal{C}.$$

### 5.4 A more general framework

So far, all inflation operations we employed to obtain polynomial-size extended formulations (i.e., Theorem 6, and Propositions 2–4) have a simple form: we inflated a number of signed edges to the same set. This in turn enabled us to use Theorems 1 and 2 to obtain our extended formulations. It is important to remark that the inflation of signed edges can be useful in a more general setting. Consider a signed hypergraph $H = (V, S)$. Assume that $S$ is not totally ordered. Let $\mathcal{E}$ denote a set of subsets of $V$ that is totally ordered. Moreover suppose that for each $s \in S$, there exists some $e \in \mathcal{E}$ such that $e \supseteq s$. Now inflate each $s \in S$ to some element in $\mathcal{E}$, which we denote by $e(s)$. Define

$$I(S, \mathcal{E}) := \bigcup_{s \in S} I(s, e(s)).$$

Define the signed hypergraph $H' = (V, I(S, \mathcal{E}))$. By Part 2 of Remark 1, the pseudo-Boolean polytope $\text{PBP}(H')$ has a polynomial-size extended formulation in $|V|, |I(S, \mathcal{E})|$. Therefore, from Lemma 3 we deduce that if $|I(S, \mathcal{E})|$ is a polynomial in $|V|, |S|$, we can obtain a polynomial-size extended formulation for $\text{PBP}(H)$ as well. The following proposition is an illustration of how this general setting can be used to obtain a polynomial-size extended formulation for the pseudo-Boolean polytope.

**Proposition 5.** *Consider a signed hypergraph $H = (V, S)$. Let $S = S_1 \cup S_2$, such that each $s \in S_1$ contains at least $|V| - k_1$ nodes, whereas each $s \in S_2$ is contained in $U \subset V$, where $|U| = k_2$. Then PBP$(H)$ has an extended formulation with $O(2^{\max\{k_1,k_2\}}|S||V|)$ variables and inequalities. In particular, if $\max\{k_1, k_2\} = O(\log \text{poly}(|V|, |S|))$, then PBP$(H)$ has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP$(H)$ are $0, \pm 1$.*

*Proof.* Consider the signed hypergraph $H' = (V, S')$ obtained from $H$ by inflating every $s \in S_1$ with $|s| < |V|$ to $V$, and by inflating every $s \in S_2$ with $|s| < |U|$ to $U$. We then have $|S'| \leq 2^{k_1}|S_1| + 2^{k_2}|S_2| \leq 2^{\max\{k_1,k_2\}}|S|$. The remainder of the proof is identical to the proof of Proposition 2. □

We conclude the paper by remarking that while we presented several classes of signed hypergraphs for which our proposed RID framework provides polynomial-size extended formulations for the corresponding pseudo-Boolean polytopes, a complete characterization of such signed hypergraphs remains an open question and is a subject of future research.

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